PRESENCE OR ABSENCE OF ANALYTIC STRUCTURE IN MAXIMAL IDEAL SPACES

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ABSTRACT. We study extensions of Wermer's maximality theorem to several complex variables. We exhibit various smoothly embedded manifolds in complex Euclidean space whose hulls are non-trivial but contain no analytic disks. We answer a question posed by Lee Stout concerning the existence of analytic structure for a uniform algebra whose maximal ideal space is a manifold.

1. INTRODUCTION

A central theme in the theory uniform algebras is to find analytic structure in the maximal ideal space of a given algebra. For M a compact space and $f_1, \ldots, f_N$ continuous complex-valued functions on M, we will denote by $[f_1, \ldots, f_N]_M$ the uniform algebra on M generated by $f_1, \ldots, f_N$. Setting $f = (f_1, \ldots, f_N)\colon M \to \mathbb{C}^N$, the uniform algebra $[f_1, \ldots, f_N]_M$ is isomorphic to $[z_1, \ldots, z_N]_{f(M)}$, *i.e.*, to the uniform algebra on $f(M) \subset \mathbb{C}^N$ generated by the complex coordinate functions. Hence, the maximal ideal space of $[f_1, \ldots, f_N]_M$ is isomorphic to the polynomially convex hull $\widehat{f(M)}$ of $f(M)$. In particular, notice that analytic structure in $\widehat{f(M)}$ will prevent $[f_1, \ldots, f_N]_M$ from being the algebra $\mathcal{C}(M)$ of all continuous functions on M, and that the maximal ideal space of $[f_1, \ldots, f_N]_M$ is M if and only if $f(M)$ is polynomially convex. In this paper we will mainly be concerned with the case when M is the boundary of some domain in $\mathbb{C}^n$ with polynomially convex closure. For notational convenience, we will denote $[z_1, \ldots, z_n, f_1, \ldots, f_N]$ by $[z, f]$, and we will denote the graph of f over M (*i.e.*, the image of M under the map (z, f)) by $\mathcal{G}_f(M)$. Throughout the paper the word "smooth" will mean of class $\mathcal{C}^\infty$ except where explicitly indicated otherwise.

Recall Wermer's maximality theorem [23]: If f is a continuous function on the unit circle $b\mathbb{D} \subset \mathbb{C}$, then either f is the boundary value of a holomorphic function or else the uniform algebra $[z, f]_{b\mathbb{D}}$ generated by z and f is equal

2000 *Mathematics Subject Classification.* Primary 32E20; Secondary 32A38, 32A65, 32E30, 32V10, 46J10, 46J15.

The first author was partially supported by NFR grant 209751/F20.
The second author was partially supported by the Swedish Research Council.
The third author was supported by NFR grant 209751/F20.

to $\mathcal{C}(b\mathbb{D})$. Since $[z, \frac{1}{z}]_{b\mathbb{D}} = \mathcal{C}(b\mathbb{D})$ it is clear from the Oka-Weil theorem that $[z, f]_{b\mathbb{D}} = \mathcal{C}(b\mathbb{D})$ if and only if the graph $\mathcal{G}_f(b\mathbb{D})$ of f is polynomially convex. Thus for a continuous function f on $b\mathbb{D}$ the following four conditions are equivalent:

(1) $\mathcal{G}_f(b\mathbb{D})$ is polynomially convex.
(2) $\widehat{\mathcal{G}_f(b\mathbb{D})} \setminus \mathcal{G}_f(b\mathbb{D})$ contains no analytic disk.
(3) f does not extend continuously to a holomorphic function on the unit disk $\mathbb{D}$.
(4) $[z, f]_{b\mathbb{D}} = \mathcal{C}(b\mathbb{D})$.

We sketch here a proof of (3) $\implies$ (1) that illustrates our approach in this paper. Let $\tilde{f}$ be the harmonic extension of f to $\mathbb{D}$. Using harmonic conjugates and the fact that $\mathbb{D}$ is starshaped, it is not hard to see that $\mathcal{G}_{\tilde{f}}(\overline{\mathbb{D}})$ is polynomially convex. Assume that $\tilde{f}$ is not holomorphic; then the set $A \subset \mathbb{D}$ of points where $\overline{\partial}\tilde{f} = 0$ is discrete. One can show that every point in $\mathbb{D} \setminus A$ is a local peak point for the algebra $[z, \tilde{f}]_{\overline{\mathbb{D}}}$. It then follows from Rossi's local maximum principle (see, *e.g.*, [17] or [13, Theorem III.8.2]) that $(z, f(z)) \notin \widehat{\mathcal{G}_f(b\mathbb{D})}$ for every $z \in \mathbb{D} \setminus A$. Since A is discrete it follows that $\mathcal{G}_f(b\mathbb{D})$ is polynomially convex.

One could ask about the possibility of carrying over the equivalence of (1), (2), (3), and (4) above to the setting of several complex variables. Specifically, one could ask whether for $\Omega \subset \mathbb{C}^n$ a sufficiently nice domain and $f_1, \ldots, f_n$ continuous functions on $\mathcal{C}(b\Omega)$ the following four conditions are equivalent:

(1) $\mathcal{G}_f(b\Omega)$ is polynomially convex.
(2) $\widehat{\mathcal{G}_f(b\Omega)} \setminus \mathcal{G}_f(b\Omega)$ contains no analytic disk.
(3) There does not exist an analytic set[1] "attached" to $b\Omega$ to which f extends continuously as a holomorphic map.
(4) $[z, f]_{b\Omega} = \mathcal{C}(b\Omega)$.

Of course it is always true that (4) $\implies$ (1) $\implies$ (2) $\implies$ (3). That these implications are not reversible is shown in the work of Richard Basener [5]. Specifically, letting $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$, Basener showed that there exist smooth functions $f_1, \ldots, f_4$ on $S^3 = b\mathbb{B}_2 \subset \mathbb{C}^2$ such that $\mathcal{G}_f(b\mathbb{B}_2)$ is polynomially convex but $[z, f]_{b\mathbb{B}_2} \neq \mathcal{C}(b\mathbb{B}_2)$, and he also observed that there exist different smooth functions $f_1, \ldots, f_3$ such that the polynomially convex hull of $\mathcal{G}_f(b\mathbb{B}_2) \subset \mathbb{C}^5$ is non-trivial but contains no analytic disk. Theorems 1.6, 1.7, and 1.8 exhibit further instances of this phenomenon of smooth manifolds in $\mathbb{C}^n$ with non-trivial polynomially convex hull without analytic structure, and in particular, we strengthen the second result of Basener by reducing the number of functions needed.

[1]Throughout the paper, by an *analytic set* we mean a subset of $\mathbb{C}^n$ that is locally the common zero set of finitely many holomorphic functions. Such sets are often referred to as analytic varieties or holomorphic varieties.

In view of the results of Basener, we need some conditions on the f_j's if we are to obtain multivariable versions of Wermer's maximality theorem. A difference between one and several complex variables is that in $\mathbb{C}^n$ for $n \geq 2$, the Dirichlet problem, while solvable for harmonic functions, is not in general solvable with *pluriharmonic* functions, and it is only the pluriharmonic functions that have conjugates. In seeking extensions of Wermer's maximality theorem to several complex variables, it is therefore natural to restrict consideration to those functions that are boundary values of pluriharmonic functions. Uniform algebras generated by holomorphic and pluriharmonic functions of several complex variables were studied by E. M. Čirka in [8], the first author in [14] and [15], and the second and third authors in [22]. We will denote the set of all complex-valued pluriharmonic functions on a domain Ω by $PH(\Omega)$. Our first result is closely related to a result in [22].

Theorem 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\mathcal{C}^1$-smooth boundary and polynomially convex closure, and let $h_j \in PH(\Omega) \cap \mathcal{C}(\overline{\Omega})$ for $j = 1, \ldots, N$. Then the following four conditions are equivalent:*

1. $\mathcal{G}_h(b\Omega)$ *is polynomially convex.*
2. $\widehat{\mathcal{G}_h(b\Omega)} \setminus \mathcal{G}_h(b\Omega)$ *contains no analytic disk.*
3. *There does not exist a nontrivial analytic disk $\triangle \hookrightarrow \Omega$ on which all the h_j's are holomorphic.*
4. $[z, h]_{\overline{\Omega}} = \{ f \in \mathcal{C}(\overline{\Omega}) : f|_{b\Omega} \in [z, h]_{b\Omega} \}$.

As an immediate corollary we have the following.

Corollary 1.2. *Let Ω and $h_1, \ldots, h_N$ be as in Theorem 1.1. Then*

$$[z, h]_{b\Omega} = \mathcal{C}(b\Omega) \quad \Longrightarrow \quad [z, h]_{\overline{\Omega}} = \mathcal{C}(\overline{\Omega}). \tag{1.1}$$

This result sharpens Theorem 1.3 in [22], where it is proved that if $[z, h]_{b\Omega} = \mathcal{C}(b\Omega)$, then either there exists an analytic disk in Ω on which each h_j is holomorphic, or else $[z, h]_{\overline{\Omega}} = \mathcal{C}(\overline{\Omega})$. A related version of "the maximality theorem" for the distinguished boundary Γ of the bi-disk $\mathbb{D}^2$ is also proved in [22]: Let $h_j \in PH(\mathbb{D}^2) \cap \mathcal{C}(\overline{\mathbb{D}}^2)$. Then either there exists an algebraic variety $Z \subset \mathbb{C}^2$ with $Z \cap b\mathbb{D}^2 \subset \Gamma$ on which all h_j are holomorphic, or $[z, h]_\Gamma = \mathcal{C}(\Gamma)$. In particular, starting with continuous functions f_j on Γ, the only obstruction to $[z, f]_\Gamma = \mathcal{C}(\Gamma)$ is the presence of analytic structure in the maximal ideal space of the algebra $[z, f]_{\overline{\mathbb{D}}^2}$ *provided* we assume that the f_j's extend to pluriharmonic functions on the bi-disk. Theorem 1.6 below shows that this becomes false if the plurihamonicity condition is dropped.

Theorem 1.1 implies another extension of Wermer's maximality theorem to several complex variables. A result of Lee Stout [20] asserts that the complex polynomials are uniformly dense in the continuous functions on any compact, polynomially convex, real-analytic subvariety of complex Euclidean space. In combination with Theorem 1.1 this gives that in the case when the h_j's are *real-analytic* on $b\Omega$ we can replace condition (4) in Theorem 1.1 by the condition that $[z,h]_{\overline{\Omega}} = \mathcal{C}(\overline{\Omega})$. Explicitly we obtain the following result.[2]

Corollary 1.3. *Let Ω and $h_1,\dots,h_N$ be as in Theorem 1.1. Suppose in addition that $b\Omega$ is real-analytic and the $h_1,\dots,h_N$ are real-analytic on $b\Omega$. Then the following four conditions are equivalent:*

(1) *$\mathcal{G}_h(b\Omega)$ is polynomially convex.*
(2) *$\widehat{\mathcal{G}_h(b\Omega)} \setminus \mathcal{G}_h(b\Omega)$ contains no analytic disk.*
(3) *There does not exist a nontrivial analytic disk $\triangle \hookrightarrow \Omega$ on which all the h_j's are holomorphic.*
(4) *$[z,h]_{\overline{\Omega}} = \mathcal{C}(\overline{\Omega})$.*

Corollary 1.3 becomes false, in general, if the real-analyticity hypotheses are dropped. On the (nonsmooth) bi-disk $\mathbb{D}^2$ with $h_1 = \overline{z_1 - 1}$ and $h_2 = \overline{z_2(z_1-1)}$, condition (3) is satisfied while condition (4) fails since h_1 and h_2 are both identically zero on the analytic disk $\{z_1 = 1, |z_2| \leq 1\}$ lying in $b\mathbb{D}^2$. Replacing the bi-disk by a smoothly bounded domain Ω such that $\{z_1 = 1, |z_2| \leq 1\} \subset b\Omega$ and $\{z_1 = 1\} \cap \Omega$ is empty gives a counterexample to Corollary 1.3 without the real-analyticity hypotheses. We conjecture, however, that for *strictly pseudoconvex* domains, Corollary 1.3 remains true without the real-analyticity hypotheses. Note that this conjecture constitues an n-dimensional generalization of Wermer's maximality theorem.

When $\mathcal{G}_h(b\Omega)$ fails to be polynomially convex we can say more than just that there is an analytic disk in $\widehat{\mathcal{G}_h(b\Omega)} \setminus \mathcal{G}_h(b\Omega)$; we show that there is an analytic set $Z \subset \Omega$ that can be foliated in such a way the h_j's are holomorphic along the plaques, and the graph of h over Z is contained in $\widehat{\mathcal{G}_h(b\Omega)} \setminus \mathcal{G}_h(b\Omega)$. However, this "foliation" can have singularities, so it is not a regular foliation in the usual sense. We therefore make the following definition.

Definition 1.4. A *singular foliation* $\{(U_\alpha, F_\alpha)\}$ *of an analytic set* Z *by nontrivial varieties* is a cover of Z by open sets U_α together with holomorphic maps F_α on U_α such that each level set of each F_α has dimension ≥ 1 and in each nonempty intersection $U_\alpha \cap U_\beta$ the collection of components of the level sets of F_α coincides with the collection of components of the level sets of F_β. For each α, the level sets of F_α are called *plaques.*

Given a singular foliation of an analytic set Z, it is possible to piece together the plaques to obtain a partition of Z into disjoint "leaves" that are "immersed analytic sets", in a manner similar to how, given a regular

[2] When every point of $b\Omega$ is known to be a peak point for $A(\Omega)$ [e.g., for strictly pseudoconvex domains] one can invoke a weaker result of Anderson, Izzo, and Wermer [4] in place of Stout's theorem.

foliation, one obtains leaves that are immersed submanifolds. Since we will not need the leaves of a singular foliation, we omit the proof.

Theorem 1.5. *Let Ω and $h_1, \ldots, h_N$ be as in Theorem 1.1. Suppose that $\mathcal{G}_h(b\Omega)$ is not polynomially convex. Then there is an analytic set $Z \subset \Omega$ of dimension $d \geq 1$ and a singular foliation $\mathcal{F}$ of Z by non-trivial varieties such that all the h_j are strongly holomorphic along the plaques of $\mathcal{F}$. The graph $\mathcal{G}_h(Z)$ is contained in $\widehat{\mathcal{G}_h(b\Omega)} \setminus \mathcal{G}_h(b\Omega)$.*

As mentioned above, without the assumption that our functions $f_j : b\Omega \to \mathbb{C}$ have pluriharmonic extensions to Ω, the presence or absence of analytic structure in the maximal ideal space of $[z, f]_{b\Omega}$ is more delicate. Our next three results show that it is not uncommon that the maximal ideal space of $[z, f]_{b\Omega}$ strictly contains $b\Omega$ but lacks analytic structure. As mentioned above the first of these results shows that pluriharmonicity cannot be omitted in [22, Theorem 1.3], and the second improves a result of Basener by decreasing the dimension of the ambient space. The third shows that every smooth manifold of dimension at least three smoothly embeds in some complex Euclidean space so as to have a non-trivial hull without analytic structure.

Theorem 1.6. *There exists a real-valued smooth function f on $\Gamma = (b\mathbb{D})^2 \subset \mathbb{C}^2$ such that the polynomially convex hull of $\mathcal{G}_f(\Gamma) \subset \mathbb{C}^3$ is non-trivial but contains no analytic disk.*

Theorem 1.7. *There exist real-valued smooth functions f_1 and f_2 on $S^3 = b\mathbb{B}_2 \subset \mathbb{C}^2$ such that the polynomially convex hull of $\mathcal{G}_f(S^3) \subset \mathbb{C}^4$ is non-trivial but contains no analytic disk.*

Theorem 1.8. *If M is a smooth compact manifold-with-boundary of real dimension $m \geq 3$, then there is a smooth embedding $F : M \to \mathbb{C}^{2m+4}$ such that the polynomially convex hull of $F(M)$ is nontrivial but contains no analytic disk.*

Even in dimension 1, it is trivial that Corollary 1.2 fails without the pluriharmonicity hypothesis since h could be holomorphic on a nonempty proper open subset of Ω while not agreeing with the boundary values of a holomorphic function on $b\Omega$. The next result provides a more interesting illustration of what can go wrong in Corollary 1.2, even in dimension 1, without the pluriharmonicity hypothesis. (Here we denote by $\mathcal{R}(K)$ the uniform closure on K of the rational functions holomorphic on a neighborhood of K.)

Theorem 1.9. *Let $\Omega \subset \mathbb{C}^1$ be a bounded open set. There exist functions $f_1, f_2, f_3 \in \mathcal{C}^\infty(\overline{\Omega})$ and a compact set $K \subset \Omega$ such that the following hold:*

1. *$\mathcal{G}_f(\overline{\Omega})$ is polynomially convex,*
2. *the Shilov boundary of $[z, f]_{\overline{\Omega}}$ is $\overline{\Omega}$, so in particular there is no disk where all the f_j's are holomorphic,*
3. *$\mathcal{C}(\overline{\Omega}) \cap \mathcal{R}(K) \subset [z, f]_{\overline{\Omega}}$, and*

(4) $[z,f]_{\overline{\Omega}} \neq \mathcal{C}(\overline{\Omega})$.

A theorem of John Anderson and the first author [3] shows that it is not possible to strengthen condition (2) above to require that every point of $\overline{\Omega}$ be a peak point for $[z,f]_{\overline{\Omega}}$. However, in $\mathbb{C}^n$, for $n \geq 2$, this stronger condition can be achieved as well.

Theorem 1.10. *Let $\Omega \subset \mathbb{C}^n$, with $n \geq 2$, be a bounded open set. There exist functions $f_1, \dots, f_N \in \mathcal{C}^\infty(\overline{\Omega})$ and a compact set $K \subset \Omega$ such that the conditions in Theorem 1.9 are satisfied with condition (2) replaced by:*

(2′) *every point of $\overline{\Omega}$ is a peak point for $[z,f]_{\overline{\Omega}}$, so in particular there is no analytic disk in Ω on which all the f_j's are holomorphic.*

Although a nontrivial polynomially convex hull need not contain analytic structure, one could ask whether the presence of a (smooth) manifold of dimension at least 2 in $\hat{X} \setminus X$ (X a compact set in $\mathbb{C}^n$) implies the existence of analytic structure. The answer is no. In fact, given positive integers $k < n$, there is a compact set $X \subset b\mathbb{B}_n \subset \mathbb{C}^n$ such that $\hat{X} \setminus X$ contains a smooth k-manifold but contains no analytic disk. This follows immediately from the following result of Julien Duval and Norman Levenberg by taking the set K there to be defined by $K = \{(x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{R}^n \subset \mathbb{C}^n : \sum_{j=1}^k |x_j|^2 \leq 1/2\}$.

Theorem 1.11 (Duval–Levenberg [10])**.** *If K is a compact, polynomially convex subset of the ball $\mathbb{B}_n \subset \mathbb{C}^n$, $n \geq 2$, then there is a compact subset X of $b\mathbb{B}_n$ such that $\hat{X} \supset K$ and such that the set $\hat{X} \setminus K$ contains no analytic disk.*

However, we show that if there is an *open* subset of the hull $\hat{X} \setminus X$ that is a smooth manifold, then there necessarily is analytic structure in the hull. More precisely we have the following result.

Theorem 1.12. *Let X be a compact set in $\mathbb{C}^n$. Suppose U is an open subset of $\hat{X} \setminus X$ and is a $\mathcal{C}^1$-smooth submanifold of $\mathbb{C}^n$ of dimension at least 2. Then there exists a dense open subset Ω of U each component of which is an integrable CR-submanifold of $\mathbb{C}^n$.*

More generally we have the following result about uniform algebras.

Theorem 1.13. *Let $\mathcal{A}$ be a uniform algebra. Suppose that U is an open subset of the maximal ideal space of $\mathcal{A}$ disjoint from the Shilov boundary of $\mathcal{A}$ and that U is a manifold of real dimension $n \geq 2$ with a differentiable structure such that the collection of functions in $\mathcal{A}$ that are $\mathcal{C}^1$ on U is dense in $\mathcal{A}$. Then there exists a dense open subset Ω of U each component of which has an integrable CR-structure $\mathcal{F}$ such that every function in $\mathcal{A}$ is holomorphic along the leaves of $\mathcal{F}$.*

In [19] Stout studied uniform algebras whose maximal ideal spaces are $\mathcal{C}^1$-smooth surfaces and which admit sets of $\mathcal{C}^1$-smooth generators, and he showed that such algebras consist of functions holomorphic off their Shilov boundaries. In connection with this result he raised the question whether there is also always analytic structure when the maximal ideal space is a manifold of higher dimension [21, Section 5, Question 2]. The above result answers this question of Stout in the affirmative. Specifically specializing Theorem 1.13 to the case in which the maximal ideal space is a manifold gives the following.

Theorem 1.14. *Let M be a compact $\mathcal{C}^1$-smooth manifold (possibly with boundary) of real dimension $n \geq 2$, and let $\mathcal{A}$ be a uniform algebra on M generated by $\mathcal{C}^1$-smooth functions. Assume further that the maximal ideal space of $\mathcal{A}$ is M, and let $\Gamma_{\mathcal{A}}$ denote the Shilov boundary of $\mathcal{A}$. Then there exists a dense open subset Ω of $M \setminus \Gamma_{\mathcal{A}}$ each component of which has an integrable CR-structure $\mathcal{F}$ such that every function in $\mathcal{A}$ is holomorphic along the leaves of $\mathcal{F}$.*

We remark that the dense open set Ω in Theorems 1.12–1.14 may have several connected components and the CR-dimension of different components may be different.

2. Proof of Theorems 1.1 and 1.5

We will prove the equivalence of conditions (1), (3), and (4) in Theorem 1.1 first. Theorem 1.5 will follow essentially as a bi-product of this proof, and the equivalence of conditions (1) and (2) in Theorem 1.1 is an immediate consequence of Theorem 1.5. We begin with the following proposition which is the key observation. (Given an analytic set $Z \subset \Omega$, we denote the set of regular points of Z by Z_{reg}, the set of singular points by Z_{sing}, and the inclusion map of Z_{reg} into Ω by $i_{Z_{\mathrm{reg}}}$.)

Proposition 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $h_j \in PH(\Omega) \cap \mathcal{C}(\overline{\Omega})$ for $j = 1, \ldots, N$. Suppose there is an irreducible analytic set $Z \subset \Omega$ of dimension $d \geq 1$ such that $i^*_{Z_{\mathrm{reg}}}(\overline{\partial} h_{i_1} \wedge \cdots \wedge \overline{\partial} h_{i_d}) \equiv 0$ for all $(i_1, \ldots, i_d)$. Then $\mathcal{G}_h(b\Omega)$ is not polynomially convex.*

Lemma 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain, let $h_j \in PH(\Omega)$ for $j = 1, \ldots, N$, and let $Z \subset \Omega$ be an irreducible analytic set of dimension $d \geq 1$. Let $1 \leq m \leq d$, fix $(i_1, \ldots, i_m)$, and define*

$$Z' := \{z \in Z_{\mathrm{reg}} : i^*_{Z_{\mathrm{reg}}}(\overline{\partial} h_{i_1} \wedge \cdots \wedge \overline{\partial} h_{i_m})(z) = 0\}.$$

Then $\tilde{Z} := Z' \cup Z_{\mathrm{sing}}$ is an analytic subset of Ω.

Proof. Let $z_0 \in \Omega$, and let U_0 be a simply connected neighborhood of z_0. Then for each $j = 1, \ldots, N$ there is a $g_j \in \mathcal{O}(U_0)$ such that $h_j + g_j$ is real and then also $f_j \in \mathcal{O}(U_0)$ such that $Re(f_j) = h_j + g_j$. Notice that

$$\bar\partial h_j = \bar\partial Re(f_j) = \overline{df_j}/2, \tag{2.1}$$

so that

$$i^*_{Z_{\rm reg}}(\bar\partial h_{i_1} \wedge \cdots \wedge \bar\partial h_{i_m}) = 0 \text{ if and only if } i^*_{Z_{\rm reg}}(df_{i_1} \wedge \cdots \wedge df_{i_m}) = 0. \tag{2.2}$$

Hence, Z' is an analytic subset of $Z_{\rm reg}$.

Assume now that $z_0 \in Z_{\rm sing}$ and choose (possibly after shrinking U_0) generators $\psi_1, \dots, \psi_\ell$ for the radical ideal sheaf in U_0 of holomorphic functions vanishing on Z. Let

$$W = \{\, z \in U_0 : df_{i_1} \wedge \cdots \wedge df_{i_m} \wedge d\psi_{j_1} \wedge \cdots \wedge d\psi_{j_{n-d}}(z) = 0 \quad \forall\, (j_1, \dots, j_{n-d})\}.$$

Then W is an analytic subset of U_0. A point $z \in Z \cap U_0$ lies in $Z_{\rm reg}$ if and only there is some choice of $(j_1, \dots, j_{n-d})$ such that $d\psi_{j_1} \wedge \cdots \wedge d\psi_{j_{n-d}}(z) \neq 0$. It follows that W contains $Z_{\rm sing} \cap U_0$. It also follows that for $z \in Z_{\rm reg} \cap U_0$ we have $i^*_{Z_{\rm reg}}(df_{i_1} \wedge \cdots \wedge df_{i_m})(z) = 0$ if and only if $(df_{i_1} \wedge \cdots \wedge df_{i_m} \wedge d\psi_{j_1} \wedge \cdots \wedge d\psi_{j_{n-d}})(z) = 0$ for all $(j_1, \dots, j_{n-d})$. Consequently, (2.2) gives $W \cap Z_{\rm reg} = Z' \cap U_0$, and the lemma follows. □

Proof of Proposition 2.1. Let m be the largest integer such that there exist $h_{i_1}, \dots, h_{i_m}$ such that $i^*_{Z_{\rm reg}}(\bar\partial h_{i_1} \wedge \cdots \wedge \bar\partial h_{i_m})$ does not vanish identically; by hypothesis, $m < d$. If all h_i are holomorphic on Z then the hull of $\mathcal{G}_h(b\Omega)$ contains $\mathcal{G}_h(Z)$ and we are done; we may thus assume that $m \geq 1$.

We will first define a singular foliation $\mathcal{F}$ of Z by non-trivial varieties such that all the h_i are holomorphic along the plaques; see Definition 1.4 for the precise meaning of this. As in the beginning of the proof of Lemma 2.2 every $z_0 \in \Omega$ has a neighborhood U_0 such that for all j, $Re(f_j) = h_j + g_j$, where the f_j and the g_j are holomorphic in U_0. Notice that the f_j (and the g_j) are uniquely defined up to adding constants; the level sets of the map $F = (f_1, \dots, f_N) \colon U_0 \to \mathbb{C}^N$ thus unambiguously defines a partitioning of U_0 by varieties. We define $\mathcal{F}$ by intersecting with Z. Clearly all the h_j are holomorphic along each plaque. We need to show that each plaque has dimension ≥ 1. Let

$$\tilde Z = Z_{\rm sing} \cup \bigcap_{(i_1, \dots, i_m)} \{z \in Z_{\rm reg} : i^*_{Z_{\rm reg}}(\bar\partial h_{i_1} \wedge \cdots \wedge \bar\partial h_{i_m}) = 0\},$$

which is a proper analytic subset of Z by Lemma 2.2. Let $z_0 \in Z \setminus \tilde Z \subset Z_{\rm reg}$ and assume that $i^*_{Z_{\rm reg}}(\bar\partial h_1 \wedge \cdots \wedge \bar\partial h_m)(z_0) \neq 0$. Then by (2.1), $i^*_{Z_{\rm reg}}(df_1 \wedge \cdots \wedge df_m)(z_0) \neq 0$ and so we can choose local coordinates $\zeta_1, \dots, \zeta_d$ for $Z_{\rm reg}$ centered at z_0 such that $f_1 - f_1(z_0) = \zeta_1, \dots, f_m - f_m(z_0) = \zeta_m$. By (2.1) and the choice of m we have $i^*_{Z_{\rm reg}}(df_1 \wedge \cdots \wedge df_m \wedge df_j) \equiv 0$ for all j, and so $\partial f_j / \partial \zeta_k \equiv 0$ for $j = m+1, \dots, N$ and $k = m, \dots, d$. Hence, the level set of $f_j|_Z$ through z_0 contains the common level set of $f_1|_Z, \dots, f_m|_Z$ through z_0; the plaque through z_0 thus equals the latter set which has dimension $d - m \geq 1$. Now, the function $Z \ni z \mapsto \dim_z Z \cap (F^{-1}(F(z)))$ is ≥ 1 on

$Z \setminus \tilde{Z}$ and upper semicontinuous, see, *e.g.*, [9, Ch. 2, Prop. 8.2]. Hence, the plaque through any point of Z has dimension ≥ 1.

Assume now to get a contradiction that $\mathcal{G}_h(b\Omega)$ is polynomially convex. Then there is a point $z_0 \in Z$ and a polynomial $P(z,w)$ in $\mathbb{C}^{n+N}$ such that

$$\sup_{z \in \overline{Z}} |P(z,h(z))| = |P(z_0,h(z_0))| > \sup_{z \in b\Omega} |P(z,h(z))|.$$

It follows that the set $K := \{z \in Z : P(z,h(z)) = P(z_0,h(z_0))\}$ is a compact subset of Ω. Since the h_j are holomorphic along the plaques of $\mathcal{F}$ it follows from the maximum principle (see, *e.g.*, [24, Ch. 4, Thm. 2G]) applied to $z \mapsto P(z,h(z))$ that K is a union of plaques of $\mathcal{F}$. Let $(p_1,\dots,p_n) \in K$ be a point where $|z_1|$ attains its maximum on K. Then $K_1 := K \cap \{z_1 = p_1\}$ is compact and non-empty and is, again by the maximum principle, a union of plaques of $\mathcal{F}$. Repeating for the rest of the coordinate functions $z_2,\dots,z_n$ we see that there is a plaque of $\mathcal{F}$ contained in a point, a contradiction. □

Proof of Theorem 1.1. (1) ⇒ (3): Assume that $\mathcal{G}_h(b\Omega)$ is polynomially convex and let $\varphi\colon \triangle \hookrightarrow \Omega$ be a holomorphic embedding. We will show that there exist $h_{i_1},\dots,h_{i_d}$ and an irreducible analytic set $Z \subset \Omega$ of dimension d such that $\dim(\varphi(\triangle) \cap (Z \setminus \tilde{Z})) = 1$, where

$$\tilde{Z} := \{z \in Z_{\mathrm{reg}} : i^*_{Z_{\mathrm{reg}}}(\overline{\partial} h_{i_1} \wedge \cdots \wedge \overline{\partial} h_{i_d})(z) = 0\} \cup Z_{\mathrm{sing}}.$$

In that case, clearly all of the h_{i_j} cannot be holomorphic along $\varphi(\triangle)$.

To obtain the analytic set Z, first we let

$$Z_1 = \{z \in \Omega : \overline{\partial} h_{i_1} \wedge \cdots \wedge \overline{\partial} h_{i_n}(z) = 0 \quad \forall (i_1,\dots,i_n)\}. \tag{2.3}$$

Then $\dim(Z_1) < n$ by Proposition 2.1 since $\mathcal{G}_h(b\Omega)$ is polynomially convex. If $\varphi(\triangle)$ is not contained in Z_1 then Ω works as Z and we are done. Otherwise $\varphi(\triangle)$ is contained in an irreducible component of Z_1; abusing notation we denote this component by Z_1 and we define

$$Z_2 = \{z \in (Z_1)_{\mathrm{reg}} : i^*_{Z_1}(\overline{\partial} h_{i_1} \wedge \cdots \wedge \overline{\partial} h_{i_{d_1}})(z) = 0 \quad \forall (i_1,\dots,i_{d_1})\} \cup (Z_1)_{\mathrm{sing}},$$

where $d_1 = \dim(Z_1)$. By Lemma 2.2 we have that Z_2 is an analytic subset of Ω and by Proposition 2.1 we have that $\dim(Z_2) < \dim(Z_1)$. If $\varphi(\triangle)$ is not contained in Z_2 then Z_1 works as Z and we are done. Repeating this process we eventually find the desired analytic set Z.

(3) ⇒ (4): The proof of [22, Theorem 1.3] given in [22] in fact shows that this implication holds.

(4) ⇒ (1): By [22, Lemma 4.6] $\mathcal{G}_h(\overline{\Omega})$ is polynomially convex. Thus $\widehat{\mathcal{G}_h(b\Omega)}$ is contained in $\mathcal{G}_h(\overline{\Omega})$. For any point a in Ω, by (3) there is a function g in $[z,h]_{\overline{\Omega}}$ such that $g(p) = 1$ and $g|b\Omega = 0$. Consequently there is a polynomial p in $\mathbb{C}^{n+N}$ such that $p\big(a,h(a)\big) > \sup_{z \in b\Omega} p\big(z,h(z)\big)$, so $\big(a,h(a)\big)$ is not in $\widehat{\mathcal{G}_h(b\Omega)}$. Thus $\mathcal{G}_h(b\Omega)$ is polynomially convex.

(1) ⇒ (2): Obvious.

(2) $\Rightarrow$ (1): This is immediate from Theorem 1.5 which we prove next. □

Proof of Theorem 1.5. Assume that $\mathcal{G}_h(b\Omega)$ is not polynomially convex. Then by Theorem 1.1 (2) there is an embedded analytic disk $\varphi\colon \triangle \to \Omega$ such that each h_j is holomorphic along $\varphi(\triangle)$. The procedure in the proof of Theorem 1.1, (1) $\Rightarrow$ (2), gives a decreasing sequence $Z_1 \supset Z_2 \supset \cdots$ of irreducible analytic subsets of Ω such that $\varphi(\triangle) \subset Z_j$ for all j. For dimensional reasons it follows that for some k, $Z_k = Z_{k+1} = \cdots$ and dim $Z_k \geq 1$. Moreover, from the construction of the sets Z_j we have that

$$i^*_{Z_k}\left(\overline{\partial} h_{i_1} \wedge \cdots \wedge \overline{\partial} h_{i_{d_k}}\right) \equiv 0, \quad \forall (i_1, \dots, i_{d_k}),$$

where $d_k = \dim Z_k$. As in the first part of the proof of Proposition 2.1 we then get a singular foliation of Z_k by non-trivial varieties such that all the h_j are strongly holomorphic along the plaques. That $\mathcal{G}_h(Z)$ is contained in $\widehat{\mathcal{G}_h(b\Omega)} \setminus \mathcal{G}_h(b\Omega)$ is proved by repeating the argument in the last paragraph of the proof of Proposition 2.1 □

3. Algebras generated by smooth functions

In this section we prove Theorems 1.6–1.10. We will use the following version of [22, Proposition 4.7].

Proposition 3.1. *Let $K \subset \mathbb{C}^n$ be a compact set, let $F\colon K \to \mathbb{C}^k$ be the restriction to K of a polynomial map, and assume that $[w_1, \dots, w_k]_{F(K)} = \mathcal{C}(F(K))$. Then the following holds:*

(i) $\widehat{K} = \bigcup_{w\in\mathbb{C}^k} \widehat{F^{-1}(w)}$
(ii) *If $[z_1, \dots, z_n]_{F^{-1}(w)} = \mathcal{C}(F^{-1}(w))$ for all $w \in \mathbb{C}^k$, then $[z_1, \dots, z_n]_K = \mathcal{C}(K)$.*

Recall that we denote by Γ the distinguished boundary of the bidisk.

Lemma 3.2. *Let $K \subset \Gamma$ be a closed subset such that for some $c \in S^1$ the set K is disjoint from the circle $\{\,(z_1, z_2) \in \Gamma : z_1 = c\,\}$, and there is no $a \in S^1$ such that K contains the full circle $\{\,(z_1, z_2) \in \Gamma : z_1 = a\,\}$. Then $[z_1, z_2]_K = \mathcal{C}(K)$, and in particular, K is polynomially convex.*

Proof. First note that if L is a proper closed subset of $S^1 \subset \mathbb{C}$, then $[z]_L = \mathcal{C}(L)$. (See for instance [12, pp. 81–82].) Let F be the restriction to K of the projection $(z_1, z_2) \mapsto z_1$. The assumptions on K imply that $F(K)$ is a proper closed subset of S^1 and that $F^{-1}(a)$ is a proper closed subset of $\{a\} \times S^1$ for each $a \in \mathbb{C}$. The hypotheses of Proposition 3.1 (ii) are thus satisfied and the result follows. □

Lemma 3.3. *If K is a closed subset of a smooth manifold M such that $M \setminus K$ is disconnected and U and V form a separation of $M \setminus K$, then there is a smooth function f on M that is identically zero on K, strictly positive everywhere on U, and strictly negative everywhere on V.*

Proof. Every closed subset of a smooth manifold is the zero set of some smooth real-valued function on the manifold (see [18, pp. 76–77]), so let g be a smooth real-valued function on M with K its zero set. By replacing g by g^2, we may assume that $g \geq 0$. Let $\gamma\colon \mathbb{R} \to \mathbb{R}$ be a smooth function that is identically zero for $x \leq 0$ and strictly positive for $x > 0$. Then the function $\gamma \circ g : M \to \mathbb{R}$ is smooth and has K as its zero set, and it is easily verified that all partial derivatives of $\gamma \circ g$ vanish identically on K. It follows that the function f defined by setting f equal to $\gamma \circ g$ on $K \cup U$ and equal to $-\gamma \circ g$ on V has the required properties. □

The key to our proof of Theorem 1.6 is the result of Herbert Alexander [2] that asserts the existence of a closed set $E \subset \Gamma$ such that $\hat{E} \setminus E$ is nonempty but contains no analytic disk. However, we need to show that the set can be taken to have a certain additional property. This is achieved in the following lemma.

Lemma 3.4. *There is a closed subset X of the torus Γ such that*
(i) *$\hat{X} \setminus X$ is nontrivial but contains no analytic disk, and*
(ii) *$\Gamma \setminus X$ is the disjoint union of open sets U and V each of which is disjoint from some circle $\{(z_1, z_2) \in \Gamma : z_1 = c\}$ (depending on U and V).*

We postpone the proof of this lemma and first use it to prove Theorem 1.6.

Proof of Theorem 1.6. Consider the graph $K := \mathcal{G}_f(\Gamma) \subset \mathbb{C}^3$ and let F be the restriction to K of the projection $(z_1, z_2, z_3) \mapsto z_3$. Then $F(K)$ is a compact subset of $\mathbb{R}$ so, by Weierstrass' approximation theorem, the hypotheses of Proposition 3.1 (i) are fulfilled and we can determine the hull of K by determining the hulls of the fibers $F^{-1}(r) = \{f = r\} \times \{r\}$. To do this, first notice that X has to intersect every circle $\{(z_1, z_2) \in \Gamma : z_1 = c\}$. Indeed, X cannot contain such a full circle since $\hat{X} \setminus X$ contains no analytic disk and so if X were disjoint from some such circle, then Corollary 3.2 would imply that X were polynomially convex. It follows that $\{f = r\}$, $r \neq 0$, contains no full circle $\{(z_1, z_2) \in \Gamma : z_1 = c\}$. Furthermore, since $\{f = r\}$ is contained in either U or V for $r \neq 0$, $\{f = r\}$ is disjoint from some such circle. Hence, for $r \neq 0$, the fiber $F^{-1}(r) = \{f = r\} \times \{r\}$ is polynomially convex by Corollary 3.2. The hull of $F^{-1}(0) = X \times \{0\}$ is $\hat{X} \times \{0\}$ and from Proposition 3.1 we get that

$$\widehat{\mathcal{G}_f(\Gamma)} \setminus \mathcal{G}_f(\Gamma) = (\hat{X} \setminus X) \times \{0\},$$

which is nonempty but contains no analytic disk. □

The proof of Lemma 3.4 uses the following lemma.

Lemma 3.5. *If $K \subset \Gamma$ and $C \subset \{(z_1, z_2) \in \Gamma : z_1 = c\}$ are closed sets such that $K \cup C$ does not contain the full circle $\{(z_1, z_2) \in \Gamma : z_1 = c\}$, then $(K \cup C)\widehat{\ } = \widehat{K} \cup C$. The same holds with z_1 replaced by z_2.*

Proof. Clearly $\overline{\mathbb{D}}^2 \supset (K\cup C)\widehat{} \supset \widehat{K}\cup C$ so we are to show that if $a = (a_1, a_2) \in \overline{\mathbb{D}}^2 \setminus (\widehat{K} \cup C)$, then a is not in $(K \cup C)\widehat{}$. Assume first that $a_1 \neq c$ and let $q(z) = z_1 - c$; then $q = 0$ on C and $q(a) \neq 0$. Since a is not in $\hat{K}$, there is a polynomial p such that $p(a) = 1$ and $\sup_{z\in K} |p(z)| < 1$. Some power p^n of p satisfies $\sup_{z\in K} |p^n(z)| < |q(a)|/(1 + \sup_{z\in K} |q(z)|)$ and it follows that $q \cdot p^n$ separates a from $K \cup C$.

Assume now that $a_1 = c$. Since each point of Γ is a peak point for the bidisk algebra, we may assume that $|a_2| < 1$. Choose θ such that $e^{i\theta} = c$ and choose $\alpha_1, \beta_1, \alpha_2, \beta_2$ with $\alpha_1 < \theta < \beta_1$ such that the "rectangle" $\{(e^{is}, e^{it}) : s \in [\alpha_1, \beta_1], t \in [\alpha_2, \beta_2]\}$ is disjoint from $K\cup C$; this is possible since $K\cup C$ is closed and does not contain the full circle $\{(z_1, z_2) \in \Gamma : z_1 = c\}$. Since the set $J = S^1 \setminus \{e^{it} : t \in [\alpha_2, \beta_2]\}$ is polynomially convex, there is a polynomial q of one variable such that $q(a_2) = 1$ and $\sup_{w\in J} |q(w)| < 1$. Let M be the supremum of $|q|$ over the unit disk. For sufficiently large n, the supremum of $|(1 + \bar{c}z)/2|^n$ over $S^1 \setminus \{e^{is} : s \in [\alpha_1, \beta_1]\}$ is strictly less than $1/M$ and so

$$\sup_{(z_1,z_2)\in K\cup C} \left| \left(\frac{1 + \bar{c}z_1}{2}\right)^n q(z_2) \right| < 1 = \left(\frac{1 + \bar{c}z_1}{2}\right)^n q(z_2)\Big|_{(z_1,z_2)=(a_1,a_2)}.$$

Obviously the roles of z_1 and z_2 can be reversed. □

Proof of Lemma 3.4. By [2] there exists a closed set $E \subset \Gamma$ such that $\hat{E} \setminus E$ is nonempty but contains no analytic disk. We will obtain the set X by taking the union of E with certain circle segments.

Since E is a proper closed subset of Γ there exist $\alpha_1, \beta_1, \alpha_2, \beta_2$ with $-\pi < \alpha_1 < \beta_1 < \pi$ and $-\pi < \alpha_2 < \beta_2 < \pi$ such that the "rectangle" $\{(e^{is}, e^{it}) : s \in [\alpha_1, \beta_1], t \in [\alpha_2, \beta_2]\}$ is disjoint from E. Choose $\varphi_1, \psi_1, \varphi_2, \psi_2$ and $\varphi_1', \psi_1', \varphi_2', \psi_2'$ such that $\alpha_1 < \varphi_1 < \psi_1 < \varphi_1' < \psi_1' < \beta_1$ and $\alpha_2 < \varphi_2 = \varphi_2' < \psi_2 = \psi_2' < \beta_2$. Let

$$\begin{aligned} \Sigma \;=\; & \big\{(e^{i\varphi_1}, e^{it}) : t \in [-\pi, \pi] \setminus [\varphi_2, \psi_2]\big\} \cup \big\{(e^{i\psi_1}, e^{it}) : t \in [\varphi_2, \psi_2]\big\} \\ & \cup\big\{(e^{is}, e^{i\varphi_2}) : s \in [\varphi_1, \psi_1]\big\} \cup \big\{(e^{is}, e^{i\psi_2}) : s \in [\varphi_1, \psi_1]\big\}, \end{aligned}$$

let Σ' be given by the same expression but with primes throughout, and let $X = E \cup \Sigma \cup \Sigma'$. It is easily seen that $\Gamma \setminus X$ is disconnected and can be written as the disjoint union of open sets U and V each of which is disjoint from some circle $\{(z_1, z_2) \in \Gamma : z_1 = c\}$. By repeated use of Lemma 3.5 it follows that $\hat{X} \setminus X = \hat{E} \setminus E$ and so $\hat{X} \setminus X$ is nonempty but contains no analytic disk. □

The proof of Theorem 1.7 uses the following lemma.

Lemma 3.6. *Let $X \subset b\mathbb{B}_2$ be a compact set that is disjoint from the circle $\{ (z_1, z_2) \in b\mathbb{B}_2 : |z_1| = 1,\ z_2 = 0 \}$ and contains no full circle $\{ (z_1, z_2) \in$*

$b\mathbb{B}_2 : z_1 = c\,\}$ for c a constant with $|c| < 1$. Then there is a smooth real-valued function g on $b\mathbb{B}_2$ with X as its zero set and that is nonconstant on each circle $\{\,(z_1, z_2) \in b\mathbb{B}_2 : z_1 = c\,\}$ for $|c| < 1$.

We postpone the proof of this lemma and first use it to prove Theorem 1.7.

Proof of Theorem 1.7. By the theorem of Duval and Levenberg [10] quoted in the introduction as Theorem 1.11 there is a compact set $X \subset b\mathbb{B}_2$ that is not polynomially convex but whose polynomially convex hull contains no analytic disk. This example is also presented in [19]. Examination of the proof as presented in [19] reveals that one can easily arrange to have X be disjoint from the circle $\{\,(z_1, z_2) \in b\mathbb{B}_2 : |z_1| = 1,\ z_2 = 0\,\}$. (In Theorem 1.7.2 take the set K to be disjoint from the set $\{\,z_2 = 0\,\}$ and in the proof take z_2 to be among the polynomials Q_j.)

Let g be the function given by Lemma 3.6, and set $f_1 = g$ and $f_2 = x_1 \cdot g$. Consider the graph $K := \mathcal{G}_f(b\mathbb{B}_2) \subset \mathbb{C}^4$ and let F be the restriction to K of the projection $(z_1, \ldots, z_4) \mapsto (z_3, z_4)$. Then $F(K)$ is a compact subset of $\mathbb{R}^2$ so, by Weierstrass' approximation theorem, the hypotheses of Proposition 3.1 (i) are fulfilled and we can determine the hull of K by determining the hulls of the fibers $F^{-1}(r)$. Since the hull of $F^{-1}(0,0) = X \times \{(0,0)\}$ is $\hat{X} \times \{(0,0)\}$, and the set $\hat{X} \setminus X$ is nonempty but contains no analytic disk, it suffices to show that each of the other fibers is polynomially convex.

Let $E = F^{-1}(r_1, r_2)$ be some other fiber. Note that on E we have $z_3 = r_1 \neq 0$ and $x_1 = z_4/z_3 = r_2/r_1$. In particular E is contained in a level set of x_1. Let G be the restriction to E of the mapping $(z_1, \ldots, z_4) \mapsto (z_3, (z_1 - (r_2/r_1))/i)$. We will establish polynomial convexity of E by applying Proposition 3.1 (i) again, this time K replaced by E and F replaced by G. Since $(z_1 - (r_2/r_1))/i = y_1$ on E, the set $G(E)$ is contained in $\mathbb{R}^2$ so, by Weierstrass' approximation theorem again, the hypotheses of Proposition 3.1 (i) are fulfilled. On each fiber $G^{-1}(w_1, w_2)$, the functions z_1, z_3, and z_4 are constant. Because g is nonconstant on each circle $\{\,(z_1, z_2) \in b\mathbb{B}_2 : z_1 = c\,\}$ for $|c| < 1$, it follows that each fiber is either a *proper* subset of some circle where z_1, z_3, and z_4 are constant or else is a single point. Thus each fiber is polynomially convex and hence so is E.

□

The proof of Lemma 3.6 uses the following calculus lemma for which the reader can easily supply a proof.

Lemma 3.7. *Suppose that f is a smooth function on $\mathbb{R}^n \setminus \{0\}$ and to each partial derivative $\partial^{k_1+\cdots+k_n} f/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$ of any order there corresponds an integer k such that $\partial^{k_1+\cdots+k_n} f/\partial x_1^{k_1} \cdots \partial x_n^{k_n}$ blows up at the origin no faster than $1/r^k$. Suppose also that α is a smooth function on $\mathbb{R}^n$ with all partial derivatives of all orders equal to zero at the origin. Then the function $\alpha \cdot f$ is smooth on $\mathbb{R}^n$. (Of course here we define $(\alpha \cdot f)(0)$ to be zero.)*

Proof of Lemma 3.6. Let ρ be a smooth function on $b\mathbb{B}^2$ such that $X = \{\rho = 0\}$ (which recall is possible by [18, pp. 76–77]). By replacing ρ by ρ^2 and rescaling, we may assume that $0 \le \rho \le 1$. Let $\pi\colon b\mathbb{B}^2 \to \mathbb{C}$ be the restriction to $b\mathbb{B}^2$ of the projection $\pi(z_1, z_2) = z_1$ and let $X' = \pi(X)$. By the hypotheses on X, note that X' is a compact subset of the open unit disk $\Delta \subset \mathbb{C}$. For each $c \in \Delta$, denote the circle $b\mathbb{B}^2 \cap \{z_1 = c\}$ by S_c. For functions defined on $b\mathbb{B}^2$, let "subscript c" denote restriction to S_c, so for instance ρ_c denotes the restriction of ρ to S_c. Notice that if $c \in X'$, then ρ is non-constant on the circle S_c since X does not contain S_c. By compactness of X' and continuity of ρ there is a $\delta > 0$ and a neighborhood $U \subset\subset \Delta$ of X' such that

$$\max \rho_c - \min \rho_c > \delta, \quad \forall c \in U. \tag{3.1}$$

Let $\chi_1 = \chi_1(z_1)$ be a smooth function such that $\chi_1 = 1$ in a neighborhood of X', the support of χ_1 is contained in U, and $0 \le \chi_1 \le 1$; let $\chi_2 = 1 - \chi_1$. We will consider χ_j as a function on $\mathbb{C}^2$ or $b\mathbb{B}^2$ that is independent of z_2. For $k \in \mathbb{N}$ let

$$\psi_k(z_2) = 1 + e^{-1/|z_2|} \sin(k \cdot \arg(z_2)).$$

Notice that ψ_k is smooth, by Lemma 3.7, and that $0 < \psi_k < 2$. We will show, that for sufficiently large k, the function $g\colon b\mathbb{B}^2 \to \mathbb{R}$ defined by

$$g := \rho\chi_1 + \psi_k\chi_2$$

has the desired properties.

Clearly the zero set of g equals X. Given $c \in \Delta$ we must show that g_c is non-constant. For notational convenience we write $z_2 = re^{i\theta}$; on the circle S_c thus $z_2 = \sqrt{1-|c|^2}\, e^{i\theta}$. If $c \in \Delta \setminus U$ then $g_c = 1 + e^{-1/\sqrt{1-|c|^2}} \sin(k \cdot \theta)$, which is non-constant. Assume that $c \in U$ is such that $\chi_2(c) \ge \delta/3$. Differentiating g_c with respect to θ we get

$$\frac{d\rho_c}{d\theta}\chi_1(c) + k e^{-1/\sqrt{1-|c|^2}} \cos(k\theta)\chi_2(c). \tag{3.2}$$

Since $c \in U \subset\subset \Delta$, $\chi_2(c) \ge \delta/3$, and $d\rho_c/d\theta$ is uniformly bounded for $c \in U$, it follows that, taking $\theta = 0$, the second term in (3.2) dominates the first one for k sufficiently large. Hence, $dg_c/d\theta$ cannot be identically zero and so g_c is non-constant. Finally, assume that $c \in \{\chi_2 < \delta/3\} \subset U$. Then

$$\max g_c > \max \rho_c \cdot \chi_1(c) > \max \rho_c \cdot (1 - \delta/3) \ge \max \rho_c - \delta/3, \tag{3.3}$$

$$\min g_c < \min \rho_c \cdot \chi_1(c) + 2\chi_2(c) < \min \rho_c + 2\delta/3. \tag{3.4}$$

From (3.1), (3.3), and (3.4) it thus follow that

$$\max g_c - \min g_c > \max \rho_c - \min \rho_c - \delta = 0$$

and so g_c is non-constant.

□

The proofs of Theorems 1.9 and 1.10 use the following easy lemma. The same principle has been used by the first author to construct other counterexamples to the peak point conjecture [16].

Lemma 3.8. *Suppose X is a compact Hausdorff space, $E \subset X$ is closed, B is a uniform algebra on E, and $A = \{ f \in \mathcal{C}(X) : f|E \in B \}$. If $\{f_\alpha\}$ is a collection of functions in A such that $\{f_\alpha|E\}$ generates B, and $\{g_\beta\}$ is a collection of real-valued functions that generates $\mathcal{C}(X)$, and $\rho \in \mathcal{C}(X)$ is a real-valued function that vanishes precisely on E, then the collection $\{f_\alpha\} \cup \{\rho\} \cup \{\rho g_\beta\}$ generates the algebra A.*

Proof. One trivially verifies that the collection of functions $\{\rho\} \cup \{\rho g_\beta\}$ has E as its common zero set and separates points off of E, so these functions induce real-valued functions that separate points on the quotient space X/E obtained from X by identifying E to a point. Consequently, the Stone-Weierstrass theorem shows that the induced functions generated $\mathcal{C}(X/E)$. Hence the collection $\{\rho\} \cup \{\rho g_\beta\}$ generates the algebra $\{ f \in \mathcal{C}(X) : f \text{ is constant on } E\}$. It follows that the collection $\{f_\alpha\} \cup \{\rho\} \cup \{\rho g_\beta\}$ generates the algebra A. $\square$

Proof of Theorem 1.9. This is essentially proved in [16]. Let K be a compact planar set that has no interior and is contained in Ω such that $\mathcal{R}(K) \neq \mathcal{C}(K)$. (See, for instance, [13, p. 25–26] for the existence of such a set.) By [5, Lemma 11] there is a C^∞ function g such that the functions z and g generate $\mathcal{R}(K)$. Let ρ be a C^∞ function on $\overline{\Omega}$ that vanishes precisely on K. Let $\mathcal{A} = \{ f \in \mathcal{C}(\overline{\Omega}) : f|K \in \mathcal{R}(K) \}$. By Lemma 3.8, $[z, g, \rho, \rho\overline{z}] = [z, g, \rho, \rho x, \rho y] = \mathcal{A}$. The maximal ideal space of $\mathcal{A}$ is $\overline{\Omega}$ by [6, Theorem 4], and hence $\mathcal{G}_{(g,\rho,\overline{z}\rho)}(\overline{\Omega})$ is polynomially convex. Taking $\{f_1, f_2, f_3\} = \{g, \rho, \overline{z}\rho\}$ yields the theorem. $\square$

Proof of Theorem 1.10. The proof is similar to the previous one. By [5, Theorem 4] (or see [19, Example 19.8]) there is a compact set $K \subset \mathbb{C}^2 \subset \mathbb{C}^n$ such that K is rationally convex and every point of K is a peak point for $\mathcal{R}(K)$ but $\mathcal{R}(K) \neq \mathcal{C}(K)$. By translating and rescaling, we may assume that K is contained in Ω. By [5, Lemma 11] there is a C^∞ function g such that the functions $z_1, \ldots, z_n, g$ generate $\mathcal{R}(K)$. Let ρ be a C^∞ function on $\overline{\Omega}$ that vanishes precisely on K. Let $\mathcal{A} = \{ f \in \mathcal{C}(\overline{\Omega}) : f|K \in \mathcal{R}(K) \}$. By Lemma 3.8, $[z_1, \ldots, z_n, g, \rho, \rho\overline{z}_1, \ldots, \rho\overline{z}_n] = [z_1, \ldots, z_n, g, \rho, \rho x_1, \ldots, \rho x_n, \rho y_1, \ldots, \rho y_n] = \mathcal{A}$. The maximal ideal space of $\mathcal{A}$ is $\overline{\Omega}$ by [6, Theorem 4]. Now taking $\{f_1, \ldots, f_N\} = \{z_1, \ldots, z_n, g, \rho, \rho\overline{z}_1, \ldots, \rho\overline{z}_n\}$ yields the theorem. $\square$

Proof of Theorem 1.8. We divide the proof into steps.

Step 1: We show that for some N there is an embedding Φ of M into $\mathbb{C}^2 \times \mathbb{R}^N \subset \mathbb{C}^{2+N}$ such that $\Phi(M) \cap (\mathbb{C}^2 \times \{0\}^N) = \Gamma \times \{0\}^N$ and $\Phi(M) \cap (\mathbb{C}^2 \times \{r\})$ contains at most one point for each $r \in \mathbb{R}^N$ with $r \neq 0$; recall that Γ is the distinguished boundary of the bidisk in $\mathbb{C}^2$.

First choose a 2-torus T that is a smoothly embedded submanifold of M, and choose complex-valued functions g_1, g_2 such that $(g_1, g_2) : T \to \mathbb{C}^2$ gives a diffeomorphism of T onto Γ. Extend g_1, g_2 to smooth functions defined on all of M which we continue to denote by g_1, g_2. We will show that for each point p in M there are real-valued smooth functions $\phi_1, \ldots, \phi_k$ for some k such that the map $(g_1, g_2, \phi_1, \ldots, \phi_k) : M \to \mathbb{C}^2 \times \mathbb{R}^k$ is an immersion in a neighborhood of p and $\phi_1, \ldots, \phi_k$ are identically zero on T. If p is not in T, we simply take real-valued functions $x_1, \ldots, x_m$ that form a local coordinate system on a neighborhood of p and multiply them by a smooth function that is equal to 1 on a neighborhood of p and 0 outside of a closed neighborhood of p contained in the coordinate patch and disjoint from T. The resulting functions $\phi_1, \ldots, \phi_m$ then have the required properties. If p is in T then there is a local coordinate system $x_1, \ldots, x_m$ on a neighborhood U of p in M such that $U \cap T = \{x_1 = 0, \ldots, x_{m-2} = 0\}$. Then $(g_1, g_2, x_1, \ldots, x_{m-2}) : U \to \mathbb{C}^2 \times \mathbb{R}^{m-2}$ is an immersion in a neighborhood of p. By multiplying each of $x_1, \ldots, x_{m-2}$ by a smooth function that is equal to 1 on a neighborhood of p in M and 0 on a neighborhood of $M \setminus U$, we obtain functions $\phi_1, \ldots, \phi_{m-2}$ with the required properties.

By the result of the preceding paragraph and a compactness argument, there exist real-valued smooth functions $\phi_1, \ldots, \phi_k$ for some k such that $(g_1, g_2, \phi_1, \ldots, \phi_k) : M \to \mathbb{C}^2 \times \mathbb{R}^k$ is an immersion and the functions $\phi_1, \ldots, \phi_k$ are identically 0 on T. Now choose finitely many real-valued smooth functions $f_1, \ldots, f_n$ that separate points on M. Finally choose a smooth real-valued function ρ on M whose zero-set is exactly T. It is straightforward to verify that the mapping $(g_1, g_2, \rho, \rho f_1, \ldots, \rho f_n) : M \to \mathbb{C}^2 \times \mathbb{R}^{n+3}$ is injective. Thus, the mapping $(g_1, g_2, \rho, \rho f_1, \ldots, \rho f_n, \phi_1, \ldots, \phi_k) : M \to \mathbb{C}^2 \times \mathbb{R}^{3+n+k}$ is an embedding with the desired properties.

Step 2: We show that the N in Step 1 can be taken to be $2m + 1$.

The proof is similar to the proof of a version of the Whitney embedding theorem given in [11]. Suppose we can show that whenever $N > 2m + 1$, then there is a nonzero vector $a \in \mathbb{R}^N$ such that the composition of Φ with the orthogonal projection of $\mathbb{C}^2 \times \mathbb{R}^N$ onto the orthogonal complement of $(0, a)$ in $\mathbb{C}^2 \times \mathbb{R}^N$ is an injective immersion (and hence an embedding). The orthogonal complement of $(0, a)$ in $\mathbb{C}^2 \times \mathbb{R}^N$ is of the form $\mathbb{C}^2 \times V$ where V is an $(N - 1)$-dimensional real vector subspace of $\mathbb{R}^N$, so we would get an embedding of M into $\mathbb{C}^2 \times \mathbb{R}^{N-1}$. Let $p : \mathbb{C}^2 \times \mathbb{R}^N \to \mathbb{C}^2 \times \mathbb{R}^{N-1}$ denote the map obtained from the orthogonal projection and an identification of V with $\mathbb{R}^{N-1}$. Since the projection is the identity on $\mathbb{C}^2 \times \{0\}^N$, we get $(p \circ \Phi)(M) \cap (\mathbb{C}^2 \times \{0\}^{N-1}) \supset \Gamma \times \{0\}^{N-1}$. If in addition $\Phi(M) \cap (\mathbb{C}^2 \times \operatorname{span}\{a\}) = \Gamma \times \{0\}^N$, then the inclusion is an equality. Also $(p \circ \Phi)(M) \cap (\mathbb{C}^2 \times \{r\})$ will contain at most one point for each $r \neq 0$ in $\mathbb{R}^{N-1}$ provided $\Phi(M) \cap (\mathbb{C}^2 \times (\{s\} + \operatorname{span}\{a\})$ contains at most one point for each s in $\mathbb{R}^N \setminus \operatorname{span}\{a\}$. So if we can choose $a \in \mathbb{R}^N$ so that these conditions are satisfied, then we get an embedding of the required sort into $\mathbb{C}^2 \times \mathbb{R}^{N-1}$.

The existence of the desired embedding into $\mathbb{C}^2 \times \mathbb{R}^{2m+1}$ then follows by induction.

Let $\tilde{\Phi}$ be the composition of Φ with the map of $\mathbb{C}^2 \times \mathbb{R}^N$ to $\mathbb{R}^N$ that projects onto the second factor. Define a map $h : M \times M \times \mathbb{R} \to \mathbb{R}^N$ by $h(x, y, t) = t[\tilde{\Phi}(x) - \tilde{\Phi}(y)]$. Also, letting $T(M)$ denote the (real) tangent bundle to M, define a map $g : T(M) \to \mathbb{R}^N$ by $g(x, v) = d\tilde{\Phi}_x(v)$. Since $N > 2m + 1$, Sard's theorem shows that there exists a point a in $\mathbb{R}^N$ belonging to neither the image of h nor the image of g. Note that $a \neq 0$.

Let π be the projection of $\mathbb{C}^2 \times \mathbb{R}^N$ onto the orthogonal complement H of $(0, a)$. We want to show that $\pi \circ \Phi : M \to H$ is injective. Suppose $(\pi \circ \Phi)(x) = (\pi \circ \Phi)(y)$. Then $\Phi(x) - \Phi(y) = t(0, a)$ for some scalar t, or equivalently $\tilde{\Phi}(x) - \tilde{\Phi}(y) = ta$. If $x \neq y$, then $t \neq 0$, because Φ is injective. But then $h(x, y, 1/t) = a$, contradicting the choice of a.

Next we want to show that $\pi \circ \Phi : M \to H$ is an immersion. Suppose v is a nonzero vector in $T_x(M)$ for which $d(\pi \circ \Phi)_x(v) = 0$. By the chain rule $d(\pi \circ \Phi)_x = \pi \circ d\Phi_x$. Thus $\pi \circ d\Phi_x(v) = 0$, so $d\Phi_x(v) = t(0, a)$ for some scalar t, and $d\tilde{\Phi}_x(v) = ta$. Since Φ is an immersion, $t \neq 0$. Thus $g(x, (1/t)v) = a$, again contradiction the choice of a.

Finally we need to consider $\Phi(M) \cap \big(\mathbb{C}^2 \times (\{s\} + \mathrm{span}\{a\})\big)$ for $s \in \mathbb{R}^N$. Because $h(x, y, t) = t[\tilde{\Phi}(x) - \tilde{\Phi}(y)]$ is never equal to a, we have that $\tilde{\Phi}(x) - \tilde{\Phi}(y)$ is never in $\mathrm{span}\{a\}$ unless it is zero. Thus $\tilde{\Phi}(M)$ intersects $\{s\} + \mathrm{span}\{a\}$ in at most one point for each s. Because $\Phi(M) \cap (\mathbb{C}^2 \times \{r\})$ contains at most one point for each nonzero $r \in \mathbb{R}^N$, this gives that $\Phi(M) \cap \big(\mathbb{C}^2 \times (\{s\} + \mathrm{span}\{a\})\big)$ contains at most one point for s nonzero. We also get that $\tilde{\Phi}(M)$ intersects $\mathrm{span}\{a\}$ only in the point 0 so that $\Phi(M) \cap (\mathbb{C}^2 \times \mathrm{span}\{a\}) = \Phi(M) \cap (\mathbb{C}^2 \times \{0\}) = \Gamma \times \{0\}$.

Step 3: We complete the proof of the theorem.

We now have an embedding $\Phi : M \to \mathbb{C}^2 \times \mathbb{R}^{2m+1} \subset \mathbb{C}^{2m+3}$ such that $\Phi(M) \cap (\mathbb{C}^2 \times \{0\}^{2m+1}) = \Gamma \times \{0\}^{2m+1}$ and $\Phi(M) \cap (\mathbb{C}^2 \times \{r\})$ contains at most one point for each $r \in \mathbb{R}^{2m+1}$ with $r \neq 0$. Let f denote the function on the standard 2-torus given in Theorem 1.6 whose graph has hull without analytic structure. Pull f back to a function on $\Phi^{-1}(\Gamma \times \{0\}^{2m+1})$ by precomposing with Φ, and extend the resulting function to a smooth function on M which we will denote by h. Then $(\Phi, h) : M \to \mathbb{C}^2 \times \mathbb{R}^{2m+2} \subset \mathbb{C}^{2m+4}$ is a smooth embedding. Let $K = (\Phi, h)(M)$ and $G = (\Phi, h)(\Phi^{-1}(\Gamma \times \{0\}^{2m+2}))$. Let $K_r = K \cap (\mathbb{C}^2 \times \{r\})$. Proposition 3.1 gives that $\hat{K} = \bigcup_{r \in \mathbb{R}^{2m+2}} \widehat{K_r}$. For each $r = (r_1, \ldots, r_{2m+2}) \in \mathbb{C}^{2m+2}$ with $(r_1, \ldots, r_{2m+1}) \neq 0$, we know K_r contains at most one point and hence makes no contribution to $\hat{K} \setminus K$. For $r \in \mathbb{C}^{2m+2}$ with $(r_1, \ldots, r_{2m+1}) = 0$ and r_{2m+2} arbitrary, K_r is contained in G. It is now easily seen that $\hat{K} \setminus K = \widehat{G} \setminus G$. Since G is the image of the graph of f under the embedding of $\mathbb{C}^3$ into $\mathbb{C}^{2m+4}$ given by $(z_1, z_2, z_3) \mapsto (z_1, z_2, 0, \ldots, 0, z_3)$, we know that $\widehat{G}\setminus G$ is non-trivial but contains no analytic subset of positive dimension by Theorem 1.6. $\square$

4. Foliation structure for uniform algebras on manifolds

Let M be a $\mathcal{C}^1$-smooth manifold of real dimension $n \geq 2$. Recall that a CR-structure on M is a subbundle $\mathbb{L}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ that is involutive and such that $\mathbb{L}_p \cap \overline{\mathbb{L}}_p = \{0\}$ for each $p \in M$. Let $N = \{L+\overline{L} : L \in \mathbb{L}\}$. Then N is a subbundle of the real tangent space TM, and there is a complex structure map J on N (i.e., a bundle isomorphism $J : N \to N$ such that $J^2 = -\mathrm{Id}$) so that $\mathbb{L}$ and $\overline{\mathbb{L}}$ are respectively the $+i$ and $-i$ eigenspaces of the extension of J to $\mathbb{L} \oplus \overline{\mathbb{L}}$. We say that a $\mathcal{C}^1$-smooth function $f \colon M \to \mathbb{C}$ is CR, $f \in \mathrm{CR}(M)$, if $df(Jv) = idf(v)$.

If $f \colon M \to \mathbb{C}^k$ is an embedding and the dimension of $H_p f(M) := T_p f(M) \cap J_{st} T_p f(M)$, where J_{st} is the standard complex structure on $\mathbb{C}^k$, is independent of $p \in f(M)$, then there is a CR-structure $\mathbb{L}$ on $f(M)$ whose associated real subbundle is $Hf(M)$. Moreover, since $Df \colon TM \to Tf(M)$ is an isomorphism, we get an induced CR-structure on M with associated real subbundle $N := (Df)^{-1} Hf(M) \subset TM$.

We say that a CR-structure with associated real subbundle $N \subset TM$ is integrable on M if for each point $p \in M$ there is a neighborhood $U \ni p$ and a $\mathcal{C}^1$-smooth mapping $\rho = (\rho_1, \ldots, \rho_{n-2m}) \colon U \to \mathbb{R}^{n-2m}$ such that $d\rho_1 \wedge \cdots \wedge d\rho_{n-2m}$ is non-vanishing in U and such that for every $y \in U$, the tangent space of $Z_y := \{x \in U : \rho(x) = \rho(y)\}$ equals N_y. The local submanifolds Z_y then define a foliation $\mathcal{F}$ on M, and M has the structure of a Leviflat CR-manifold with the Levi foliation $\mathcal{F}$ inducing the given CR-structure.

Proof of Theorem 1.13. Let $\mathcal{A}_0$ denote the collection of functions in $\mathcal{A}$ that are $\mathcal{C}^1$ on U, and let $\tilde{\Omega} \subset U$ be the set of points $x \in U$ such that there exists an open neighborhood U_x of x, and $f_j \in \mathcal{A}_0, j = 1, \ldots, n$, with $f|_{U_x} : U_x \to \mathbb{C}^n$ being an embedding. We begin by showing that $\tilde{\Omega}$ is dense in U; this will depend only on the fact that $\mathcal{A}_0$ is point separating.

For every $f \in \mathcal{A}_0$ we write $f = u_f + iv_f$, where u_f and v_f are real. By induction we will pick $f_1, \ldots, f_s, g_1, \ldots, g_t \in \mathcal{A}_0$, $s + t = n$, such that

$$du_{f_1} \wedge \cdots \wedge du_{f_s} \wedge dv_{g_1} \wedge \cdots \wedge dv_{g_t} \tag{4.1}$$

is not identically zero: Since $\mathcal{A}_0$ separates points, $\mathcal{A}_0$ must contain a function whose differential is not identically zero. Assume that we have found $f_1, \ldots, f_s, g_1, \ldots, g_t \in \mathcal{A}_0$, $s + t \leq n$, such that (4.1) is non-vanishing on some open set Ω_{s+t}. If $s + t < n$ then some level set of the map (u_f, v_g) defines a $\mathcal{C}^1$-smooth submanifold Y of Ω_{s+t} of positive dimension. Since $\mathcal{A}_0$ is point separating, all functions in $\mathcal{A}_0$ cannot be constant on Y and so there is a function in $\mathcal{A}_0$ such that the wedge product of its differential with (4.1) is not identically zero. The resulting map $(f, g)\colon M \to \mathbb{C}^n$ now gives an embedding of some neighborhood of some point in U and hence, $\tilde{\Omega}$ is nonempty. If $\tilde{\Omega}$ were not dense in U, then we could repeat the argument and show that there is a map in $\mathcal{A}_0^n$ giving an embedding of some neighborhood of some point in $U \setminus \tilde{\Omega}$.

For each $x \in \tilde{\Omega}$ we let m_x be the smallest integer such there exists a neighborhood U_x of x and an embedding $f : U_x \to \mathbb{C}^k$, $f \in \mathcal{A}_0^k$, for some k, with the property that the complex dimension of $H_{f(x)}f(U_x)$ is m_x. We claim that there exists a dense open subset Ω of $\tilde{\Omega}$ such that the map $x \mapsto m_x$ is locally constant on Ω and strictly greater than zero. To see this let $m_1 = \min_{x\in\tilde{\Omega}}\{m_x\}$. Note that if $m_x = 0$ then there exists an embedding $f : U_x \to \mathbb{C}^k$ such that $f(U_x)$ is totally real. Hence x is a local peak point for $\mathcal{A}$, and so x is in the Shilov boundary. Thus, $m_1 > 0$. Let $\Omega_1 := \{x \in \tilde{\Omega} : m_x = m_1\}$; then Ω_1 is open by upper semi-continuity of the dimension of the maximal complex tangent space. If Ω_1 is dense we are done; otherwise let $m_2 = \min_{x\in\tilde{\Omega}\setminus\overline{\Omega}_1}\{m_x\}$. Let $\Omega_2 := \{x \in \tilde{\Omega} \setminus \overline{\Omega}_1 : m_x = m_2\}$; then Ω_2 is open. It is now clear how to proceed to obtain $\Omega := \Omega_1 \cup \cdots \cup \Omega_s$.

Next we define a CR-structure on Ω via local embeddings into $\mathbb{C}^k$ as explained in the beginning of this section. For $x \in \Omega$ pick an embedding $f : U_x \to \mathbb{C}^k$ such that $\dim_{\mathbb{C}} H_{f(y)}f(U_x) = m_y = m_x$ for all $y \in U_x$. Then $f(U_x)$ defines a CR-structure CR_f on U_x. Let $N_f \subset TU_x$ denote the associated real subbundle. We claim that if $g \in \mathcal{A}_0$ then $g \in \mathrm{CR}_f(U_x)$. If not, consider the embedding $h = (f, g) : U_x \to \mathbb{C}^{k+1}$; it then has the property that there is a $y \in U_x$ such that $\dim_{\mathbb{C}} H_{h(y)}h(U_x) < m_x = m_y$, which is a contradiction. It now follows that CR_f is in fact independent of the choice of such an embedding f. Indeed, if g is another choice of such an embedding at x, then, since $g \in \mathrm{CR}_f(U_x)$, we have that $N_f \subset N_g$, and then, since $\dim N_f = \dim N_g$, it follows that $N_g = N_f$. Hence, we have a well defined CR-structure on each component of Ω, and the functions in $\mathcal{A}_0$ are in $\mathrm{CR}(\Omega)$.

Finally we show that the obtained CR-structure on Ω is integrable. (Once this is done, it is immediate that every function in $\mathcal{A}$ is holomorphic along the leaves since the functions in $\mathcal{A}_0$ are CR and $\mathcal{A}_0$ is dense in $\mathcal{A}$.) By a result of R. A. Airapetian [1] it suffices to show that each image $f(U_x) \subset \mathbb{C}^k$ of an embedding as above is locally polynomially convex. We now consider each $g \in \mathcal{A}$ as a function on $f(U_x)$. If U_x is sufficiently small it follows from the approximation theorem of Salah Baouendi and François Treves[3] that each $g \in \mathcal{A}$ is uniformly approximable by polynomials on $f(U_x)$. Thus for each closed neighborhood V_x of x contained in U_x, and continuing to regard the functions in $\mathcal{A}$ as functions on $f(U_x)$, we have that $\mathcal{A}|_{f(V_x)} = [z_1, \ldots, z_k]$. Consequently $f(V_x)$ will be polynomially convex provided the maximal ideal

[3]Although this approximation theorem often is formulated in the $\mathcal{C}^2$-category, it holds also in $\mathcal{C}^1$.

space of $\mathcal{A}|_{V_x}$ is V_x. The proof is thus concluded by invoking the following lemma. □

Lemma 4.1. *Let A be a uniform algebra on a compact Hausdorff space X, and suppose that X is the maximal ideal space of A. Given a point $p \in X$ and a neighborhood U of p, there is a closed neighborhood V of p such that the maximal ideal space of the restriction algebra $\overline{A|_V}$ is V.*

Proof. A simple compactness argument shows that there exists a finite collection $\{f_1, \ldots, f_n\}$ of functions in A such that the set $V := \{\, x \in X : |f_j| \leq 1 \quad \forall j = 1, \ldots, n\}$ is a (closed) neighborhood of p contained in U. Clearly V is A-convex in the terminology of [13, II.6]. Thus by [13, II.6.1], the maximal ideal space of $\overline{A|_V}$ is V. □

For the proof of Theorem 1.12, note that taking $\mathcal{A} = [z_1, \ldots, z_n]_X$ and applying Theorem 1.13 gives the desired dense open set Ω with a CR-structure on each component. Since the CR-structure is obtained from local embeddings and was shown to be independent of the choice of embedding, we can now use the canonical global embedding. Hence the obtained CR-structure makes the components of Ω into CR-submanifolds of $\mathbb{C}^n$.

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