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Deformed Richardson-Gaudin model

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Abstract. The Richardson-Gaudin model describes strong pairing correlations of fermions confined to a finite chain. The integrability of the Hamiltonian allows the algebraic construction of its eigenstates. In this work we show that the quantum group theory provides a possibility to deform the Hamiltonian preserving integrability. More precisely, we use the so-called Jordanian r-matrix to deform the Hamiltonian of the Richardson-Gaudin model. In order to preserve its integrability, we need to insert a special nilpotent term into the auxiliary L-operator which generates integrals of motion of the system. Moreover, the quantum inverse scattering method enables us to construct the exact eigenstates of the deformed Hamiltonian. These states have a highly complex entanglement structure which require further investigation.

The Richardson-Gaudin model [1, 2] is an integrable spin-$\frac{1}{2}$ periodic chain with Hamiltonian

$$H = \sum_{j=1}^{N} \epsilon_j S_j^z + g \sum_{j,\,k=1}^{N} S_j^- S_k^+ \tag{1}$$

where g is a coupling constant and $S_l^{\pm} = S_l^x \pm i S_l^y$, with N copies of the Lie algebra $su(2)$ generators S_l^α,

$$[S_l^\alpha, S_{l'}^\beta] = i\varepsilon^{\alpha\,\beta\,\gamma} S^\gamma \delta_{l\,l'}\,, \quad \alpha, \beta = x, y, z$$

As shown by Cambiaggio *et al* [3], by introducing the fermion operators $c_{lm}^\dagger$ and c_{lm} related to the $sl(2)$ generators by

$$S_l^z = 1/2 \sum_m c_{lm}^\dagger c_{lm} - 1/2, \quad S_l^+ = \frac{1}{2} \sum_m c_{lm}^\dagger c_{l\bar{m}}^\dagger = (S_l^-)^\dagger$$

the Richardson-Gaudin model in Eq. (1) gets mapped onto the pairing model Hamiltonian

$$H_P = \sum_l \epsilon_l \hat{n}_l + g/2 \sum_{l,l'} A_l^\dagger A_{l'} \tag{2}$$

Here $c_{lm}^\dagger$ (c_{lm}) creates (annihilates) a fermion in the state $|lm\rangle$ (with $|l\bar{m}\rangle$ in the time reversed state of $|lm\rangle$) and

$$n_l = \sum_m c_{lm}^\dagger c_{lm}, \quad A_l^\dagger = (A_l)^\dagger = \sum_m c_{lm}^\dagger c_{l\bar{m}}^\dagger$$

are the corresponding number- and pair-creation operators. The pairing strengths $g_{ll'}$ are here approximated by a single constant g, with ϵ_l the single-particle level corresponding to the m-fold degenerate states $|lm\rangle$.

As it is well-known, the pairing model in Eq. (2) is central in the theory of superconductivity. Richardson's exact solution of the model [1], exploiting its integrability, has been important for applications in mesoscopic and nuclear physics where the small number of fermions prohibits the use of conventional BCS theory [4]. Moreover, its (pseudo)spin representation in the guise of the Richardson-Gaudin model, Eq. (1), provides a striking link between quantum magnetism and pairing phenomena, both central concepts in the physics of quantum matter.

The eigenstates of the Richardson-Gaudin Hamiltonian, eq. (1), can be constructed algebraically using the quantum inverse scattering method (QISM) [5, 6]. The main objects of this method are the classical r-matrix

$$r(\lambda,\,\mu) = \frac{4}{\lambda-\mu}\sum_{\alpha} S^{\alpha}\otimes S^{\alpha}\Big|_{s=\frac{1}{2}} \simeq \frac{1}{\lambda-\mu}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0\\ 0 & 2 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{3}$$

where $h(\lambda)$, $X^{+}(\lambda)$, $X^{-}(\lambda)$ are the generators of the loop algebra $\mathcal{L}(sl(2))$ whereas the L-matrix is

$$L(\lambda) = \begin{pmatrix} h(\lambda) & 2X^{-}(\lambda)\\ 2X^{+}(\lambda) & -h(\lambda)\end{pmatrix}$$

The commutation relations (CR) of loop algebra generators are given in compact matrix form

$$[L_1(\lambda), L_2(\mu)] = -[r_{1\,2}(\lambda,\mu), L_1(\lambda) + L_2(\mu)]$$

where

$$L_1(\lambda) = L(\lambda)\otimes\mathbb{I}, \quad L_2(\mu) = \mathbb{I}\otimes L(\mu)$$

and $r(\lambda,\,\mu)$ is the 4×4 c-number matrix in Eq. (3). A consequence of this form is the commutativity of transfer matrices,

$$t(\lambda) = \frac{1}{2}\mathrm{tr}_0(L^2(\lambda)) \quad \in \mathcal{L}(sl(2)), \quad [t(\lambda), t(\mu)] = 0 \tag{4}$$

The corresponding mutually commuting operators extracted from the decomposition of $t(\lambda)$ define a Gaudin model [2, 7]. However, to get Richardson Hamiltonian a mild change of the L-operator is necessary

$$L(\lambda) \to L(\lambda; c) := c\,h_0 + L(\lambda)$$

where $h_0 = \sigma_0^z$ in auxiliary space $\mathbb{C}_0^2$ of spin 1/2. This transformation does not change the CR of matrix elements of this matrix $L(\lambda; c)$ due to the symmetry of the r-matrix (3):

$$[Y\otimes\mathbb{I} + \mathbb{I}\otimes Y, r(\lambda,\,\mu)] = 0, \quad Y\in sl(2)$$

The resulting transfer matrix obtains some extra terms

$$t(\lambda; c) = \frac{1}{2}\mathrm{tr}_0\,(L(\lambda; c))^2 = c^2\mathbf{1} + c\,h(\lambda) + h^2(\lambda) + 2\left(X^{+}(\lambda)X^{-}(\lambda) + X^{-}(\lambda)X^{+}(\lambda)\right)$$

Let us consider a spin-$\frac{1}{2}$ representation on auxiliary space $V_0 \simeq \mathbb{C}^2$ and spin ℓ_k representations on quantum spaces $V_k \simeq \mathbb{C}^{\ell_k+1}$ with extra parameters ϵ_k corresponding to site $k = 1, 2, \ldots, N$.

The whole space of quantum states is $\mathcal{H} = \otimes_1^N V_k$ and the highest weight vector (highest spin, "ferromagnetic state") $|\Omega_+\rangle$ satisfies

$$X^+(\lambda)\,|\Omega_+\rangle = 0, \quad h(\lambda)\,|\Omega_+\rangle = \rho(\lambda)\,|\Omega_+\rangle \tag{5}$$

where

$$\rho(\lambda) = \sum_{k=1}^{N} l_k/(\lambda - \epsilon_k)$$

It is useful to introduce notation for global operators of $sl(2)$-representation $Y_{gl} := \sum_{k=1}^{N} Y_k$. To find the eigenvectors and spectrum of $t(\lambda)$ on $\mathcal{H}$ one requires that vectors of the form

$$|\mu_1, \ldots, \mu_M\rangle = \prod_{j=1}^{M} X^-(\mu_j)\,|\Omega_+\rangle$$

are eigenvectors of $t(\lambda)$,

$$t(\lambda)\,|\{\mu_j\}_{j=1}^M\rangle = \Lambda\left(\lambda; \{\mu_j\}_{j=1}^M\right)|\{\mu_j\}_{j=1}^M\rangle$$

provided that the parameters μ_j satisfy the Bethe equations:

$$2c + \sum_{k=1}^{N} \ell_k/(\mu_i - \epsilon_k) - \sum_{j\neq i}^{M} 2/(\mu_i - \mu_j) = 0, \quad i = 1, \ldots, M \tag{6}$$

The realization of the loop algebra generators on the space $\mathcal{H}$ takes the form

$$h(\lambda) = \sum_{k=1}^{N} \frac{h_k}{\lambda - \epsilon_k}, \quad X^-(\lambda) = \sum_{k=1}^{N} \frac{X_k^-}{\lambda - \epsilon_k}, \quad X^+(\lambda) = \sum_{k=1}^{N} \frac{X_k^+}{\lambda - \epsilon_k} \tag{7}$$

The coupling constant g of (1) is connected with parameter $c = 1/g$ while the Hamiltonian (1) is obtained as operator coefficient of term $1/\lambda^2$ in the expansion of $t(\lambda; c)$ at $\lambda \to \infty$.

The quantum group theory provides a possibility to deform a Hamiltonian preserving integrability [8, 9]. Specifically, we use the so-called Jordanian r-matrix to quantum deform the Hamiltonian of Richardson-Gaudin model (1). We add to $sl(2)$ symmetric r-matrix (3) the Jordanian part

$$r^J(\lambda,\, \mu) = \frac{C_2^{\otimes}}{\lambda - \mu} + \xi\left(h \otimes X^+ - X^+ \otimes h\right)$$

with Casimir element $C_2^{\otimes}$ in the tensor product of two copies of $sl(2)$,

$$C_2^{\otimes} = h \otimes h + 2\left(X^+ \otimes X^- + X^- \otimes X^+\right)$$

After Jordanian twist the r-matrix (14) is commuting with the generator X_0^+ only

$$\left[X_0^+ \otimes \mathbb{I} + \mathbb{I} \otimes X_0^+, r^{(J)}(\lambda, \mu)\right] = 0$$

Hence, one can add the term $cX_0^+ + L(\lambda, \xi)$ to the L-operator. This yields the twisted transfer-matrix

$$t^{(J)}(\lambda) = \frac{1}{2}\mathrm{tr}_0(cX_0^+ + L(\lambda,\xi))^2 = cX^+(\lambda) + h(\lambda)^2 - 2h'(\lambda) + 2(2X^-(\lambda)+\xi)X^+(\lambda) \tag{8}$$

The corresponding commutation relations between the generators of the twisted loop algebra are explicitly given by

$$\begin{aligned}
&[h(\lambda), h(\mu)] = 2\xi\left(X^+(\lambda) - X^+(\mu)\right), && \left[X^-(\lambda), X^-(\mu)\right] = -\xi\left(X^-(\lambda) - X^-(\mu)\right)\\
&\left[X^+(\lambda), X^-(\mu)\right] = -\frac{h(\lambda)-h(\mu)}{\lambda-\mu} + \xi X^+(\lambda), && \left[X^+(\lambda), X^+(\mu)\right] = 0\\
&\left[h(\lambda), X^-(\mu)\right] = 2\frac{X^-(\lambda)-X^-(\mu)}{\lambda-\mu} + \xi h(\mu), && \left[h(\lambda), X^+(\mu)\right] = -2\frac{X^+(\lambda)-X^+(\mu)}{\lambda-\mu}
\end{aligned} \tag{9}$$

The realization of the Jordanian twisted loop algebra $\mathcal{L}_J(sl(2))$ with CR (9) is given similar to (7) with extra terms proportional to the deformation parameter ξ

$$h(\lambda) = \sum_{k=1}^{N}\left(\frac{h_k}{\lambda-\epsilon_k} + \xi X_k^+\right), \quad X^-(\lambda) = \sum_{k=1}^{N}\left(\frac{X_k^-}{\lambda-\epsilon_k} - \frac{\xi}{2}h_k\right), \quad X^+(\lambda) = \sum_{k=1}^{N}\frac{X_k^+}{\lambda-\epsilon_k} \tag{10}$$

To construct eigenstates for the twisted model one has to use operators of the form [9, 10]

$$B_M(\mu_1, \ldots, \mu_M) = X^-(\mu_1)\left(X^-(\mu_2)+\xi\right)\ldots\left(X^-(\mu_M) + \xi(M-1)\right)$$

acting by these operators on the ferromagnetic state $|\Omega_+\rangle$.

The deformed Richardson-Gaudin model Hamiltonian can now be extracted from the transfer-matrix $t^{(J)}(\lambda)$ as the operator coefficient in its expansion $\lambda \to \infty$.

According to (4) and (8) one can also extract quantum integrals of motion J_k using the realization (10). It would yield rather cumbersome expressions for J_k :

$$t^{(J)}(\lambda) = J_0 + \frac{1}{\lambda}J_1 + \frac{1}{\lambda^2}J_2 + \ldots$$

The corresponding quantum deformed Hamiltonian reads

$$H \simeq J_2 = c\sum_{j=1}^{N}\epsilon_j X_j^+ + 2\xi\left\{\left(\sum_{j=1}^{N}\epsilon_j h_j\right)X_{gl}^+ - h_{gl}\sum_{j=1}^{N}\epsilon_j X_j^+\right\} + \left(h_{gl}^2 + 2h_{gl} + 4X_{gl}^-X_{gl}^+\right)$$

It is instructive to write down a simplified case without the Jordanian twist: $\xi = 0$. One thus obtains

$$J_0 = 0, \quad J_1 = X_{gl}^+, \quad J_2 \simeq \sum_{k=1}^{N}\epsilon_k X_k^+ + g/2\left(h_{gl}^2 + 2h_{gl} + 4X_{gl}^-X_{gl}^+\right)$$

The case $\xi = 0$ can also be obtained by taken off from the inhomogeneous XXX spin chain. The model can be described by a 2×2 monodromy matrix [5]

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}$$

and entries of this matrix satisfy quadratic relations

$$R(\lambda,\mu)T(\lambda)\otimes T(\mu) = (I\otimes T(\mu))\,(T(\lambda)\otimes I)\,R(\lambda,\mu) \tag{11}$$

If we multiplay $T(\lambda)$ by a constant 2×2 matrix $M(\varepsilon)$ the resulting matrix $\widetilde{T}(\lambda) = M(\varepsilon)\cdot T(\lambda)$ will satisfy the same relation (11). Choosing a triangular matrix $M(\varepsilon) = \left(\begin{smallmatrix}1 & \varepsilon\\ 0 & 1\end{smallmatrix}\right)$the entries of monodromy matrices become simply related:

$$\widetilde{A} = A + \varepsilon C, \quad \widetilde{B} = B + \varepsilon D, \quad \widetilde{C} = C, \quad \widetilde{D} = D.$$

This choice of $M(\varepsilon)$ (of the same type as considered in [11]) permits us to use the same reference state $|\,\Omega_+\rangle \in \mathcal{H}$ (5) and $\widetilde{B}$ as a creation operator of the algebraic Bethe ansatz [5].

Bethe states are given by the same action of product operators $\widetilde{B}(\mu_j) = B(\mu_j) + \varepsilon D(\mu_j)$ although operators $B(\mu_j)$ do not commute with $D(\mu_j)$:

$$D(\lambda)B(\mu) = \alpha(\lambda,\mu)B(\mu)D(\lambda) + \beta(\lambda,\mu)B(\lambda)D(\mu)$$

where

$$\alpha(\lambda,\mu) = (\lambda-\mu+\eta)/(\lambda-\mu), \quad \beta(\lambda,\mu) = -\eta/(\lambda-\mu)$$

For a 3 magnon state one gets due to B-D ordering

$$\prod_{j=1}^{3}\widetilde{B}(\mu_j) = \prod_{j=1}^{3}B(\mu_j) + \varepsilon\sum_{s=1}^{3}\alpha(\mu_k,\mu_s)\alpha(\mu_s,\mu_l)B(\mu_k)B(\mu_l)D(\mu_s)$$
$$+\,\varepsilon^2\sum_{s=1}^{3}\alpha(\mu_k,\mu_s)\alpha(\mu_l,\mu_s)B(\mu_s)D(\mu_k)D(\mu_l) + \varepsilon^3\prod_{j=1}^{3}D(\mu_j)$$

Similar formula is valid for M-magnon state. Hence, acting on ferromagnet state $|\,\Omega_+\rangle$, we obtain filtration of states with eigenvalues of $S^z : \frac{N}{2}, \frac{N}{2}-1, \frac{N}{2}-2, \frac{N}{2}-3$.

More complicated deformations of the Richardson-Gaudin model can be obtained using r-matrices related to the higher rank Lie algebras [12]. The structure of the eigenstates of the transfer matrix and their entanglement properties [13] are under investigation.

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