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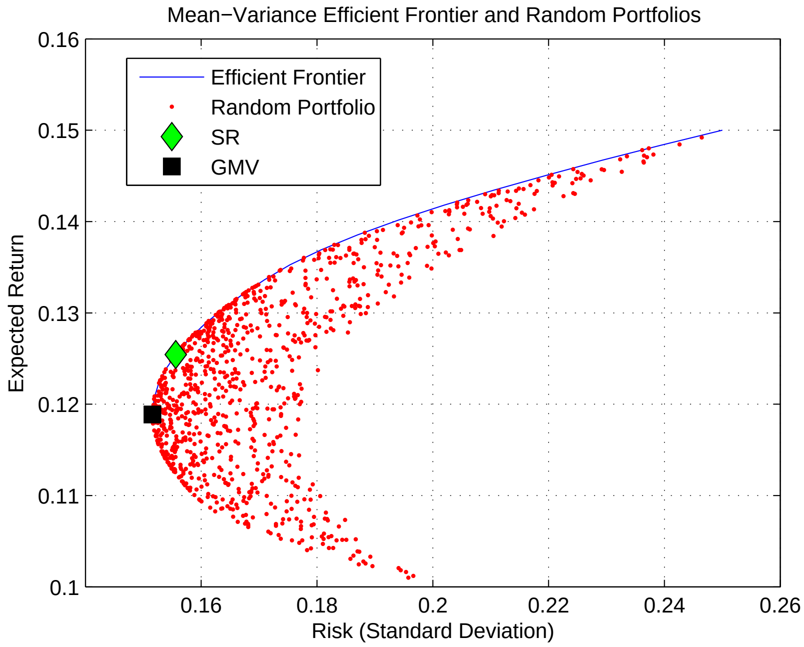
Stability of Traditional Portfolio Models



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Thesis for the Degree of Master of Science

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Abstract

The idea behind this thesis came from, at that time, a fellow colleague at SEB. They had evaluated the resampling model when doing portfolio optimizations and knew that it was more stable than the original model by Markowitz. What they hadn't studied, was to what extent and also the instability of the resampling model itself. Therefore the research questions for this thesis were

- Research question 1: How sensitive are the two models' optimal portfolio weights to changes in expected return, risk and correlation.
- Research question 2: How does the inherit portfolio characteristic affect the results of research question 1.

To study these questions reference portfolios were derived based on historical asset returns of several multi asset portfolios at different risk levels. To investigate the instability of the models the input parameters were stressed in different combinations to see how the portfolios weights changed.

In short, the Markowitz model was more unstable then the resampling model, the dispersion increases with portfolio risk level and the expected return parameter is the parameter that has the largest impact on stability for both models. A few exceptions was seen on the upper end of the risk scale. The results in this study confirms the result of previous studies but also challenges other.

Acknowledgements

I would like to express my special thanks to my supervisor at SEB, Fredrik Sundvall, who provided this research area based on thoughts he had in his daily work. He has also supported me with valuable insights and ideas along the progress of the thesis. Further, he has also reminded me to continue and finishing, despite all delays, which has been really helpful.

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Finally, Carl Lindberg has also been helpful and provided his insights on my research topic and provided me with several articles for my literature study.

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1 Introduction

In this introductory section, a background and a problem discussion of the thesis are presented in order to let the reader get acquainted with the problem. Thereafter, the problem formulation with limitations and purpose are presented. Finally, a subsection regarding the outline of the rest of the thesis is presented.

1.1 Background and problem discussion

According to Harry Max Markowitz, Nobel laureate in economics, the process of selecting portfolios is divided into two stages. The first stage focuses on observations, experiences and future performance of available securities. While the second stage focuses on relevant beliefs about future performance and portfolio selection. In 1952, Markowitz's paper Portfolio Selection, was published in the Journal of Finance. The paper deals with the second stage in the process of selecting a portfolio. It is based on the rule that an investor considers expected return as something desirable and variance of the returns undesirable, and that the investor does not seek only to maximize anticipated returns. This is called the "expected returns-variance of returns" rule according to Markowitz [18]. Developing the ideas in the paper further, Markowitz wrote a book, Portfolio Selection: Efficient Diversification of Investments, published in 1959, giving more thoughts, insights and examples [19].

From Markowitz's pioneering research, a new concept in portfolio management was introduced. The investors should focus on selecting optimal portfolios as opposed to optimal assets. He was the first to consider risk alongside return in portfolio management. For a given level of expected return, μ, the risk is minimized when the covariance of asset returns, Σ, within the portfolio is minimized. This is known as the mean-variance theory which is a single-period theory for the decision of portfolio weights which provides an optimal trade-off between return and risk of the portfolio [19]. Markowitz's work pioneered the modern portfolio theory and according to many people, this was the first major breakthrough in the field of modern financial theory and can be regarded as one of the foundational theories in financial economics. Merton continued on Markowitz's work and showed that the set of all optimal portfolios with respect to risk and return is lying on a parabola in the mean-variance space. The upper part of this parabola is well known under the name, the efficient frontier [20].

Markowitz's seminal work provides an insight into the early thinking and development of portfolio management and theory. Even today, it is a strong reference for individuals and financial institutions selecting optimal portfolios. The work also provided the foundation for financial and computational economics, and the basis for other important financial concepts such as the capital asset pricing model, efficient markets hypothesis, and behavioural finance [21]. The importance of Markowitz's work can be summarized in the Swedish Riksbank's prize in economic Science in memory of Albert Nobel which he received in 1990 and shared equally with Merton H. Miller and William F. Sharpe with the motivation: **"for their pioneering work in the theory of financial economics"** [29].

However, even if Markowitz's work has been met with a lot of appreciation there is a lot of criticism as well. Potentially the toughest comes from Nassim Nicholas Taleb, author of the famous book Fooled by Randomness and The Black Swan - The impact of the Highly Improbable among other books. In these books, Taleb blames portfolio theory for a lot of problems in the financial history, and he wants, if possible, to ban portfolio theory [28]. The more composed critique however consists of problems in estimating the parameters in the model, expected return ($\boldsymbol{\mu}$) and covariance ($\boldsymbol{\Sigma}$). In Markowitz's model these parameters are assumed to be known. Since this is usually not the case in real life, the parameters are estimated from historical data assuming that the returns are i.i.d. and normality. The maximum likelihood estimates (MLE) of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are the sample mean, $\hat{\boldsymbol{\mu}}$, and the sample covariance matrix, $\hat{\boldsymbol{\Sigma}}$. These are also method-of-moments estimates without the normality and the i.i.d. assumption is replaced by weak stationarity [7]. The problem is that is difficult to estimate the parameters with sufficient accuracy from historical data, due to noise in the data among other problems [16]. Several researchers have studied the effect of errors in means and covariances as well as the sensitivity with respect to changes in optimal portfolio weights. The conclusion is that the model may perform poorly when $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are replaced with their sample counterparts, $\hat{\boldsymbol{\mu}}$ and the sample covariance matrix $\hat{\boldsymbol{\Sigma}}$ [3] [7] [8].

A major direction within research in portfolio theory has therefore been to find other, better performing estimators. Different approaches have been proposed to solve the problems e.g. Bayes and shrinkage estimators, multifactor models, the Black-Litterman model and Michaud's resampling model to name a few [5] [7] [21]. Both Bayes and shrinkage estimators can be derived for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, accounting for uncertainties respectively shrinking the MLE of the covariance matrix. The multifactor model reduces the dimension in estimating $\boldsymbol{\Sigma}$ and using e.g. arbitrage pricing theory expected returns can be calculated. The Black-Litterman model introduces a possibility for subjective opinions, the investor can incorporate his/her own views on assets. The resampling model is attempting to handle the "Markowitz optimization enigma" by incorporating sampling variability of $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ via bootstrap.

Another explanation to why the MLE of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ may perform poorly is the assumption of normality of the return distribution. The efficiency of MLE, based on the normality assumptions, are highly sensitive to deviations from the assumed distribution, even if the deviations are small. In portfolio theory this is important since numerous studies have shown that asset returns are not normally distributed. These problems may be tackled by introducing robust estimators which gives meaningful information even when the returns deviates from the assumed normality distribution [10].

There are a lot of things which can be studied within portfolio theory but in this thesis the focus is to study the sensitivity of the weights in Markowitz's original model and Michaud's resampling model in the same manner. Finally the two models will be compared and their differences analysed.

1.2 Problem statement

In this report, the focus is to analyse the sensitivity of the two models primarily discussed in the previous section, Markowitz and Michaud's resampling model, with respect to changes in the three parameters; expected return, risk and correlation. By isolating changes in one parameter at a time it is possible to see what parameter has the largest impact on the optimization process. The analysis is done for several portfolios at different risk levels and also consisting of different asset classes to find out how the inherit characteristic affects the analysis. To summarize, the main research questions are

- Research question 1: How sensitive are the two models' optimal portfolio weights to changes in expected return, risk and correlation.
- Research question 2: How does the inherit portfolio characteristic affect the results of research question 1.

1.3 Purpose

Today the resampling model is one of the portfolio optimization models which has been continuously evaluated at Skandinaviska Enskilda Banken (SEB). However SEB has not done any in-depth analysis of the model itself. Previous studies such as Best and Grauer which have studied the sensitivity of portfolio model have not been focusing on the impact of each parameter while other studies such as Yarema and Schmid which have focused more on parameter impact have only used portfolios with few asset classes [3] [25]. Using more assets is an attempt to closer replicate a trading portfolio compared to a more simple academical portfolio[1] used in Campbell R. H. [6]. Therefore this analysis provides additional information to the once already published. Other studies such as Chopra and Ziemba studied the effect of each parameter but based their analysis on how each parameter was stressed instead of the looking at different risk levels [8].

According to Michaud's reasoning behind the resampling model it should be less sensitive to changes compared to the Markowitz model and with better performance [21]. Kohli agrees with the stability while he cannot confirm the performance [15]. Becker et al. on the other hand argue that the Markowitz model is superior to the resampling model [2].

In addition, these results can be used to act as guidelines for further sensitivity analysis for future portfolios. In portfolio theory it is common to use parameter estimates based on assumptions on the future. This results of this thesis can therefore be used how weights depends on erroneous estimations at different risk levels and input parameters.

[1] Here the meaning of academical portfolio is a simple portfolio with few assets

1.4 Limitations

In this analysis the portfolios are limited to nine assets each to represent a true aggregated portfolio. However in reality, a real trading portfolio may consist of many more assets depending on structure and strategy. This study is also limited to the different cases described within the thesis. Depending on the results of research question 2 it might be hard to draw general conclusions but the results are at least indicative. Simulations were done for both the unconstrained and constrained models but the focus will be on the constrained model. There are two reasons for this, first, SEB advises mainly clients with long-only strategic allocation. Second, the resampling model is primarily to be used in a constrained fashion [22].

1.5 Thesis outline

The rest of the thesis is outlined as follows, in chapter two the theoretical framework for the thesis is presented, focusing on portfolio theory. Positive and negative aspects will be highlighted and explained. In chapter three the methodology is presented, such as data description and the empirical procedure and other relevant information for understanding how the analysis were done. In chapter four the results from the simulations will be presented and discussed. In the last chapter, chapter five, the findings of the thesis will be summarized and the conclusions outlined and suggestions for further analysis.

2 Theory

In this chapter the theoretical aspects of the mean-variance theory and the Michaud's resampled efficient frontier are covered. Within each subject various subtopics will be introduced and explained.

2.1 Mean-variance theory

In this section the mean-variance framework is presented in three parts. First, an introduction to the framework, then some of the criticism towards the model is presented and finally an analytical sensitivity analysis.

2.1.1 Introduction to Markowitz mean-variance framework

Markowitz presented his ideas in his paper, Portfolio Selection, and the book, Portfolio Selection: Efficient Diversification of Investments, which is based on the article but more extensive, as mentioned earlier. Markowitz suggestion is that investors should not seek to find optimal assets, they should seek the optimal portfolios instead. He considered both risk and return in his approach, earlier only return was considered [18]. In Markowitz's model the goal is to optimize the investors' utility function[2] at a given level of risk aversion [19]. The resulting portfolio is the investor's mean-varianceportfolio, i.e. optimal portfolio in relation to return and risk. Before the model is introduced some important concepts and formulas are explained.

First, assume that a portfolio with N assets is available and a fraction, w_i, of the total wealth, is invested in each asset with an expected return, $\mathrm{E}[R_i]$. The expected return on

[2]The utility unit is a economic measurement of the total satisfaction received where more is better introduced by Daniel Bernoulli, for more information see e.g. the St. Petersburg Paradox. The utility function is nowadays used in microeconomics and the Theory of the Consumer

the portfolio, $E[R_p]$, is a linear combination of the individual expected asset returns [18]

$$E[R_p] = \sum_{i=1}^{N} w_i E[R_i] = \sum_{i=1}^{N} w_i \mu_i = \mathbf{w}^T \boldsymbol{\mu},$$

where $\boldsymbol{\mu}$ is the expected return vector for all assets. The portfolio risk is measured via the variance (or standard deviation) of the return on the portfolio [18]

$$Var[R_p] = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \sigma_{ij} w_j = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} = \sigma_p^2,$$

where $\boldsymbol{\Sigma}$ is the $N \times N$ covariance vector with elements

$$\sigma_{ij} = E\Big[(R_i - \mu_i)(R_j - \mu_j)\Big].$$

With expected returns and covariances of the assets, Markowitz's mean-variance model, optimizes the weights for an optimal portfolio with respect to risk and retur. Mathematically the model can be formulated as

$$\max_{\mathbf{w}} \quad U(\mathbf{w}) = \mathbf{w}^T \boldsymbol{\mu} - \frac{\lambda}{2} \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \quad \text{s.t} \quad \mathbf{w}^T \mathbf{1} = 1, \tag{1}$$

where $U(\mathbf{w})$ denotes the utility function, λ, the investors risk aversion, $\mathbf{1}$, a vector with all elements equal to one and $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as before. The constraint ensures that the total wealth is invested, neither more or less. Note that global minimum variance (GMV) portfolio is the solution to the following optimization problem [25]

$$\min_{\mathbf{w}} \quad \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \quad \text{s.t.} \quad \mathbf{w}^T \mathbf{1} = 1. \tag{2}$$

The risk aversion parameter, λ, is a measure of how much more risk the investor is willing to be exposed to in order to increase the expected return. A larger λ in Equation 1 implies a more risk averse investor. Within the resulting portfolio the GMV can be identified and a more risk-averse investor will invest a larger portion in his/her wealth in the GMV portfolio. An analytical solution to the GMV problems is also available and is presented in Equation 3 [25]

$$\mathbf{w}_{GMV} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{\Sigma}^{-1}\mathbf{1}}. \tag{3}$$

In both Equations 1 and 2, short-selling is allowed. To prevent this, an additional constraint is included in the model, not allowing any portfolio weight to be less than zero. Equation 4 represents Equation 1 with the additional no short-selling constraint

$$\max_{\mathbf{w}} \quad U(\mathbf{w}) = \mathbf{w}^T\boldsymbol{\mu} - \frac{\lambda}{2}\mathbf{w}^T\mathbf{\Sigma}\mathbf{w} \quad \text{s.t} \begin{cases} \mathbf{w}^T\mathbf{1} = 1 \\ w_i \ \geq \ 0 \ \forall \ i, \end{cases} \tag{4}$$

the same can be done for Equation 2.

Other constraints which might be useful in a practice are constraints controlling the weight of any investments not to be larger (or smaller) than a predetermined level, e.g. $w_i \leq f_i$ $\forall \ i$.

The risk aversion parameter, λ, can be somewhat hard to quantify and understand. A more useful approach to Markowitz's model and Equation 1 can be formulated excluding λ, and seek to minimize the risk for a given level of portfolio return, R_p, or maximize the return at a given level of portfolio risk (expressed as variance), σ_p^2, expressed in Equations 5 and 6

$$\min_{\mathbf{w}} \quad \mathbf{w}^T\mathbf{\Sigma}\mathbf{w} \quad \text{s.t} \begin{cases} \mathbf{w}^T\boldsymbol{\mu} = R_p \\ \mathbf{w}^T\mathbf{1} = 1, \end{cases} \tag{5}$$

$$\max_{\mathbf{w}} \quad \mathbf{w}^T\boldsymbol{\mu} \quad \text{s.t} \begin{cases} \mathbf{w}^T\mathbf{\Sigma}\mathbf{w} = \sigma_p^2 \\ \mathbf{w}^T\mathbf{1} = 1. \end{cases} \tag{6}$$

From a mathematical optimization perspective there is a difference between Equations 5 and 6. That is in Equation 5 the objective function is nonlinear and the constraints are linear while in Equation 6 the objective function is linear but there are nonlinear constraints.

Another important and popular measurement of portfolio and asset performance is the Sharpe ratio (SR) which has its origin in Markowitz framework. The SR is a risk-adjusted performance measurement, measuring the best return to risk ratio. It is found by solving the following optimization problem[3] [25]

$$\max_{\mathbf{w}} \quad \frac{\mathbf{w}^T \boldsymbol{\mu}}{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}} \quad \text{s.t.} \quad \mathbf{w}^T \mathbf{1} = 1,$$

with the analytical solution

$$\mathbf{w}_{SR} = \frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}.$$

A portfolio derived from Markowitz optimization is called optimal or efficient since the portfolio has the best expected return at the each risk level. Thinking in terms of Equation 5, changing R_p (or λ in Equation 1) a new efficient portfolio can be found. Merton showed that the set of all efficient and feasible portfolios are lying on a parabola, the efficient frontier [20]. In Figure 1, an example of the efficient frontier is shown together with random portfolios with arbitrary weights. Also, the GMV and SR-portfolios are shown. The GMV-portfolio is obviously all the way to the left of the risk-scale. The SR-portfolio is placed where the reward to risk ratio is highest.

Mean-variance optimization works well in theory, however it is assumed that both the expected return, $\boldsymbol{\mu}$, and the covariance matrix, $\boldsymbol{\Sigma}$, are known and correct. In practice this is not the case, therefore these two parameters have to be estimated. The usual and easiest way is to use historical data and approximate the parameters with their MLE, the sample mean, $\hat{\boldsymbol{\mu}}$, and sample covariance matrix, $\hat{\boldsymbol{\Sigma}}$. Other more sophisticated approaches were

[3] Assuming the risk-free interest rate is 0.

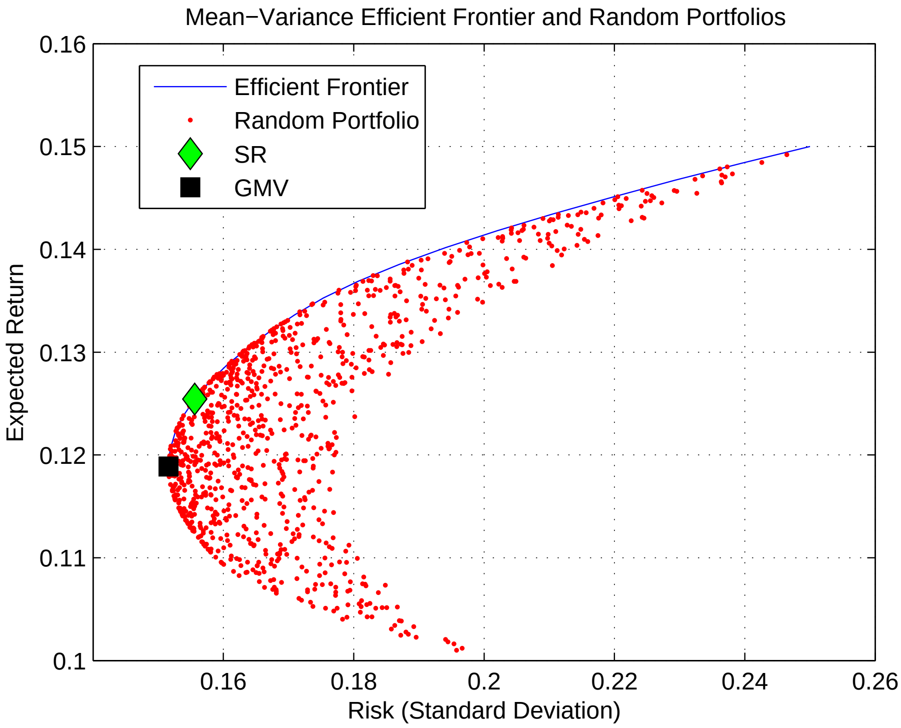

Figure 1: Illustration of Markowitz mean-variance efficient frontier and GMV and the maximum SR portfolios.

mentioned earlier in the background. At last, it is worth mentioning that previous studies e.g. Fusai and Roncoroni [12], stresses the fact that mean-variance optimization will overweight assets with large expected return, low variance and low correlation. Unfortunately, these assets are probably those which will be mostly affected by estimation errors.

2.1.2 Criticism of Markowitz mean-variance framework

There have been numerous studies showing flaws and problems with the framework. In this section the criticism has been divided into four different areas: non variance risk measure, utility function optimization, multiple investment horizons and instability.

Non variance risk measure: There are many different ways to measure risk, in finance the usual way is to use the variance or standard deviation of asset returns. For stocks, this approach is well used in the industry, however for portfolios it is not a uniformly accepted risk measurement. Since the Markowitz model is based on variance as a risk measurement, the critique is indirectly towards the model. One suggested risk measurement is the semivariance, a measurement of downside risk. It is similar to variance but only considers observations below the mean or a specified target level.

$$Semivariance = \frac{1}{N} \sum_{r_t < M}^{N} (M - r_t)^2,$$

where N is the number of observations, r_t, which are below the "target level", M [22]. Other important measurements are mean-absolute deviation and range measures. Pros and cons of a risk measure is dependent on the nature of the return distribution. The more asymmetric the return distribution is, the more different the efficient frontiers will be. Instruments like options and asset classes like fixed income are not as symmetric as equity returns [22].

Utility function optimization: Markowitz's model is not generally optimizing expected utility, only under either of two conditions, normally distributed asset returns or quadratic utility functions. There are numerous studies proving that asset returns are not normally distributed. In some cases the returns are symmetrical but not normally distributed. The limitation of a quadratic utility function is that it is not a monotone increasing function of wealth. This is a problem when representing investor behavior since the quadratic function has a maximum and thereafter decreases as a function of increasing wealth which is not the case of a rational investor [22].

Multiple investment horizons: The basic Markowitz model is only a one-period model while the majority of investors have a longer time horizons for their investments. Markowitz showed that the mean-variance efficient portfolios does not need to be efficient in the long-run. They work better in the short-run or single period [19]. However, one solution might be to reallocate the portfolio.

Instability: From empirical studies, mean-variance portfolios have shown to be quite sensitive to changes in input parameters. One study which had a large impact was On the sensitivity of Mean-Variance-Efficient Portfolios to changes in Asset Means: Some Analytical and Computational results by Best and Grauer [3]. They investigated the sensitivity of mean-variance efficient portfolios to changes in the asset means of individual assets and found that with only a budget constraint implemented, the portfolio's mean, variance and weights all were sensitive to changes in asset means. When a no short-selling constrained were implemented as well, weights were sensitive but not portfolio mean and variance. Chopra and Ziemba [8], argue that "errors in means are 10 and 20 times as damaging as error in variance and covariance respectively." Palczewski and Palczewski [26] on the other hand shows that errors due to error in the covariance estimation account for 20 % - 30 % of the variability of the portfolio weights and up to 50 % in practically important cases.

2.1.3 Sensitivity analysis of Markowitz mean-variance framework

Starting with Equation 5, the goal is to derive an analytical solutions for how the weights changes when the input data, the expected return, $\boldsymbol{\mu}$, and the covariance matrix $\boldsymbol{\Sigma}$, are stressed. First, changes in $\boldsymbol{\mu}$ is studied, define $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\mu} + \boldsymbol{q}$ where $\boldsymbol{q}$ is a vector corresponding to the change in the expected return with the same dimensions as $\boldsymbol{\mu}$, this implies $\widetilde{\boldsymbol{\mu}}, \boldsymbol{q}, \boldsymbol{\mu} \in \Re^{n\times 1}$ and $\boldsymbol{\Sigma} \in \Re^{n\times n}$, which is positive-definite symmetric. Assume that the objective function in Equation 5 is divided with two, without loss of generality. Using Lagrange multipliers[4] a number of equations can be derived

$$L = \frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w} - \lambda(R_p - \mathbf{w}^T\widetilde{\boldsymbol{\mu}}) - \gamma(1 - \mathbf{w}^T\mathbf{1}),$$

with the following first partial derivatives

$$\begin{cases} \dfrac{\partial L}{\partial \mathbf{w}} &= \boldsymbol{\Sigma}\mathbf{w} - \lambda\widetilde{\boldsymbol{\mu}} - \gamma\mathbf{1} = 0, \\ \dfrac{\partial L}{\partial \lambda} &= R_p - \mathbf{w}^T\widetilde{\boldsymbol{\mu}} = 0, \\ \dfrac{\partial L}{\partial \gamma} &= 1 - \mathbf{w}^T\mathbf{1} = 0. \end{cases} \tag{7}$$

Use the first partial derivative of Equation 7 and solving for $\mathbf{w}$

$$\begin{aligned} \mathbf{w} &= \boldsymbol{\Sigma}^{-1}(\lambda\widetilde{\boldsymbol{\mu}} + \gamma\mathbf{1}) \\ &= \boldsymbol{\Sigma}^{-1}(\lambda\boldsymbol{\mu} + \lambda q + \gamma\mathbf{1}). \end{aligned} \tag{8}$$

[4]Consult a textbook in Optimization if needed, e.g. An Introduction to Continuous Optimization by Andréasson et al. [1].

Using the following substitutions

$$\begin{cases} A = & \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \\ \widetilde{A} = & \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{q}, \\ B = & \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \\ \widetilde{B} = & \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{q}, \\ C = & \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}, \\ Q = & \boldsymbol{q}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{q}, \end{cases} \tag{9}$$

multiplying Equation 8 with $\tilde{\boldsymbol{\mu}}$ and $\mathbf{1}$ respectively and comparing to the two other partial derivatives from Equation 7 gives

$$\begin{cases} \mathbf{w}^T\tilde{\boldsymbol{\mu}} = & R_p = & \lambda(B + 2\widehat{B}) + \gamma(A + \widehat{A}) + \lambda Q, \\ \mathbf{1}^T\mathbf{w} = & 1 = & \lambda(A + \widehat{A}) + \gamma C. \end{cases}$$

which can be formulated in the following system

$$\begin{bmatrix} R_p \\ 1 \end{bmatrix} = \begin{bmatrix} B + 2\widehat{B} + Q & A + \widehat{A} \\ A + \widehat{A} & C \end{bmatrix} \times \begin{bmatrix} \lambda \\ \gamma \end{bmatrix},$$

which is solved for γ and λ. The result can then be substitute into Equation 8

$$\begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \frac{1}{\Psi} \begin{bmatrix} C & -A - \widehat{A} \\ -A - \widehat{A} & B + 2\widehat{B} + Q \end{bmatrix} \times \begin{bmatrix} R_p \\ 1 \end{bmatrix},$$

where $\Psi = C(B + 2\widehat{B} + Q) - (A + \widehat{A})^2$ and assuming $\Psi \neq 0$ the following equation is derived

$$\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}}{\Psi} \left(R_p \left(C(\boldsymbol{\mu} + \boldsymbol{q}) - (A + \widehat{A})\mathbf{1} \right) - (A + \widehat{A})(\boldsymbol{\mu} + \boldsymbol{q}) + (B + 2\widehat{B} + Q)\mathbf{1} \right). \tag{10}$$

To find the the sensitivity, take the partial derivative of Equation 10 with respect to $\boldsymbol{q}$. Before that some important statements are introduced which simplifies the calculations. Show that $\mathbf{\Sigma}^{-1} = (\mathbf{\Sigma}^{-1})^T$ is true, because of symmetry we know

$$
\begin{aligned}
\mathbf{\Sigma} &= \mathbf{\Sigma}^T \\
(\mathbf{\Sigma}^{-1})^T\mathbf{\Sigma}\mathbf{\Sigma}^{-1} &= (\mathbf{\Sigma}^{-1})^T\mathbf{\Sigma}^T\mathbf{\Sigma}^{-1} \\
\mathbf{\Sigma}^{-1} &= (\mathbf{\Sigma}^{-1})^T \\
\mathbf{\Sigma}^{-1} &= (\mathbf{\Sigma}^{-1})^T \qquad \square.
\end{aligned}
$$

Taking the partial derivative, $\partial/\partial\boldsymbol{q}$, of the equations in Equation 9, $\hat{A}$,$\hat{B}$ and Q all depends on $\mathbf{w}$ and results in

$$
\begin{cases}
\dfrac{\partial \widehat{A}}{\partial \boldsymbol{q}} = & \mathbf{\Sigma}^{-1}\mathbf{1}, \\
\dfrac{\partial \widehat{B}}{\partial \boldsymbol{q}} = & \mathbf{\Sigma}^{-1}\boldsymbol{\mu}, \\
\dfrac{\partial Q}{\partial \boldsymbol{q}} = & 2\mathbf{\Sigma}^{-1}\boldsymbol{q}.
\end{cases}
$$

Note that for a given function, F, dependent on $\boldsymbol{q}$

$$
\frac{\partial}{\partial \mathbf{q}}\left(F(\mathbf{q})\mathbf{v}(\mathbf{q})\right) = \mathbf{v}(\mathbf{q})\nabla F(\mathbf{q})^T + \nabla\mathbf{v}(\mathbf{q})F(\mathbf{q}), \tag{11}
$$

and hence the special case

$$
\frac{\partial}{\partial \mathbf{q}}\left(F(\mathbf{q})\mathbf{q}\right) = \mathbf{q}\nabla F(\mathbf{q})^T + \mathbb{I}^{n\times n}F(\mathbf{q}),
$$

where $\nabla \equiv \partial/\partial\mathbf{q}$ and $\mathbb{I}^{n\times n}$ is the identity matrix. Further, denoting $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ and using the fact that $(\nabla\mathbf{v})_{ij} = \partial v_i/\partial q_j$. With the gradients of $\hat{A}$, $\hat{B}$ and Q given earlier, the chain rule implies that the sensitivity of Equation 10 with respect to $\boldsymbol{q}$ is equal to

$$
\begin{aligned}
\frac{\partial \mathbf{w}}{\partial \mathbf{q}} &= \Psi^{-1}\Sigma^{-1}\left[R_p(C\mathbb{I}^{n\times n} - \mathbf{1}\mathbf{1}^T\Sigma^{-1}) - (\mathbf{q}+\boldsymbol{\mu})\mathbf{1}^T\Sigma^{-1}\right. \\
&\quad \left. -(A+\hat{A})\mathbb{I}^{n\times n} + 2\mathbf{1}(\mu+\mathbf{q})^T\Sigma^{-1}\right] \\
&\quad -2\mathbf{w}\Psi^{-1}\left[C(\mathbf{q}+\mu)^T - (A+\hat{A})\mathbf{1}^T\right]\Sigma^{-1},
\end{aligned}
$$

where the expressions of $\mathbf{w}$ and Φ can be inserted.

Earlier, it was assumed that $\Psi \neq 0$. To ensure that this assumption is fulfilled it can be expanded as

$$
\Psi = \mathbf{1}^T\Sigma^{-1}\left[\mathbf{1}(\mathbf{q}+\mu)^T - (\mathbf{q}+\mu)\mathbf{1}^T\right]\Sigma^{-1}(\mathbf{q}+\mu).
$$

Denoting $S = \mathbf{1}(\mathbf{q}+\mu)^T$ the following equation is given

$$
\Psi = \mathbf{1}^T\Sigma^{-1}\left[S - S^T\right]\Sigma^{-1}(\mathbf{q}+\mu),
$$

and $S - S^T$ is a skew matrix and it is known that if, n is odd, then $\det(S - S^T) = 0$ hence the null space is not the trivial set. However, it is not clear whether $\Sigma^{-1}(\mathbf{q}+\mu)$ is in the null space since S is a function of the vector. It is however clear that if $S - S^T = 0^{n\times n}$ then $\Psi = 0$ and the problem is not uniquely solvable. So, $S - S^T \neq 0^{n\times n}$ gives $q_i + \mu_i \neq k, \forall i$ where $k \in \mathbb{R}$ is the same constant for all i. This implies $\mathbf{q}+\mu \neq k\mathbf{1}$, or that all assets have the same expected return. If all assets have the same initial return and being stressed in the same way the weights would not change and $\Psi = 0$. To clarify, if all assets would have the same return then $S - S^T = 0$ and the problem would not be solvable.

2.2 The Michaud resampled frontier

In this section, Michaud's resampled frontier is presented in three sections. First, an introduction to the framework followed by some of the advantages and disadvantages. Finally, the underlying statistical properties the model is based on.

2.2.1 Introduction to the resampled frontier

Michaud invented the so called resampled efficiency method as an answer to some of the problems with Markowitz' model, primarily the problem of instability and uncertainty in forecasting inputs [22]. The method is based on resampling the optimization in a Monte Carlo simulation procedure. The procedure of a portfolio with N assets can be summarized in the following steps [6], [12]:

Step 1: Estimate the expected return, $\hat{\boldsymbol{\mu}}$, and the covariance matrix $\hat{\boldsymbol{\Sigma}}$, for the N assets e.g. from historical data.

Step 2: Solve the optimization problem for the minimum-variance portfolio and denote the expected return for this portfolio $R_{min-variance}$

Step 3: Solve the optimization problem for the maximum return portfolio and denote the expected return for this portfolio $R_{max\ return}$

Step 4: Choose a number of discrete increments, in returns, for the frontier, k.

Step 5: Set $R_{min-variance} = a$, $R_{max\ return} = b$ and $\delta = (b-a)/k$. The front is evaluated at all δ number of returns.

Step 6: One frontier is represented by F_k, consisting of k row vectors with weights for all portfolios on the frontier. Since there is N asset, F_k is of size $k \times N$.

Step 7: Assuming that the return distribution is a multivariate normal distribution with mean return vector, $\hat{\boldsymbol{\mu}}$, and covariance matrix, $\hat{\boldsymbol{\Sigma}}$. Draw a sufficient large enough sample and estimate a new mean return vector, $\boldsymbol{\mu}^*$, and a new covariance matrix, $\boldsymbol{\Sigma}^*$.

Step 8: Use the new estimates to derive a new frontier, F_k^*, and the corresponding weights.

Step 9: According to Michaud $\hat{\boldsymbol{\Sigma}}$ and $\boldsymbol{\Sigma}^*$ are now "statistically equivalent"[5].

Step 10: Using Monte Carlo simulations, repeat **Step 7-8** S times to get S number of new frontiers.

[5]Explained below after the procedure

Step 11: To calculate the resampled weights, average over all available frontiers, F_k. Derive the resampled frontier using the resampled weights and the original estimates for mean, $\hat{\boldsymbol{\mu}}$, and the covariance matrix, $\hat{\boldsymbol{\Sigma}}$.

One concept that is very important for the resampled frontier is statistically equivalent. Michaud defines two statistically equivalent portfolios as having the same risk-reward trade-off. The portfolios are not necessarily equivalent in risk, expected return nor in the weight space [22]. As seen in Figure 2, the resampled frontier is to the right in relation to the Markowitz frontier. Since the resampled frontier uses the average weights but the original mean, $\hat{\boldsymbol{\mu}}$, and covariance, $\hat{\boldsymbol{\Sigma}}$, the weights used are not the optimal weights. Therefore, for any given expected return the risk has to be higher, the resampled frontier is therefore to the right. At the low-risk end, resampled efficient frontier portfolios are similar to their associated mean-variance efficient portfolios. As portfolio risk increases, the similarities diminish. As a consequence, the "length" of the two frontiers differs, the resampled frontier is shorter than the mean-variance frontier. Fusai and Roncoroni and many others shows or argue that resampling rules out both extreme and bad diversified portfolios. This implies that the method is implicitly providing investors with a sensible and "safer" set of portfolios, the implicit protection is increasing in the horizontal axis [12].

2.2.2 Advantages and disadvantages with the resampled frontier

Earlier studies of the resampling model have showed both on advantages and disadvantages. Below some of these are introduced and explained.

Advantages: The main advantage of Michaud's resampling model is obviously the fact that it is less sensitive to perturbations in input variables. This is the main idea behind the model and the purpose for which it was created [22]. Resampled portfolios are the result of an averaging process which makes the result more stable to small changes in input parameters. Michaud argues that a resampled portfolio is more diversified and intuitively less risky than the corresponding Markowitz portfolio. Greater diversification is proved in earlier studies e.g. by Scherer in Portfolio Resampling: Review and Critique [24]. Scherer

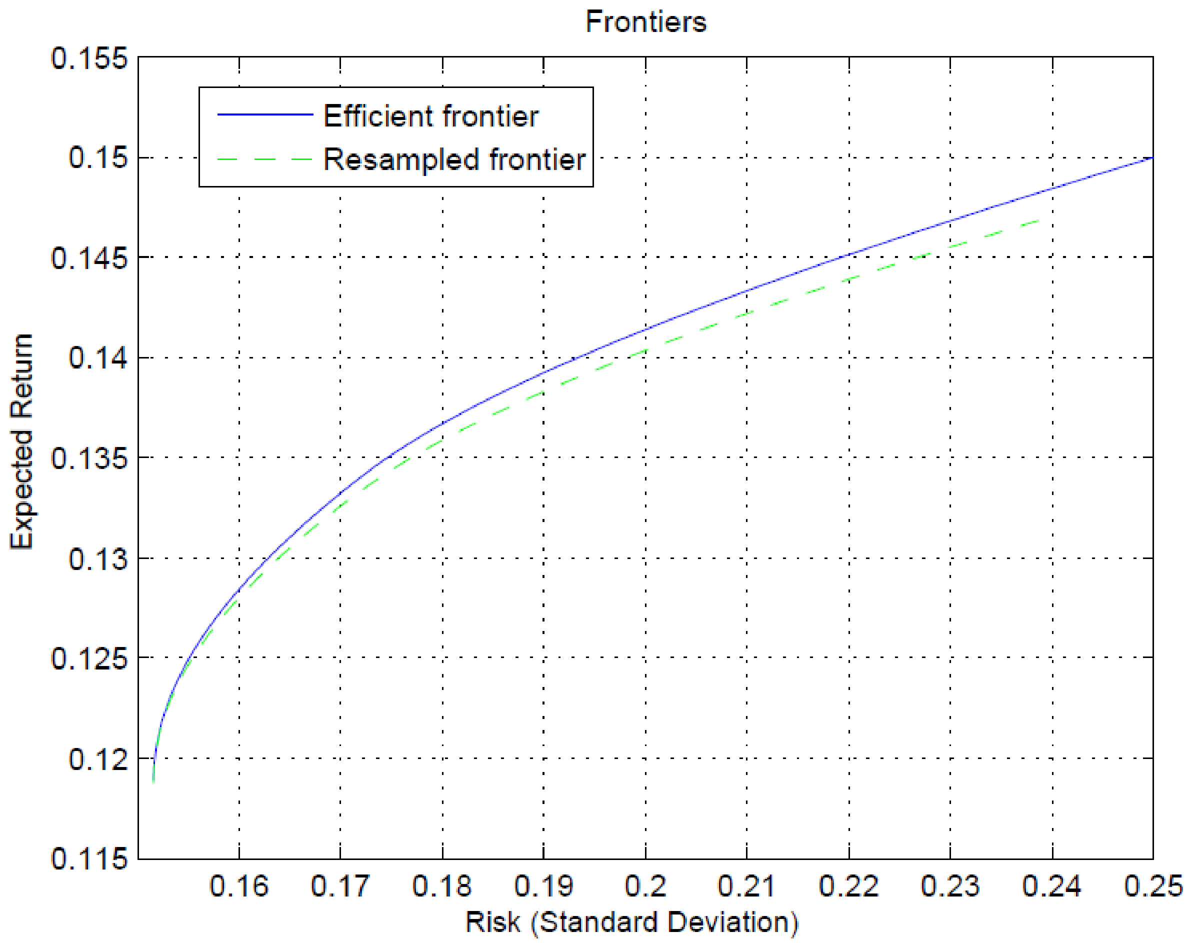

Figure 2: Comparison between Markowitz efficient frontier and the resampled frontier

also points out that diversification and less-sudden shifts in allocation changes as the given level of return changes are two aspects which appeals to practitioners.

Disadvantages: Compared to the Markowitz model, there exists no "theoretical foundation" for the resampling model. Further, there is no economic rationale derived from the optimizing behavior of rational arguments that supports this method [24]. Also, there are no theoretical arguments on why the resampled portfolio should outperform the mean-variance efficient portfolio. Hence, it is a heuristic solution to the problem. Michaud argues that the two portfolios are "statistically equivalent", however these portfolios are neither identical in risk nor return. One problem with the resampling model is that a few extreme frontiers can result in strange portfolio weights. Another example is when combining assets with different return and volatility. Varying the worst performing asset's (with respect to expected return) volatility and considering the allocation in the maximum return portfolio. Even though the asset has the lowest return, the allocation peaks in the maximum return portfolio. When the volatility increases, the corresponding result in the resampling model is that the allocation in the maximum return portfolio increases. As a result, deterioration in the risk-return trade-off, Sharpe ratio. The result however, is not derived from higher volatility leading to a higher estimation error but instead directly from averaging over long-only portfolios. This phenomenon does not arise in long-short portfolios [24].

2.2.3 Statistical properties underlying the resampled frontier

The central problem in statistical analysis is the problem of uncertainty and as a direct consequence, how to account for uncertainty. The resampling model is a way to handle the uncertainty of the estimated parameters in the Makowitz model. There are, in general, several ways to formalize uncertainty and adjust such as prior information, probability models, likelihood, standard error and confidence intervals. In simple situations, these methods can be sufficient for calculating and measuring the uncertainty of estimations. However, in more complex problems, these measures can be a lot harder to obtain and sometimes be misleading due to inappropriate assumption or simplifications [9]. Due to lack of analytical properties for the resampling model it falls under the latter category.

The resulting sub-problem is now, how to obtain reliable measures of errors in complex situations. The solution is various resampling methods and in Efron's article, Bootstrap Methods: Another Look at the Jackknife, a new revolutionizing approach was introduced [11]. What Efron did was to combine earlier ideas of resampling methods and thus establishing a new framework for simulation-based statistics. The methodology behind resampling models is to create resampled data sets from the original data, directly or via fitted models and the all statistical analysis is performed on the "new" data. A frequently used set of these methodologies are called computer-intensive methods due to repetitive process of replicating data or bootstrap method since it uses the original data to generate more data[6] [9]. There are in general two types of bootstrap methods, parametric estimation of the underlying distribution or resample from the data with replacements, non-parametric [30].

Recalling step seven in resampling model, resampling new data using the parameters estimated from historical data. This is an example of parametric bootstrap with an underlying normal distribution. Each dataset generated, is a bootstrap sample and the efficient weights are an average of all bootstrap weight vectors [7]. The entire procedure of the Michaud model is a special case of "bootstrap aggregating" [30]. Bootstrap aggregating is a statistical technique invented by Breiman and is probably more know under the acronym, Bagging [4]. This technique is useful when small changes in data can lead to significant changes in estimations. First, resample data via bootstrap, calculate the desired estimates and repeating this. Then aggregating the results and averaging hopefully yields in estimates with more stable performance then the original estimates [30]. Breiman however, showed that there is no guarantee that the resampled estimates works better [4]. These estimates can be better, worse or equal and if the technique is useful has to be determined from case-to-case [30]. The theoretical foundations for the resampling model is thin, it is based on a special case of model which is also a special case[7]. Further, there exists no analytical ways to evaluate the model, it can only be done in empirical studies [24].

[6]Recall the stories of Baron Munchahusen who used his bootstraps to pull himself out of a swamp. The technique of generating more data from a limited set is analogues to the trick used by Baron Munchahusen.

[7]Bootstrap aggregating is a special case of the model moving average approach [4]

3 Methodology

In this section the methodology is presented in four parts. First, a description of the data, then conversion of data, thereafter the empirical procedure and finally the programming code is explained.

3.1 Data description

In a normal asset allocation study, the number of assets are normally between 3-20, rarely exceeding 50. Usually a broad range of asset classes are included, such as bonds, equities (international and domestic) and commodities [22]. In this thesis, the analysis has been based on a multi asset portfolio with nine assets from different asset classes such as bonds, equities (international and domestic) and properties among others. The portfolio is well diversified, as in a normal asset allocation study. In Table 1, a presentation of the assets in the portfolio is provided. Tables 2 and 3 show additional statistics for the portfolio, descriptive statistics and correlations. The portfolio statistics are based on weekly observations between 2001-12-18 and 2010-04-14, 434 observations of asset prices.

The reasons why weekly observations were used instead of daily, monthly or annual are several. First, the portfolio should be based and evaluated on a reasonable time range, second, less frequent data the better since it minimizes the noise and third, there has to be sufficient data for estimation of the parameters. Remembering that for the covariance matrix there are $n(n-1)/2+n$ different parameters to estimate and with nine asset there are 45 parameters to estimate. The portfolio has 433 observations, there are almost 10 data points for each parameter which can be regarded as sufficient from a statistical point-of-view. It would not be possible to use less frequent data since there would not be enough data for each parameter.

From the asset prices, the simple returns were calculated and used for all other calculations. The reason for using simple returns rather than logarithmic returns is because the data contains several large daily movements, up but mostly down and that simple returns are usually used for discrete portfolio optimization. All data for both portfolios are gathered from the Bloomberg API. The prices are adjusted for capital changes defaults such as stock splits/consolidations, stock dividend/bonus and rights offerings/entitlements and all forms of cash dividends, cash dividend intraday, normal cash dividends and abnormal cash dividends.

Comparing returns and risks in Table 2 the risk may seem to be relatively higher than the return. However, if the values are annualized, see Section 3.2, the relations between the values are more intuitive and logical. This misconception depends on that the data is based on weekly observations instead of annual which is a more common way to present data.

Table 1: Presentation of assets in the portfolio

Asset	**Description**
LD12TRUU Index (LD12)	Barclays Capital 1-3 Month U.S. Treasury Bill Index.
LBUSTRUU Index (LBUS)	Barclays Capital Index represents the securities of the U.S. dollar denominated investment grade bond market
LHVLTRUU Index (LHVL)	Barclays Capital High Yield Bond Index in U.S. dollar.
GDDUWI Index (GDDU)	The MSCI World Index is a free float-adjusted market capitalization weighted index that is designed to measure the equity market performance of developed markets/countries, gross of tax total return index.
SBX Index (SBX)	The OMX Stockholm Benchmark Index is a capitalization-weighted total return index designed as an indicator of the Stockholm Exchange. This index includes only the share capital, which is freely available for trading in the market, the so called free float. Dividends are reinvested back into the index.
RUGL Index (RUGL)	The RUGL Index or FTSE EPRA/NAREIT Global Real Estate Index Series is designed to represent general trends in eligible real estate equities worldwide. Relevant real estate activities are defined as the ownership, disposure and development of income-producing real estate. The index series now covers Global, Developed and Emerging indices, as well the UK's AIM market, total return index.
SPGSLETR Index (SPGS)	S&P GSCI Light Energy Total Return, tracks the performance of a rolling basket of front-month commodity futures. Uses 1/4 of the S&P GSCI contract production weights for the energy components.
USDSEK Currency (SEK)	Foreign Exchange Rate between the US dollar and the Swedish Krona.
USDEUR Currency (EUR)	Foreign Exchange Rate between the US dollar and the Euro.

Table 2: Descriptive statistics for the portfolio based on weekly observations

Company	**# Obs**	**Mean** [10^{-1}%]	**Std. dev.** [%]	**Min** [%]	**Max** [%]
LD12	433	0.42	0.04	-0.09	0.21
LBUS	433	1.04	0.53	-1.97	1.69
LHVL	433	1.67	1.62	-11.97	7.64
GDDU	433	1.29	2.63	-20.01	12.41
SBX	433	1.80	3.20	-20.98	12.68
RUGL	433	2.54	3.19	-16.80	19.17
SPGS	433	1.33	2.74	-15.70	8.38
SEK	433	-0.75	1.74	-6.26	6.97
EUR	433	-0.87	1.35	-5.21	6.05

Table 3: Correlations within the portfolio based on returns

	LD12	LBUS	LHVL	GDDU	SBX	RUGL	SPGS	SEK	EUR
LD12	1.00								
LBUS	0.03	1.00							
LHVL	- 0.12	0.17	1.00						
GDDU	- 0.10	- 0.11	0.58	1.00					
SBX	- 0.11	- 0.17	0.47	0.83	1.00				
RUGL	- 0.08	0.01	0.57	0.84	0.67	1.00			
SPGS	- 0.02	- 0.05	0.27	0.42	0.28	0.38	1.00		
SEK	0.00	- 0.12	- 0.28	- 0.45	- 0.21	- 0.49	- 0.47	1.00	
EUR	- 0.03	- 0.23	- 0.17	- 0.31	-0.10	- 0.36	- 0.41	0.82	1.00

3.2 Data conversion

It is more convenient to express returns as “annualized returns”, even when observations are of another frequency. Multiplying return with a frequency factor, f, which depends on the frequency of the data[8], it is possible to annualize the data. For risk, standard deviation, there is a similar relationship, the risk is multiplied with square root of the frequency multiplier. Regarding correlation, there is no need to include any frequency multiplier since it is cancelled out anyway. In Equations 12-14 a mathematical presentation is given.

For the return the relationship is quite easy and intuitive, if the return one day is π % assuming that the return is measured on one year the return should be π % times the number of trading days, f, in one year[9]. Expressed mathematically with the subscripts A for annual and D for daily

$$r_A = r_D \times f \Rightarrow E[r_A] = E[r_D] \times f. \tag{12}$$

For the variance, the relationship is derived using Equation 12, with the subscripts X and Y for different assets

$$\begin{aligned} \sigma^2(X)_A &= E[r_{X_A}^2] - E[r_{X_A}]^2 \\ &= f \times (E[r_{X_D}^2] - E[r_{X_D}]^2) \\ &= f \times \sigma^2(X)_D, \end{aligned}$$

where the rule for variance calculations, $\sigma^2(aX) = a^2\sigma^2(X)$, is used. The same relationship can be used for the standard deviation

[8]Daily, weekly, monthly or annual
[9]Assuming simple returns

$$\begin{aligned}\sigma(X)_A &= \sqrt{\sigma^2(X)_A} \\ &= \sqrt{f \times \sigma^2(X)_D} \\ &= \sqrt{f} \times \sigma(X)_D. \end{aligned} \tag{13}$$

Using the above relationships, the relationship for the correlation can be derived as

$$Corr(X,Y)_A = \frac{Cov(X,Y)_A}{\sigma(X)_A \sigma(Y)_A} = \frac{Cov(X,Y)_D}{\sigma(X)_D \sigma(Y)_D} \frac{f}{\sqrt{(f)}\sqrt{(f)}} = Corr(X,Y)_D, \tag{14}$$

where $Cov(aX,bY) = abCov(X,Y)$ and $Cov(X,X) = \sigma^2(X)$ are used.

For more information about conversions, see almost any textbook in finance covering conversions e.g. Options, Futures and Other Derivatives by Hull [13].

3.3 Empirical procedure

In the first part of the empirical testing phase, the true returns and covariances are calculated for all assets based on weekly observations. On the resulting mean-variance frontier, ten portfolios at different return and risk levels where chosen. All portfolios where chosen with respect to the constrained portfolio since it is the most common one for the Michaud model. The return of the portfolios are in the range between 3 % and 13 % and the risk of the portfolios are in the range between 1 % and 23 %, all numbers are annualized. Ten unique equidistant risk levels over the range were selected as a reference portfolios. The weights in the efficient portfolios at these levels are denoted as the “true weights”, $\mathbf{w}_{true}$ and are the benchmark weights. The set of true weights is thus the set of weights from optimizations based on the return, volatility and correlation from the historical returns.

The ambition was to study the stability of the two portfolio models, Markowitz and Michaud’s resampling model in five different cases with respect to input parameters.

1. Change in all input parameters: expected return, risk, correlation
2. Change in expected return only
3. Change in risk only
4. Change in correlations only
5. Change in covariance, combination of risk and correlation

Initially simulations were done for both the different risk and return levels. However, to limit the analysis the focus from here on is on the risk levels only. Further, the constrained case is considered the most important while the unconstrained will only be considered as a comparison when necessary.

To measure the sensitivity in a such a way that all scenarios would be comparable, all parameters were changed in the same way. For every asset, each input parameter was varied according to $IP + IP \cdot 10\% \cdot N(1{,}1)$ where IP is the input parameter based on historical values and $N(1{,}1)$ is a normal distributed variable with mean and standard deviation equal to one. The mean and standard deviation were chosen so there would be changes in the parameters on average. It can be argued that a standard normal distribution could also be used which is true but since the purpose was to study the stability when the input parameters were varied a mean of one would imply an average change of 10%. For return and risk a $M^{100\times9}$ matrix while for correlation and covariance 100 $M^{9\times9}$ matrices where used to represent the changes. Then for each case, $1-5$, the corresponding series and/or original values where used so the random numbers where the same so the results would not be affected by the randomness between each case.

In total, 100 efficient frontiers were calculated for each scenario and model. On each of the 100 frontier the portfolio weights at each risk level were saved for further analysis.

Another question was how the instability should be measured. What was sought after was to see how much the simulated weights deviated from the true known weights. Also to be able to compare between different models with an easy measure. No interests if the weights were larger or smaller, only the variation. Therefore after careful consideration and evaluation the most efficient way is to calculate the standard deviation of the difference for each asset weight relative to the true weight and then the summarizing the deviations to get the result

$$Asset\ Instability_{k,l} = \sum_{s=1}^{m} \sigma(w_{s,k,l} - w_{t,k,l}), \tag{15}$$

$$Portfolio\ Instability_{l} = \sum_{k=1}^{n} Asset\ Stability_{k,l}, \tag{16}$$

where $l =$ denotes the risk level, $k =$ denotes the asset, $w_{s,k,l} =$ is the simulated weight from simulation s at risk level l for asset k and $w_{t,k,l} =$ is the true weight at risk level l for asset k.

Ideas that were discarded where to scale with the true weight since a deviation in any weight is equally important. Also if positive and negative deviations would be treated in different ways. The standard deviation measure is well recognized, easy to compare and replicable and therefore the best measure.

In the cases where the covariance matrix is altered, it is important to be careful. The covariance matrix must be at least positive semi definite, all eigenvalues ≥ 0. When the covariance matrix is estimated from historical values it is per definition always positive definite, however now when the matrix is disturbed it has to be made sure that it still is at least positive semi definite. This can be done with the following perturbation [27]

$$A \rightarrow A + \epsilon(I - A) = B, \tag{17}$$

where ϵ is a scalar and I is the identity matrix. The question is how to choose ϵ. A large ϵ implies a large change of the matrix A, ϵ should therefore be as little as possible but large enough to ensure that the eigenvalues of A is non-negative. Denoting the smallest eigenvalue of A with λ_A and using the Rayleigh-Ritz theorem it is known that

$$\lambda_A = \min_{x^T x = 1} x^T A x. \tag{18}$$

Combining Equations 17 and 18

$$\lambda_B = \min_{x^T x = 1} x^T B x = \min_{x^T x = 1} [(1-\epsilon) x^T A x + \epsilon x^T x]. \tag{19}$$

If $\lambda_A \leq 0$ Equation 19 implies that $\lambda_B \geq 0$ if $\epsilon \geq -\lambda_A/(1-\lambda_A)$. Using the derived λ it is ensured that the matrix is positive semi definite. It is also possible to measure the difference between A and B with the following formula

$$\eta = \frac{||B - A||_2}{||A||_2},$$

where $|| \cdot ||_2$ denotes the L^2-norm.

All these steps in the analysis have been done for both the Markowitz portfolio optimization framework and the Michaud resampling model. In addition, the analysis is made on both a constrained and an unconstrained portfolio. Michaud argues that weights should be constrained, however to compare and measure the true sensitivity in the model the analysis should be done for both types of portfolios [2]. Furthermore, a constrained portfolio can falsely be interpreted as more stable, invoking constraints reduces the number of feasible solutions.

3.4 Programming

All coding used to make the analysis have been written in the numerical computational program MATLAB™. Previous studies e.g. the two Master Thesis of Jiao and Kohil which both of whom used MATLAB™ for similar analyses have given helpful programming tips and ideas [14], [15]. Further, the book Implementing Models in Quantitative Finance: Methods and Cases by Fusai and Roncoroni was helpful in validating the functionality of all programs [12].

For the Markowitz model the portfolio optimization equations have been solved with two different methods. First, with the fmincon routine, it finds the minimum of constrained nonlinear multivariable function with the option of using nonlinear constraints. To solve the desired equations, several function files, different input arguments and constraints were combined with fmincon. The constraints can be for a specific risk or return level but also to solve for a constrained or unconstrained portfolio. However, in some cases there exists an analytical solution, these were compared to the result of fmincon which could validate that the routine with all additional function files worked properly. The second approach was to use the frontcon routine in the financial toolbox. First, it finds the minimum risk portfolio and the maximum return portfolio. Second, it calculates a specified number of efficient portfolios in between the return of the minimum risk portfolio and the maximum return, equidistant with respect to return. The routine also has optionally of boundedness on the weights. An empirical testing study showed that it was sufficient with weights in the range of $w_i \leq |2|$, for both portfolios in the case of unconstrained portfolios. Frontcon does not give portfolios weights for specific risk or return which is desired in many cases. To improve the accuracy a larger number of portfolios can be used in the calculations. For this analysis the number of portfolios, N_{port1} and N_{port2}, for the constrained and unconstrained case respectively was set to $N_{port1} = 1000$ and $N_{port2} = 4000$. The value of N_{port1} can be compared to the values of Jiao and Kohil, both of whom only used constrained portfolios in their analyses and used 25 and 30 portfolios respectively.

For the resampling model, the portfolio optimization equations have been solved only with the frontcon routine. Since this routine is similar to the first steps in the Michaud framework it is perfect to use for calculating the resampled frontier. To estimate the new parameters, return and covariance 1000 random values were used. This can be compared to Kohil and Jiao who used 30 and 50 random values respectively. To ensure the proper covariance between the random variables, a Cholesky decomposition of the "true covariance matrix" is used. With Monte Carlo simulations this process is then repeated 500 times, this was decreased from 1000. From a time perspective, there is a trade-off between the number of portfolios and the number of Monte Carlo and to be able to do the simulations in a reasonable time frame, there has to be compromise between the parameter values. Once again this value can be compared with those of Jiao and Kohil who did 200 and 100 Monte Carlos simulations respectively while Fusai and Roncoroni did 500 simulations as well.

The reason why two approaches are used for the Markowitz model and only one for the Michaud model is that only the frontcon routine is suitable for the resampling model. To be able to compare the models, they should be calculated using the same routines to minimize errors from the routines.

4 Result and discussion

In this section the results and findings of this thesis are presented and discussed. The first part is an introduction and gives the reader all starting values. In the second part the results from all simulations are presented and deviations are discussed.

4.1 Introduction

To better be able to understand the results from the simulations there are a some base line information worth considering. First, see Table 4 for the different risk levels for all portfolios. Also the true weights from the constrained optimization for the two models are shown in the Tables 5 and 6 below. These optimizations used the original historical time series as inputs. Initially ten risk levels were defined but only eight used in the analysis. This is due to the inherent characteristics of the resampling model, see Figure 2. Since the model is an average of the result and the high risk portfolios are more unlikely, it becomes harder to achieve these portfolios.

Table 4: Risk level for the sample portfolios, annualized standard deviation

	Portf 1	**Portf 2**	**Portf 3**	**Portf 4**	**Portf 5**	**Portf 6**	**Portf 7**	**Portf 8**
Risk level σ	0.3	2.8	5.3	7.8	10.3	12.7	15.2	17.7

Table 5: True weights in the Markowitz constrained case

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
Portf 1	0.96	0.03	-	-	-	-	-	-	-
Portf 2	0.28	0.62	0.04	-	0.03	0.02	0.01	-	-
Portf 3	-	0.72	0.15	-	-	0.13	-	-	-
Portf 4	-	0.52	0.25	-	-	0.23	-	-	-
Portf 5	-	0.34	0.34	-	-	0.32	-	-	-
Portf 6	-	0.18	0.42	-	-	0.4	-	-	-
Portf 7	-	0.01	0.51	-	-	0.48	-	-	-
Portf 8	-	-	0.34	-	-	0.66	-	-	-

Table 6: True weights in the resampled constrained case

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
Portf 1	0.96	0.03	-	-	-	-	-	-	-
Portf 2	0.29	0.58	0.05	-	0.03	0.03	0.02	0.01	-
Portf 3	0.04	0.64	0.12	-	0.04	0.12	0.04	-	-
Portf 4	0.02	0.48	0.20	-	0.05	0.20	0.06	-	-
Portf 5	0.01	0.30	0.28	-	0.06	0.27	0.07	-	-
Portf 6	-	0.17	0.30	-	0.08	0.36	0.08	-	-
Portf 7	-	0.09	0.23	-	0.10	0.48	0.10	-	-
Portf 8	-	0.04	0.14	-	0.12	0.61	0.09	-	-

Comparing the two tables, Table 5 and 6, the first difference between the models is that the resampling model includes more assets than the Markowitz with exception for the first portfolio. This was expected due to the diversifying effect that the resampling model has. The intra-portfolio relationships are similar i.e. the large asset weights in one model are the large in the other model as well while assets not included in both modes have small weights which also is quite reasonable from a fundamental perspective. Also asset 4 and 9 are neither included in any portfolio in either model while asset 5 and 7 are included in all but one Michaud portfolios but only in one of the Markowitz portfolios. Further looking at the table over the correlations within in the portfolio, Table 3, and Table 7 which shows the risk-adjusted return, more things can be explained.

Table 7: Historical risk-adjusted returns

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Risk-adjusted returns	1.05	0.20	0.10	0.05	0.06	0.08	0.05	-0.04	-0.06
Volatility	0.04	0.53	1.62	2.63	3.20	3.19	2.74	1.74	1.35

The reason that asset 4 is not included in any portfolio is because of its relatively bad risk-adjusted return and high correlation with high return assets. Asset 8 and 9 are not included due to negative expected return, both assets have negative correlations with most of the assets but the diversifying effects are not sufficient to motivate inclusion. One of the major difference between the models is the allocation to assets 5 and 7. The risk-adjusted returns are not the most attractive but the correlations with other assets with a large allocation are attractive from a portfolio construction perspective in the resampling model together with the volatility component. High volatility is considered favourable in the resampling model since there is a probability for positive high returns in with bootstrapping. In a constrained model this is more obvious, high volatility implies high positive and negative returns. With positive returns it will be included in the portfolio but with a high negative return it will not be included but not negative ($0 \leq w_i$). Therefore the relationship is asymmetric and the reason why high volatility assets gets positively skewed by the constrained version of the resampling model.

4.2 Portfolio and asset instability

The five cases described earlier are listed below.

1. Case 1: Change in all input parameters: expected return, risk, correlation
2. Case 2: Change in expected return only
3. Case 3: Change in risk only
4. Case 4: Change in correlations only
5. Case 5: Change in covariance, combination of risk and correlation

These five cases will be referred to back during this section. The main measures used in this thesis are the portfolio and asset instability derived in Equations 15 and 16. The asset instability is as defined earlier, the standard deviation of the variation from the true weight and the portfolio instability the sum of the asset instability for each asset in the portfolio. In Tables 8 and 9 the asset instabilities are shown for both models in case 1[10] scaled with the true weight.

Table 8: Asset instability of Markowitz model

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.7	0.7	0.2	0	0.1	0.1	0	0.1	0
Portf 2	5	6.6	3.4	0	1.5	2	0.8	0.1	0
Portf 3	0	6.2	10.3	0	2.7	5.2	1.5	0	0
Portf 4	0	10.4	18	0	3.1	8.8	1.8	0	0
Portf 5	0	14.2	24.9	0	3.6	12	2.2	0	0
Portf 6	0	16.5	28.1	0	5.1	13.5	2.5	0	0
Portf 7	0	13.6	21.8	0	7.1	11	2	0	0
Portf 8	0	8.8	14.1	0	8.6	10.3	1.7	0	0

Values in the table multiplied with 100 and rounded to two decimals.

[10] Other cases in appendix A

Table 9: Asset instability of resampling model

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
Portf 1	0.7	0.6	0.2	0	0.1	0.1	0	0	0
Portf 2	4.7	6.1	2.4	0.2	1	1.4	0.6	0.4	0.1
Portf 3	1.1	3.1	5.4	0.3	2.1	3.1	1.3	0.3	0
Portf 4	0.6	5.1	9.2	0.3	3.1	5.2	2	0.2	0
Portf 5	0.4	6.7	11.8	0.4	4.1	6.7	2.6	0.2	0
Portf 6	0.2	6	10	0.5	5.4	7	3.2	0.1	0
Portf 7	0.1	4.4	7.2	0.6	7	7.4	3.5	0.1	0
Portf 8	0	2.6	5.7	0.5	8.1	8.8	3.3	0.1	0

Values in the table multiplied with 100 and rounded to two decimals.

Comparing the two tables, there are two important differences. First, the asset instabilities for the Markowitz model are significantly larger on a overall basis. Second, there are more non-zero asset instability values for the resampling model. This also illustrates the nature of the models, Markowitz is the concentrated model and resampling is the diversified model. The effect of diversifying in the resampling model in terms of asset sensitivity is the smoothing effect on distributing changes on more assets. For the remaining cases[11] the difference between the models are similar.

The instability becomes easier to analyse when looking at the figures instead of tables. Figures 3 and 4 shows the portfolios' instability on the y-axis, denoted in percent, against the different risk levels on the x-axis, denoted as σ or standard deviation, for the Markowitz model and the resampling model respectively. The five cases, different parameters stressed, are represented with different lines. While in Figure 5 both models are compared on a relative basis, where case "i" represents instability in Markowitz case "i" - instability in resampling case "i".

First, looking at Figure 3 and consider case 1, when all variables are stressed. For the Markowitz model the portfolio instability increases with increasing risk level up until the 6^{th} risk level and is then decreasing. It can largely be explained by the number of of assets included in the true portfolio, see Table 5. Comparing the 6^{th}, 7^{th} and 8^{th} portfolio, the

[11] These tables are shown in Appendix A

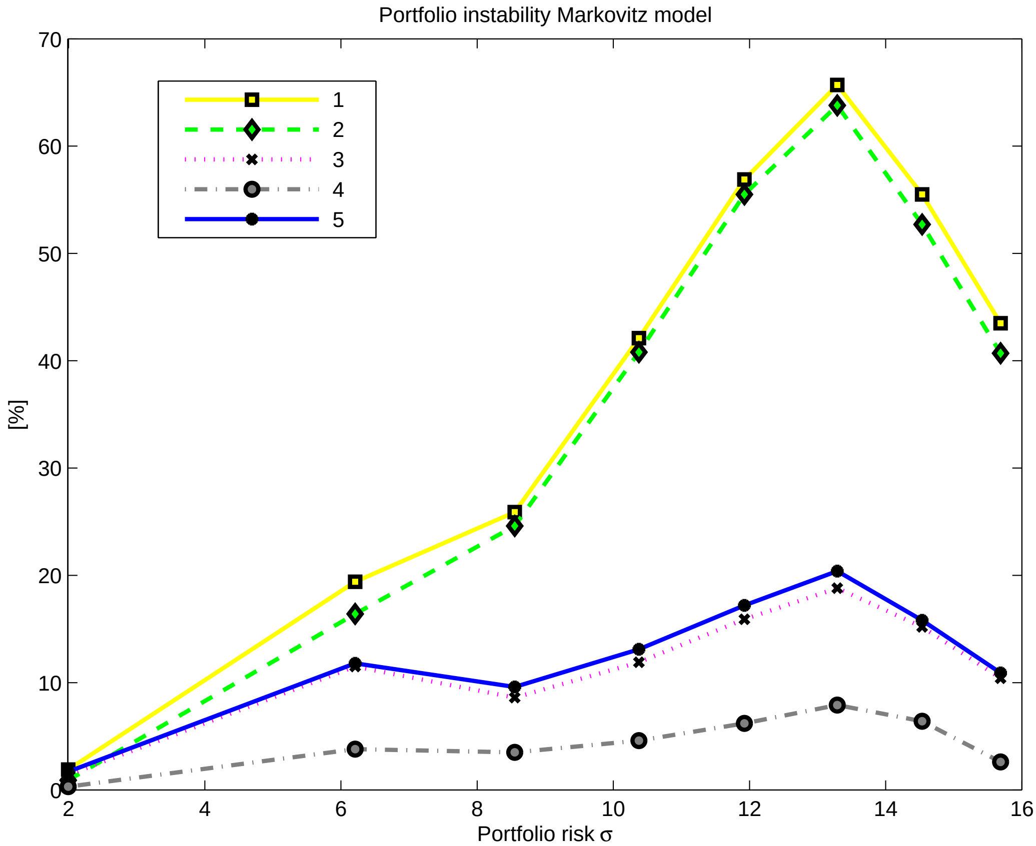

Figure 3: Portfolio instability for the five cases in the Markowitz model. The x-axis shows the risk level, standard deviation. The y-axis show the instability measure as defined earlier. Each dot in the graph represents one of the portfolios.

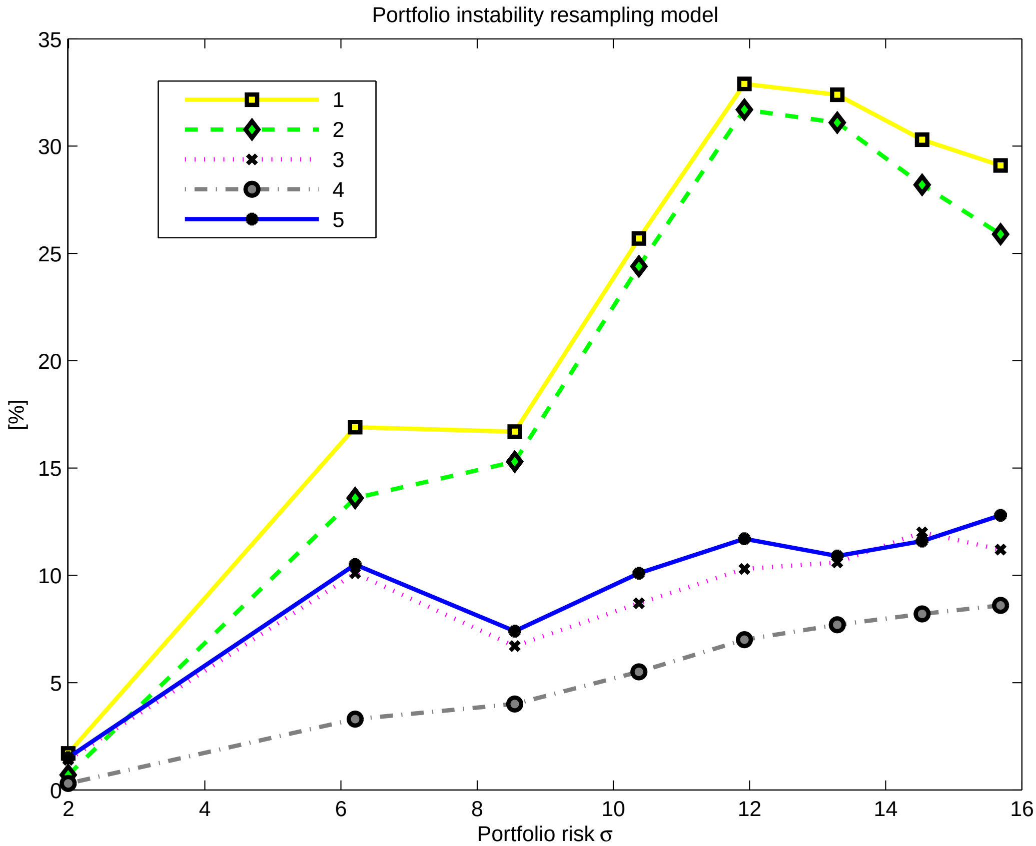

Figure 4: Portfolio instability for the five cases in the resampled model. The x-axis shows the risk level, standard deviation. The y-axis show the instability measure as defined earlier. Each dot in the graph represents one of the portfolios.

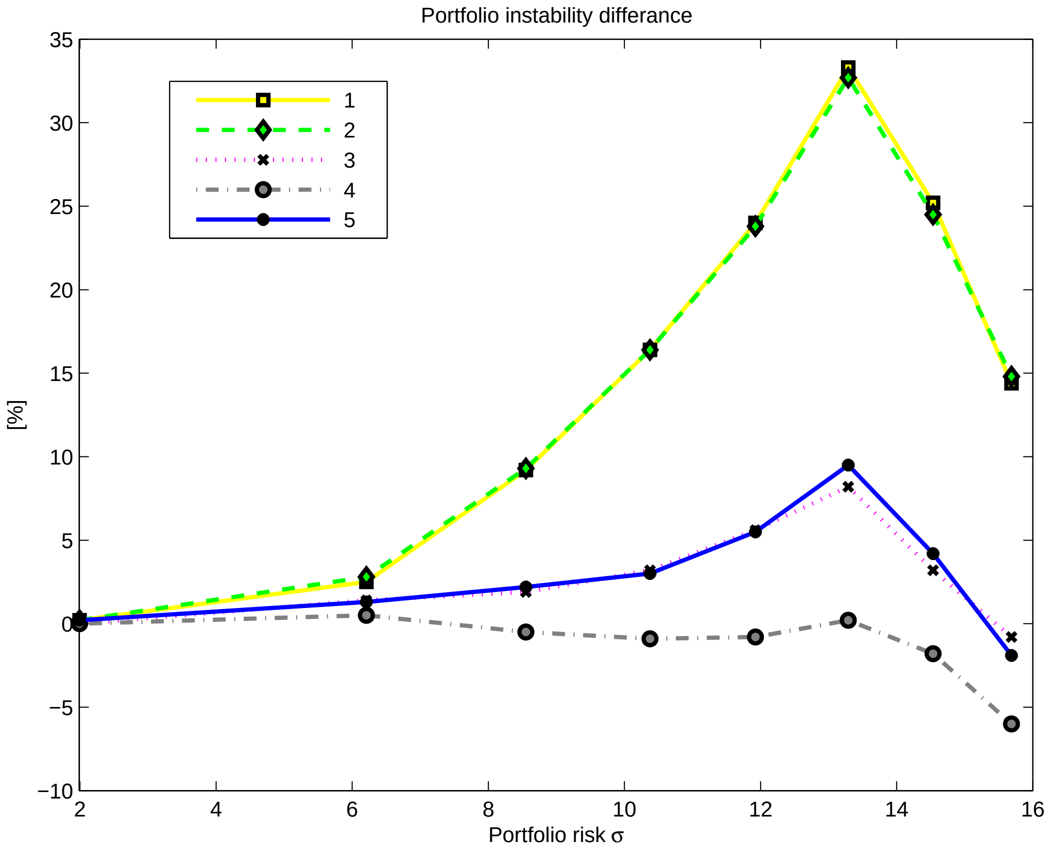

Figure 5: Difference in portfolio instability, Markowitz model - resampling model. The x-axis shows the risk level, standard deviation. The y-axis show the instability measure as defined earlier. Each dot in the graph represents the difference between the models for portfolios with the same risk level.

first two includes three assets while the last one only includes two assets. One of the assets in the 7^{th} portfolio has a weight of 1% and therefore has a limited impact. These results are quite intuitive since low risk portfolios only includes low risky assets and high risky portfolios will only include high risky assets. This limits the available asset combinations and also the instability.

Second, comparing the other cases for the Markowitz model. The instability is largest when all parameters are stressed. The return component is the most crucial parameter, followed by covariance, risk and lastly correlation. For case 2, the return component has the same characteristics as the case 1 described above. For case 5 the risk is the main driver of covariance. Interesting for case 3 to 5 is also that the 2^{nd} portfolio has higher instability than the 3^{rd}. The explanation of this is that the instability impact of risk, covariance and correlation is less then the effect of additional assets in the portfolio. The two factors that impacts the instability is risk level and number of assets in the true portfolio.

Shifting focus to the resampling model, Figure 4. When all variables are stressed the instability is increasing up until the 5^{th} risk level and is then decreasing, while the 2^{nd} and 3^{rd} portfolio are almost equal. The rational is the same as for the Markowitz model with the number of assets as seem in Table 6. To give an explanation for the 2^{nd} and 3^{rd} risk level, see the other cases, 2-5. The effect is more obvious for case 3 and 5 which results that there are some effect of instability diversification between the input factors. The ranking between the cases is the same as in the Markowitz model, interestingly the impact of correlation is strictly increasing.

Comparing the models, one interesting result is the impact of each parameter relative to stress in all parameters. The impact hierarchy for both models was the same but there are some underlying differences. For the return parameter the impact on the Makrowitz model is relatively more important while the other input parameters have a larger impact for the resampling model as seen in Figures 3 and 4.

To conclude, in Figure 5 the difference between the two models is shown. This is to be able to compare the models on a relative basis. In the figure, case ”i” represents instability in Markowitz case ”i” - instability in resampling case ”i”.

For the low risk portfolios there is no real difference between the models. The difference then increases with risk except for the last two portfolios. For both models the instability is decreasing but the marginal decrease for the Markowitz model is bigger and hence this result. The figure over the instability difference for case 1 and 2 resembles Figure 3 and it is the Markowitz model which is driving the difference. The resampling model outperforms the Markowtiz model in most instances, the exception is the last risk level for the risk and covariance, case 3 and 5 but also correlation on most risk levels, case 4.

Since the difference in instability is positive, the conclusion is that the Markowitz model is more sensitive to uncertainty in the input variables. Also that the difference is a increasing function of risk level conditioned on the number of assets.

5 Summary and conclusions

In this section the thesis is summarized and the major three findings are highlighted and discussed in separate sections. The findings are compared to results from previous studies. The sections ends with a section on suggestions for further studies.

5.1 Introduction

The three findings to highlight from this thesis are summarized in the bullets below and then discussed in separate sections.

- The Markowitz model is more sensitive than the resampling model to uncertainty in input parameters.
- The dispersion between the two models increase with risk subject to the number of assets in the portfolio.
- The expected return is the main driver of instability for parameter optimizations.

5.2 The Markowitz model is more sensitive than the resampling model to uncertainty in input parameters

The first finding is quite obvious after analysing Figure 5. Reviewing previous studies this was the most expected result since the purpose with the tesampling model was to handle uncertainty in the input parameters as described in Michaud's studies [21] and [22]. The results in this study then confirms the results in Michaud's study and also to some extent the findings of Kohli who argues that the resampling model is more stable [15].

In general, the resampling model includes more assets than the Markowitz model at the same risk level. Despite this, due to its characteristics, the volatility of each weight is less than the corresponding volatility for Markowitz model. Therefore the model is more stable than the Markowitz model. Even though there is no solid analytical proof for the model, only empirical proof, and this study is contributing to those. Michaud followers usually refer to the out-of-sample performance as the most compelling evidence to why the resampling model is superior compared to the Markowitz model [21] and [22]. However other studies such as Kohli shows that there is no conclusive advantage or disadvantage of using the resampling model [15]. Becker, F. et. al. go even further and argue that the Markovitz model is superior which illustrate that there is not one answer to the question [2]. The two aspects, performance and stability, are not necessary synonyms for what model is the most stable. A model can be less stable but have a better out-of sample performance e.g. if the out-of-sample parameters are exactly as the in-sample the Markowitz model will have better performance.

5.3 The dispersion between the two models increase with risk subject to the number of assets in the portfolio

Recalling Figure 3, the instability increases with increasing risk level, σ, in the Markowitz model. This statement is almost accurate for all input parameters[12]. Comparing with the results with studies of Okhrin and Schmid, de Roon and MacLean et. al., this study validates that the sensitivity appears to diminish for higher risk aversion [25] [23] [17].

Recalling Figure 4, the same conclusion is not as obvious be said. The trend is however that the instability is increasing with risk level but the diminishing effect is seen earlier but not as big as in the Markowitz model. There are no other studies on this topic which can confirm or deny the results but the results are similar for both models.

To summarize, for the two models, there is not sufficient results to state that the instability increases with risk without also considering the number of assets. What is true is that the increase in instability for the Markowtiz model is relatively more than for the resampling model and it drives the dispersion between the models, recall Figure 5. However when the number of assets differ substantially, in this case > 3, one has to pay extra attention since fewer assets reduce instability. This is the case at the end of the frontier, at the end of the risk spectra when fewer assets are usually included in the Markowitz optimal

[12] Exceptions highlighted in Section 4.2

portfolio as seen in Figure 5.

5.4 The expected return is the main driver of instability for parameter optimizations

For both models, see Figure 3 and Figure 4, the order in terms of impact on instability is the same. By far, returns are the main source of instability, then covariance, followed by risk and then finally correlation. The covariance is mainly driven by the risk parameter.

The parameter uncertainty was simulated from a distribution with equal relative standard deviation for each parameter. This was done to be able to compare the parameters on a relative basis. However, a few limitations on some parameters were needed to ensure that values were not mathematically incorrect. The risk parameter cannot be negative and thus limited at 0. The correlation between two variables must satisfy $-1 \leq correlation \leq 1$ and was bounded to this interval.

The return is a standalone input parameter while the risk and correlation parameters are combined into a covariance matrix. Therefore the return for one asset can be changed without affecting any other assets directly. If the risk for one asset is changed it will affect the covariance matrix and its relationship with all other assets. The correlation matrix must be positive semi-definite matrix. Therefore if the correlation between two variables is changed it might result in a non positive semi-definite matrix and then the relationship between many assets will be changed in order to adjust the matrix. All these aspects do also contribute to the result that the return is the main driver.

Comparing to previous studies, Best and Grauer also states that changes in mean is the main source of instability [3]. Chopra and Ziemba also argue that mean is the main source of instability [8]. However they argue that variance contributes more than covariance to the instability which is not the case in this study. Okhrin and Schmid argue that the sensitivity related to the covariance matrix is substantial and robust [25]. Palczewski and Palczewski have a similar argumentation that the effect of covariance is neglected [26]. Also that it depends on the type of portfolio e.g. in a multi asset portfolio the effect of covariance can account for up to 50% of the instability. They also argue that the statement mean is the main source of instability is exaggerated. In this study the result is rather that the relative importance of the covariance matrix is diminishing with increasing risk level and is not anywhere near the impact on a multi asset portfolio.

5.5 Summarizing the conclusions

Two of the three conclusions above have been discussed mainly from the perspective of the overall result, uncertainty simulated in all parameters, while the third conclusion is focusing on one parameter only. Equal for all three is that the result considers the uncertainty in all assets and does not reflect on the individual assets' contribution to uncertainty. If cases 2 to 5 are considered for the statements listed above.

For the first conclusion, the exceptions are in the last risk level for the covariance and risk, case 5 and 3 as well as most of the correlation portfolios, case 4, when consider the five different cases one by one. However, the correlation has a small impact on the instability for both models compared to the other cases and in absolute value of the difference for all this cases are negligible compared to the main and return case, 1 and 2.

For the second conclusion, there are additional exceptions as seen in Figure 5. In case 2,3 and 5 the last two portfolios are are the reason why the number of asset comment has to be added to the statement since the instability is decreasing for the 7^{th} and 8^{th} portfolio. This has been discussed in Section 4.2. For case 4 is not possible to confirm the statement case sine it is close to zero and does not increase with the risk level.

For the third conclusion there are really anything in the result of this thesis that contradict in any way. The conclusion is indifferent of the models and only focusing on case 1.

To summarize, recall the two research questions defined in the beginning of the thesis.

- Research question 1: How sensitive are the two models' optimal portfolio weights to changes in expected return, risk and correlation.
- Research question 2: How does the inherit portfolio characteristic affect the results of research question 1.

The first and third findings are the answers to the first research question while the second findings gives a partial answer to the second question. The instability for a portfolio increases with risk with the exception at the end of the frontier where the possible combinations are reduced. Palczewski and Palczewski argument that the covariance has a larger impact than in a multi asset portfolio compared to an equity portfolio is not possible to give any color on that in this thesis [26].

5.6 Further studies

There are a number of different ways to expand this study. One interesting topic would be the measure to study the uncertainty as it could be defined in many different ways. In this study, the measure only considered relative deviations from the optimal portfolio weights. One alternative way would be to penalize high volatility assets higher since wrong decisions in these assets tend to have a larger impact on performance.

A second topic could be to work with the stress levels. In times of big market shocks, substantial risk-off environments, ordinary correlations tend to break down and the impact of this could also be an interesting topic to study.

A third topic would be to consider the individual assets' contribution to uncertainty. What constitutes an asset with large respective low contribution in terms of expected return, risk and correlation.

A final topic, which only focuses on the resampling model would be to remove eventual outliers from the sample, scale with variance or to use the median instead of the sample average to improve the model and the performance.

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Appendix

Asset instability

Table 10: Asset instability of Markowitz model - case 2

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.2	0.3	0.2	0	0.1	0.1	0	0	0
Portf 2	3.7	5.4	3.4	0	1.3	1.9	0.7	0	0
Portf 3	0	6	10.3	0	2.2	4.9	1.2	0	0
Portf 4	0	10.4	18.4	0	2.2	8.5	1.3	0	0
Portf 5	0	14.4	25.6	0	2.3	11.8	1.4	0	0
Portf 6	0	17	28.8	0	3.7	12.7	1.6	0	0
Portf 7	0	14	22	0	5.8	9.7	1.2	0	0
Portf 8	0	9.4	14.2	0	7.5	8.5	1.1	0	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 11: Asset instability of Markowitz model - case 3

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.7	0.6	0.1	0	0	0	0	0	0
Portf 2	3.9	4.5	1.4	0	0.6	0.8	0.3	0	0
Portf 3	0	2.4	3.5	0	0.6	1.9	0.2	0	0
Portf 4	0	3.5	5.6	0	0	2.8	0	0	0
Portf 5	0	4.6	7.5	0	0	3.8	0	0	0
Portf 6	0	5.4	8.8	0	0	4.6	0	0	0
Portf 7	0	3.4	6.9	0	0	4.9	0	0	0
Portf 8	0	0	5.2	0	0	5.2	0	0	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 12: Asset instability of Markowitz model - case 4

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.1	0.1	0.1	0	0	0	0	0	0
Portf 2	0.8	1	0.8	0	0.5	0.5	0.2	0	0
Portf 3	0	1.1	1.3	0	0.6	0.3	0.2	0	0
Portf 4	0	1.8	2.2	0	0.2	0.4	0	0	0
Portf 5	0	2.5	3	0	0.1	0.6	0	0	0
Portf 6	0	3.2	3.9	0	0	0.8	0	0	0
Portf 7	0	2.4	2.9	0	0	1.1	0	0	0
Portf 8	0	0	1.3	0	0	1.3	0	0	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 13: Asset instability of Markowitz model - case 5

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.7	0.7	0.1	0	0.1	0.1	0	0	0
Portf 2	3.8	4.4	1.5	0	0.8	0.9	0.4	0	0
Portf 3	0	2.7	3.7	0	0.9	1.9	0.4	0	0
Portf 4	0	4	6	0	0.3	2.8	0	0	0
Portf 5	0	5.3	8.1	0	0	3.8	0	0	0
Portf 6	0	6.2	9.6	0	0	4.6	0	0	0
Portf 7	0	4.1	7	0	0	4.7	0	0	0
Portf 8	0	0.3	5.3	0	0	5.3	0	0	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 14: Asset instability of resampling model - case 2

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.2	0.3	0.1	0	0	0.1	0	0	0
Portf 2	3.4	4.9	2.3	0.2	0.8	1.2	0.4	0.3	0.1
Portf 3	0.8	2.9	5.3	0.2	1.9	2.8	1.1	0.3	0
Portf 4	0.6	5	9.1	0.3	2.8	4.7	1.7	0.2	0
Portf 5	0.4	6.9	11.8	0.4	3.7	6	2.3	0.2	0
Portf 6	0.3	6.6	9.8	0.5	4.9	6	2.9	0.1	0
Portf 7	0.1	4.2	6.8	0.5	6.4	6.8	3.3	0.1	0
Portf 8	0	2.4	4.8	0.3	7.7	7.6	3	0.1	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 15: Asset instability of resampling model - case 3

	Asset 1	**Asset 2**	**Asset 3**	**Asset 4**	**Asset 5**	**Asset 6**	**Asset 7**	**Asset 8**	**Asset 9**
Portf 1	0.7	0.6	0.1	0	0	0	0	0	0
Portf 2	3.6	4.1	0.9	0.1	0.4	0.5	0.3	0.2	0
Portf 3	0.7	1.6	1.9	0.1	0.7	1.2	0.4	0.1	0
Portf 4	0.3	2	2.7	0.1	1	1.9	0.6	0.1	0
Portf 5	0.2	2.4	3	0.1	1.2	2.5	0.8	0.1	0
Portf 6	0.2	2.8	1.9	0.1	1.3	3.3	0.9	0.1	0
Portf 7	0.1	1.5	2.8	0.1	1.5	4.8	1.1	0.1	0
Portf 8	0	1.2	2.6	0.1	1.4	4.4	1.4	0.1	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 16: Asset instability of resampling model - case 4

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
Portf 1	0.1	0.1	0.1	0	0	0	0	0	0
Portf 2	0.7	0.8	0.5	0.1	0.4	0.4	0.2	0.2	0
Portf 3	0.5	0.6	0.8	0.2	0.8	0.6	0.4	0.1	0
Portf 4	0.3	0.9	1.3	0.2	1.2	0.9	0.6	0.1	0
Portf 5	0.2	1.2	1.8	0.2	1.5	1.2	0.8	0.1	0
Portf 6	0.1	1.3	1.7	0.3	1.9	1.3	1	0.1	0
Portf 7	0.1	1	1.6	0.4	2.4	1.4	1.2	0.1	0
Portf 8	0	0.8	1.7	0.4	2.8	1.5	1.3	0.1	0

Values in the table multiplied with 100 and rounded to two decimals.

Table 17: Asset instability of resampling model - case 5

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7	Asset 8	Asset 9
Portf 1	0.7	0.6	0.1	0	0.1	0	0	0	0
Portf 2	3.6	4	1	0.2	0.5	0.6	0.3	0.2	0.1
Portf 3	0.7	1.6	2	0.2	0.9	1.3	0.6	0.1	0
Portf 4	0.4	2.2	2.9	0.3	1.3	2	0.9	0.1	0
Portf 5	0.2	2.5	3.3	0.3	1.6	2.6	1.1	0.1	0
Portf 6	0.1	1.9	2.1	0.4	2	3	1.3	0.1	0
Portf 7	0.1	1.5	2.1	0.4	2.4	3.7	1.3	0.1	0
Portf 8	0	1.2	2.6	0.4	2.8	4.3	1.4	0.1	0

Values in the table multiplied with 100 and rounded to two decimals.