

EXPLICIT SERRE DUALITY ON COMPLEX SPACES

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ABSTRACT. In this paper we use recently developed calculus of residue currents together with integral formulas to give a new explicit analytic realization, as well as a new analytic proof of Serre duality on any reduced pure n -dimensional paracompact complex space X . At the core of the paper is the introduction of concrete fine sheaves $\mathcal{A}_X^{n,q}$ of certain currents on X of bidegree (n, q) , such that the Dolbeault complex $(\mathcal{A}_X^{n,\bullet}, \bar{\partial})$ becomes, in a certain sense, a dualizing complex. In particular, if X is Cohen-Macaulay (e.g., Gorenstein or a complete intersection) then $(\mathcal{A}_X^{n,\bullet}, \bar{\partial})$ is an explicit fine resolution of the Grothendieck dualizing sheaf.

1. INTRODUCTION

Let X be a complex n -dimensional manifold and let $F \rightarrow X$ be a complex vector bundle. Let $\mathcal{E}^{0,q}(X, F)$ denote the space of smooth F -valued $(0, q)$ -forms on X and let $\mathcal{E}_c^{n,q}(X, F^*)$ denote the space of smooth compactly supported (n, q) -forms on X with values in the dual vector bundle F^* . Serre duality, [28], can be formulated analytically as follows: *There is a non-degenerate pairing*

$$(1.1) \quad H^q(\mathcal{E}^{0,\bullet}(X, F), \bar{\partial}) \times H^{n-q}(\mathcal{E}_c^{n,\bullet}(X, F^*), \bar{\partial}) \rightarrow \mathbb{C},$$

$$([\varphi]_{\bar{\partial}}, [\psi]_{\bar{\partial}}) \mapsto \int_X \varphi \wedge \psi,$$

provided that $H^q(\mathcal{E}^{0,\bullet}(X, F), \bar{\partial})$ and $H^{q+1}(\mathcal{E}^{0,\bullet}(X, F), \bar{\partial})$ are Hausdorff considered as topological vector spaces. If we set $\mathcal{F} := \mathcal{O}(F)$ and $\mathcal{F}^* := \mathcal{O}(F^*)$ and let Ω_X^n denote the sheaf of holomorphic n -forms on X , then one can, via the Dolbeault isomorphism, rephrase Serre duality more algebraically: There is a non-degenerate pairing

$$(1.2) \quad H^q(X, \mathcal{F}) \times H_c^{n-q}(X, \mathcal{F}^* \otimes \Omega_X^n) \rightarrow \mathbb{C},$$

realized by the cup product, provided that $H^q(X, \mathcal{F})$ and $H^{q+1}(X, \mathcal{F})$ are Hausdorff. In this formulation Serre duality has been generalized to complex spaces, see, e.g., Hartshorne [19], [20], and Conrad [15] for the algebraic setting and Ramis-Ruget [26] and Andreotti-Kas [11] for the analytic. In fact, if X is a pure n -dimensional paracompact complex space that in addition is Cohen-Macaulay, then again there is a perfect pairing (1.2) if we construe Ω_X^n as the *Grothendieck dualizing sheaf* that we will get back to shortly. If X is not Cohen-Macaulay things get more involved and $H_c^{n-q}(X, \mathcal{F}^* \otimes \Omega_X^n)$ is replaced by $\text{Ext}_c^{-q}(X; \mathcal{F}, \mathbf{K}^\bullet)$, where \mathbf{K}^\bullet is the *dualizing complex* in the sense of [26]; a certain complex of \mathcal{O}_X -modules with coherent cohomology.

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To our knowledge there is no such explicit analytic realization of Serre duality as (1.1) in the case of singular spaces. In fact, *verbatim* the pairing (1.1) cannot realize Serre duality in general since the Dolbeault complex $(\mathcal{E}_X^{0,\bullet}, \bar{\partial})^1$ in general does not provide a resolution of \mathcal{O}_X . In this paper we replace the sheaves of smooth forms by concrete fine sheaves of certain currents $\mathcal{A}_X^{p,q}$, $p = 0$ or $p = n$, that are smooth on X_{reg} and such that (1.1) with \mathcal{E} replaced by \mathcal{A} indeed realizes Serre duality.

We will say that a complex $(\mathcal{D}_X^\bullet, \delta)$ of fine sheaves is a *dualizing Dolbeault complex* for a coherent sheaf \mathcal{F} if $(\mathcal{D}_X^\bullet, \delta)$ has coherent cohomology and if there is a non-degenerate pairing $H^q(X, \mathcal{F}) \times H^{n-q}(\mathcal{D}_c^\bullet(X), \delta) \rightarrow \mathbb{C}$. In this terminology, $(\mathcal{A}_X^{n,\bullet}, \bar{\partial})$ thus is a dualizing Dolbeault complex for \mathcal{O}_X .

At this point it is appropriate to mention that Ruget in [27] shows, using Coleff-Herrera residue theory, that there is an injective morphism $\mathbf{K}_X^\bullet \rightarrow \mathcal{C}_X^{n,\bullet}$, where $\mathcal{C}_X^{n,\bullet}$ is the sheaf of germs of currents on X of bidegree (n, \bullet) .

Let X be a reduced complex space of pure dimension n . Recall that every point in X has a neighborhood V that can be embedded into some pseudoconvex domain $D \subset \mathbb{C}^N$, $i: V \rightarrow D$, and that $\mathcal{O}_V \cong \mathcal{O}_D/\mathcal{I}_V$, where \mathcal{I}_V is the radical ideal sheaf in D defining $i(V)$. Similarly, a (p, q) -form φ on V_{reg} is said to be smooth on V if there is a smooth (p, q) -form $\tilde{\varphi}$ in D such that $\varphi = i^*\tilde{\varphi}$ on V_{reg} . It is well known that the so defined smooth forms on V define an intrinsic sheaf $\mathcal{E}_X^{p,q}$ on X . The currents of bidegree (p, q) on X are defined as the dual of the space of compactly supported smooth $(n-p, n-q)$ -forms on X with a certain topology. More concretely, given a local embedding $i: V \rightarrow D$, for any (p, q) -current μ on V , $\tilde{\mu} := i_*\mu$ is a current of bidegree $(p+N-n, q+N-n)$ in D with the property that $\tilde{\mu} \cdot \xi = 0$ for every test form ξ in D such that $i^*\xi|_{V_{reg}} = 0$. Conversely, if $\tilde{\mu}$ is a current in D with this property, then it defines a current on V (with a shift in bidegrees). We will often suggestively write $\int \mu \wedge \xi$ for the action of the current μ on the test form ξ .

A current μ on X is said to have the *standard extension property* (SEP) with respect to a subvariety $Z \subset X$ if $\chi(|h|/\epsilon)\mu \rightarrow \mu$ as $\epsilon \rightarrow 0$, where χ is a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$ and h is a holomorphic tuple such that $\{h = 0\}$ has positive codimension and intersects Z properly; if $Z = X$ we simply say that μ has the SEP on X . In particular, two currents with the SEP on X are equal on X if and only if they are equal on X_{reg} .

We will say that a current μ on X has *principal value-type singularities* if μ is locally integrable outside a hypersurface and has the SEP on X . Notice that if μ has principal value-type singularities and h is a generically non-vanishing holomorphic tuple such that μ is locally integrable outside $\{h = 0\}$, then the action of μ on a test form ξ can be computed as

$$\lim_{\epsilon \rightarrow 0} \int_X \chi(|h|/\epsilon) \mu \wedge \xi,$$

where the integral now is an honest integral of an integrable form on the manifold X_{reg} .

By using integral formulas and residue theory, Andersson and the second author introduced in [7] fine sheaves $\mathcal{A}_X^{0,q}$ (i.e., modules over $\mathcal{E}_X^{0,0}$) of $(0, q)$ -currents with the SEP on X , containing $\mathcal{E}_X^{0,q}$, and coinciding with $\mathcal{E}_{X_{reg}}^{0,q}$ on X_{reg} , such that the

¹See below for the definition of $\mathcal{E}_X^{p,q}$; the sheaf of smooth (p, q) -forms on X .

associated Dolbeault complex yields a resolution of \mathcal{O}_X . We introduce our sheaves $\mathcal{A}_X^{n,q}$ of (n, q) -currents in a similar way and we show that such currents have the SEP on X , that $\mathcal{E}_X^{n,q} \subset \mathcal{A}_X^{n,q}$, and that $\mathcal{A}_X^{n,q}$ coincides with $\mathcal{E}_X^{n,q}$ on X_{reg} ; cf. Proposition 4.3. Moreover, by Theorem 4.4, $\bar{\partial}: \mathcal{A}_X^{n,q} \rightarrow \mathcal{A}_X^{n,q+1}$, where of course $\bar{\partial}$ is defined by duality; $\int \bar{\partial}\mu \wedge \xi := \pm \int \mu \wedge \bar{\partial}\xi$ for currents μ and test forms ξ on X . By adapting the constructions in [7] to the setting of (n, q) -forms we get the following semi-global homotopy formula for $\bar{\partial}$.

Theorem 1.1. *Let V be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$, and put $V' = V \cap D'$. There are integral operators*

$$\check{\mathcal{K}}: \mathcal{A}^{n,q}(V) \rightarrow \mathcal{A}^{n,q-1}(V'), \quad \check{\mathcal{P}}: \mathcal{A}^{n,q}(V) \rightarrow \mathcal{A}^{n,q}(V'),$$

such that if $\psi \in \mathcal{A}^{n,q}(V)$, then the homotopy formula

$$\psi = \bar{\partial}\check{\mathcal{K}}\psi + \check{\mathcal{K}}(\bar{\partial}\psi) + \check{\mathcal{P}}\psi$$

holds on V' .

The integral operators $\check{\mathcal{K}}$ and $\check{\mathcal{P}}$ are given by kernels $k(z, \zeta)$ and $p(z, \zeta)$ that are respectively integrable and smooth on $Reg(V_z) \times Reg(V'_\zeta)$ and that have principal value-type singularities at the singular locus of $V \times V'$. In particular, one can compute $\check{\mathcal{K}}\psi$ and $\check{\mathcal{P}}\psi$ as

$$\check{\mathcal{K}}\psi(\zeta) = \lim_{\epsilon \rightarrow 0} \int_{V_z} \chi(|h(z)|/\epsilon) k(z, \zeta) \wedge \psi(z), \quad \check{\mathcal{P}}\psi(\zeta) = \lim_{\epsilon \rightarrow 0} \int_{V_z} \chi(|h(z)|/\epsilon) p(z, \zeta) \wedge \psi(z),$$

where χ is a smooth approximation of the characteristic function of $[1, \infty) \subset \mathbb{R}$, h is a holomorphic tuple cutting out V_{sing} , and where the limit is understood in the sense of currents. We use our integral operators to prove the following result.

Theorem 1.2. *Let X be a reduced complex space of pure dimension n . The cohomology sheaves $\omega_X^{n,q} := \mathcal{H}^q(\mathcal{A}_X^{n,\bullet}, \bar{\partial})$ of the sheaf complex*

$$(1.3) \quad 0 \rightarrow \mathcal{A}_X^{n,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{n,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{n,n} \rightarrow 0$$

are coherent. If X is Cohen-Macaulay, then

$$(1.4) \quad 0 \rightarrow \omega_X^{n,0} \hookrightarrow \mathcal{A}_X^{n,0} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{n,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}_X^{n,n} \rightarrow 0$$

is exact.

In fact, our proof of Theorem 1.2 shows that if $V \subset X$ is identified with an analytic codimension p subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, then $\omega_V^{n,q} \cong \mathcal{E}xt^{p+q}(\mathcal{O}_D/\mathcal{I}_V, \Omega_D^N)$, where Ω_D^N is the canonical sheaf on D . Hence, we get a concrete analytic realization of these $\mathcal{E}xt$ -sheaves.

The sheaf $\omega_V^{n,0}$ of $\bar{\partial}$ -closed currents in $\mathcal{A}_V^{n,0}$ is in fact equal to the sheaf of $\bar{\partial}$ -closed meromorphic currents on V in the sense of Henkin-Passare [21, Definition 2], cf. [7, Example 2.8]. This sheaf was introduced earlier by Barlet in a different way in [12]; cf. also [21, Remark 5]. In case X is Cohen-Macaulay $\mathcal{E}xt^p(\mathcal{O}_D/\mathcal{I}_V, \Omega_D^N)$ is by definition the Grothendieck dualizing sheaf. Thus, (1.4) can be viewed as a concrete analytic fine resolution of the Grothendieck dualizing sheaf in the Cohen-Macaulay case.

Let φ and ψ be sections of $\mathcal{A}_X^{0,q}$ and $\mathcal{A}_X^{n,q'}$ respectively. Since φ and ψ then are smooth on the regular part of X , the exterior product $\varphi|_{X_{reg}} \wedge \psi|_{X_{reg}}$ is a smooth $(n, q + q')$ -form on X_{reg} . In Theorem 5.1 we show that $\varphi|_{X_{reg}} \wedge \psi|_{X_{reg}}$ has a natural extension across X_{sing} as a current with principal value-type singularities; we denote this current by $\varphi \wedge \psi$. Moreover, it turns out that the Leibniz rule $\bar{\partial}(\varphi \wedge \psi) = \bar{\partial}\varphi \wedge \psi + (-1)^q \varphi \wedge \bar{\partial}\psi$ holds. Now, if $q' = n - q$ and either φ or ψ has compact support, then $\int \varphi \wedge \psi$ (i.e., the action of $\varphi \wedge \psi$ on 1) gives us a complex number. Since the Leibniz rule holds we thus get a pairing, a *trace map*, on cohomology level:

$$Tr: H^q(\mathcal{A}^{0,\bullet}(X), \bar{\partial}) \times H^{n-q}(\mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \rightarrow \mathbb{C},$$

$$Tr([\varphi]_{\bar{\partial}}, [\psi]_{\bar{\partial}}) = \int_X \varphi \wedge \psi,$$

where $\mathcal{A}^{0,q}(X)$ denotes the global sections of $\mathcal{A}_X^{0,q}$ and $\mathcal{A}_c^{n,q}(X)$ denotes the global sections of $\mathcal{A}_X^{n,q}$ with compact support. It causes no problems to insert a locally free sheaf: If $F \rightarrow X$ is a vector bundle, $\mathcal{F} = \mathcal{O}(F)$ the associated locally free sheaf, and $\mathcal{F}^* = \mathcal{O}(F^*)$ the dual sheaf, then the trace map gives a pairing $\mathcal{F} \otimes \mathcal{A}^{0,q}(X) \times \mathcal{F}^* \otimes \mathcal{A}_c^{n,n-q}(X) \rightarrow \mathbb{C}$.

Theorem 1.3. *Let X be a paracompact reduced complex space of pure dimension n and \mathcal{F} a locally free sheaf on X . If $H^q(X, \mathcal{F})$ and $H^{q+1}(X, \mathcal{F})$, considered as topological vector spaces, are Hausdorff (e.g., finite dimensional) then the pairing*

$$H^q(\mathcal{F} \otimes \mathcal{A}^{0,\bullet}(X), \bar{\partial}) \times H^{n-q}(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \rightarrow \mathbb{C}, \quad ([\varphi], [\psi]) \mapsto \int_X \varphi \wedge \psi$$

is non-degenerate.

By [7, Corollary 1.3], the complex $(\mathcal{F} \otimes \mathcal{A}_X^{0,\bullet}, \bar{\partial})$ is a fine resolution of \mathcal{F} and so, via the Dolbeault isomorphism, Theorem 1.3 gives us a non-degenerate pairing

$$H^q(X, \mathcal{F}) \times H^{n-q}(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \rightarrow \mathbb{C}.$$

The complex $(\mathcal{F}^* \otimes \mathcal{A}_X^{n,\bullet}, \bar{\partial})$ is thus a concrete analytic dualizing Dolbeault complex for \mathcal{F} . If X is Cohen-Macaulay, then $(\mathcal{F}^* \otimes \mathcal{A}_X^{n,\bullet}, \bar{\partial})$ is, by Theorem 1.2, a fine resolution of the sheaf $\mathcal{F}^* \otimes \omega_X^{n,0}$ and so Theorem 1.3 yields in this case a non-degenerate pairing

$$H^q(X, \mathcal{F}) \times H_c^{n-q}(X, \mathcal{F}^* \otimes \omega_X^{n,0}) \rightarrow \mathbb{C}.$$

In Section 7 we show that this pairing also can be realized as the cup product in Čech cohomology.

Remark 1.4. By [26, Théorème 2] there is another non-degenerate pairing

$$H_c^q(X, \mathcal{F}) \times \text{Ext}^{-q}(X; \mathcal{F}, \mathbf{K}_X^\bullet) \rightarrow \mathbb{C}$$

if $H_c^q(X, \mathcal{F})$ and $H_c^{q+1}(X, \mathcal{F})$ are Hausdorff. In view of this we believe that one can show that, under the same assumption, the pairing

$$H^q(\mathcal{F} \otimes \mathcal{A}_c^{0,\bullet}(X), \bar{\partial}) \times H^{n-q}(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(X), \bar{\partial}) \rightarrow \mathbb{C}, \quad ([\varphi], [\psi]) \mapsto \int_X \varphi \wedge \psi$$

is non-degenerate but we do not pursue this question in this paper.

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2. PRELIMINARIES

Our considerations here are local or semi-global so let V be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$.

2.1. Pseudomeromorphic currents on a complex space. In \mathbb{C}_z the principal value current $1/z^m$ can be defined, e.g., as the limit as $\epsilon \rightarrow 0$ in the sense of currents of $\chi(|h(z)|/\epsilon)/z^m$, where χ is a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$ and h is a holomorphic function vanishing at $z = 0$, or as the value at $\lambda = 0$ of the analytic continuation of the current-valued function $\lambda \mapsto |h(z)|^{2\lambda}/z^m$. Regularizations of the form $\chi(|h|/\epsilon)\mu$ of a current μ occur frequently in this paper and throughout χ will denote a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$. The *residue current* $\bar{\partial}(1/z^m)$ can be computed as the limit of $\bar{\partial}\chi(|h(z)|/\epsilon)/z^m$ or as the value at $\lambda = 0$ of $\lambda \mapsto \bar{\partial}|h(z)|^{2\lambda}/z^m$. Since tensor products of currents are well-defined we can form the current

$$(2.1) \quad \tau = \bar{\partial} \frac{1}{z_1^{m_1}} \wedge \cdots \wedge \bar{\partial} \frac{1}{z_r^{m_r}} \wedge \frac{\gamma(z)}{z_{r+1}^{m_{r+1}} \cdots z_n^{m_n}}$$

in \mathbb{C}_z^n , where m_1, \dots, m_r are positive integers, m_{r+1}, \dots, m_n are nonnegative integers, and γ is a smooth compactly supported form. Notice that τ is anti-commuting in the residue factors $\bar{\partial}(1/z_j^{m_j})$ and commuting in the principal value factors $1/z_k^{m_k}$. We say that a current of the form (2.1) is called an *elementary pseudomeromorphic current* and we say that a current μ on V is *pseudomeromorphic*, $\mu \in \mathcal{PM}(V)$, if it is a locally finite sum of pushforwards $\pi_*\tau = \pi_*^1 \cdots \pi_*^\ell \tau$ under maps

$$V^\ell \xrightarrow{\pi^\ell} \cdots \xrightarrow{\pi^2} V^1 \xrightarrow{\pi^1} V^0 = V,$$

where each π^j is either a modification, a simple projection $V^j = V^{j-1} \times Z \rightarrow V^{j-1}$, or an open inclusion, and τ is an elementary pseudomeromorphic current on V^ℓ . The sheaf of pseudomeromorphic currents on V is denoted \mathcal{PM}_V . Pseudomeromorphic currents were originally introduced in [10] but with a more restrictive definition; simple projections were not allowed. In this paper we adopt the definition of pseudomeromorphic currents in [7].

Example 2.1. Let $f \in \mathcal{O}(V)$ be generically non-vanishing and let α be a smooth form on V . Then α/f is a semi-meromorphic form on V and it defines a *semi-meromorphic current*, also denoted α/f , on V by

$$(2.2) \quad \xi \mapsto \lim_{\epsilon \rightarrow 0} \int_V \chi(|h|/\epsilon) \frac{\alpha}{f} \wedge \xi,$$

where ξ is a test form on V and $h \in \mathcal{O}(V)$ is generically non-vanishing and vanishes on $\{f = 0\}$. That (2.2) indeed gives a well-defined current is proved in [22]; the existence of the limit in (2.2) relies on Hironaka's theorem on resolution of singularities. Let $\pi: \tilde{V} \rightarrow V$ be a smooth modification such that $\{\pi^*f = 0\}$ is a normal crossings divisor. Locally on \tilde{V} one can thus choose coordinates so that π^*f is a monomial. One can then show that the semi-meromorphic current α/f is the push forward under π of elementary pseudomeromorphic currents (2.1) with $r = 0$; hence, $\alpha/f \in \mathcal{PM}(V)$.

The $(0, 1)$ -current $\bar{\partial}(1/f)$ is the residue current of f . Since the class of elementary pseudomeromorphic currents is closed under $\bar{\partial}$ it follows that also $\bar{\partial}(1/f) \in \mathcal{PM}(V)$. Moreover, since the action of $1/f$ on test forms is given by (2.2) with $\alpha = 1$ it follows

from Stokes' theorem that

$$\bar{\partial} \frac{1}{f} \cdot \xi = \lim_{\epsilon \rightarrow 0} \int_V \frac{\bar{\partial} \chi(|h|/\epsilon)}{f} \wedge \xi.$$

□

One crucial property of pseudomeromorphic currents is the following, see, e.g., [7, Proposition 2.3].

Dimension principle. *Let $\mu \in \mathcal{PM}(V)$ and assume that μ has support on the subvariety $Z \subset V$. If $\dim V - \dim Z > q$ and μ has bidegree $(*, q)$, then $\mu = 0$.*

The subsheaf of \mathcal{PM}_V of currents with the SEP is denoted \mathcal{W}_V . It is closed under multiplication by smooth forms and if $\pi: \tilde{V} \rightarrow V$ is either a modification or a simple projection then $\pi_*: \mathcal{W}(\tilde{V}) \rightarrow \mathcal{W}(V)$. A natural subclass of $\mathcal{W}(V)$ is the class of *almost semi-meromorphic currents* on V ; a current μ on V is said to be almost semi-meromorphic if there is a smooth modification $\pi: \tilde{V} \rightarrow V$ and a semi-meromorphic current $\tilde{\mu}$ on \tilde{V} such that $\pi_* \tilde{\mu} = \mu$, see [7]. Notice that almost semi-meromorphic currents are generically smooth and have principal value-type singularities.

Proposition 2.2 (Proposition 2.7 in [7]). *Let α be an almost semi-meromorphic current on V and let $\mu \in \mathcal{W}(V)$. Then the current $\alpha \wedge \mu$, a priori defined where α is smooth, has a unique extension to a current in $\mathcal{W}(V)$.*

We will also have use for the following slight variation of [6, Theorem 1.1 (ii)].

Proposition 2.3. *Let $Z \subset V$ be a pure dimensional analytic subset and let $\mathcal{J} \subset \mathcal{O}_V$ be the ideal sheaf of holomorphic functions vanishing on Z . Assume that $\tau \in \mathcal{PM}(V)$ has the SEP with respect to Z and that $h\tau = dh \wedge \tau = 0$ for all $h \in \mathcal{J}$. Then there is a current $\mu \in \mathcal{PM}(Z)$ with the SEP such that $\iota_* \mu = \tau$, where $\iota: Z \hookrightarrow V$ is the inclusion.*

Proof. Let $i: V \hookrightarrow D$ be the inclusion. By [6, Theorem 1.1 (i)] we have that $i_* \tau \in \mathcal{PM}(D)$. It is straightforward to verify that $i_* \tau$ has the SEP with respect to Z considered now as a subset of D and that $hi_* \tau = dh \wedge i_* \tau = 0$ for all $h \in \mathcal{J}$, where we now consider \mathcal{J} as the ideal sheaf of Z in D . Hence, it is sufficient to show the proposition when V is smooth. To this end, we will see that there is a current μ on Z such that $\iota_* \mu = \tau$; then the proposition follows from [6, Theorem 1.1 (ii)].

The existence of such a μ is equivalent to that $\tau \cdot \xi = 0$ for all test forms ξ such that $\iota^* \xi = 0$ on Z_{reg} . By, e.g., [7, Proposition 2.3] and the assumption on τ it follows that $\bar{h}\tau = d\bar{h} \wedge \tau = h\tau = dh \wedge \tau = 0$ for every $h \in \mathcal{J}$. Using this it is straightforward to check that if $p \in Z_{reg}$ and ξ is a smooth form such that $\iota^* \xi = 0$ in a neighborhood of p , then $\xi \wedge \tau = 0$ in a neighborhood of p . Thus, if g is a holomorphic tuple in V cutting out Z_{sing} , then $\chi(|g|/\epsilon)\tau \cdot \xi = 0$ for any test form ξ such that $\iota^* \xi = 0$ on Z_{reg} . Since τ has the SEP with respect to Z it follows that $\tau \cdot \xi = 0$ for all test forms ξ such that $\iota^* \xi = 0$ on Z_{reg} . □

2.2. Residue currents. We briefly recall the the construction in [9] of a residue current associated to a generically exact complex of Hermitian vector bundles.

Let \mathcal{J} be the radical ideal sheaf in D associated with $V \subset D$. Possibly after shrinking D somewhat there is a free resolution

$$(2.3) \quad 0 \rightarrow \mathcal{O}(E_m) \xrightarrow{f_m} \dots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0),$$

of $\mathcal{O}_D/\mathcal{J}$, where E_k are trivial vector bundles, E_0 is the trivial line bundle, f_k are holomorphic mappings, and $m \leq N$. The resolution (2.3) induces a complex of vector bundles

$$0 \rightarrow E_m \xrightarrow{f_m} \dots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0$$

that is pointwise exact outside V . Let p be the codimension of V in D , let for $r \geq 1$ V^r be the set where $f_{p+r}: E_{p+r} \rightarrow E_{p+r-1}$ does not have optimal rank², and let $V^0 := V_{\text{sing}}$. Then

$$(2.4) \quad \dots \subset V^{k+1} \subset V^k \subset \dots \subset V^1 \subset V^0 \subset V$$

and these sets are in fact independent of the choice of resolution (2.3) and of the embedding $V \hookrightarrow D$, i.e., they are invariants of the sheaf $\mathcal{O}_V = \mathcal{O}_D/\mathcal{J}$, and they somehow measure the singularities of V . Since V has pure dimension it follows from Corollary 20.14 in [17] that

$$\dim V^r < n - r, \quad r \geq 0.$$

Hence, $V^n = \emptyset$ and so f_N has optimal rank everywhere; we may thus assume that $m \leq N - 1$ in (2.3). Notice also that $V^r = \emptyset$ for $r \geq 1$ if and only if there is a resolution (2.3) with $m = p$ of \mathcal{O}_V , i.e., if and only if V is Cohen-Macaulay.

Given Hermitian metrics on the E_j , following [9], one can construct a smooth form $u = \sum_{k \geq 1} u_k$ in $D \setminus V$, where u_k is a $(0, k - 1)$ -form taking values in E_k , such that

$$(2.5) \quad f_1 u_1 = 1, \quad f_{k+1} u_{k+1} = \bar{\partial} u_k, \quad k = 1, \dots, m - 1, \quad \bar{\partial} u_m = 0 \quad \text{in } D \setminus V.$$

Moreover, if F is a holomorphic tuple in D vanishing on V , then it is proved that

$$(2.6) \quad \lambda \mapsto |F|^{2\lambda} u,$$

a priori defined for $\Re \lambda \gg 0$, has an analytic continuation as a current-valued function to a neighborhood of the origin. The value at $\lambda = 0$ is a pseudomeromorphic current $U = \sum_{k \geq 1} U_k$, where U_k is a $(0, k - 1)$ -current taking values in E_k , that one should think of as a generalization of the meromorphic current $1/f$ in D when $V = f^{-1}(0)$ is a hypersurface. The residue current $R = \sum_{k \geq 0} R_k$ associated with V is then defined by

$$R_0 = 1 - f_1 U_1, \quad R_k = \bar{\partial} U_k - f_{k+1} U_{k+1}, \quad k = 1, \dots, m - 1, \quad R_m = \bar{\partial} U_m.$$

Hence, R_k is a pseudomeromorphic $(0, k)$ -current in D with values in E_k , and from (2.5) it follows that R_k has support on V . By the dimension principle, thus $R = R_p + \dots + R_m$. Notice that if V is Cohen-Macaulay then $R = R_p$ and $\bar{\partial} R = 0$. By [9, Theorem 1.1] we have that if $h \in \mathcal{O}_D$ then

$$(2.7) \quad hR = 0 \quad \text{if and only if} \quad h \in \mathcal{J}.$$

For future reference we note that

$$(2.8) \quad \lambda \mapsto \bar{\partial} |F|^{2\lambda} \wedge u,$$

a priori defined for $\Re \lambda \gg 0$, has an analytic continuation as a current-valued function to a neighborhood of the origin and the value at $\lambda = 0$ is R ; cf. (2.6).

²For $j \leq p$, the set where f_j does not have optimal rank is V .

Example 2.4. Let $V = f^{-1}(0)$ be a hypersurface in D . Then $0 \rightarrow \mathcal{O}(E_1) \xrightarrow{f} \mathcal{O}(E_0)$ is a resolution of $\mathcal{O}/\langle f \rangle$, where E_1 and E_0 are auxiliary trivial line bundles. The associated current U then becomes $(1/f) \otimes e_1$, where e_1 is a holomorphic frame for E_1 , and the associated residue current R is $\bar{\partial}(1/f) \otimes e_1$.

Let $g_1, \dots, g_p \in \mathcal{O}(D)$ be a regular sequence. Then the Koszul complex associated to the g_j is a free resolution of $\mathcal{O}_D/\langle g_1, \dots, g_p \rangle$. The associated residue current R then becomes the Coleff-Herrera product [14]

$$\bar{\partial} \frac{1}{g_1} \wedge \dots \wedge \bar{\partial} \frac{1}{g_p}$$

times an auxiliary frame element, see [2, Theorem 1.7]. \square

2.3. Structure forms of a complex space. Assume first that V is a reduced hypersurface, i.e., $V = f^{-1}(0) \subset D \subset \mathbb{C}^N$, $N = n + 1$, where $f \in \mathcal{O}(D)$ and $df \neq 0$ on V_{reg} . Let ω' be a meromorphic $(n, 0)$ -form in $D \subset \mathbb{C}^{n+1}$ such that

$$df \wedge \omega' = 2\pi i dz_1 \wedge \dots \wedge dz_{n+1} \quad \text{on } V_{reg}.$$

Then $\omega := i^* \omega'$, where $i: V \hookrightarrow D$ is the inclusion, is a meromorphic form on V that is uniquely determined by f ; ω is the Poincaré residue of the meromorphic form $2\pi i dz_1 \wedge \dots \wedge dz_{n+1}/f(z)$. For brevity we will sometimes write dz for $dz_1 \wedge \dots \wedge dz_N$. Leray's residue formula can be formulated as

$$\int \bar{\partial} \frac{1}{f} \wedge dz \wedge \xi = \lim_{\epsilon \rightarrow 0} \int_V \chi(|h|/\epsilon) \omega \wedge i^* \xi,$$

where ξ is a $(0, n)$ -test form in D , the left hand side is the action of $\bar{\partial}(1/f)$ on $dz \wedge \xi$ and h is a holomorphic tuple cutting out V_{sing} . If we consider ω as a meromorphic current on V we can rephrase this as

$$(2.9) \quad \bar{\partial} \frac{1}{f} \wedge dz = i_* \omega.$$

Assume now that $V \xrightarrow{i} D \subset \mathbb{C}^N$ is an arbitrary pure n -dimensional analytic subset. From Section 2.2 we have, given a free resolution (2.3) of $\mathcal{O}_D/\mathcal{I}_V$ and a choice of Hermitian metrics on the involved bundles E_j , an associated residue current R that plays the role of $\bar{\partial}(1/f)$. By the following result, which is an abbreviated version of [7, Proposition 3.3], there is an almost semi-meromorphic current ω on V such that $R \wedge dz = i_* \omega$; such a current will be called a *structure form* of V .

Proposition 2.5. *Let (2.3) be a Hermitian free resolution of $\mathcal{O}_D/\mathcal{I}_V$ in D and let R be the associated residue current. Then there is a unique almost semi-meromorphic current*

$$\omega = \omega_0 + \omega_1 + \dots + \omega_{n-1}$$

on V , where ω_r is smooth on V_{reg} , has bidegree (n, r) , and takes values in $E_{p+r}|_V$, such that

$$(2.10) \quad R \wedge dz_1 \wedge \dots \wedge dz_N = i_* \omega.$$

Moreover,

$$f_p|_V \omega_0 = 0, \quad f_{p+r}|_V \omega_r = \bar{\partial} \omega_{r-1}, \quad r \geq 1,$$

in the sense of currents on V , and there are $(0, 1)$ -forms α_k , $k \geq 1$, that are smooth outside V^k and that take values in $\text{Hom}(E_{p+k-1}|_V, E_{p+k}|_V)$, such that

$$\omega_k = \alpha_k \omega_{k-1}, \quad k \geq 1.$$

It is sometimes useful to reformulate (2.10) suggestively as

$$(2.11) \quad R \wedge dz_1 \wedge \cdots \wedge dz_N = \omega \wedge [V],$$

where $[V]$ is the current of integration along V .

The following result will be useful for us when defining our dualizing complex.

Proposition 2.6 (Lemma 3.5 in [7]). *If ψ is a smooth (n, q) -form on V , then there is a smooth $(0, q)$ -form ψ' on V with values in $E_p^*|_V$ such that $\psi = \omega_0 \wedge \psi'$.*

2.4. Koppelman formulas in \mathbb{C}^N . We recall some basic constructions from [1] and [4].

Let $D \subset \mathbb{C}^N$ be a domain (not necessarily pseudoconvex at this point), let $k(z, \zeta)$ be an integrable $(N, N-1)$ -form in $D \times D$, and let $p(z, \zeta)$ be a smooth (N, N) -form in $D \times D$. Assume that k and p satisfy the equation of currents

$$(2.12) \quad \bar{\partial}k(z, \zeta) = [\Delta^D] - p(z, \zeta)$$

in $D \times D$, where $[\Delta^D]$ is the current of integration along the diagonal. Applying this current equation to test forms $\psi(z) \wedge \varphi(\zeta)$ it is straightforward to verify that for any compactly supported (p, q) -form φ in D one has the following Koppelman formula

$$\varphi(z) = \bar{\partial}_z \int_{D_\zeta} k(z, \zeta) \wedge \varphi(\zeta) + \int_{D_\zeta} k(z, \zeta) \wedge \bar{\partial}\varphi(\zeta) + \int_{D_\zeta} p(z, \zeta) \wedge \varphi(\zeta).$$

In [1] Andersson introduced a very flexible method of producing solutions to (2.12). Let $\eta = (\eta_1, \dots, \eta_N)$ be a holomorphic tuple in $D \times D$ that defines the diagonal and let Λ_η be the exterior algebra spanned by $T_{0,1}^*(D \times D)$ and the $(1, 0)$ -forms $d\eta_1, \dots, d\eta_N$. On forms with values in Λ_η interior multiplication with $2\pi i \sum \eta_j \partial / \partial \eta_j$, denoted δ_η , is defined; put $\nabla_\eta = \delta_\eta - \bar{\partial}$.

Let s be a smooth $(1, 0)$ -form in Λ_η such that $|s| \lesssim |\eta|$ and $|\eta|^2 \lesssim |\delta_\eta s|$ and let $B = \sum_{k=1}^N s \wedge (\bar{\partial}s)^{k-1} / (\delta_\eta s)^k$. It is proved in [1] that then $\nabla_\eta B = 1 - [\Delta^D]$. Identifying terms of top degree we see that $\bar{\partial}B_{N,N-1} = [\Delta^D]$ and we have found a solution to (2.12). For instance, if we take $s = \partial|\zeta - z|^2$ and $\eta = \zeta - z$, then the resulting B is sometimes called the full Bochner-Martinelli form and the term of top degree is the classical Bochner-Martinelli kernel.

A smooth section $g(z, \zeta) = g_{0,0} + \cdots + g_{N,N}$ of Λ_η , defined for $z \in D' \Subset D$ and $\zeta \in D$, such that $\nabla_\eta g = 0$ and $g_{0,0}|_{\Delta^D} = 1$ is called a *weight* with respect to $z \in D'$. It follows that $\nabla_\eta(g \wedge B) = g - [\Delta^D]$ and, identifying terms of bidegree $(N, N-1)$, we get that

$$(2.13) \quad \bar{\partial}(g \wedge B)_{N,N-1} = [\Delta^D] - g_{N,N}$$

in $D' \times D$ and hence another solution to (2.12). If D is pseudoconvex and K is a holomorphically convex compact subset, then one can find a weight g with respect to z in some neighborhood $D' \Subset D$ of K such that $z \mapsto g(z, \zeta)$ is holomorphic in D' and $\zeta \mapsto g(z, \zeta)$ has compact support in D ; see, e.g., Example 2 in [4].

2.5. Koppelman formulas for $(0, q)$ -forms on a complex space. We briefly recall from [7] the construction of Koppelman formulas for $(0, q)$ -forms on $V \subset D$. The basic idea is to use the currents U and R discussed in Section 2.2 to construct a weight that will yield an integral formula of division/interpolation type in the same spirit as in [13].

Let (2.3) be a resolution of $\mathcal{O}_D/\mathcal{J}$, where as before \mathcal{J} is the sheaf in D associated to $V \xrightarrow{i} D$. One can find, see [4, Proposition 5.3], holomorphic $(k-\ell, 0)$ -form-valued Hefer morphisms, i.e., matrices $H_k^\ell: E_k \rightarrow E_\ell$ depending holomorphically on z and ζ such that $H_k^k = I_{E_k}$ and

$$\delta_\eta H_k^\ell = H_{k-1}^\ell f_k(\zeta) - f_{\ell+1}(z) H_k^{\ell+1}, \quad k > 1.$$

Let $U^\lambda = |F(\zeta)|^{2\lambda} u(\zeta)$ and let $R^\lambda = \bar{\partial}|F(\zeta)|^{2\lambda} \wedge u(\zeta)$, cf. (2.6) and (2.8). Then

$$(2.14) \quad \gamma^\lambda := \sum_{k=0}^N H_k^0 R_k^\lambda + f_1(z) \sum_{k=1}^N H_k^1 U_k^\lambda.$$

is a weight if $\Re \lambda \gg 0$. Let also g be an arbitrary weight. Then $\gamma^\lambda \wedge g$ is again a weight and we get

$$(2.15) \quad \bar{\partial}(\gamma^\lambda \wedge g \wedge B)_{N, N-1} = [\Delta^D] - (\gamma^\lambda \wedge g)_{N, N}$$

in the current sense in $D \times D$, cf. (2.13). Let us proceed formally and, also, let us temporarily assume that V is Cohen-Macaulay so that R is $\bar{\partial}$ -closed. Then, multiplying (2.15) with $R(z) \wedge dz$ and using (2.7) so that $f_1(z)R(z) = 0$, we get that

$$(2.16) \quad \bar{\partial} \left(R(z) \wedge dz \wedge (HR^\lambda \wedge g \wedge B)_{N, N-1} \right) = R(z) \wedge dz \wedge [\Delta^D] - R(z) \wedge dz \wedge (HR^\lambda \wedge g)_{N, N},$$

where $HR^\lambda = \sum_{k=0}^N H_k^0 R_k^\lambda$, cf. (2.14). In view of (2.11) we have $R(z) \wedge dz \wedge [\Delta^D] = \omega \wedge [\Delta^V]$, where $[\Delta^V]$ is the integration current along the diagonal $\Delta^V \subset V \times V \subset D \times D$, and formally letting $\lambda = 0$ in (2.16) we thus get

$$(2.17) \quad \bar{\partial} \left(\omega(z) \wedge [V_z] \wedge (HR \wedge g \wedge B)_{N, N-1} \right) = \omega \wedge [\Delta^V] - \omega(z) \wedge [V_z] \wedge (HR \wedge g)_{N, N}.$$

To see what this means we will use (2.11). Notice first that one can factor out $d\eta = d\eta_1 \wedge \cdots \wedge d\eta_N$ from $(HR \wedge g \wedge B)_{N, N-1}$ and $(HR \wedge g)_{N, N}$. After making these factorization in (2.17) we may replace $d\eta$ by $C_\eta(z, \zeta) d\zeta$, where $C_\eta(z, \zeta) = N! \det(\partial\eta_j/\zeta_k)$, since $\omega(z) \wedge [V_z]$ has full degree in dz_j . More precisely, let $\epsilon_1, \dots, \epsilon_N$ be a basis for an auxiliary trivial complex vector bundle over $D \times D$ and replace all occurrences of $d\eta_j$ in H , g , and B by ϵ_j . Denote the resulting forms by \hat{H} , \hat{g} , and \hat{B} respectively and let

$$(2.18) \quad k(z, \zeta) = C_\eta(z, \zeta) \epsilon_N^* \wedge \cdots \wedge \epsilon_1^* \lrcorner \sum_{k=0}^n \hat{H}_{p+k}^0 \omega_k(\zeta) \wedge (\hat{g} \wedge \hat{B})_{n-k, n-k-1}$$

$$(2.19) \quad p(z, \zeta) = C_\eta(z, \zeta) \epsilon_N^* \wedge \cdots \wedge \epsilon_1^* \lrcorner \sum_{k=0}^n \hat{H}_{p+k}^0 \omega_k(\zeta) \wedge \hat{g}_{n-k, n-k}.$$

Notice that k and p have bidegrees $(n, n-1)$ and (n, n) respectively. In view of (2.11) we can replace $(HR \wedge g \wedge B)_{N, N-1}$ and $(HR \wedge g)_{N, N}$ with $[V_\zeta] \wedge k(z, \zeta)$ and $[V_\zeta] \wedge p(z, \zeta)$ respectively in (2.17). It follows that

$$\bar{\partial}(\omega(z) \wedge k(z, \zeta)) = \omega \wedge [\Delta^V] - \omega(z) \wedge p(z, \zeta)$$

holds in the current sense at least on $V_{reg} \times V_{reg}$. The formal computations above can be made rigorous, see [7, Section 5], and combined with Proposition 2.6 we get Proposition 2.7 below; notice that $\omega = \omega_0$ and $\bar{\partial}\omega = 0$ since we are assuming that V is Cohen-Macaulay.

The following result will be the starting point of the next section and it holds without any assumption about Cohen-Macaulay.

Proposition 2.7 (Lemma 5.3 in [7]). *With $k(z, \zeta)$ and $p(z, \zeta)$ defined by (2.18) and (2.19) respectively we have*

$$\bar{\partial}k(z, \zeta) = [\Delta^V] - p(z, \zeta)$$

in the sense of currents on $V_{reg} \times V_{reg}$.

Remark 2.8. In [7] it is assumed that the weight g in k and p has compact support in D_ζ but the proof goes through for any weight.

The integral operators \mathcal{K} and \mathcal{P} for forms in $\mathcal{W}^{0,q}$ introduced in [7] are defined as follows. Let $\tilde{\pi}: V_z \times V_\zeta \rightarrow V_z$ be the natural projection onto V_z , let g in (2.18) and (2.19) be a weight with respect to z in some $D' \Subset D$ and with compact support in D_ζ , and let $\mu \in \mathcal{W}^{0,q}(D)$. Since ω and B are almost semi-meromorphic $k(z, \zeta)$ and $p(z, \zeta)$ are also almost semi-meromorphic and it follows from Proposition 2.2 that $k(z, \zeta) \wedge \mu(\zeta)$ and $p(z, \zeta) \wedge \mu(\zeta)$ are in $\mathcal{W}(V' \times V)$, where $V' = D' \cap V$. It follows that

$$\mathcal{K}\mu(z) := \tilde{\pi}_*(k(z, \zeta) \wedge \mu(\zeta)),$$

$$\mathcal{P}\mu(z) := \tilde{\pi}_*(p(z, \zeta) \wedge \mu(\zeta)),$$

are in $\mathcal{W}(V'_z)$. The sheaves $\mathcal{A}_V^{0,\bullet}$ are then morally defined to be the smallest sheaves that contain $\mathcal{E}_V^{0,\bullet}$ and are closed under operators \mathcal{K} and under multiplication with $\mathcal{E}_V^{0,\bullet}$. More precisely, the stalk $\mathcal{A}_{V,x}^{0,q}$ consists of those germs of currents which can be written as a finite sum of of terms

$$\xi_m \wedge \mathcal{K}_m(\cdots \xi_1 \wedge \mathcal{K}_1(\xi_0) \cdots),$$

where ξ_j are smooth $(0,*)$ -forms and \mathcal{K}_j are integral operators at x of the above form; cf. [7, Definition 7.1].

Theorem 2.9 (Theorem 1.2 [7]). *Let X be a reduced complex space of pure dimension n . The sheaves $\mathcal{A}_X^{0,q}$ are fine sheaves of $(0, q)$ -currents on X , they contain $\mathcal{E}_X^{0,q}$, and moreover*

- (i) $\oplus_q \mathcal{A}_X^{0,q}$ is a module over $\oplus_q \mathcal{E}_X^{0,q}$,
- (ii) $\mathcal{A}_X^{0,q}|_{X_{reg}} = \mathcal{E}_X^{0,q}|_{X_{reg}}$,
- (iii) the complex $(\mathcal{A}_X^{0,\bullet}, \bar{\partial})$ is a resolution of \mathcal{O}_X .

3. KOPPELMAN FORMULAS FOR (n, q) -FORMS

Let V be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$ and let ω be a structure form on V . Let $k(z, \zeta)$ and $p(z, \zeta)$ be the kernels defined respectively in (2.18) and (2.19). Since k and p are almost semi-meromorphic it follows from Proposition 2.2 that if $\mu = \mu(z) \in \mathcal{W}^{n,q}(V)$, then $k(z, \zeta) \wedge \mu(z)$ and $p(z, \zeta) \wedge \mu(z)$ are well-defined currents in $\mathcal{W}(V \times V)$. Assume that the weight g in (2.18) and (2.19) has compact support in V_z or that μ has compact support in V_z . Let $\pi: V_z \times V_\zeta \rightarrow V_\zeta$ be the natural projection and define

$$(3.1) \quad \check{\mathcal{K}}\mu(\zeta) := \pi_*(k(z, \zeta) \wedge \mu(z))$$

$$(3.2) \quad \check{\mathcal{P}}\mu(\zeta) := \pi_*(p(z, \zeta) \wedge \mu(z)).$$

It follows that $\check{\mathcal{K}}\mu$ and $\check{\mathcal{P}}\mu$ are well-defined currents in \mathcal{W} . Notice that $\check{\mathcal{P}}\mu$ is of the form $\sum_r \omega_r \wedge \xi_r$, where ξ_r is a smooth $(0, *)$ -form (with values in an appropriate bundle) in general, and holomorphic if the weight $g(z, \zeta)$ is chosen holomorphic in ζ ; cf. (2.19). It is natural to write

$$\check{\mathcal{K}}\mu(\zeta) = \int_{V_z} k(z, \zeta) \wedge \mu(z), \quad \check{\mathcal{P}}\mu(\zeta) = \int_{V_z} p(z, \zeta) \wedge \mu(z).$$

We have the following analogue of Proposition 6.3 in [7].

Proposition 3.1. *Let $\mu(z) \in \mathcal{W}^{n,q}(V)$ and assume that $\bar{\partial}\mu \in \mathcal{W}^{n,q+1}(V)$. Let g in (2.18) and (2.19) be a weight with respect to ζ in some $D' \Subset D$. If either μ has compact support in V or g has compact support in D_z , then*

$$(3.3) \quad \mu = \bar{\partial}\check{\mathcal{K}}\mu + \check{\mathcal{K}}(\bar{\partial}\mu) + \check{\mathcal{P}}\mu$$

in the sense of currents on $V'_{reg} = D' \cap V_{reg}$.

Proof. If $\varphi = \varphi(\zeta)$ is a $(0, n - q)$ -test form on V'_{reg} it follows, cf. the beginning of Section 2.4, from Proposition 2.7 that

$$\varphi(z) = \bar{\partial}_z \int_{V_\zeta} k(z, \zeta) \wedge \varphi(\zeta) + \int_{V_\zeta} k(z, \zeta) \wedge \bar{\partial}\varphi(\zeta) + \int_{V_\zeta} p(z, \zeta) \wedge \varphi(\zeta)$$

for $z \in V_{reg}$. By [7, Lemma 6.1]³ and since $p(z, \zeta)$ is smooth in z each term on the right hand side is smooth on V . Moreover, since k and p have compact support in z each term is in fact a test form on V so that μ acts on each term. Thus (3.3) follows provided that μ has compact support in V_{reg} .

For the general case, let $h = h(z)$ be a holomorphic tuple cutting out V_{sing} and let $\chi_\epsilon = \chi(|h|/\epsilon)$. Then the proposition holds for $\chi_\epsilon\mu$ (since k and p have compact support in z). Since $k(z, \zeta) \wedge \mu(z)$ and $p(z, \zeta) \wedge \mu(z)$ are in $\mathcal{W}(V \times V')$ it follows that $\check{\mathcal{K}}(\chi_\epsilon\mu) \rightarrow \check{\mathcal{K}}\mu$ and that $\check{\mathcal{P}}(\chi_\epsilon\mu) \rightarrow \check{\mathcal{P}}\mu$ in the sense of currents, and consequently $\bar{\partial}\check{\mathcal{K}}(\chi_\epsilon\mu) \rightarrow \bar{\partial}\check{\mathcal{K}}\mu$ in the current sense. It remains to see that $\lim_{\epsilon \rightarrow 0} \check{\mathcal{K}}(\bar{\partial}(\chi_\epsilon\mu)) = \check{\mathcal{K}}(\bar{\partial}\mu)$. In fact, since by assumption $\bar{\partial}\mu \in \mathcal{W}(V)$ it follows that $\check{\mathcal{K}}(\chi_\epsilon\bar{\partial}\mu) \rightarrow \check{\mathcal{K}}(\bar{\partial}\mu)$ and so

$$(3.4) \quad \lim_{\epsilon \rightarrow 0} \check{\mathcal{K}}(\bar{\partial}(\chi_\epsilon\mu)) = \check{\mathcal{K}}(\bar{\partial}\mu) + \lim_{\epsilon \rightarrow 0} \check{\mathcal{K}}(\bar{\partial}\chi_\epsilon \wedge \mu);$$

it also follows that

$$(3.5) \quad \bar{\partial}\chi_\epsilon \wedge \mu = \bar{\partial}(\chi_\epsilon\mu) - \chi_\epsilon\bar{\partial}\mu \rightarrow \bar{\partial}\mu - \bar{\partial}\mu = 0.$$

Now, if ζ is in a compact subset of V'_{reg} and ϵ is sufficiently small, then $k(z, \zeta) \wedge \bar{\partial}\chi_\epsilon$ is a smooth form times $\omega = \omega(\zeta)$. Since $\mu(z) \wedge \omega(\zeta)$ is just a tensor product it follows from (3.5) that $\bar{\partial}\chi_\epsilon \wedge \mu(z) \wedge \omega(\zeta) \rightarrow 0$. Hence, $\check{\mathcal{K}}(\bar{\partial}\chi_\epsilon \wedge \mu) \rightarrow 0$ as a current on V'_{reg} and so by (3.4) we have $\lim_{\epsilon \rightarrow 0} \check{\mathcal{K}}(\bar{\partial}(\chi_\epsilon\mu)) = \check{\mathcal{K}}(\bar{\partial}\mu)$. \square

4. THE DUALIZING DOLBEAULT COMPLEX OF $\mathcal{A}_X^{n,q}$ -CURRENTS

Let X be a reduced complex space of pure dimension n . We define our sheaves $\mathcal{A}_X^{n,\bullet}$ in a way similar to the definition of $\mathcal{A}_X^{0,\bullet}$; see the end of Section 2.5. In a moral sense $\oplus_q \mathcal{A}_X^{n,q}$ then becomes the smallest sheaf that contains $\oplus_q \mathcal{E}_X^{n,q}$ and that is closed under integral operators $\check{\mathcal{K}}$ and exterior products with elements of $\oplus_q \mathcal{E}_X^{0,q}$.

³The proof goes through also in our setting, i.e., when g not necessarily has compact support in D_ζ but $\varphi(\zeta)$ has.

Definition 4.1. We say that an (n, q) -current ψ on an open set $V \subset X$ is a section of $\mathcal{A}_X^{n, q}$, $\psi \in \mathcal{A}_X^{n, q}(V)$, if, for every $x \in V$, the germ ψ_x can be written as a finite sum of terms

$$(4.1) \quad \xi_m \wedge \check{\mathcal{K}}_m (\cdots \xi_1 \wedge \check{\mathcal{K}}_1 (\omega \wedge \xi_0) \cdots),$$

where ξ_j are smooth $(0, *)$ -forms, $\check{\mathcal{K}}_j$ are integral operators at x given by (3.1) with kernels of the form (2.18), and ω is a structure form at x .

Notice that ω takes values in some bundle $\oplus_j E_j$ so we let ξ_0 take values in $\oplus_j E_j^*$ to make $\omega \wedge \xi_0$ scalar valued.

It is clear that $\check{\mathcal{K}}$ preserves $\oplus_q \mathcal{A}_X^{n, q}$. Notice that we allow $m = 0$ in the definition above so that $\mathcal{A}_X^{n, \bullet}$ contains all currents of the form $\omega \wedge \xi_0$, where ξ_0 is smooth with values in $\oplus_j E_j^*$. Since $\check{\mathcal{P}}\mu$ is of the form $\omega \wedge \xi$ for a smooth ξ , also $\check{\mathcal{P}}$ preserves $\oplus_q \mathcal{A}_X^{n, q}$.

Recall that if $\mu \in \mathcal{W}^{n, *}(V)$, then $\check{\mathcal{K}}\mu \in \mathcal{W}^{n, *}(V')$, where V' is a relatively compact subset of V . Since $\omega \wedge \xi_0 \in \mathcal{W}_X^{n, *}$ it follows that $\mathcal{A}_X^{n, q}$ is a subsheaf of $\mathcal{W}_X^{n, q}$. In fact, by Proposition 4.3 below we can say more.

Definition 4.2. A current $\mu \in \oplus_q \mathcal{W}_X^{n, q}$ is said to be in *the domain of $\bar{\partial}$* , $\mu \in \text{Dom } \bar{\partial}$, if $\bar{\partial}\mu \in \oplus_q \mathcal{W}_X^{n, q}$.

Assume that $\mu \in \mathcal{W}_X^{n, q}$ is smooth on X_{reg} , let h be a holomorphic tuple cutting out X_{sing} , and let $\chi_\epsilon = \chi(|h|/\epsilon)$. Then $\bar{\partial}(\chi_\epsilon \mu) \rightarrow \bar{\partial}\mu$ since μ has the SEP. In view of the first equality in (3.5) it follows that $\bar{\partial}\mu$ has the SEP if and only if $\bar{\partial}\chi_\epsilon \wedge \mu \rightarrow 0$ as $\epsilon \rightarrow 0$; this last condition can be interpreted as a “boundary condition” on μ at X_{sing} .

Proposition 4.3. *Let X be a reduced complex space of pure dimension n . Then*

- (i) $\mathcal{A}_X^{n, q}|_{X_{reg}} = \mathcal{E}_X^{n, q}|_{X_{reg}}$,
- (ii) $\mathcal{E}_X^{n, q} \subset \mathcal{A}_X^{n, q} \subset \text{Dom } \bar{\partial}$.

Proof. Part (i) is proved in the same way as part (i) of Lemma 6.1 in [7].

Let ψ be a smooth (n, q) -form on X and let $\omega = \sum_r \omega_r$ be a structure form. Then, by Proposition 2.6, there is smooth $(0, q)$ -form ξ (with values in the appropriate bundle) such that $\psi = \omega_0 \wedge \xi$ and so $\mathcal{E}_X^{n, q} \subset \mathcal{A}_X^{n, q}$.

To prove the second inclusion of (ii) we may assume that μ is of the form (4.1). Let $k_j(w^{j-1}, w^j)$, $j = 1, \dots, m$, be the integral kernel corresponding to $\check{\mathcal{K}}_j$; w^j are coordinates on V for each j . We define an almost semi-meromorphic current T on V^{m+1} (the $m + 1$ -fold Cartesian product) by

$$(4.2) \quad T := \bigwedge_{j=1}^m k_j(w^{j-1}, w^j) \wedge \omega(w^0),$$

and we let T_r be the term of T corresponding to ω_r . Notice that $\pi_*(\xi \wedge T) = \mu$ for a suitable smooth $(0, *)$ -form ξ on V^{m+1} , where $\pi: V^{m+1} \rightarrow V_{w^m}$ is the natural projection. We will prove that

$$(4.3) \quad \lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h(w^m)|/\epsilon) \wedge T_r = 0,$$

where h is a holomorphic tuple cutting out V_{sing} , by double induction over m and r ; cf. the discussion after Definition 4.2.

If $m = 0$ then $T = \omega(w^0)$ and, since $\bar{\partial}\omega_r = f_{r+1}|_V \omega_{r+1}$ by (2.5), it follows that $\bar{\partial}T$ has the SEP, i.e., $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi(|h|/\epsilon) \wedge T = 0$.

Assume that (4.3) holds for $m \leq k - 1$ and all r . The left hand side of (4.3), with $m = k$, defines a pseudomeromorphic current τ_r of bidegree $(*, kn - k + r + 1)$ since each k_j has bidegree $(*, n - 1)$ and clearly $\text{supp } \tau_r \subset \text{Sing}(V_{w^m}) \times V^m$. If $w^j \neq w^{j-1}$, then $k_j(w^{j-1}, w^j)$ is a smooth form times some structure form $\tilde{\omega}(w^j)$. Thus T , with $m = k$, is a smooth form times the *tensor product* of two currents, each of which is of the form (4.2) with $m < k$. By the induction hypothesis, it follows that (4.3), with $m = k$, holds outside $\{w^j = w^{j-1}\}$ for all j . Hence, τ_r has support in $\{w^1 = \dots = w^k\} \cap (\text{Sing}(V_{w^m}) \times V^m)$, which has codimension at least $kn + 1$ in V^{k+1} . Since τ_0 has bidegree $(*, kn - k + 1)$, $k \geq 1$, it follows from the dimension principle that $\tau_0 = 0$.

By Proposition 2.5, there is a $(0, 1)$ -form α_1 such that $\omega_1 = \alpha_1 \omega_0$ and α_1 is smooth outside V^1 (cf. (2.4)) which has codimension at least 2 in V . Since $\tau_1 = \alpha_1(w^0)\tau_0$ outside $V_{w^0}^1$ and $\tau_0 = 0$ it follows that τ_1 has support in $\{w^1 = \dots = w^k\} \cap (V_{w^0}^1 \times V^m)$. This set has codimension at least $kn + 2$ in V^{m+1} and τ_1 has bidegree $(*, kn - k + 2)$ so the dimension principle shows that $\tau_1 = 0$. Continuing in this way we get that $\tau_r = 0$ for all r and hence, (4.3) holds with $m = k$. \square

Theorem 4.4. *Let X be a reduced complex space of pure dimension n . Then $\bar{\partial}: \mathcal{A}_X^{n,q} \rightarrow \mathcal{A}_X^{n,q+1}$.*

Proof. Let ψ be a germ of a current in $\mathcal{A}_X^{n,q}$ at some point x ; we may assume that

$$\psi = \xi_m \wedge \check{\mathcal{K}}_m (\dots \xi_1 \wedge \check{\mathcal{K}}_1 (\omega \wedge \xi_0) \dots),$$

see Definition 4.1.

We will prove the theorem by induction over m . Assume first that $m = 0$ so that $\psi = \omega \wedge \xi_0$; recall that ξ_0 takes values in $\oplus_j E_j^*$ so that ψ is scalar valued. Then, by Proposition 2.5, we have that

$$\bar{\partial}\psi = \bar{\partial}\omega \wedge \xi_0 \pm \omega \wedge \bar{\partial}\xi_0 = f\omega \wedge \xi_0 \pm \omega \wedge \bar{\partial}\xi_0 = \omega \wedge f^* \xi_0 \pm \omega \wedge \bar{\partial}\xi_0,$$

where $f = \oplus_{r=0}^n f_{p+r}|_V$ and f^* is the transpose of f . Hence, $\bar{\partial}\psi$ is in $\mathcal{A}_X^{n,q+1}$. Assume now that $\bar{\partial}\psi' \in \oplus_q \mathcal{A}_X^{n,q}$, where

$$\psi' = \xi_{m-1} \wedge \check{\mathcal{K}}_{m-1} (\dots \xi_1 \wedge \check{\mathcal{K}}_1 (\omega \wedge \xi_0) \dots).$$

Then $\psi' \in \text{Dom } \bar{\partial} \subset \mathcal{W}_X$ and by Proposition 4.3 ψ' is smooth on X_{reg} . Thus, from Proposition 3.1 it follows that

$$(4.4) \quad \psi' = \bar{\partial}\check{\mathcal{K}}_m \psi' + \check{\mathcal{K}}_m (\bar{\partial}\psi') + \check{\mathcal{P}}_m \psi'$$

in the current sense on V_{reg} , where V is some neighborhood of x . By the induction hypothesis, $\bar{\partial}\psi' \in \oplus_q \mathcal{A}_X^{n,q}$ and since $\check{\mathcal{K}}_m$ and $\check{\mathcal{P}}_m$ preserve $\oplus_q \mathcal{A}_X^{n,q}$ and furthermore $\oplus_q \mathcal{A}_X^{n,q} \subset \text{Dom } \bar{\partial}$ it follows that every term of (4.4) has the SEP. Thus, (4.4) holds in fact on V . Finally, notice that $\psi = \xi_m \wedge \check{\mathcal{K}}_m \psi'$ and so, since ψ' , $\check{\mathcal{K}}_m (\bar{\partial}\psi')$, and $\check{\mathcal{P}}_m \psi'$ all are in $\oplus_q \mathcal{A}_X^{n,q}$, it follows that $\bar{\partial}\psi \in \mathcal{A}_X^{n,q+1}$. \square

Proof of Theorem 1.1. Choose a weight g with respect to $\zeta \in D'$ and with compact support in D_z in the kernels $k(z, \zeta)$ and $p(z, \zeta)$, cf. (2.18) and (2.19), and let $\check{\mathcal{K}}$ and $\check{\mathcal{P}}$ be the associated integral operators.

Let $\psi \in \mathcal{A}^{n,q}(V)$. By Proposition 3.1,

$$(4.5) \quad \psi = \bar{\partial}\check{\mathcal{K}}\psi + \check{\mathcal{K}}(\bar{\partial}\psi) + \check{\mathcal{P}}\psi$$

holds on V'_{reg} . Since $\check{\mathcal{K}}$ and $\check{\mathcal{P}}$ map $\oplus_q \mathcal{A}^{n,q}(V)$ to $\oplus_q \mathcal{A}^{n,q}(V')$ it follows from Theorem 4.4 that every term of (4.5) has the SEP. Hence, (4.5) holds on V' and the theorem follows. \square

Proof of Theorem 1.2. Let V be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let \mathcal{J} be the sheaf in D defined by V , let $i: V \hookrightarrow D$ be the inclusion, and let $p = N - n$ be the codimension of V in D . Let (2.3) be a resolution of $\mathcal{O}_D/\mathcal{J}$ in (possibly a slightly smaller domain still denoted) D and let $\omega = \sum_r \omega_r$ be an associated structure form.

Taking $\mathcal{H}om$'s from the complex (2.3) into \mathcal{O}_D and tensoring with the invertible sheaf Ω_D^N gives the complex

$$(4.6) \quad 0 \rightarrow \mathcal{O}(E_0^*) \otimes_{\mathcal{O}_D} \Omega_D^N \xrightarrow{f_1^*} \dots \xrightarrow{f_m^*} \mathcal{O}(E_m^*) \otimes_{\mathcal{O}_D} \Omega_D^N \rightarrow 0.$$

It is well-known that the cohomology sheaves of (4.6) are isomorphic to $\mathcal{E}xt^\bullet(\mathcal{O}_D/\mathcal{J}, \Omega_D^N)$ and that $\mathcal{E}xt^k(\mathcal{O}_D/\mathcal{J}, \Omega_D^N) = 0$ for $k < p$. Notice that if V is Cohen-Macaulay, i.e., if we can take $m = p = \text{codim } V$ in (2.3), then $\mathcal{E}xt^k(\mathcal{O}_D/\mathcal{J}, \Omega_D^N) = 0$ for $k \neq p$.

We define mappings $\varrho_k: \mathcal{O}(E_{p+k}^*) \otimes \Omega_D^N \rightarrow \mathcal{A}_V^{n,k}$ by letting $\varrho_k(hdz) = 0$ for $k < 0$ and $\varrho_k(hdz) = \omega_k \cdot h$ for $k \geq 0$; here we let $\mathcal{A}_V^{n,k} := 0$ for $k < 0$ and $\mathcal{O}(E_k^*) \otimes \Omega_D^N := 0$ for $k > m$. We get a map

$$(4.7) \quad \varrho_\bullet: (\mathcal{O}(E_{p+\bullet}^*) \otimes \Omega_D^N, f_{p+\bullet}^*) \longrightarrow (\mathcal{A}_V^{n,\bullet}, \bar{\partial})$$

which is a morphism of complexes since if $h \in \mathcal{O}(E_{p+k}^*)$, then, by Proposition 2.5,

$$\bar{\partial}\varrho_k(hdz) = \bar{\partial}\omega_k \cdot h = f_{p+k+1}\omega_{k+1} \cdot h = \omega_{k+1} \cdot f_{p+k+1}^*h = \varrho_{k+1}(f_{p+k+1}^*h).$$

Hence, (4.7) induces a map on cohomology. We claim that ϱ_\bullet in fact is a quasi-isomorphism, i.e., that ϱ_\bullet induces an isomorphism on cohomology level. Given the claim it follows that $\mathcal{H}^k(\mathcal{A}_V^{n,\bullet})$ is coherent since the corresponding cohomology sheaf of $(\mathcal{O}(E_{p+\bullet}^*) \otimes \Omega_D^N, f_{p+\bullet}^*)$ is $\mathcal{E}xt^{p+k}(\mathcal{O}_D/\mathcal{J}, \Omega_D^N)$, which is coherent.

To prove the claim, recall first that $i_*\omega_k = R_k \wedge dz$. Thus, by [5, Theorem 7.1] the mapping on cohomology is injective. For the surjectivity, choose integral operators $\check{\mathcal{K}}$ and $\check{\mathcal{P}}$ corresponding to integral kernels (2.18) and (2.19) respectively, where g is a weight with respect to $\zeta \in D'$ that is holomorphic in ζ and has compact support in D_z . Let $\psi \in \mathcal{A}^{n,k}(V)$ be $\bar{\partial}$ -closed. By Theorem 1.1 we get

$$\psi(\zeta) = \bar{\partial} \int_{V_z} k(z, \zeta) \wedge \psi(z) + \int_{V_z} p(z, \zeta) \wedge \psi(z)$$

in $V \cap D'$. Hence, the $\bar{\partial}$ -cohomology class of ψ is represented by the last integral. For degree reasons it follows from (2.19) that this integral is of the form

$$\omega_k(\zeta) \wedge \int_{V_z} G(z, \zeta) \wedge \psi(z),$$

where G takes values in E_k^* and $G(z, \zeta)$ is holomorphic in ζ since we have chosen the weight g to be. Thus, the class of ψ is in the image of ϱ_k .

If V is Cohen-Macaulay, then (4.6) is exact except for at level p and so $(\mathcal{A}_V^{n,\bullet}, \bar{\partial})$ is exact except for at level 0 where the cohomology is $\omega_V^{n,0} = \ker(\bar{\partial}: \mathcal{A}_V^{n,0} \rightarrow \mathcal{A}_V^{n,1})$. Thus, (1.4) is exact. \square

5. THE TRACE MAP

The basic result of this section is the following theorem. It is the key to define our trace map.

Theorem 5.1. *Let X be a reduced complex space of pure dimension n . There is a unique map*

$$\wedge: \mathcal{A}_X^{n,q} \times \mathcal{A}_X^{0,q'} \rightarrow \mathcal{W}_X^{n,q+q'} \cap \text{Dom } \bar{\partial}$$

extending the exterior product on X_{reg} .

It follows that one can compute this product in the following way: Let $\psi \in \mathcal{A}_{X,x}^{n,q}$, let $\varphi \in \mathcal{A}_{X,x}^{0,q'}$ and let $V \subset X$ be an open set where both ψ and φ are defined. Then, if h is a generically non-vanishing holomorphic tuple such that $V_{\text{sing}} \subset \{h = 0\}$, we have that

$$\psi \wedge \varphi = \lim_{\epsilon \rightarrow 0} \chi(|h|/\epsilon) \psi \wedge \varphi,$$

where the limit is understood in the sense of currents.

Let $\psi \in \mathcal{A}^{n,q}(X)$ and $\varphi \in \mathcal{A}^{0,n-q}(X)$ and assume that at least one of ψ and φ has compact support. Then, by Theorem 5.1, we can define our trace map on the level of currents by mapping (ψ, φ) to the action of $\psi \wedge \varphi$ on the function 1; explicitly, if h is a generically non-vanishing holomorphic tuple such that $X_{\text{sing}} \subset \{h = 0\}$, then

$$(5.1) \quad (\psi, \varphi) \mapsto \lim_{\epsilon \rightarrow 0} \int_X \chi(|h|/\epsilon) \psi \wedge \varphi.$$

This map induces a trace map on cohomology. Indeed, assume that ψ and φ are $\bar{\partial}$ -closed and that one of them, say φ , is $\bar{\partial}$ -exact so that there is a $\tilde{\varphi} \in \mathcal{A}^{0,n-q-1}(X)$, which has compact support if φ has, such that $\varphi = \bar{\partial}\tilde{\varphi}$. By Theorem 5.1, $\psi \wedge \tilde{\varphi}$ is in the domain of $\bar{\partial}$, which implies that $\bar{\partial}\chi_\epsilon \wedge \psi \wedge \tilde{\varphi} \rightarrow 0$ as $\epsilon \rightarrow 0$, where $\chi_\epsilon = \chi(|h|/\epsilon)$; cf. the first equality in (3.5). Hence

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_X \chi_\epsilon \psi \wedge \varphi &= \lim_{\epsilon \rightarrow 0} \int_X \chi_\epsilon \psi \wedge \bar{\partial}\tilde{\varphi} \\ &= (-1)^{n+q} \lim_{\epsilon \rightarrow 0} \left(\int_X \bar{\partial}(\chi_\epsilon \psi \wedge \tilde{\varphi}) - \int_X \bar{\partial}\chi_\epsilon \wedge \psi \wedge \tilde{\varphi} \right) = 0. \end{aligned}$$

Proof of Theorem 5.1. Notice first that if there is a current $\mu \in \mathcal{W}_X$ coinciding with $\psi|_{X_{\text{reg}}} \wedge \varphi|_{X_{\text{reg}}}$ on X_{reg} , then it must be unique. Moreover, then $\mu = \lim_{\epsilon \rightarrow 0} \chi_\epsilon \mu = \lim_{\epsilon \rightarrow 0} \chi_\epsilon \psi \wedge \varphi$. We now prove that such a μ exists.

Let V be a relatively compact open subset of a pure n -dimensional analytic subset of some pseudoconvex domain in some \mathbb{C}^N . Let $\psi \in \mathcal{A}^{n,q}(V)$ and $\varphi \in \mathcal{A}^{0,q'}(V)$. The tensor product $\psi(w) \wedge \varphi(z)$ is of course well defined on $V \times V$. Let $\phi = (\phi_1, \dots, \phi_s)$ be generators for the radical ideal sheaf over $V \times V$ associated to the diagonal $\Delta^V \subset V \times V$. Let

$$M^\lambda = \bar{\partial}|\phi|^{2\lambda} \wedge \frac{\partial \log |\phi|^2}{2\pi i} \wedge (dd^c \log |\phi|^2)^{n-1},$$

where $\lambda \in \mathbb{C}$, $\Re \lambda \gg 0$, and $dd^c = i\bar{\partial}\partial/2\pi$. It is proved in [3] that $\lambda \mapsto M^\lambda$ has an analytic continuation as a current-valued function to a neighborhood of $\lambda = 0$.

Moreover, by [8, Theorem 1.2], $M := M^\lambda|_{\lambda=0} = \beta[\Delta^V]$, where β is the generic multiplicity of Δ^V in $V \times V$. Hence, $M = [\Delta^V]$.

Claim:

$$\lambda \mapsto M^\lambda \wedge \psi(w) \wedge \varphi(z)$$

has an analytic continuation to a neighborhood of $\lambda = 0$ and $M^\lambda \wedge \psi(w) \wedge \varphi(z)|_{\lambda=0}$ defines an intrinsic \mathcal{PM} -current on $\Delta^V \simeq V$ with the SEP, i.e., there is a current $\mu \in \mathcal{W}(\Delta^V)$ such that $i_*\mu = M^\lambda \wedge \psi(w) \wedge \varphi(z)|_{\lambda=0}$, where $i: \Delta^V \rightarrow V \times V$ is the inclusion.

Assume the claim for the moment. Since $M^\lambda|_{\lambda=0} = [\Delta^V]$ and ψ and φ are smooth on $V_{reg} \times V_{reg}$ it follows, after making the identification $\Delta^V \simeq V$, that $\mu = \psi \wedge \varphi$ on V_{reg} . Thus $\psi \wedge \varphi|_{V_{reg}}$ has a $\mathcal{W}(V)$ -extension to V (namely μ); we denote the extension by $\psi \wedge \varphi$ as well.

To prove the claim we may assume, cf. Definition 4.1 and the end of Section 2.5, that

$$\psi = \xi_m \wedge \check{\mathcal{K}}_m(\cdots \xi_1 \wedge \check{\mathcal{K}}_1(\omega \wedge \xi_0) \cdots), \quad \varphi = \tilde{\xi}_\ell \wedge \mathcal{K}_\ell(\cdots \tilde{\xi}_1 \wedge \mathcal{K}_1(\tilde{\xi}_0) \cdots),$$

where ξ_i and $\tilde{\xi}_j$ are smooth $(0, *)$ -forms, $\omega = \sum_k \omega_k$ is a structure form, and $\check{\mathcal{K}}_i$ and \mathcal{K}_j are integral operators for $(n, *)$ -forms and $(0, *)$ -forms respectively. Let $\check{k}_j(w^{j-1}, w^j)$ be the integral kernel corresponding to $\check{\mathcal{K}}_j$ and let $k_j(z^{j-1}, z^j)$ be the integral kernel corresponding to \mathcal{K}_j ; w^j and z^j are coordinates on V . We will assume that for each j , $z^j \mapsto k_{j+1}(z^j, z^{j+1})$ has compact support where $z^j \mapsto k_j(z^{j-1}, z^j)$ is defined and similarly for \check{k}_j ; possibly we will have to multiply by a smooth cut-off function that we however will suppress. The kernels \check{k}_i and k_j are almost semi-meromorphic and M^λ is as smooth as we want if $\Re \lambda$ is sufficiently large and hence, cf. Proposition 2.2,

$$(5.2) \quad T^\lambda := M^\lambda(z^\ell, w^m) \wedge \bigwedge_{j=1}^m \check{k}_j(w^{j-1}, w^j) \wedge \omega(w^0) \wedge \bigwedge_{j=1}^\ell k_j(z^{j-1}, z^j)$$

is an almost semi-meromorphic current on $V^{\ell+m+2}$.⁴ We will consider $\phi = \phi(z^\ell, w^m)$, $\check{k}_j(w^{j-1}, w^j)$, etc. as functions (or forms) on $V^{\ell+m+2}$. By resolution of singularities, there is a modification $\Pi: Y \rightarrow V^{\ell+m+2}$, with Y smooth, such that (locally on Y) we have $\Pi^*\phi = \phi^0 \phi'$, where ϕ^0 is a holomorphic function and ϕ' is a non-vanishing holomorphic tuple, and

$$\Pi^* \left(\bigwedge_{j=1}^m \check{k}_j(w^{j-1}, w^j) \wedge \omega(w^0) \wedge \bigwedge_{j=1}^\ell k_j(z^{j-1}, z^j) \right) = s_0/a,$$

where s_0 is smooth and a is a holomorphic function. A straightforward computation then shows that (locally on Y) we have

$$\Pi^* M^\lambda = \frac{\bar{\partial}|\phi^0 \phi'|^{2\lambda}}{\phi^0} \wedge s_1 + \bar{\partial}|\phi^0 \phi'|^{2\lambda} \wedge s_2, \quad \Re \lambda \gg 0,$$

where the s_1 and s_2 are smooth. From, e.g., [24, Lemma 6] it follows that $\lambda \mapsto \bar{\partial}|\phi^0 \phi'|^{2\lambda}/(\phi^0 a)$ has an analytic continuation as a current-valued function to a neighborhood of $\lambda = 0$ and that the value at $\lambda = 0$ is a \mathcal{PM} -current on Y . Hence,

⁴In this proof V^j will mean either the Cartesian product of j copies of V or the j^{th} set in (2.4). We hope that it will be clear from the context what we are aiming at.

$\lambda \mapsto \Pi^*(M^\lambda) \wedge s_0/a = \Pi^*T^\lambda$ has an analytic continuation to a neighborhood of $\lambda = 0$ and so $\lambda \mapsto T^\lambda$ has an analytic continuation to a neighborhood of $\lambda = 0$ and

$$(5.3) \quad T := T^\lambda|_{\lambda=0}$$

is a \mathcal{PM} -current on $V^{\ell+m+2}$. Moreover, it is clear that the support of T must be contained in $\{z^\ell = w^m\}$.

Let $\pi: V^{\ell+m+2} \rightarrow V_{w^m} \times V_{z^\ell}$ be the natural projection. Since $M^\lambda \wedge \psi(w) \wedge \varphi(z)$ is $\pi_*(T^\lambda)$ times a smooth form it is sufficient to prove our claim with $M^\lambda \wedge \psi(w) \wedge \varphi(z)$ replaced by $\pi_*(T^\lambda)$. We know already that $\lambda \mapsto \pi_*(T^\lambda)$ has an analytic continuation to a neighborhood of $\lambda = 0$ and that $\tau := \pi_*(T^\lambda)|_{\lambda=0}$ is a \mathcal{PM} -current on $V_{w^m} \times V_{z^\ell}$ with support in Δ^V . We will now use the following lemma.

Lemma 5.2. *Let $h = h(z^\ell, w^m)$ be a holomorphic tuple such that $H = \{h = 0\} \subset V^{\ell+m+2}$ intersects $\{z^\ell = w^m\}$ properly and let $\chi_\epsilon = \chi(|h|/\epsilon)$. Let also $g = g(z^\ell, w^m)$ be a holomorphic function vanishing on $\{z^\ell = w^m\}$. Then*

- (i) $\mathbf{1}_H T := T - \lim_{\epsilon \rightarrow 0} \chi_\epsilon T = 0$
- (ii) $\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_\epsilon \wedge T = 0$
- (iii) $gT = 0$.

From part (i) of the lemma it follows that τ has the SEP with respect to Δ^V and from part (iii) it follows that $h\tau = 0$ for any holomorphic function vanishing on Δ^V . Moreover, in $\text{Reg}(V_{w^m}) \times \text{Reg}(V_{z^\ell})$ we have that τ equals $[\Delta^V]$ times a smooth form and since $dh \wedge [\Delta^V] = d(h[\Delta^V]) = 0$ for any holomorphic function vanishing on Δ^V it follows that $dh \wedge \tau = 0$ in $\text{Reg}(V_{w^m}) \times \text{Reg}(V_{z^\ell})$ for any holomorphic function vanishing on Δ^V . Since τ has the SEP with respect to Δ^V this holds in fact on $V_{w^m} \times V_{z^\ell}$. Thus, by Proposition 2.3, there is a $\mu \in \mathcal{W}(\Delta^V)$ such that $i_*\mu = \tau$ and the claim follows.

It remains to show that our current $\psi \wedge \varphi \in \mathcal{W}(V)$ is in the domain of $\bar{\partial}$. Let h and χ_ϵ be as in Lemma 5.2. From part (ii) of Lemma 5.2 it follows that $\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_\epsilon \wedge \tau = 0$. Since $\tau = i_*\mu$ and $\psi \wedge \varphi = \mu$ (after identifying $V \simeq \Delta^V$) we see that $\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi(|\tilde{h}|/\epsilon) \wedge \psi \wedge \varphi = 0$ for any generically non-vanishing holomorphic tuple \tilde{h} on V such that $\{\tilde{h} = 0\} \supset V_{\text{sing}}$. From the discussion after Definition 4.2 it follows that $\psi \wedge \varphi$ indeed is in the domain of $\bar{\partial}$. \square

Proof of Lemma 5.2. Let T_k be the component of T corresponding to $\omega_k(w^0)$. We will show the lemma by double induction over k and $\ell + m$ by using the dimension principle, cf. the proof of Proposition 4.3. Notice first that $\mathbf{1}_H T$ and $\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_\epsilon \wedge T$ have support contained in $\{z^\ell = w^m\} \cap H$.

Consider first the case $\ell + m = 0$; then $T = M^\lambda(z^0, w^0) \wedge \omega(w^0)|_{\lambda=0}$ and part (i) means precisely that T has the SEP with respect to the diagonal $\Delta^V \subset V_{z^0} \times V_{w^0}$. The \mathcal{PM} -current $\mathbf{1}_H T_0$ has bidegree $(2n, n)$ and support contained $\{z^0 = w^0\} \cap H$, which has codimension at least $n + 1$ in $V_{z^0} \times V_{w^0}$, and hence $\mathbf{1}_H T_0 = 0$ by the dimension principle. By Proposition 2.5 there are α_k that are smooth outside V^k such that $\omega_k = \alpha_k \omega_{k-1}$. Thus, since $\mathbf{1}_H T_0 = 0$ it follows that $\mathbf{1}_H T_1 = 0$ outside $V_{z^0} \times V_{w^0}^1$. Moreover, since $\mathbf{1}_H T_1$ is also 0 outside $\{z^0 = w^0\}$, the support of $\mathbf{1}_H T_1$ is contained in $V_{z^0} \times V_{w^0}^1 \cap \{z^0 = w^0\}$, which has codimension $\geq n + 2$ in $V_{z^0} \times V_{w^0}$. Noticing that $\mathbf{1}_H T_1$ has bidegree $(2n, n + 1)$ the dimension principle again shows that $\mathbf{1}_H T_1 = 0$. Continuing in this way we see that $\mathbf{1}_H T_k = 0$ for all k , and so (i) holds in the case $\ell + m = 0$.

To see that (iii) holds in the case $\ell + m = 0$, notice that if $w^0 \in V_{reg}$ then T_0 is the integration current over $\{z^0 = w^0\}$ times a smooth form. Since $g = 0$ on $\{z^0 = w^0\}$ it follows that $gT_0 = 0$ at least outside of $\{z^0 = w^0, w^0 \in V_{sing}\}$. But then, as above, $gT_0 = 0$ and $gT_k = 0$ inductively by the dimension principle.

To see that also (ii) holds in the case $\ell + m = 0$ we proceed as follows: Let \tilde{h} be a generically non-vanishing holomorphic function on V such that $\{\tilde{h} = 0\} \supset V_{sing}$ and let $\tilde{\chi}_\delta = \chi(|\tilde{h}(w^0)|/\delta)$. We have proved that T_k has the SEP with respect to $\Delta^V \subset V_{z^0} \times V_{w^0}$ and so

$$T_k = \lim_{\delta \rightarrow 0} \tilde{\chi}_\delta M(z^0, w^0) \wedge \omega_k(w^0).$$

Let $i: \Delta^V \hookrightarrow V_{z^0} \times V_{w^0}$ be the inclusion of the diagonal. Since ω is smooth outside V_{sing} and M is the integration current over $\{z^0 = w^0\}$ we have

$$(5.4) \quad \bar{\partial}\chi_\epsilon \wedge \tilde{\chi}_\delta M(z^0, w^0) \wedge \omega_k(w^0) = i_* (\bar{\partial}\chi_\epsilon|_{\Delta^V} \wedge \tilde{\chi}_\delta \omega_k).$$

As $\delta \rightarrow 0$, the left hand side of (5.4) goes to $\bar{\partial}\chi_\epsilon \wedge T_k$. Since ω has the SEP with respect to $V \simeq \Delta^V$ it follows that the right hand side goes to $i_*(\bar{\partial}\chi_\epsilon|_{\Delta^V} \wedge \omega_k)$ as $\delta \rightarrow 0$. A straightforward computation and Proposition 2.5 show that

$$\bar{\partial}\chi_\epsilon|_{\Delta^V} \wedge \omega_k = \bar{\partial}(\chi_\epsilon|_{\Delta^V} \omega_k) - \chi_\epsilon|_{\Delta^V} f_{k+1} \omega_{k+1} \rightarrow \bar{\partial}\omega_k - f_{k+1} \omega_{k+1} = 0, \quad \epsilon \rightarrow 0.$$

Hence, $i_*(\bar{\partial}\chi_\epsilon|_{\Delta^V} \wedge \omega_k) \rightarrow 0$ as $\epsilon \rightarrow 0$ and (ii) follows for $\ell + m = 0$.

Assume now that the lemma holds for $\ell + m \leq s - 1$, where $s \geq 1$, and let T be given by (5.2) and (5.3) with $\ell + m = s$. Let $1 \leq j \leq \ell$; if $z^{j-1} \neq z^j$ then $k_j(z^{j-1}, z^j)$ is a smooth form times some structure form $\tilde{\omega}(z^j)$. Hence, outside $\{z^j = z^{j-1}\}$, T is (ignoring smooth factors) the *tensor product* of

$$\tilde{\omega}(z^j) \bigwedge_{i=1}^{j-1} k_i(z^{i-1}, z^i),$$

and some current \tilde{T} , where \tilde{T} is of the form (5.2) and (5.3) but with $\ell + m = s - j$. From the induction hypothesis it thus follows that $\mathbf{1}_H T$, $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi_\epsilon \wedge T$, and gT have supports contained in $\{z^0 = \dots = z^\ell\}$. Similarly, let $1 \leq j \leq m$. If $w^{j-1} \neq w^j$ then $\tilde{k}_j(w^{j-1}, w^j)$ is a smooth form times some structure form $\tilde{\omega}(w^j)$ and so, outside $\{w^j = w^{j+1}\}$, T is (again ignoring smooth factors) the tensor product of

$$\bigwedge_{i=1}^{j-1} \tilde{k}_i(w^{i-1}, w^i) \wedge \omega(w^0)$$

and a current of the form (5.2) and (5.3) with $\ell + m = s - j$. Thus, again from the induction hypothesis, it follows that $\mathbf{1}_H T$, $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi_\epsilon \wedge T$, and gT have supports contained in $\{w^0 = \dots = w^m\}$. In addition, since T vanishes outside $\{z^\ell = w^m\}$, we have that the supports of $\mathbf{1}_H T$, $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi_\epsilon \wedge T$, and gT must be contained in the diagonal $\Delta^V \subset V^{\ell+m+2}$.

The currents $\mathbf{1}_H T_0$ and gT_0 both have bidegree $(*, n(\ell + m + 1) - (\ell + m))$ and since $\Delta^V \simeq V$ has codimension $n(\ell + m + 1)$, the dimension principle shows that $\mathbf{1}_H T_0 = gT_0 = 0$. As in the beginning of the proof, one inductively shows that $\mathbf{1}_H T_k = gT_k = 0$ using that $\omega_k(w^0) = \alpha_k(w^0) \omega_{k-1}(w^0)$ and the dimension principle. Hence (i) and (iii) hold.

To prove (ii) we notice that $\lim_{\epsilon \rightarrow 0} \bar{\partial}\chi_\epsilon \wedge T$ vanishes outside H . Its support is thus contained in $H \cap \Delta^V$, which has codimension at least $n(\ell + m + 1) + 1$. Since

$\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_\epsilon \wedge T_0$ has bidegree $(*, n(\ell+m+1) - (\ell+m) + 1)$ the dimension principle shows that $\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_\epsilon \wedge T_0 = 0$. As above, it follows inductively that $\lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_\epsilon \wedge T_k = 0$. \square

6. SERRE DUALITY

6.1. Local duality. Let V be a pure n -dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$ be a strictly pseudoconvex subdomain, and let $V' = V \cap D'$. Consider the complexes

$$(6.1) \quad 0 \rightarrow \mathcal{A}^{0,0}(V') \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(V') \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}^{0,n}(V') \rightarrow 0$$

$$(6.2) \quad 0 \rightarrow \mathcal{A}_c^{n,0}(V') \xrightarrow{\bar{\partial}} \mathcal{A}_c^{n,1}(V') \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{A}_c^{n,n}(V') \rightarrow 0.$$

From Section 5 we know that we have a well-defined trace map on the level of currents and that it induces a trace map on the level of cohomology

$$(6.3) \quad Tr: H^0(\mathcal{A}^{0,\bullet}(V')) \times H^n(\mathcal{A}_c^{n,\bullet}(V')) \rightarrow \mathbb{C}, \quad Tr([\varphi], [\psi]) = \int_{V'} \varphi \psi.$$

By Theorem 2.9 (iii) the complex (6.1) is exact except for at the level 0 where the cohomology is $\mathcal{O}(V')$.

Theorem 6.1. *The complex (6.2) is exact except for at the top level and the pairing (6.3) makes $H^n(\mathcal{A}_c^{n,\bullet}(V'))$ the topological dual of the Fréchet space $H^0(\mathcal{A}^{0,\bullet}(V')) = \mathcal{O}(V')$; in particular (6.3) is non-degenerate.*

Proof. Let $\psi \in \mathcal{A}_c^{n,q}(V')$ be $\bar{\partial}$ -closed. We choose integral kernels $k(z, \zeta)$ and $p(z, \zeta)$ corresponding to a weight g , with respect to z in some neighborhood of $\text{supp } \psi$ in D' , such that g depends holomorphically on z and has compact support in D'_ζ . Since ψ has compact support in V' , Theorem 1.1 shows that

$$(6.4) \quad \psi(\zeta) = \bar{\partial}_\zeta \int_{V'_z} k(z, \zeta) \wedge \psi(z) + \int_{V'_z} k(z, \zeta) \wedge \bar{\partial} \psi(z) + \int_{V'_z} p(z, \zeta) \wedge \psi(z),$$

holds on V' . The second term on the right hand side vanishes since $\bar{\partial} \psi = 0$. Since g is holomorphic in z the kernel p has degree 0 in $d\bar{z}_j$ and hence, also the last term vanishes if $q \neq n$. The first integral on the right hand side is in $\mathcal{A}_c^{n,q-1}(V')$ since g has compact support in D'_ζ and so (6.2) is exact except for at level n .

To see that $H^n(\mathcal{A}_c^{n,\bullet}(V'))$ is the topological dual of $\mathcal{O}(V')$, recall that the topology on $\mathcal{O}(V') \cong \mathcal{O}(D')/\mathcal{J}(D')$ is the quotient topology, where \mathcal{J} be the sheaf in D associated with $V \subset D$. It is clear that each $[\psi] \in H^n(\mathcal{A}_c^{n,\bullet}(V'))$ yields a continuous linear functional on $\mathcal{O}(V')$ via (6.3). Moreover, if $q = n$ and $\int_{V'} \varphi \psi = 0$ for all $\varphi \in \mathcal{O}(V')$ then, since $p(z, \zeta)$ is holomorphic in z by the choice of g , the last integral on the right hand side of (6.4) vanishes and thus $[\psi] = 0$. Hence, $H^n(\mathcal{A}_c^{n,\bullet}(V'))$ is a subset of the topological dual of $\mathcal{O}(V')$.

To see that there is equality, let λ be a continuous linear functional on $\mathcal{O}(V')$. By composing with the projection $\mathcal{O}(D') \rightarrow \mathcal{O}(D')/\mathcal{J}(D')$ we get a continuous functional $\tilde{\lambda}$ on $\mathcal{O}(D')$. By definition of the topology on $\mathcal{O}(D')$, $\tilde{\lambda}$ is carried by some compact subset $K \Subset D'$. By the Hahn-Banach theorem, $\tilde{\lambda}$ can be extended to a continuous linear functional on $C^0(D')$ and so it is given as integration against some measure μ on D' that has support in a neighborhood $U(K) \Subset D'$ of K . Let $\tilde{p}(z, \zeta)$ be an integral kernel, as in (2.19), corresponding to a weight \tilde{g} with respect to $z \in U(K)$

such that \tilde{g} has compact support in D'_ζ and depends holomorphically on $z \in U(K)$. Let $f \in \mathcal{O}(V')$ and define the sequence $f_\epsilon(z) \in \mathcal{O}(K)$ by

$$f_\epsilon(z) = \int_{V'_\zeta} \chi(|h|/\epsilon) \tilde{p}(z, \zeta) f(\zeta),$$

where $h = h(\zeta)$ is a holomorphic tuple cutting out V_{sing} . For each z in a neighborhood in V' of $K \cap V'$ we have that $\lim_{\epsilon \rightarrow 0} f_\epsilon(z) = \mathcal{P}f(z) = f(z)$ by [7, Theorem 1.4]. We claim that f_ϵ in fact converges uniformly in a neighborhood of K in D' to some $\tilde{f} \in \mathcal{O}(K)$, which then is an extension of f to a neighborhood in D' of K . To see this, first notice by (2.19) that $\tilde{p}(z, \zeta)$ is a sum of terms $\omega_k(\zeta) \wedge p_k(z, \zeta)$ where $p_k(z, \zeta)$ is smooth in both variables and holomorphic for $z \in U(K)$. By Proposition 2.5, the ω_k are almost semi-meromorphic. The claim then follows from a simple instance of [18, Theorem 1]⁵. We now get

$$\begin{aligned} \lambda(f) &= \lim_{\epsilon \rightarrow 0} \int_z f_\epsilon(z) d\mu(z) = \lim_{\epsilon \rightarrow 0} \int_z \int_{V'_\zeta} \chi(|h|/\epsilon) \tilde{p}(z, \zeta) f(\zeta) d\mu(z) \\ &= \lim_{\epsilon \rightarrow 0} \int_{V'_\zeta} f(\zeta) \chi(|h|/\epsilon) \int_z \tilde{p}(z, \zeta) d\mu(z) \\ &= \lim_{\epsilon \rightarrow 0} \int_{V'_\zeta} f(\zeta) \chi(|h|/\epsilon) \sum_k \omega_k(\zeta) \wedge \int_z p_k(z, \zeta) d\mu(z) \\ &= \int_{V'_\zeta} f(\zeta) \sum_k \omega_k(\zeta) \wedge \int_z p_k(z, \zeta) d\mu(z). \end{aligned}$$

But $\zeta \mapsto \int_z p_k(z, \zeta) d\mu(z)$ is smooth and compactly supported in D' and so λ is given as integration against some element $\psi \in \mathcal{A}_c^{n,n}(V')$; hence λ is realized by the cohomology class $[\psi]$ and the theorem follows. \square

Corollary 6.2. *Let $F \rightarrow V$ be a vector bundle, $\mathcal{F} = \mathcal{O}(F)$ the associated locally free \mathcal{O} -module, and $\mathcal{F}^* = \mathcal{O}(F^*)$. Then the following pairing is non-degenerate*

$$Tr: H^0(V', \mathcal{F}) \times H^n(\mathcal{F}^* \otimes \mathcal{A}_c^{n,\bullet}(V')) \rightarrow \mathbb{C}, \quad ([\varphi], [\psi]) \mapsto \int_{V'} \varphi \psi.$$

By Theorem 1.2, if X is Cohen-Macaulay, then the complex $(\mathcal{F}^* \otimes \mathcal{A}_V^{n,\bullet}, \bar{\partial})$ is a resolution of $\mathcal{F}^* \otimes \omega_V^{n,0}$ and so we get a non-degenerate pairing

$$H^0(V', \mathcal{F}) \times H_c^n(V', \mathcal{F}^* \otimes \omega_V^{n,0}) \rightarrow \mathbb{C}.$$

6.2. Global duality. The global duality follows from the local one by an abstract patching argument, see [26], cf. also [11, Theorem (I)]. We will need to make this argument explicit and for this we will use the following perhaps non-standard formalism for Čech cohomology; cf. [23, Section 7.3]

Let \mathcal{F} be a sheaf on X and let $\mathcal{V} = \{V_j\}$ be a locally finite covering of X . We let $C^k(\mathcal{V}, \mathcal{F})$ be the group of formal sums

$$\sum_{i_0 \cdots i_k} f_{i_0 \cdots i_k} V_{i_0} \wedge \cdots \wedge V_{i_k}, \quad f_{i_0 \cdots i_k} \in \mathcal{F}(V_{i_0} \cap \cdots \cap V_{i_k})$$

⁵Take $p = 0$, $q = 1$, and $\mu = 1$ in this theorem.

with the suggestive computation rules, e.g., $f_{12}V_1 \wedge V_2 + f_{21}V_2 \wedge V_1 = (f_{12} - f_{21})V_1 \wedge V_2$. Each element of $C^k(\mathcal{V}, \mathcal{F})$ thus has a unique representation of the form

$$\sum_{i_0 < \dots < i_k} f_{i_0 \dots i_k} V_{i_0} \wedge \dots \wedge V_{i_k}$$

that we will abbreviate as $\sum'_{|I|=k+1} f_I V_I$. The coboundary operator $\delta: C^k(\mathcal{V}, \mathcal{F}) \rightarrow C^{k+1}(\mathcal{V}, \mathcal{F})$ can in this formalism be taken to be the formal wedge product

$$\delta\left(\sum'_{|I|=k+1} f_I V_I\right) = \left(\sum'_{|I|=k+1} f_I V_I\right) \wedge \left(\sum_j V_j\right).$$

If \mathcal{V} is a Leray covering for \mathcal{F} , then $H^k(C^\bullet(\mathcal{V}, \mathcal{F}), \delta) \cong H^k(X, \mathcal{F})$. Indeed, let (\mathcal{F}^\bullet, d) be a flabby resolution of \mathcal{F} . Then $H^k(X, \mathcal{F}) = H^k(\mathcal{F}^\bullet(X), d)$ and applying standard homological algebra to the double complex $C^\bullet(\mathcal{V}, \mathcal{F}^\bullet)$ one shows that $H^k(C^\bullet(\mathcal{V}, \mathcal{F}), \delta) \simeq H^k(\mathcal{F}^\bullet(X), d)$. If \mathcal{F} is fine, i.e., a $\mathcal{E}_X^{0,0}$ -module, then the complex $(C^\bullet(\mathcal{V}, \mathcal{F}), \delta)$ is exact except for at level 0 where $H^0(C^\bullet(\mathcal{V}, \mathcal{F}), \delta) \cong H^0(X, \mathcal{F})$.

Let \mathcal{G}' be a precosheaf on X . Recall, see, e.g., [11, Section 3], that a precosheaf of abelian groups is an assignment that to each open set V associates an abelian group $\mathcal{G}'(V)$, together with inclusion maps $i_W^V: \mathcal{G}'(V) \rightarrow \mathcal{G}'(W)$ for $V \subset W$ such that $i_W^{V'} = i_W^V i_V^{V'}$ if $V' \subset V \subset W$. We define $C_c^{-k}(\mathcal{V}, \mathcal{G}')$ to be the group of formal sums

$$\sum_{i_0 \dots i_k} g_{i_0 \dots i_k} V_{i_0}^* \wedge \dots \wedge V_{i_k}^*,$$

where $g_{i_0 \dots i_k} \in \mathcal{G}'(V_{i_0} \cap \dots \cap V_{i_k})$ and only finitely many $g_{i_0 \dots i_k}$ are non-zero; for $k < 0$ we let $C_c^{-k}(\mathcal{V}, \mathcal{G}') = 0$. We define a coboundary operator $\delta^*: C_c^{-k}(\mathcal{V}, \mathcal{G}') \rightarrow C_c^{-k+1}(\mathcal{V}, \mathcal{G}')$ by formal contraction

$$\delta^*\left(\sum'_{|I|=k+1} g_I V_I^*\right) = \sum_j V_j \lrcorner \sum'_{|I|=k+1} g_I V_I^*,$$

see (6.5) and (6.6) below. If \mathcal{G} is a sheaf (of abelian groups), then $V \rightarrow \mathcal{G}_c(V)$ is a precosheaf \mathcal{G}' by extending sections by 0. We will write $C_c^{-k}(\mathcal{V}, \mathcal{G})$ in place of $C_c^{-k}(\mathcal{V}, \mathcal{G}')$.

Assume now that there, for every open $V \subset X$, is a map $\mathcal{F}(V) \otimes \mathcal{G}'(V) \rightarrow \mathcal{F}'(V)$ where \mathcal{F}' and \mathcal{G}' are precosheaves on X . We then define a contraction map $\lrcorner: C^k(\mathcal{V}, \mathcal{F}) \times C_c^{-\ell}(\mathcal{V}, \mathcal{G}') \rightarrow C_c^{k-\ell}(\mathcal{V}, \mathcal{F}')$ by using the following computation rules.

$$(6.5) \quad V_i \lrcorner V_j^* = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

$$(6.6) \quad V_i \lrcorner (V_{j_0}^* \wedge \dots \wedge V_{j_\ell}^*) = \sum_{m=0}^{\ell} (-1)^m V_{j_0}^* \wedge \dots \wedge (V_i \lrcorner V_{j_m}^*) \wedge \dots \wedge V_{j_\ell}^*,$$

$$(V_{i_0} \wedge \dots \wedge V_{i_k}) \lrcorner V_J^* = \begin{cases} 0, & k > |J| \\ ((V_{i_0} \wedge \dots \wedge V_{i_{k-1}}) \lrcorner (V_i \lrcorner V_J^*)), & k \leq |J| \end{cases}.$$

If \mathcal{F}' and \mathcal{G}' are sheaves we define in a similar way also the contraction $\lrcorner: C_c^{-k}(\mathcal{V}, \mathcal{G}') \times C^\ell(\mathcal{V}, \mathcal{F}) \rightarrow C^{\ell-k}(\mathcal{V}, \mathcal{F}')$. If $g = g_I V_I^*$ and $f = f_J V_J$, then $g \lrcorner f = g_I f_J V_I^* \lrcorner V_J$, where $g_I f_J$ is the extension to $\bigcap_{i \in J \setminus I} V_i$ by 0; this is well-defined since $g_I f_J$ is 0 in a neighborhood of the boundary of $\bigcap_{j \in J} V_j$ in $\bigcap_{i \in J \setminus I} V_i$.

Lemma 6.3. *If \mathcal{G} is a fine sheaf, then*

$$H^{-k}(C_c^\bullet(\mathcal{V}, \mathcal{G}), \delta^*) = \begin{cases} 0, & k \neq 0 \\ H_c^0(X, \mathcal{G}), & k = 0 \end{cases}.$$

Proof. Let $\{\chi_j\}$ be a smooth partition of unity subordinate to \mathcal{V} and let $\chi = \sum_j \chi_j V_j^*$. Since $\delta^* \chi = \sum \chi_j = 1$ we have

$$\delta^*(\chi \wedge g) = \delta^*(\chi) \cdot g - \chi \wedge \delta^*(g) = g - \chi \wedge \delta^*(g)$$

for $g \in C_c^{-k}(\mathcal{V}, \mathcal{G})$. Hence, if g is δ^* -closed, then g is δ^* -exact. It follows that the complex

$$\dots \xrightarrow{\delta^*} C_c^{-1}(\mathcal{V}, \mathcal{F}^*) \xrightarrow{\delta^*} C_c^0(\mathcal{V}, \mathcal{G}) \xrightarrow{\delta^*} H_c^0(X, \mathcal{G}) \rightarrow 0$$

is exact and so the lemma follows. \square

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Let X be a paracompact reduced complex space of pure dimension n . Let \aleph be the precosheaf on X defined by

$$\aleph(V) = H^n(\mathcal{A}_c^{n, \bullet}(V), \bar{\partial}),$$

$$i_W^V: \aleph(V) \rightarrow \aleph(W), \quad i_W^V([\psi]) = [\tilde{\psi}],$$

where $\psi \in \mathcal{A}_c^{n, n}(V)$ and $\tilde{\psi}$ is the extension of ψ by 0.⁶ Let $\mathcal{V} = \{V_j\}$ be a suitable locally finite Leray covering of X and consider the complexes

$$(6.7) \quad 0 \rightarrow C^0(\mathcal{V}, \mathcal{O}_X) \xrightarrow{\delta} C^1(\mathcal{V}, \mathcal{O}_X) \xrightarrow{\delta} \dots$$

$$(6.8) \quad \dots \xrightarrow{\delta^*} C_c^{-1}(\mathcal{V}, \aleph) \xrightarrow{\delta^*} C_c^0(\mathcal{V}, \aleph) \rightarrow 0.$$

By Theorem 6.1 we have non-degenerate pairings

$$Tr: C^k(\mathcal{V}, \mathcal{O}_X) \times C_c^{-k}(\mathcal{V}, \aleph) \rightarrow \mathbb{C}, \quad Tr(f, g) = \int_X f \lrcorner g,$$

induced by the trace map (6.3); in fact, Theorem 6.1 shows that these pairings make the complex (6.8) the topological dual of the complex of Frechét spaces (6.7). Moreover, if $f \in C^{k-1}(\mathcal{V}, \mathcal{O}_X)$ and $g \in C_c^{-k}(\mathcal{V}, \aleph)$ we have

$$(6.9) \quad \begin{aligned} Tr(\delta f, g) &= \int_X (\delta f) \lrcorner g = \int_X (f \wedge \sum_j V_j) \lrcorner g = \int_X f \lrcorner ((\sum_j V_j) \lrcorner g) \\ &= \int_X f \lrcorner (\delta^* g) = Tr(f, \delta^* g). \end{aligned}$$

Hence, we get a well-defined pairing on cohomology level

$$(6.10) \quad Tr: H^k(C^\bullet(\mathcal{V}, \mathcal{O}_X)) \times H^{-k}(C_c^\bullet(\mathcal{V}, \aleph)) \rightarrow \mathbb{C}, \quad Tr([f], [g]) = \int_X f \lrcorner g.$$

Since \mathcal{V} is a Leray covering we have

$$(6.11) \quad H^k(C^\bullet(\mathcal{V}, \mathcal{O}_X)) \cong H^k(X, \mathcal{O}_X) \cong H^k(\mathcal{A}^{0, \bullet}(X)),$$

and these isomorphisms induce canonical topologies on $H^k(X, \mathcal{O}_X)$ and $H^k(\mathcal{A}^{0, \bullet}(X))$; cf. [26, Lemma 1]. To understand $H^{-k}(C_c^\bullet(\mathcal{V}, \aleph))$, consider the double complex

$$K^{-i, j} := C_c^{-i}(\mathcal{V}, \mathcal{A}_X^{n, j}),$$

⁶In view of Theorem 6.1 and [11, Proposition 8 (a)], \aleph is in fact a cosheaf.

where the map $K^{-i,j} \rightarrow K^{-i+1,j}$ is the coboundary operator δ^* and the map $K^{-i,j} \rightarrow K^{-i,j+1}$ is $\bar{\delta}$. We have that $K^{-i,j} = 0$ if $i < 0$ or $j < 0$ or $j > n$. Moreover, the “rows” $K^{-i,\bullet}$ are, by Theorem 6.1, exact except for at the n^{th} level where the cohomology is $C_c^{-i}(\mathcal{V}, \mathbb{N})$; the “columns” $K^{\bullet,j}$ are exact except for at level 0 where the cohomology is $\mathcal{A}_c^{n,j}(X)$ by Lemma 6.3 since the sheaf $\mathcal{A}_X^{n,j}$ is fine. By standard homological algebra (e.g., a spectral sequence argument) it follows that

$$(6.12) \quad H^{-k}(C_c^\bullet(\mathcal{V}, \mathbb{N})) \cong H^{n-k}(\mathcal{A}_c^{n,\bullet}(X), \bar{\delta}),$$

cf. also the proof of Theorem 1.3 below. The vector space $C_c^{-k}(\mathcal{V}, \mathbb{N})$ has a natural topology since it is the topological dual of the Frechét space $C^k(\mathcal{V}, \mathcal{O}_X)$; therefore (6.12) gives a natural topology on $H^{n-k}(\mathcal{A}_c^{n,\bullet}(X))$.

Lemma 6.4. *Assume that $H^k(X, \mathcal{O}_X)$ and $H^{k+1}(X, \mathcal{O}_X)$, considered as topological vector spaces, are Hausdorff. Then the pairing (6.10) is non-degenerate.*

Proof. Since (6.8) is the topological dual of (6.7) it follows (see, e.g., [26, Lemma 2]) that the topological dual of

$$(6.13) \quad \text{Ker}(\delta: C^k(\mathcal{V}, \mathcal{O}_X) \rightarrow C^{k+1}(\mathcal{V}, \mathcal{O}_X)) / \overline{\text{Im}(\delta: C^{k-1}(\mathcal{V}, \mathcal{O}_X) \rightarrow C^k(\mathcal{V}, \mathcal{O}_X))}$$

equals

$$(6.14) \quad \text{Ker}(\delta^*: C_c^{-k}(\mathcal{V}, \omega_X^{n,n}) \rightarrow C_c^{-k+1}(\mathcal{V}, \omega_X^{n,n})) / \overline{\text{Im}(\delta^*: C_c^{-k-1}(\mathcal{V}, \omega_X^{n,n}) \rightarrow C_c^{-k}(\mathcal{V}, \omega_X^{n,n}))}.$$

Since $H^k(X, \mathcal{O}_X)$ and $H^{k+1}(X, \mathcal{O}_X)$ are Hausdorff it follows that the images of $\delta: C^{k-1} \rightarrow C^k$ and $\delta: C^k \rightarrow C^{k+1}$ are closed. Since the image of the latter map is closed it follows from the open mapping theorem and the Hahn-Banach theorem that also the image of $\delta^*: C_c^{-k-1} \rightarrow C_c^{-k}$ is closed. The images of δ and δ^* in (6.13) and (6.14) are thus closed and so the closure signs may be removed. Hence, (6.10) makes $H^{-k}(C_c^\bullet(\mathcal{V}, \omega_X^{n,n}))$ the topological dual of $H^k(X, \mathcal{O}_X)$. \square

Remark 6.5. If X is compact the Cartan-Serre theorem says that the cohomology of coherent sheaves on X is finite dimensional, in particular Hausdorff. In the compact case the pairing (6.10) is thus always non-degenerate. The pairing (6.10) is also always non-degenerate if X is holomorphically convex since then, by [25, Lemma II.1], $H^k(X, \mathcal{S})$ is Hausdorff for any coherent sheaf \mathcal{S} .

If X is q -convex it follows from the Andreotti-Grauert theorem that for any coherent sheaf \mathcal{S} , $H^k(X, \mathcal{S})$ is Hausdorff for $k \geq q$. Hence, in this case, (6.10) is non-degenerate for $k \geq q$.

Proof of Theorem 1.3. For notational convenience we assume that $\mathcal{F} = \mathcal{O}_X$. By Lemma 6.4 we know that (6.10) is non-degenerate. In view of the Dolbeault isomorphisms (6.11) and (6.12) we get an induced non-degenerate pairing

$$\text{Tr}: H^k(\mathcal{A}^{0,\bullet}(X)) \times H^{n-k}(\mathcal{A}_c^{n,\bullet}(X)) \rightarrow \mathbb{C}.$$

It remains to see that this induced trace map is realized by $([\varphi], [\psi]) \mapsto \int_X \varphi \wedge \psi$; for this we will make (6.11) and (6.12) explicit.

Let $\{\chi_j\}$ be a partition of unity subordinate to \mathcal{V} , and let $\chi = \sum_j \chi_j V_j^*$. We will use the convention that forms commute with all V_i^* and V_j , i.e., if ξ is a differential form then

$$\xi V_I^* = V_I^* \xi, \quad V_I^* \lrcorner (\xi V_J) = \xi V_I^* \lrcorner V_J.$$

Moreover, we let $\bar{\partial}(\xi V_I^*) = \bar{\partial}\xi V_I^*$. We now let

$$T_{k,j} : C^k(\mathcal{V}, \mathcal{O}_X) \rightarrow C^{k-j-1}(\mathcal{V}, \mathcal{A}_X^{0,j}), \quad T_{k,j}(f) = (\chi \wedge (\bar{\partial}\chi)^j) \lrcorner f,$$

where we put $C^{-1}(\mathcal{V}, \mathcal{A}_X^{0,k}) = \mathcal{A}_X^{0,k}(X)$ and $C^\ell(\mathcal{V}, \mathcal{A}_X^{0,k}) = 0$ for $\ell < -1$.⁷ Using that $\chi \lrcorner V = 1$ it is straightforward to verify that

$$(6.15) \quad T_{k,j}(\delta\tilde{f}) = \delta T_{k-1,j}(\tilde{f}) + (-1)^{k-j} \bar{\partial} T_{k-1,j-1}(\tilde{f}), \quad \tilde{f} \in C^{k-1}(\mathcal{V}, \mathcal{O}_X).$$

It follows that if $f \in C^k(\mathcal{V}, \mathcal{O}_X)$ is δ -closed then $T_{k,k}(f)$ is $\bar{\partial}$ -closed and if f is δ -exact then $T_{k,k}(f)$ is $\bar{\partial}$ -exact. Thus $T_{k,k}$ induces a map

$$\text{Dol} : H^k(C^\bullet(\mathcal{V}, \mathcal{O}_X)) \rightarrow H^k(\mathcal{A}_X^{0,\bullet}(X)), \quad \text{Dol}([f]_\delta) = [T_{k,k}(f)]_{\bar{\partial}};$$

this is a realization of the composed isomorphism (6.11).

To make (6.12) explicit, let $[g] \in C_c^{-k}(\mathcal{V}, \mathfrak{N})$, where $g \in C_c^{-k}(\mathcal{V}, \mathcal{A}_X^{n,n})$, be δ^* -closed. This means that there is a $\tau^{n-1} \in C_c^{-k+1}(\mathcal{V}, \mathcal{A}_X^{n,n-1})$ such that $\delta^*g = \bar{\partial}\tau^{n-1}$. Hence, $\bar{\partial}\delta^*\tau^{n-1} = \delta^*\bar{\partial}\tau^{n-1} = \delta^*\delta^*g = 0$ and so by Theorem 6.1 there is a $\tau^{n-2} \in C_c^{-k+2}(\mathcal{V}, \mathcal{A}_X^{n,n-2})$ such that $\delta^*\tau^{n-1} = \bar{\partial}\tau^{n-2}$. Continuing in this way we obtain, for all j , $\tau^{n-j} \in C_c^{-k+j}(\mathcal{V}, \mathcal{A}_X^{n,n-j})$ such that $\delta^*\tau^{n-j} = \bar{\partial}\tau^{n-j-1}$. It follows that $\delta^*\tau^{n-k} \in \mathcal{A}_c^{n,n-k}(X)$, cf. the proof of Lemma 6.3, and that it is $\bar{\partial}$ -closed. One can verify that if $[g] \in C_c^{-k}(\mathcal{V}, \mathfrak{N})$ is δ^* -exact then $\delta^*\tau^{n-k}$ is $\bar{\partial}$ -exact and so we get a well-defined map

$$\text{Dol}^* : H^{-k}(C_c^\bullet(\mathcal{V}, \mathfrak{N})) \rightarrow H^{n-k}(\mathcal{A}_c^{n,\bullet}(X)), \quad \text{Dol}^*([g]_{\bar{\partial}}) = [\delta^*\tau^{n-k}]_{\bar{\partial}};$$

this is a realization of the isomorphism (6.12).

Let now $f \in C^k(\mathcal{V}, \mathcal{O}_X)$ be δ -closed and let $[g] \in C_c^{-k}(\mathcal{V}, \mathfrak{N})$ be δ^* -closed. One checks that $\delta T_{k,0}(f) = (-1)^k f$ and thus, by (6.15), we have

$$\delta T_{k,j}(f) = \begin{cases} (-1)^{k-j} \bar{\partial} T_{k,j-1}(f), & 1 \leq j \leq k \\ (-1)^k f, & j = 0 \end{cases}.$$

Using this and the computation in (6.9) we get

$$\begin{aligned} \int_X f \lrcorner g &= (-1)^k \int_X \delta T_{k,0}(f) \lrcorner g = (-1)^k \int_X T_{k,0}(f) \lrcorner \delta^* g = (-1)^k \int_X T_{k,0}(f) \lrcorner \bar{\partial} \tau^{n-1} \\ &= (-1)^{k+1} \int_X \bar{\partial} T_{k,0}(f) \lrcorner \tau^{n-1} = (-1)^{2k} \int_X \delta T_{k,1}(f) \lrcorner \tau^{n-1} \\ &= (-1)^{2k} \int_X T_{k,1}(f) \lrcorner \delta^* \tau^{n-1} = \dots = (-1)^{k(k+1)} \int_X T_{k,k}(f) \lrcorner \delta^* \tau^{n-k} \\ &= \int_X \text{Dol}([f]) \wedge \text{Dol}^*([g]). \end{aligned}$$

□

7. COMPATIBILITY WITH THE CUP PRODUCT

Assume that X is compact and Cohen-Macaulay. In view of Theorem 2.9 and Theorem 1.2 we have that

$$(7.1) \quad H^k(X, \mathcal{O}_X) \cong H^k(\mathcal{A}_X^{0,\bullet}(X), \bar{\partial}) \quad \text{and} \quad H^k(X, \omega_X^{n,0}) \cong H^k(\mathcal{A}_X^{n,\bullet}(X), \bar{\partial}).$$

⁷In fact, the image of $T_{k,j}$ is contained in $C^{k-j-1}(\mathcal{V}, \mathcal{E}_X^{0,j})$.

Now we make these Dolbeault isomorphisms explicit in a slightly different way than in the previous section: We adopt in this section the standard definition of Čech cochain groups so that now

$$C^p(\mathcal{V}, \mathcal{F}) := \prod_{\alpha_0 \neq \alpha_1 \neq \dots \neq \alpha_p} \mathcal{F}(V_{\alpha_0} \cap \dots \cap V_{\alpha_p})$$

for a sheaf \mathcal{F} on X and a locally finite open cover $\mathcal{V} = \{V_\alpha\}$.

Let \mathcal{V} be a Leray covering and let $\{\chi_\alpha\}$ be a smooth partition of unity subordinate to \mathcal{V} . Following [16, Chapter IV, §6], given Čech cocycles $c \in C^p(\mathcal{V}, \mathcal{O}_X)$ and $c' \in C^q(\mathcal{V}, \omega_X^{n,0})$ we define Čech cochains $f \in C^0(\mathcal{V}, \mathcal{A}_X^{0,p})$ and $f' \in C^0(\mathcal{V}, \mathcal{A}_X^{n,q})$ by

$$f_\alpha = \sum_{\nu_0, \dots, \nu_{p-1}} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{p-1}} \cdot c_{\nu_0 \dots \nu_{p-1} \alpha} \quad \text{in } V_\alpha,$$

$$f'_\alpha = \sum_{\nu_0, \dots, \nu_{q-1}} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{q-1}} \wedge c'_{\nu_0 \dots \nu_{q-1} \alpha} \quad \text{in } V_\alpha.$$

In fact, f and f' are cocycles and define $\bar{\partial}$ -closed global sections

$$(7.2) \quad \varphi = \sum_{\nu_p} \chi_{\nu_p} f_{\nu_p} = \sum_{\nu_0, \dots, \nu_p} \chi_{\nu_p} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{p-1}} \cdot c_{\nu_0 \dots \nu_p} \in \mathcal{A}^{0,p}(X),$$

$$(7.3) \quad \varphi' = \sum_{\nu_q} \chi_{\nu_q} f'_{\nu_q} = \sum_{\nu_0, \dots, \nu_q} \chi_{\nu_q} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{q-1}} \wedge c'_{\nu_0 \dots \nu_q} \in \mathcal{A}^{n,q}(X).$$

The Dolbeault isomorphisms (7.1) are then realized by

$$H^p(X, \mathcal{O}_X) \xrightarrow{\cong} H^p(\mathcal{A}^{0,\bullet}(X)), \quad [c] \mapsto [\varphi], \quad \text{and}$$

$$H^q(X, \omega_X^{n,0}) \xrightarrow{\cong} H^q(\mathcal{A}^{n,\bullet}(X)), \quad [c'] \mapsto [\varphi'],$$

respectively.

We can now show that the cup product is compatible with our trace map on the level of cohomology.

Proposition 7.1. *The following diagram commutes.*

$$\begin{array}{ccc} H^p(X, \mathcal{O}_X) \times H^q(X, \omega_X^{n,0}) & \xrightarrow{\cup} & H^{p+q}(X, \omega_X^{n,0}) \\ \downarrow & & \downarrow \\ H^p(\mathcal{A}^{0,\bullet}(X)) \times H^q(\mathcal{A}^{n,\bullet}(X)) & \xrightarrow{\wedge} & H^{p+q}(\mathcal{A}^{n,\bullet}(X)), \end{array}$$

where the vertical mappings are the Dolbeault isomorphisms.

Proof. Let $\mathcal{V} = \{V_\alpha\}$ be a Leray covering of X . Let $[c] \in H^p(X, \mathcal{O}_X)$ and $[c'] \in H^q(X, \omega_X^{n,0})$, where $c \in C^p(\mathcal{V}, \mathcal{O}_X)$ and $c' \in C^q(\mathcal{V}, \omega_X^{n,0})$ are cocycles. Then $c \cup c' \in C^{p+q}(\mathcal{V}, \omega_X^{n,0})$, defined by

$$(c \cup c')_{\alpha_0 \dots \alpha_{p+q}} = c_{\alpha_0 \dots \alpha_p} \cdot c'_{\alpha_p \dots \alpha_{p+q}} \quad \text{in } V_{\alpha_0} \cap \dots \cap V_{\alpha_{p+q}},$$

is a cocycle representing $[c] \cup [c'] \in \check{H}^{p+q}(X, \omega_X^{n,0})$. The image of $[c] \cup [c']$ in $H^{p+q}(\mathcal{A}^{n,\bullet}(X))$ is the cohomology class defined by the $\bar{\partial}$ -closed current

$$(7.4) \quad \sum_{\nu_0, \dots, \nu_{p+q}} \chi_{\nu_{p+q}} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{p+q-1}} \wedge c_{\nu_0 \dots \nu_p} \cdot c'_{\nu_p \dots \nu_{p+q}} \in \mathcal{A}^{n,p+q}(X).$$

The images of $[c]$ and $[c']$ in Dolbeault cohomology are, respectively, the cohomology classes of the $\bar{\partial}$ -closed currents φ and φ' defined by (7.2) and (7.3). Notice that

$$\varphi|_{V_{\nu_p}} = \sum_{\nu_0, \dots, \nu_{p-1}} \bar{\partial}\chi_{\nu_0} \wedge \cdots \wedge \bar{\partial}\chi_{\nu_{p-1}} \cdot c_{\nu_0 \cdots \nu_{p-1} \nu_p}.$$

Therefore, $\varphi \wedge \varphi'$ is given by (7.4) as well. \square

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