Symmetrization of Plurisubharmonic and Convex Functions

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ABSTRACT. We show that Schwarz symmetrization does not increase the Monge-Ampere energy for S^1-invariant plurisubharmonic functions in the ball. As a result, we derive a sharp Moser-Trudinger inequality for such functions. We also show that similar results do not hold for other balanced domains except for complex ellipsoids, and discuss related questions for convex functions.

1. INTRODUCTION

If φ is a real-valued function defined in a domain Ω in $\mathbb{R}^n$, its *Schwarz symmetrization* (see [4]) is a radial function $\hat{\varphi}(x) = f(|x|)$ with f increasing which is equidistributed with φ. The latter requirement means that, for any real t, the measure of the corresponding sublevel sets of φ and $\hat{\varphi}$ are equal; that is,

$$|\{\varphi < t\}| = |\{\hat{\varphi} < t\}| =: \sigma(t).$$

Notice that, since $\hat{\varphi}$ is radial, its natural domain of definition is a ball, B. Moreover, as t goes to infinity, $\sigma(t)$ tends to the volume of Ω and also to the volume of B. Thus, the volume of B equals the volume of Ω.

Since φ and $\hat{\varphi}$ are equidistributed, any integrals of the form

$$\int_\Omega F(\varphi)\,\mathrm{d}x \quad \text{and} \quad \int_B F(\hat{\varphi})\,\mathrm{d}x,$$

where F is a measurable function of a real variable, are equal.

One fundamental property of symmetrization is that many other quantities measuring the "size" of a function decrease under symmetrization. The prime

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examples of this are energy integrals $\int_\Omega |\nabla\varphi|^p$ for $p \geq 1$ (see [4]). By the Polya-Szegö theorem,

$$\int_B |\nabla\hat{\varphi}|^p \leq \int_\Omega |\nabla\varphi|^p, \tag{1.1}$$

if φ vanishes on the boundary of Ω. This means that, for example, the study of Sobolev-type inequalities

$$\left(\int_\Omega |\varphi|^q\right)^{1/q} \leq A\left(\int_\Omega |\nabla\varphi|^p\right)^{1/p}$$

is immediately reduced to the radial case, which is a one-variable problem.

Before we go on, we remark that the inequality (1.1) is strongly related to the isoperimetric inequality. Indeed, if we take $p = 1$ and φ to be the characteristic function of Ω, then, as noted above, the corresponding ball has the same volume as Ω. On the other hand, the L^1-norm of $\nabla\varphi$ (taken in the sense of distributions) is the area of the boundary of Ω. It follows that the area of the boundary of Ω is not smaller than the area of the sphere bounding the same volume, which is the isoperimetric inequality. The isoperimetric inequality is also the main ingredient in the proof of (1.1).

In this paper, we will investigate analogs of (1.1) for another type of energy functional which is of interest in connection with convex and plurisubharmonic functions. In the case of convex functions, the functional is

$$\mathcal{E}(\varphi) := \int (-\varphi) MA(\varphi),$$

where $MA(\varphi)$ is the *Monge-Ampere measure* of φ, defined as

$$MA(\varphi) := \det(\varphi_{jk})\,\mathrm{d}x$$

when φ is twice differentiable. We will only consider this functional when φ vanishes on the boundary. In the one-dimensional case, we can then integrate by parts, so that

$$\mathcal{E}(\varphi) = \int |\mathrm{d}\varphi|^2$$

is the classical energy. In the general case, we can also integrate by parts, and then find that $\mathcal{E}$ is still an L^2-norm of $\mathrm{d}\varphi$, but the norm of the differential is measured by the Hessian of φ. This is why this functional makes sense primarily for convex functions.

We also denote by $\mathcal{E}$ the corresponding functional for plurisubharmonic functions. Then, Ω is a domain in $\mathbb{C}^n$, and we let

$$\mathcal{E}(\varphi) := \frac{1}{n+1}\int_\Omega (-\varphi)(\mathrm{dd}^c\varphi)^n,$$

called the *pluricomplex* or *Monge-Ampere* energy. (Notice that our normalization here is slightly different from the real case. It also differs from the definition used in [1] by a sign; here, we have chosen signs so that the energy is nonnegative.) The energy is defined, but may be infinite for plurisubharmonic functions, vanishing on the boundary. Just as in the real case, the pluricomplex energy equals the classical (logarithmic) energy when the complex dimension is one.

We start with the case of plurisubharmonic functions. The first problem is that the Schwarz symmetrization of a plurisubharmonic function is not necessarily plurisubharmonic, and so the Aubin-Yau energy is not naturally defined. Indeed, already when the complex dimension is one and we take

$$\varphi(z) = \log |(z-a)/(1-\bar{a}z)|$$

to be the Green kernel, $\hat{\varphi}$ is subharmonic only if $a = 0$, so that φ is already radial. (We thank Joaquim Ortega and Pascal Thomas for providing us with this simple example.) Our first observation is that if we consider only functions (and domains) that are S^1*-invariant*, that is, invariant under the map $z \mapsto e^{i\theta}z$, then the symmetrization $\hat{\varphi}$ of a plurisubharmonic function φ is again plurisubharmonic. Thus, it is meaningful to consider its energy, and the main result we prove is that

$$\mathcal{E}(\hat{\varphi}) \leq \mathcal{E}(\varphi) \tag{1.2}$$

when Ω is a ball. Of course, the condition of S^1-symmetry makes this result trivial when $n = 1$, but notice that it is a rather weak restriction in high dimensions, as it only means invariance under a one-dimensional group.

In Section 4, we study the corresponding problems for convex functions. In that case, convexity is preserved under Schwarz symmetrization (this must be well known, but we include a proof in Section 4), and so we need no extra condition (like S^1-invariance). We then show that, for convex functions in the ball that vanish on the boundary, symmetrization decreases the Monge-Ampere energy, just as in the complex case, and following a similar argument.

It is natural to ask if these symmetrization results also hold for domains other than the ball. In the classical case of the Polya-Szegö theorem, one symmetrizes the domain and the function at the same time, and it is the symmetrization of the domain that is most clearly linked to the isoperimetric inequality. It turns out that the counterpart to this for Monge-Ampere energy does not hold. Indeed, in Section 2 we prove the somewhat surprising fact that our symmetrization result in the complex case holds if and only if the domain Ω is an ellipsoid, that is, the image of the Euclidean ball under a complex linear transformation. The proof is based on the interpretation of S^1-invariant domains as unit disk bundles of line bundles over projective space, and the proof uses the Bando-Mabuchi uniqueness theorem for Kähler-Einstein metrics on $\mathbb{P}^n$.

In the real case, the situation is a little bit more complicated. It was first shown by Tso [9] that the symmetrization inequality fails in general: there is a convex domain and a convex function vanishing on the boundary of that domain, whose Schwarz symmetrization has larger energy. In Section 4, we first give a general form of Tso's example, and relate it to Santalò's inequality. We show that, if the symmetrization inequality holds for a certain domain Ω, then Ω must be a maximizer for the *Mahler volume*, that is, for the product of the volume of Ω with the volume of its polar body Ω°. By (the converse to) Santalò's inequality, this means that Ω is an ellipsoid. Thus, we arrive at the same conclusion as in the complex case, but this time for a completely different reason that seems to have no counterpart in the complex setting. We then argue that, if we redefine the Monge-Ampere energy in the real setting by dividing by the Mahler volume, we get an energy functional that behaves more as in the complex setting, and for which the phenomenon discovered by Tso disappears. That the symmetrization inequality holds for this renormalized energy is thus a weaker statement. Nevertheless, by an argument similar to the one used in the complex case, we show that even the weaker inequality holds only for ellipsoids.

The origin of this paper is our previous article [1], where we studied Moser-Trudinger inequalities of the form

$$\log \int_\Omega e^{-\varphi} \leq A\mathcal{E}(\varphi) + B \tag{1.3}$$

for plurisubharmonic functions in Ω that vanish on the boundary. It follows immediately from (1.2) that, when Ω is a ball and φ is S^1-invariant, the proof of inequalities of this type can be reduced to the case of radial functions. In [1], we proved a Moser-Trudinger inequality using geodesics in the space of plurisubharmonic functions. Here, we will use instead symmetrization, but we point out that the proof of our main result (1.2) also uses geodesics. By the classical results of Moser [5], we then obtain in Section 3 a sharpening of the Moser-Trudinger inequality from [1]. Symmetrization was the main tool used by Moser to study the real variable Moser-Trudinger inequality, and it is interesting to note that symmetrization applied to (1.3) leads to the same one-variable inequality as in Moser's case. As a result, we deduce that if φ is S^1-invariant and has finite energy that we can normalize to be equal to one, then

$$\int_B e^{n(-\varphi)^{(n+1)/n}} < \infty.$$

We do not know if this estimate holds without the assumption of S^1-symmetry.

2. Symmetrization of Plurisubharmonic Functions

The proofs in this section are based on a result from [2] that we first recall. We consider a pseudoconvex domain $\mathcal{D}$ in $\mathbb{C}^{n+1}$ and its n-dimensional slices

$$D_t = \{z \in \mathbb{C}^n \mid (t, z) \in \mathcal{D}\},$$

where t ranges over (a domain in) $\mathbb{C}$. We say that a domain D in $\mathbb{C}^n$ is S^1-invariant if D is invariant under the map

$$z \mapsto e^{i\theta} z = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$$

for all θ in $\mathbb{R}$. A function (defined in a S^1-invariant domain) is S^1-invariant if $f(e^{i\theta} z) = f(z)$ for all real θ.

Theorem 2.1. *Assume that $\mathcal{D}$ is a pseudoconvex domain in $\mathbb{C}^{n+1}$ such that all its slices D_t are connected and S^1-invariant. Assume also that the origin belongs to D_t when t lies in a domain U in $\mathbb{C}$. Then,* $\log |D_t|$ *is a superharmonic function of t in U.*

Theorem 2.1 is a consequence of the main result in [2], which says that if $B_t(z, z)$ is the Bergman kernel on the diagonal for domain D_t, then $\log B_t(z, z)$ is plurisubharmonic in $\mathcal{D}$. The hypotheses on D_t in the theorem imply that

$$B_t(0, 0) = |D_t|^{-1},$$

which gives the theorem. Theorem 2.1 can be seen as a complex variant of the (multiplicative form of) the *Brunn-Minkowski inequality*, which says that if $\mathcal{D}$ is instead convex in $\mathbb{R}^{n+1}$, and the n-dimensional slices are defined in the same way, then $\log |D_t|$ is a concave function of t, without any extra assumptions on the slices. The Brunn-Minkowski inequality replaces the use of Theorem 2.1 when we later consider energies of convex functions. We then will use the stronger fact that even $\sigma^{1/n}$ is concave in the real setting.

In the proofs below, we will use the following lemma on symmetrizations of subharmonic functions.

Lemma 2.2. *Let u be a smooth subharmonic function defined in an open set U in $\mathbb{R}^N$, and assume that u vanishes on the boundary of U. Let*

$$\sigma(t) := |\{x : u(x) < t\}|$$

for $t < 0$. Then, σ is strictly increasing on the interval $(\min u, 0)$*, and the Schwarz symmetrization of u, $\hat{u}$ equals $g(|x|)$, where*

$$g(r) = \begin{cases} \sigma^{-1}(c_N r^N) & \text{when } c_N r^N > \sigma(\min(u)), \\ \min(u) & \text{when } c_N r^N \leq \sigma(\min(u)), \end{cases}$$

where c_N is the volume of the unit ball in $\mathbb{R}^N$.

Proof. Denote by U_t the domain where $u < t$. Assume that $|U_t| = |U_{t+\varepsilon}|$ for some $\varepsilon > 0$. By Sard's lemma, some s between t and $t + \varepsilon$ is a regular value of u, and so the boundary of U_s is smooth. By the Hopf lemma, the gradient of u does not vanish on the boundary of U_s unless u is constant in U_s. In the latter case, $s \le \min u$. If this is not the case (i.e., if $s > \min u$), the coarea formula gives

$$\sigma'(s) = \int_{\partial U_s} \mathrm{d}S/|\nabla u| > 0,$$

which contradicts that σ is constant on $(t, t+\varepsilon)$. This proves the first part of the lemma. The second part follows since

$$\sigma((g(r)) = |\{g(|x|) < g(r)\}| = c_N r^N. \qquad \square$$

We can now easily prove the next basic result. We say that a domain Ω in $\mathbb{C}^n$ is *balanced* if, for any λ in $\mathbb{C}$ with $|\lambda| \le 1$ and z in Ω, λz also lies in Ω.

Theorem 2.3. *Let Ω be a balanced domain in $\mathbb{C}^n$. Let φ be an S^1-invariant plurisubharmonic function in Ω. Then $\hat{\varphi}$, the Schwarz symmetrization of φ, is plurisubharmonic.*

Proof. We may of course assume that φ is smooth so that the previous lemma applies. By definition, $\hat{\varphi}$ can be written

$$\hat{\varphi}(z) = f(\log|z|),$$

so what we need to prove is that f is convex. Since φ and $\hat{\varphi}$ are equidistributed, for any real t, we have

$$\begin{aligned}\sigma(t) &:= |\{z \in \Omega : \varphi(z) < t\}| \\ &= |\{z \in \Omega : \hat{\varphi}(z) < t\} \\ &= |\{z : |z| < \exp(f^{-1}(t))\}|.\end{aligned}$$

Hence,

$$f^{-1}(t) = n^{-1}\log\sigma(t) + b_n.$$

Since σ is increasing, f^{-1} is also increasing. Therefore, f is convex precisely when f^{-1} is concave, that is, when $\log\sigma$ is concave.

Consider the domain in $\mathbb{C}^{n+1}$

$$\mathcal{D} = \{(\tau, z) \mid z \in \Omega \text{ and } \varphi(z) - \operatorname{Re}\tau < 0\}.$$

Then, if $t = \operatorname{Re}\tau$, $\sigma(t) = |D_\tau|$. Note that $\mathcal{D}$ is pseudoconvex since $\varphi - \operatorname{Re}\tau$ is plurisubharmonic, and we claim that $\mathcal{D}$ also satisfies all the other conditions of Theorem 2.1.

Let z lie in D_τ for some τ. The function $\gamma(\lambda) := \varphi(\lambda z)$ is then subharmonic in the unit disk, and moreover it is radial, that is, $\gamma(\lambda) = g(|\lambda|)$, where g is increasing. Therefore, the whole disk $\{\lambda z\}$ is contained in D_τ. In particular, the origin lies in any D_τ, and the origin can be connected with z by a curve, and so $\mathcal{D}_\tau$ is connected. Thus, Theorem 2.1 can be applied, and we conclude that

$$\log \sigma(\operatorname{Re}\tau) = \log |D_\tau|$$

is a superharmonic function of τ. Since this function only depends on $\operatorname{Re}\tau$, it is actually concave, and the proof is complete. □

The next theorem is the main result of this paper, and here we need to assume that Ω is a ball. See the remarks below for a discussion of the problem in considering more general domains.

Theorem 2.4. *Let φ be plurisubharmonic in the unit ball, and assume that φ extends continuously to the closed ball with zero boundary values. Assume also that φ is S^1-invariant, and let $\hat{\varphi}$ be the Schwarz symmetrization of φ. Then, $\mathcal{E}(\hat{\varphi}) \leq \mathcal{E}(\varphi)$.*

In the proof of Theorem 2.1, we used the geometrically obvious fact that the inverse of an increasing concave function is convex. We will need a generalization of this that we state as a lemma.

Lemma 2.5. *Let $a(s,t)$ be a concave function of two real variables. Assume a is strictly increasing with respect to t, and let $t = k(s,x)$ be the inverse of a with respect to the second variable for s fixed, so that $a(s,k(s,x)) = x$. Then, k is convex as a function of both variables s and x.*

Proof. Assume this is not the case. After choosing a new origin, there is then a point $p = (s_0, x_0)$ such that

$$k(0,0) > \frac{k(p) + k(-p)}{2}.$$

Since a is strictly increasing with respect to t, we have

$$\begin{aligned} 0 = a(0, k(0,0)) &> a\left(\frac{s_0 - s_0}{2}, \frac{k(p) + k(-p)}{2}\right) \\ &\geq \frac{a(s_0, k(p)) + a(-s_0, k(-p))}{2} = \frac{x_0 - x_0}{2} = 0. \end{aligned}$$

This is a contradiction. □

In the sequel, we shall use well-known facts about geodesics and subgeodesics in the space of plurisubharmonic functions on the ball (see [1] for proofs). These are curves

$$\varphi_t(z) = \varphi(t,z),$$

where t is a real parameter, here varying between 0 and 1. By definition, φ_t is a subgeodesic if $\varphi(\operatorname{Re}\tau, z)$ is plurisubharmonic as a function of (τ, z), and it is a geodesic if, moreover, this plurisubharmonic function solves the homogenuous complex Monge-Ampere equation

$$(\mathrm{dd}^c\varphi)^{n+1} = 0.$$

We also assume throughout that φ_t vanishes for $|z| = 1$. It is not hard to see that, if φ_0 and φ_1 are two continuous plurisubharmonic functions in the ball, vanishing on the boundary, then they can be connected with a bounded geodesic (see [1]). Here, we first assume that φ_0 and φ_1 are smooth and can be connected by a geodesic of class C^1, and then get the inequality for general φ by approximation.

We use the following three facts, for which we refer to [1]. First, $\mathcal{E}(\varphi_t)$ is an *affine* function of t along any bounded geodesic. Second, $\mathcal{E}(\varphi_t)$ is *concave* along a bounded subgeodesic. On the other hand, if φ_0 and φ_1 are plurisubharmonic and vanish on the boundary, and if we let $\varphi_t = t\varphi_1 + (1-t)\varphi_0$ for t between 0 and 1, then $\mathcal{E}(\varphi_t)$ is *convex*. Finally, if φ_t is of class C^1, then $\mathcal{E}(\varphi_t)$ is differentiable with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varphi_t) = \int_B -\dot{\varphi}_t(\mathrm{dd}^c_z\varphi_t)^n. \tag{2.1}$$

In the proof of Theorem 2.3, we fix in the ball a plurisubharmonic φ that we assume smooth. We put $\varphi = \varphi_1$ and connect it with φ_0, chosen to satisfy an equation

$$(\mathrm{dd}^c\varphi_0)^n = F(\varphi_0),$$

where F is some smooth function of a real variable. Actually, it is not hard to check that any smooth increasing radial function satisfies such an equation. We also first assume that $\varphi = \varphi_1$ and φ_0 can be connected with a C^1 geodesic φ_t. We then take the Schwarz symmetrization of each φ_t, and obtain another curve $\hat{\varphi}_t$. The next proposition shows that the new curve is a subgeodesic.

Proposition 2.6. *Let φ_t be a subgeodesic of S^1-invariant plurisubharmonic functions. Then, $\hat{\varphi}_t$ is also a subgeodesic.*

Proof. Let φ_s be a subgeodesic which we may assume to be smooth. Let

$$A(s,t) = |\{z, \varphi_s(z) < t\}|.$$

It follows again from Theorem 2.1 that $a := \log A$ is a concave function of s and t together. As in the proof of Theorem 2.3, all we need to prove is that the inverse of a with respect to t (for s fixed), $k(s,x)$ is convex with respect to s and t jointly. But this is precisely the content of Theorem 2.4, and so the proposition is proved. ◻

We now first sketch the principle of the argument, and then fill in some details and slightly change the setup afterwards. Consider the energy functionals along the two curves φ_t and $\hat{\varphi}_t$, $\mathcal{E}(\varphi_t) =: g(t)$ and $\mathcal{E}(\hat{\varphi}_t) = h(t)$. Since φ_0 is already radial, $g(0) = h(0)$, and we want to prove that $g(1) \geq h(1)$. We know that g is affine and that h is concave, and so this follows if we can prove that $g'(0) = h'(0)$. But

$$g'(0) = \int -\dot{\varphi}_0 (\mathrm{dd}^c \varphi_0)^n,$$

since the geodesic is C^1. We shall see below that we can arrange things so that

$$h'(0) = \int -\frac{\mathrm{d}\hat{\varphi}_t}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{dd}^c \varphi_0)^n.$$

By the choice of φ_0,

$$g'(0) = \int -\dot{\varphi}_0 F(\varphi_0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int -G(\varphi_t),$$

if $G' = F$. Similarily,

$$h'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int -G(\hat{\varphi}_t).$$

But, since φ_t and $\hat{\varphi}_t$ are equidistributed, we have that

$$\int -G(\varphi_t) = \int -G(\hat{\varphi}_t)$$

for all t. Hence, $g'(0) = h'(0)$, and the proof is complete.

It remains both to see why we can assume that the geodesic φ_t is C^1, and also to motivate the claim about the derivative of h. First, since we have assumed that φ_0 and φ_1 are smooth up to the boundary, we can by a max construction assume that they are both equal to $A \log((1 + |z|^2)/2)$ for some large $A > 0$, when $|z| > (1 - \varepsilon)$. Then, φ_0 and φ_1 can be extended to psh functions in all of $\mathbb{C}^n$, equal to $A \log((1 + |z|^2)/2)$ outside of the unit ball. In fact, we can even consider them as metrics on a line bundle $\mathcal{O}(A)$ over $\mathbb{P}^n$. It then follows from Chen's theorem [3] that they can be connected by a C^1 geodesic in the space of metrics on $\mathcal{O}(A)$. It is easy to see that this geodesic must in fact be equal to $A \log((1 + |z|^2)/2)$ for $|z| > 1 - \varepsilon$ for some positive ε. In particular, it vanishes on the boundary of the ball, and $\dot{\varphi}_t$ is identically zero near the boundary.

To handle the claim about the derivative of h, we change the setup a little bit. We have that $\hat{\varphi}_0 = \varphi_0$ is smooth, and we can approximate $\hat{\varphi}_1$ from above by a smooth radial plurisubharmonic function. Now, connect these two smooth functions by a geodesic ψ_t, which can be taken to be $C^{(1,1)}$ by the above argument. (As a matter of fact, it will even be smooth, since geodesics between radial functions come from geodesics between smooth convex functions, which are smooth). Let

$$\mathcal{E}(\psi_t) =: k(t).$$

Since $\psi_t \geq \hat{\varphi}_t$, we have that $-\dot{\psi}_0 \leq -\dot{\hat{\varphi}}_0$. We then apply the above argument to k instead of h, and find that $k(1) \leq g(1)$. Taking limits as ψ_1 tends to $\hat{\varphi}_1$, we conclude the proof.

2.1. Other domains. Let us consider a smoothly bounded balanced domain Ω in $\mathbb{C}^n$, which we can write as

$$\Omega = \{z \mid u_\Omega(z) < 0\}$$

where u_Ω is *logarithmically homogenuous*, that is, $u_\Omega(\lambda z) = \log|\lambda| + u_\Omega(z)$. Indeed, u_Ω is the logarithm of the Minkowski functional for Ω. We first claim that if Ω is pseudoconvex, then u_Ω is plurisubharmonic.

Lemma 2.7. *Let u be a smooth function such that*

$$D := \{(w, z) \mid u(z) - \operatorname{Re} w < 0\}$$

is pseudoconvex. Then, u is plurisubharmonic.

Proof. At a point z where $\mathrm{d}u = 0$, the Levi form of the boundary of D is precisely $\mathrm{dd}^c u$, and so if D is pseudoconvex, then $\mathrm{dd}^c u \geq 0$ at such points. The general case is reduced to this by subtracting a linear form $\operatorname{Re} a \cdot z$ from u and considering the biholomorphic transformation $(w, z) \mapsto (w + a \cdot z, z)$. ◻

Since the set $\{u_\Omega(z) - \operatorname{Re} w < 0\} = \{u_\Omega(ze^{-w}) < 0\}$ is pseudoconvex, it follows from the lemma that u_Ω is plurisubharmonic. Let us now consider S^1-invariant functions in Ω of the form $\varphi(z) = f(u_\Omega)$, where f is convex. If we normalize so that the volume of Ω equals the volume of the unit ball, it is clear that $\hat{\varphi}$, the Schwarz symmetrization of φ, is $f(\log|z|)$.

Proposition 2.8. *If $\varphi = f(u_\Omega)$ with f convex, the Monge-Ampere energy of φ equals*

$$\int_\Omega (-\varphi)(\mathrm{dd}^c\varphi)^n = 2^{-n}\int_{-\infty}^0 (f')^{n+1}(t)\,\mathrm{d}t.$$

In particular, the energy of φ is equal to the energy of $\hat{\varphi}$, the Schwarz symmetrization of φ.

In the proof, we use the next lemma.

Lemma 2.9. *If $\varphi = f(u_\Omega)$ with f convex, then* $\displaystyle\int_{u_\Omega < s} (\mathrm{dd}^c\varphi)^n = 2^{-n} f'(s)^n.$

Proof. We have

$$\int_{u_\Omega<s} (\mathrm{dd}^c\varphi)^n = \int_{u_\Omega=s} \mathrm{d}^c\varphi \wedge (\mathrm{dd}^c\varphi)^{n-1} = f'(s)^n \int_{u_\Omega=s} \mathrm{d}^c u_\Omega \wedge (\mathrm{dd}^c u_\Omega)^{n-1}.$$

But also

$$\int_{u_\Omega=s} \mathrm{d}^c u_\Omega \wedge (\mathrm{dd}^c u_\Omega)^{n-1} = \int_{u_\Omega<s} (\mathrm{dd}^c u_\Omega)^n.$$

Since u_Ω is log homogenous, it satisfies the homogenous Monge-Ampere equation outside of the origin, and so $(\mathrm{dd}^c u_\Omega)^n$ is a Dirac mass at the origin. But $u_\Omega - \log|z|$ is bounded near the origin, and so this point mass must be same as

$$(\mathrm{dd}^c \log|z|)^n = 2^{-n}.$$ ◻

Proof of Proposition 2.8. First, assume that $f(s)$ is constant for s sufficiently large negative. Let

$$\sigma(s) := \int_{u_\Omega < s} (dd^c \varphi)^n.$$

Then,

$$\mathcal{E}(\varphi) = \int^0 -f(s)\, d\sigma(s) = \int^0 \sigma(s) f'(s)\, ds,$$

and so the formula for the energy follows from the previous lemma. The general case, when f is not constant near $-\infty$, follows from approximation.

The last statement, that $\mathcal{E}(\varphi) = \mathcal{E}(\hat{\varphi})$, then follows if $|\Omega|$ equals the volume of the unit ball, since then $\hat{\varphi} = f(\log|z|)$. But then the same thing must hold in general, since the energy is invariant under scalings. □

Let us now define the "Ω-symmetrization" $S_\Omega(\varphi)$ of a plurisubharmonic function φ in Ω, vanishing on the boundary φ, as the unique function of the form $f(u_\Omega)$ that is equidistributed with φ. Notice that if the Ω-symmetrization of φ equals $f(u_\Omega)$, then $\hat{\varphi} = f(\log(|z|/R)$ gives the Schwarz symmetrization, if R is chosen so that the volume of Ω equals the volume of the ball of radius R. The last part of Proposition 2.8 then says that

$$\mathcal{E}_\Omega(S_\Omega(\varphi)) = \mathcal{E}_B(\hat{\varphi}),$$

where we have put subscripts on $\mathcal{E}$ to emphasize over which domain we compute the energy, and B denotes a ball of the same volume as Ω. Notice also that it follows from Theorem 2.3 that $S_\Omega(\varphi)$ is plurisubharmonic if φ is plurisubharmonic. Indeed, Theorem 2.3 says that $\hat{\varphi} = f(\log|z|)$ is plurisubharmonic, that is, that f is convex and increasing, from which it follows that $f(u_\Omega)$ is plurisubharmonic. We therefore see that, to prove that the Schwarz symmetrization of a function φ on Ω has smaller Monge-Ampere energy than φ is equivalent to proving that the Ω-symmetrization of φ has smaller energy than φ.

One might try to prove this by following the same method as in the proof of Theorem 2.4. The point where the proof breaks down, however, is that we need to choose a reference function on Ω that satisfies an equation of the form

$$(dd^c \varphi_0)^n = F(\varphi_0), \tag{2.2}$$

where φ_0 is of the form $\varphi_0 = f(u_\Omega)$. This is easy if Ω is the ball so that $u_\Omega = \log|z|$, since $(dd^c\varphi_0)^n$ then is invariant under the unitary group if φ_0 is also, and hence $(dd^c\varphi_0)^n$ must be radial. Nothing of the sort holds for other domains. Since, outside the origin,

$$(dd^c \varphi_0)^n = f''(u_\Omega) f'(u_\Omega)^{n-1} du_\Omega \wedge d^c u_\Omega \wedge (dd^c u_\Omega)^{n-1},$$

what we want is that the determinant of the Levi form

$$\mathrm{d}u_\Omega \wedge \mathrm{d}^c u_\Omega \wedge (\mathrm{dd}^c u_\Omega)^{n-1} \tag{2.3}$$

be constant on all level surfaces of u_Ω. This is clearly true if Ω is a ball, and therefore also true if Ω is the image of a ball under a complex linear transformation. We shall next see that these are the only cases in which this holds.

Proposition 2.10. *Let Ω be a balanced domain in $\mathbb{C}^n$, and let u_Ω be the uniquely determined logarithmically homogenous (plurisubharmonic) function that vanishes on the boundary of Ω. Assume u_Ω satisfies the condition that* (2.3) *be constant on some, and therefore every, level surface of u_Ω. Then, Ω is an ellipsoid*

$$\Omega = \left\{z \mid \sum a_{jk} z_j \bar{z}_k < 1\right\}$$

for some positively definite matrix $A = (a_{jk})$.

Proof. We will use the relation between logarithmically homogenous functions on $\mathbb{C}^n$ and metrics on the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^{n-1}$. Recall that $\mathbb{P}^{n-1}$ is the quotient of $\mathbb{C}^n \setminus \{0\}$ under the equivalence relation $z \sim \lambda z$ if λ is a nonzero complex number. Let $p(z) = [z]$ be the projection map from $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{P}^{n-1}$, where $[z]$ is the representation of a point in homogenous coordinates. Then, $\mathbb{C}^n \setminus \{0\}$ can be interpreted as the total space of $\mathcal{O}(-1)$, minus its zero section. A logarithmically homogenous function like u_Ω can then be written $u_\Omega = \log |z|_h$ for some metric h on $\mathcal{O}(-1)$. In an affine chart $[z] = [(1, \zeta)]$ on $\mathbb{P}^{n-1}$ with associated trivialization of $\mathcal{O}(-1)$, where $z = (\lambda, \lambda\zeta)$, we have

$$|z|_h^2 = |\lambda|^2 e^{\psi(\zeta)},$$

where $-\psi$ is a local representative for the metric h on $\mathcal{O}(-1)$. Hence,

$$u_\Omega = \log |\lambda| + \frac{1}{2}\psi(\zeta). \tag{2.4}$$

Let us now look at the form (2.3). If it is constant on level surfaces, it must be equal to $F(u)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ for some function $F(u)$. By log-homogeneity, the form is, moreover, homogenous of degree $-2n$, and so we must have

$$F(u) = Ce^{-2nu}.$$

Changing coordinates to $(\lambda, \lambda\zeta)$, we get

$$\mathrm{d}u_\Omega \wedge \mathrm{d}^c u_\Omega \wedge (\mathrm{dd}^c u_\Omega)^{n-1} = Ce^{-2nu} |\lambda|^{2n-2}\,\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \wedge \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

On the other hand, by (2.4),

$$\mathrm{d}u_\Omega \wedge \mathrm{d}^c u_\Omega \wedge (\mathrm{dd}^c u_\Omega)^{n-1} = C' |\lambda|^{-2}\,\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \wedge (\mathrm{dd}^c \psi)^{n-1}.$$

Hence,

$$(dd^c\psi)^{n-1} = C'' e^{-n\psi}\, d\zeta \wedge d\bar{\zeta}.$$

This means precisely that the metric $n\psi$ on the anticanonical line bundle $\mathcal{O}(n)$ on $\mathbb{P}^{n-1}$ solves the Kähler-Einstein equation. But all such metrics can be written (on the total space) as

$$\log|\chi|_h^2 = n\log|Az|^2,$$

where z are the standard coordinates on $\mathbb{C}^n$ and A is a positively definite matrix. (This follows from, for example, the Bando-Mabuchi uniqueness theorem, which says that any Kähler-Einstein metric can be obtained from the standard metric $\log|z|^2$ via a holomorphic automorphism.) Hence,

$$u_\Omega = \log|Az|,$$

and so Ω is an ellipsoid. ◻

We now finally show that the relevance of the form (2.3) is not just an artifact of the proof. Indeed, we show that if the symmetrization inequality

$$\mathcal{E}_B(\hat{\varphi}) \leq \mathcal{E}_\Omega(\varphi)$$

holds for all S^1-invariant plurisubharmonic functions φ in Ω that vanish on the boundary, then Ω must satisfy the hypothesis of Proposition 2.10, and therefore be an ellipsoid.

Let $\psi_0 = f_0(\log|z|)$ be a function in the ball that solves a Kahler-Einstein type equation

$$(dd^c\psi)^n = ce^{-\psi} i^{n^2}\, dz \wedge d\bar{z}.$$

Then, ψ_0 is a critical point for a functional of the type

$$\mathcal{F}_B(\varphi) := \log\int_B e^{-\varphi} - c'\mathcal{E}_B(\varphi),$$

(see [1] for more on this). From this, it follows that

$$\mathcal{F}_B(\psi') \leq \mathcal{F}_B(\psi_0) \tag{2.5}$$

for all S^1-invariant plurisubharmonic functions in the ball that vanish on the boundary. This is explained in [1], and so we just indicate the argument here. The point is that the functional $\mathcal{F}_B$ is concave along geodesics in the space of S^1-invariant plurisubharmonic functions that vanish on the boundary. This follows from two facts. First, the energy term is affine along geodesics; second, the function $\log\int_B e^{-\varphi_t}$ is concave under (sub)geodesics. The latter fact follows again from the main result in [2] on plurisubharmonic variation of Bergman kernels, since $\left(\int_B e^{-\varphi_t}\right)^{-1}$ is the Bergman kernel at the origin for the weight φ_t if φ_t is S^1-invariant. Given the concavity of $\mathcal{F}_B$, it then follows that a critical point is a maximum, that is, that (2.5) holds.

Let us now consider the analogous functional defined on functions on Ω, namely,

$$\mathcal{F}_\Omega(\varphi) := \log \int_\Omega e^{-\varphi} - c'\mathcal{E}_\Omega(\varphi).$$

Assume, to get a contradiction, that $\mathcal{E}_\Omega(S_\Omega(\varphi)) \le \mathcal{E}_\Omega(\varphi)$. Then, $\mathcal{F}_\Omega$ increases under Ω-symmetrization. Moreover, $\mathcal{F}_\Omega(S_\Omega(\varphi)) = \mathcal{F}_B(\hat{\varphi})$ by Proposition 2.8, and so

$$\mathcal{F}_\Omega(\varphi) \le \mathcal{F}_B(\hat{\varphi}) \le \mathcal{F}_B(\psi_0),$$

where ψ_0 is the Kähler-Einstein potential discussed above. Hence, the maximum of the left-hand side over all φ is attained for $\varphi = f_0(u_\Omega)$.

But then, it is easy to see that φ solves the same Kahler-Einstein equation as ψ_0. Indeed, at least if Ω is strictly pseudoconvex, φ is strictly plurisubharmonic outside the origin. Therefore, small perturbations of φ are still plurisubharmonic, and the variational equation for $\mathcal{F}_\Omega$ is just the Kähler-Einstein equation. In particular, φ solves an equation of type (2.2) in Ω, which we have seen is possible only if Ω is an ellipsoid. We summarize the discussion in the next theorem.

Theorem 2.11. *Let Ω be a strictly pseudoconvex balanced domain for which the symmetrization inequality $\mathcal{E}_B(\hat{\varphi}) \le \mathcal{E}_\Omega(\varphi)$ holds for all S^1-invariant plurisubharmonic φ that vanish on the boundary. Then, Ω is an ellipsoid.*

3. A SHARP MOSER-TRUDINGER INEQUALITY FOR S^1-INVARIANT FUNCTIONS

Our results in the previous section, together with Moser's inequality, imply rather easily the next estimate.

Theorem 3.1. *Let φ be a smooth S^1-invariant plurisubharmonic function in the unit ball that vanishes on the boundary. Let $\mathcal{E} := \mathcal{E}(\varphi)$. Then,*

$$\int_{\mathcal{B}} e^{n\mathcal{E}^{-1/n}(-\varphi)^{(n+1)/n}} \le C,$$

where C is an absolute constant.

Proof. In the proof we may, by our main result on symmetrization, assume that $\varphi(z) = f(\log|z|)$ is a radial function. The main result of Moser [5] is the following: if w is an increasing function on $(-\infty, 0)$ that vanishes when t goes to zero and satisfies

$$\int_{-\infty}^{0} (-w')^{n+1}\, dt \le 1,$$

then

$$\int_{-\infty}^{0} e^{(-w)^{(n+1)/n}} e^{t}\, \mathrm{d}t \le C,$$

where C is an absolute constant. Applying this to $w_\kappa(s) := \kappa^{n/(n+1)} w(s/\kappa)$, we obtain that

$$\int_{-\infty}^{0} e^{\kappa(-w)^{(n+1)/n}} e^{\kappa t}\, \mathrm{d}t \le \frac{C}{\kappa},$$

under the same hypothesis. Next, we have the following lemma.

Lemma 3.2. *Let f be an increasing convex function on $(-\infty, 0]$ with $f(0) = 0$, and let $\varphi(z) = f(\log|z|)$. Let F be a nonnegative measurable function of one real variable. Then,*

(a) $\int_{\mathcal{B}} F \circ \varphi = a_n \int_{-\infty}^{0} F \circ f e^{2nt}\, \mathrm{d}t$ *(with a_n being the area of the unit sphere in $\mathbb{C}^n$), and*

(b) $\mathcal{E} = 2^{-n} \int_{-\infty}^{0} (f')^{n+1}\, \mathrm{d}t.$

Proof. The first formula follows from

$$\int_B F(\varphi) = \int^{0} F \circ f\, \mathrm{d}\sigma(t),$$

where $\sigma = |\{z : |z| \le e^t\} = \pi^n/n! e^{2nt}$. The second formula is a special case of Proposition 2.8. ◻

Applying the scaled version of Moser's result with $-w = f E^{-1/(n+1)} 2^{-n/(n+1)}$ and $\kappa = 2n$, the theorem follows. ◻

To relate this to Moser-Trudinger inequalities of the form studied in [1], we start with the elementary inequality for positive numbers x and ξ:

$$x\xi \le \frac{1}{n+1} x^{n+1} + \frac{n}{n+1} \xi^{(n+1)/n}.$$

(This is valid since $(n+1)$ and $(n+1)/n$ are dual exponents.) This implies

$$\xi \le \frac{1}{n+1} x^{n+1} + \frac{n}{n+1} \xi^{(n+1)/n} / x^{(n+1)/n}.$$

Choose x so that $x^{n+1} = E/(n+1)^n$, and take $\xi = (-\varphi)$. Then,

$$-\varphi \le \frac{1}{(n+1)^{n+1}} E + n E^{-1/n} (-\varphi)^{(n+1)/n}.$$

Therefore, Theorem 3.1 implies the sharp Moser-Trudinger inequality for S^1-invariant functions from [1]:

$$\log \int e^{-\varphi} \leq \frac{1}{(n+1)^{n+1}} \mathcal{E}(\varphi) + B,$$

with $B = \log C$, C the universal constant in Moser's estimate.

4. Symmetrization of Convex functions

First, we note the following analog of Lemma 2.2.

Theorem 4.1. *Let φ be a convex function defined in a convex domain Ω in $\mathbb{R}^n$, and let $\hat{\varphi}$ be its Schwarz symmetrization. Then, $\hat{\varphi}$ is also convex.*

This fact should be well known, but we include a proof in order to emphasize the similarity with Lemma 2.2. By definition, $\hat{\varphi}(x) = g(|x|)$ for some increasing function g, and we need to prove that g is convex (notice the change in convention as compared with the complex case where we wrote $\hat{\varphi}(z) = f(\log|z|)$). As before,

$$\sigma(t) := |\{x \in \Omega : \varphi(x) < t\}| = a_n(g^{-1}(t))^n,$$

and so it suffices to prove that $\sigma^{1/n}$ is *concave*. But, if we put

$$\mathcal{D} := \{(t, x) \mid x \in \Omega \text{ and } \varphi(x) - t < 0\},$$

then $\sigma(t)$ is the volume of the slices D_t. By the Brunn-Minkowski theorem (see Section 2), it follows that $\sigma^{1/n}$ is concave, and we are done.

We next state the real variable analog of Theorem 2.3.

Theorem 4.2. *Let φ be a convex function in the ball, continuous on the closed ball and vanishing on the boundary. Let $\hat{\varphi}$ be its Schwarz symmetrization. Then,*

$$\mathcal{E}(\hat{\varphi}) \leq \mathcal{E}(\varphi).$$

This is proved in a way completely parallel to the complex case, and thus we shall not give the details. We define geodesics and subgeodesics in the space of convex functions as before. Then, the real energy is concave along subgeodesics and affine along geodesics, as before, and the analog of the formula for the first-order derivative also holds. We can therefore repeat the proof practically verbatim.

4.1. Other domains. We have already seen in Section 2 that, in the complex case, the energy does not in general decrease under Schwarz symmetrization if we consider functions defined on domains different than the ball. In the real setting, the first counterexample to the same effect was given by Tso [9]. We first discuss Tso's counterexample, and begin by giving the example in a more general form. In the next theorem, there appears the *Mahler volume* of a convex set Ω containing the origin; it is defined as

$$M(\Omega) := |\Omega|\,|\Omega^\circ|,$$

where Ω° is the polar body of Ω. In the sequel, we will write $\mathcal{E}_\Omega$ for the energy of functions defined in Ω.

Theorem 4.3. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ containing the origin, and let μ_Ω be the Minkowski functional of Ω. Let u be a convex function in Ω of the form $u(x) = f(\mu_\Omega(x))$, and let $\hat{u}$ be its Schwarz symmetrization. Then,*

$$M(\Omega)^{-1}\mathcal{E}_\Omega(u) = M(B_\Omega)^{-1}\mathcal{E}_B(\hat{u})$$

(*where B_Ω is the ball of the same volume as Ω*).

Notice that we could as well have divided by just $|\Omega^\circ|$ instead of the Mahler volume, since the volumes of Ω and B_Ω are automatically equal; however, the Mahler volume seems to simplify a little below. From the theorem, we see that if $\mathcal{E}_B(\hat{u}) \leq \mathcal{E}_\Omega(u)$, it then follows that we have an inequality for the Mahler volumes:

$$M(B) \leq M(\Omega).$$

This inequality fails in a very strong way. Indeed, if we assume that Ω is also symmetric so that $-\Omega = \Omega$, then *Santalò's inequality* [8] says that the opposite is true:

$$M(B) \geq M(\Omega).$$

(Tsos's counterexample is the case of Theorem 4.3 when Ω is a simplex.) It therefore seems that, in the real case, it is natural to normalize the energy by dividing by the Mahler volume of the domain. Notice that this is a difference as compared to the complex setting, where Theorem 4.3 holds without normalization. The reason for this is that, in the case of $\mathbb{R}^n$, the Minkowski functional of a convex domain Ω satisfies the equation

$$MA(\mu_\Omega) = |\Omega^0|.$$

On the other hand, in $\mathbb{C}^n$, we have that if Ω is a balanced domain, then

$$(\mathrm{dd}^c \log \mu_\Omega)^n = (\mathrm{dd}^c \log |z|)^n = 2^{-n}\delta_0,$$

where δ_0 is a point mass at the origin, and thus is independent of the domain.

The question then becomes whether

$$M(\Omega)^{-1}\mathcal{E}_\Omega(u) \geq M(B)^{-1}\mathcal{E}_B(\hat{u}) \tag{4.1}$$

for any convex function u on Ω that vanishes on the boundary.

Just as in the complex case, we define, for a convex function u defined on some convex domain L, its Ω-symmetrization $S_\Omega(u)$ as the unique function, equidistributed with u, that can be written

$$S_\Omega(u) = f(\mu_\Omega).$$

Note that the Ω-symmetrization of u is the same as the Ω'-symmetrization if Ω and Ω' are homothetic. Moreover, since

$$|\{S_\Omega(u) < 0\}| = |\{u < 0\}|,$$

then $S_\Omega(u)$ vanishes on the boundary of a multiple $s\Omega$ of Ω, with s chosen so that $s\Omega$ has the same volume as L, if u vanishes on the boundary of L. Notice that if Ω is a ball, centered at the origin, then S_Ω is just the Schwarz symmetrization.

In terms of Ω-symmetrizations, Theorem 4.3 says that the normalized energy of all Ω-symmetrizations coincides:

$$M(\Omega)^{-1}\mathcal{E}_\Omega(S_\Omega(u)) = M(\Omega')^{-1}\mathcal{E}_{\Omega'}(S_{\Omega'}(u)), \tag{4.2}$$

if Ω and Ω' are two convex domains.

The desired inequality (4.1) thus means that

$$\mathcal{E}_\Omega(S_\Omega(u)) \leq \mathcal{E}_\Omega(u) \tag{4.3}$$

for convex functions u on Ω that vanish on the boundary. This would be the analog of Theorem 4.2 for general convex domains, and it is precisely the same question that we discussed in the complex case. Just as in the complex case, we shall now see that this holds only for ellipsoids. Most of the argument is completely parallel to the complex case and will thus be omitted. Only the last part involving Kähler-Einstein metrics has to be changed, and we now describe how this is done.

As in the complex case, we see that if the symmetrization inequality holds, then μ_Ω, the Minkowski functional of Ω, satisfies a condition of the form: namely, there is a convex function of μ_Ω such that $u = f(\mu_\Omega)$ satisfies an equation

$$MA(u) = F(u)$$

for some function F. To see the meaning of this more explicitly, we resort to the complex formalism. Define u and μ_Ω on $\mathbb{C}^n$ by setting $u(z) = u(x)$ and so on, that is, by letting all functions involved be independent of the imaginary part of z. Then,

$$MA(u)\,\mathrm{d}\lambda(z) = C(\mathrm{dd}^c u)^n,$$

where $d\lambda$ is the standard volume form on $\mathbb{C}^n$. Since $MA(\mu_\Omega) = 0$ outside the origin, it follows if $u = f(\mu_\Omega)$ that

$$c'(\mathrm{dd}^c u)^n = (f'(\mu))^{n-1} f''(\mu)\,\mathrm{d}\mu \wedge \mathrm{d}^c\mu \wedge (\mathrm{dd}^c\mu)^{n-1}.$$

Hence, we see that

$$\mathrm{d}\mu \wedge \mathrm{d}^c\mu \wedge (\mathrm{dd}^c\mu)^{n-1} = G(\mu)\,\mathrm{d}\lambda$$

for $x \neq 0$, where we write μ instead of μ_Ω, since Ω is now fixed. Since μ is homogenous of degree 1, we have that $\mathrm{d}\mu$ is homogenous of degree zero, and $\mathrm{dd}^c\mu$ is homogenous of degree -1. Therefore, the left-hand side is homogenous of degree $(n-1)$, and so we can take $G(\mu) = \mu^{1-n}$. It also follows from this equation that *any* function $u = f(\mu)$, with f convex and strictly increasing, must satisfy an equation

$$MA(u) = F(u)$$

for some function F. Take $u = \mu^2$. Then, $F(u)$ must be homogenous of degree zero, and so $F(u)$ is a constant. All in all, $u = \mu^2$ is outside of the origin a convex function that satisfies

$$MA(u) = C.$$

Moreover, the second derivatives of u stay bounded near the origin, and so u solves the same Monge-Ampere equation on all of $\mathbb{R}^n$ in a generalized sense. By a result of Pogorelov [6], u is actually smooth. We can then apply a celebrated theorem by Jörgens, Calabi, and Pogorelov (see [7]) to conclude that u is a quadratic form. We have thus proved the next theorem.

Theorem 4.4. *Let Ω be a convex domain containing the origin. Assume that, for any convex function in Ω, v that vanishes on the boundary, the symmetrization inequality*

$$M(\Omega)^{-1}\mathcal{E}_\Omega(v) \geq M(B)^{-1}\mathcal{E}_B(\hat{v})$$

holds. Then, v is an ellipsoid.

Remark. We saw above that the condition on our domain is that $\mu = \mu_\Omega$ satisfies an equation

$$\mathrm{d}\mu \wedge \mathrm{d}^c\mu \wedge (\mathrm{dd}^c\mu)^{n-1} = C\mu^{1-n}\,\mathrm{d}\lambda.$$

One can show that this is equivalent to the condition that Ω is a stationary point for the Mahler functional

$$M(\Omega) = |\Omega|\,|\Omega^\circ|.$$

Thus, it follows from the Jörgens-Calabi-Pogorelov theorem that any such stationary point is an ellipsoid. Notice that the two results are not equivalent, though: in the case of the Mahler functional, we know beforehand that our function $u = \mu^2$ grows quadratically at infinity, whereas the Jörgens-Calabi-Pogorelov theorem applies to any convex solution. At any rate, the analogy between the Kähler-Einstein condition in the complex case and the Mahler volume in the real case seems quite interesting.

We conclude with the proof of Theorem 4.3, which is proved more or less as in the complex case. Notice that the appearance of the factor $|\Omega^\circ|$ in the lemma is the main difference as compared to Proposition 2.8.

Lemma 4.5. *Let Ω be a smoothly bounded convex domain containing the origin, with Minkowski functional μ_Ω. Let u be a smooth convex function in Ω of the form $u(x) = f(\mu_\Omega(x))$, vanishing on the boundary so that $f(1) = 0$. Then,*

$$\sigma(s) := \int_{\mu_\Omega < s} MA(u) = f'(s)^n |\Omega^\circ|.$$

Proof. We may assume that u is strictly convex. Then, the map $x \mapsto \nabla u(x)$ is a diffeomorphism from $\{\mu_\Omega \le s\}$ to a domain U_s in $\mathbb{R}^n$, and by the change of variables formula, we have that $\sigma(s) = |U_s|$.

However, U_s depends only on the gradient map restricted to the boundary of the set Ω_s where $\mu_\Omega < s$, that is, on the value of $f'(s)$. We may therefore take $f(s) = as$, and even, by homogeneity, take $a = 1$. Then, the boundary of Ω_s is mapped to the boundary of Ω°, and so the volume is $|\Omega_\circ|$. ◻

Lemma 4.6. *Under the same hypotheses as in the previous lemma,*

$$\mathcal{E}(u) = \int^1 f'(s)^{n+1} \, \mathrm{d}s |\Omega^\circ|.$$

Proof. We have

$$\mathcal{E}(u) = -\int^1 f(s) \, \mathrm{d}\sigma(s) = \int^1 f'(s)\sigma(s) \, \mathrm{d}s,$$

and so this follows from the previous lemma. ◻

Lemma 4.7. *Let Ω and u be as in the previous lemmas, and let B be a ball centered at the origin of the same volume as Ω. Then, $S_B(u) = f(\mu_B)$.*

Proof. By definition, $S_B(u) = g(\mu_B)$, and

$$|\{g(\mu_B) < t\}| = |\{f(\mu_\Omega) < t\}|.$$

The left-hand side here is $|B|(g^{-1}(t))^n$, and the right-hand side is $|\Omega|(f^{-1}(t))^n$. Since $|B| = |\Omega|$, we have $g^{-1} = f^{-1}$, and so we are done. ◻

Combining Lemma 4.6 and Lemma 4.5, we see that

$$\mathcal{E}(u)/|\Omega^\circ| = \mathcal{E}(S_B)/|B^\circ|.$$

This proves Theorem 4.3.

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