Thesis for the Degree of Doctor of Philosophy

Positive vector bundles in complex and convex geometry

Hossein Raufi

CHALMERS | UNIVERSITY OF GOTHENBURG

Division of Mathematics
Department of Mathematical Sciences
Chalmers University of Technology
and University of Gothenburg
Göteborg, Sweden 2014

Positive vector bundles in complex and convex geometry
Hossein Raufi
ISBN 978-91-7597-028-8

© Hossein Raufi, 2014.

Doktorsavhandlingar vid Chalmers tekniska högskola
Ny serie nr 3709
ISSN 0346-718X

Department of Mathematical Sciences
Chalmers University of Technology
and University of Gothenburg
412 96 Göteborg
Sweden
Phone: +46 (0)31-772 10 00

Printed in Göteborg, Sweden, 2014

Positive vector bundles in complex and convex geometry
Hossein Raufi

Abstract

This thesis concerns various aspects of the geometry of holomorphic vector bundles and their analytical theory which all, vaguely speaking, are related to the notion of positive curvature in general, and L^2-methods for the $\bar{\partial}$-equation in particular. The thesis contains four papers.

In Paper I we introduce and study the notion of singular hermitian metrics on holomorphic vector bundles. We define what it means for such metrics to be positively curved in the sense of Griffiths, and investigate the assumptions needed in order to define the curvature tensor of such metrics as currents with measure coefficients. We also investigate the regularisation of such metrics.

In Paper II we prove the Nakano vanishing theorem with Hörmander L^2-estimates on a compact Kähler manifold using Siu's $\partial\bar{\partial}$-Bochner-Kodaira method. We then introduce the singular hermitian metrics and regularisation results of Paper I, and use these to prove a Demailly-Nadel type of vanishing theorem for vector bundles over Riemann surfaces.

A fundamental tool in complex geometry closely related to the notion of positivity is the Ohsawa-Takegoshi extension theorem. In Paper III the $\partial\bar{\partial}$-Bochner-Kodaira method is applied to extend this theorem from line bundles to vector bundles over compact Kähler manifolds. Another way of obtaining a vector bundle version of this theorem is to reduce it to the line bundle setting through the useful algebraic geometric procedure of studying the projective bundle associated with the vector bundle. In Paper III we also investigate the relationship between these two different approaches.

On a trivial line bundle, a positively curved metric is the complex-analytic counterpart of a log concave function in the real-variable setting. In Paper IV we extend this link between complex and convex geometry to trivial vector bundles. We define two new notions of log concavity for real, matrix-valued functions, corresponding to Griffiths and Nakano positivity, and we prove a matrix-valued Prékopa theorem.

Keywords: holomorphic vector bundles, $\bar{\partial}$-equation, L^2-estimates, singular hermitian metrics, Griffiths positivity, Nakano positivity, vanishing theorems, Ohsawa-Takegoshi extension theorem, convex geometry, Prékopa theorem

Preface

This thesis consists of the following papers.

- ▷ Hossein Raufi,
  "*Singular hermitian metrics on holomorphic vector bundles*",
  preprint
- ▷ Hossein Raufi,
  "*The Nakano vanishing theorem and a vanishing theorem of Demailly-Nadel type for holomorphic vector bundles*",
  preprint
- ▷ Hossein Raufi,
  "*An extension theorem of Ohsawa-Takegoshi type for sections of a vector bundle*",
  preprint
- ▷ Hossein Raufi,
  "*Log concavity for matrix-valued functions and a matrix-valued Prékopa theorem*",
  preprint

Acknowledgements

Although doing a PhD in mathematics is a fun, stimulating and highly rewarding experience, it can at times also be quite frustrating, stressful and agonizing. Looking back at these last five years, it is abundantly clear to me that this thesis would not have been possible without the constant support and encouragement of a lot of friends and colleagues.

First and foremost, an enormous thank you to Bo. I have a very hard time imagining a better and more suitable advisor. Your knowledge, passion and enthusiasm for mathematics, combined with your laid-back and relaxed attitude has always impressed me, and been a constant source of inspiration. I have also really enjoyed your sense of humour, and all our discussions on economics, politics, literature and many other topics not related to mathematics.

An enormous thank you also to Oskar. By now, we have known each other and studied mathematics together for the last seven years, and during both the pleasures and hardships of this period, I have time and again been reminded of what a great friend you are.

To my officemate for most of my time as a PhD-student, Richard, a huge thank you. From the very beginning, you have always been a big support and a great friend, and I have always felt fortunate to be able to hang out and share an office with you.

A special thank you to Peter, Dawan and Magnus; long time friends and colleagues whose support and company have made these years so much more enjoyable. Thank you Peter for all the discussions about teaching, literature and life in general. Thank you Dawan for all our long walks, discussions about life, and fun facts on bicycling and football. Thank you Magnus for being such a generous and caring officemate.

There are many other friends and colleagues at the department that have created a friendly and enjoyable atmosphere: the members of the KASS-group, David, Elizabeth, Jakob, Aron, Jacob, Ragnar, Matteo, Mariana, Henrike, Jonas, Philip, Malin, Anna, Ivar, Claes, and many more. I have friends and acquaintances at many other academic departments, and over the years it has become ever more clear to me that the atmosphere and communion that exists at our department is something unique.

At the end of a PhD-period it is easy to forget the influence and importance of all the great mathematics teachers that I was fortunate enough to have at the beginning of my undergraduate studies. First and foremost among these were Jana Madjarova and Christer Borell. Thank you for opening my eyes to the power and beauty of mathematics, and for being continuous sources of inspiration and encouragement throughout my undergraduate and graduate studies.

Finally, a huge thank you to my family and relatives; my parents Ashraf and Iraj, my sister Nazli, my grandmother Mahbobeh, my uncle Sasan, and many more. Thank you for your never-ending love and support, and for always believing in me. I am indeed lucky to have been blessed with such a wonderful family.

Hossein Raufi
Göteborg, May 2014

Contents

Part I

INTRODUCTION

Introduction

1. Complex analysis

In basic, first year calculus courses it is quickly recognized that one can obtain a great deal of information about a real function $f : \mathbb{R} \to \mathbb{R}$, by studying its *derivative function*, f'. This function is defined at a point $x \in \mathbb{R}$ through

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1.1}$$

Although the precise mathematical definition of the limit in the right hand side is the big "bugaboo" of these basic calculus courses, students are nevertheless quick to grasp the importance of differentiation. In later courses, one then proceeds to define integration (the "opposite" of differentiation) and also study the several variable analogues of these concepts.

In all of these courses, it is of utmost importance that the functions depend on *real* variables. In more mathematics oriented studies, one might then turn to the study of functions that depend on one *complex* variable, $z = x + iy \in \mathbb{C}$. Hence one considers functions $f : \mathbb{C} \to \mathbb{C}$ and replace (1.1) with,

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \tag{1.2}$$

where now $h \in \mathbb{C}$. If this limit exists, one says that f is *holomorphic* at $z \in \mathbb{C}$. This innocent looking definition turns out to have quite amazing and far reaching consequences that often stand in sharp contrast to the corresponding real-variable theory. For example we have the following "miraculous" facts.

1. Contour Integration: If f is holomorphic in a domain $\Omega \subset \mathbb{C}$, (i.e. an open and connected set), then for appropriate closed paths γ in Ω,

$$\int_\gamma f(z)dz = 0.$$

2. Regularity: If f is holomorphic, then f is infinitely differentiable.

3. ANALYTIC CONTINUATION: If f and g are holomorphic functions in a domain $\Omega \subset \mathbb{C}$, which are equal in an arbitrarily small disc in Ω, then $f = g$ everywhere in Ω.

These basic features of the one-variable theory are covered in standard courses that are sometimes studied in the very first years of an undergraduate program; (we recommend [SS] for a nice introduction). However, even if one chooses to specialize in mathematics, chances are slim that one ever gets to hear anything about functions that depend on several complex variables, $f : \mathbb{C}^n \to \mathbb{C}$, at either the undergraduate or master's level. This latter subject is not at all as well-developed as the corresponding real-variable theory, and in fact, a more thorough understanding of complex analysis in several variables started to evolve in the second half of the twentieth century.

This theory, once again, starts with the following very innocent-looking definition.

Definition 1. Let Ω be a domain in $\mathbb{C}^n$. A function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic* in Ω, if it is holomorphic in each variable separately. We will denote the set of functions that are holomorphic in Ω by $\mathcal{O}(\Omega)$.

Many of the basic one complex variable properties, such as 2 and 3 above, (and even 1 if properly interpreted), also hold in several complex variables. Many other, though, do not. A salient feature of the theory of holomorphic functions of several variables is that it is not similar to neither the real variable nor the single complex variable theory. Instead, it is an independent theory with tools and methods of its own, that nevertheless overlaps with many other mathematical areas. One of the founding fathers, Kiyoshi Oka, used Figure 1 below to illustrate this, ([O1]).

In this thesis, we are concerned with the parts of the theory that fall into the 'Geometry' and 'Mathematical Analysis' groups. Before we can go on to describe this in more detail, we first need to introduce some of the main concepts and ideas of modern differential geometry.

2. Manifolds, differential forms and partitions of unity

The most fundamental objects of study in modern geometry are manifolds. From "everyday life" we are familiar with curves and surfaces, which are one and two dimensional objects in space, $\mathbb{R}^3$. The idea behind the concept of a manifold is to generalize this to arbitrary dimensions. Thus, intuitively, a manifold is a k-dimensional "surface" in $\mathbb{R}^n$, where $k < n$.

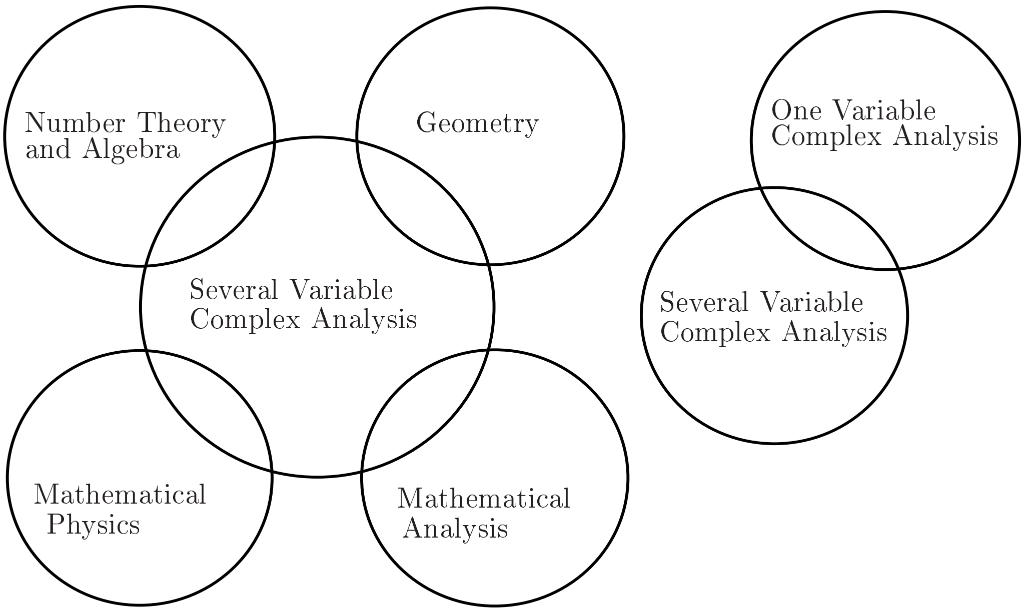

FIGURE 1. Several variable complex analysis from Kiyoshi Oka's perspective

More precisely, the characteristic property of curves and surfaces that one wants to generalize to these higher dimensional objects, is that they are "locally flat". By this we mean that if we zoom in sufficiently much on say a two dimensional surface in $\mathbb{R}^3$, it will look very much like a piece of $\mathbb{R}^2$. Hence a k-dimensional manifold M in $\mathbb{R}^n$ with $k < n$, is an object that locally looks like $\mathbb{R}^k$. Mathematically we express this in the following way.

Definition 2. A subset M of $\mathbb{R}^n$ is called a *k-dimensional manifold* if for every point $x \in M$, there exists open sets $U, V \subset \mathbb{R}^n$ with $x \in U$, and a homeomorphism $\phi : U \to V$, (i.e. a bijective continuous map with a continuous inverse), such that,

$$\phi(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V; y^{k+1} = \ldots = y^n = 0\}.$$

Strictly speaking, the objects that we have just defined are called *embedded* manifolds. The general definition is the following.

Definition 3. Let M be a topological space. We say that M is a *(topological) manifold of dimension k* if it has the following properties:

(i) M is a Hausdorff space: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.

(ii) M is second countable: there exists a countable basis for the topology of M.

(iii) M is locally Euclidean of dimension k: for every point $p \in M$ there exists an open set $U \subset M$ with $p \in U$, an open set $V \subset \mathbb{R}^k$, and a homeomorphism $\phi : U \to V$.

In the general definition, there is no mention of any ambient space $\mathbb{R}^n$. When one thinks about manifolds, one certainly imagines them to be embedded in some $\mathbb{R}^n$, and a famous theorem of Hassler Whitney, [W2], shows that such an embedding always is possible. However, in practice the ambient coordinates and the vector space structure of $\mathbb{R}^n$ are superfluous data that often are not related to the relevant problems in any way.

There are several different types of manifolds. The ones that we have just defined are the most general ones, called topological manifolds. Intuitively, these are geometric objects that are allowed to have edges, like for example a square or a cube. If one is interested in studying manifolds with the tools of calculus, like differentiation and integration, only smooth geometric objects, like for example the sphere, should be allowed. This leads to the concept of smooth manifolds.

Definition 4. A topological manifold M of dimension k, is said to be a *smooth manifold* if there exists an open covering $\{U_\alpha\}$ of M, and homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^k$ for each α, with the property that for any two open sets U_α, U_β with $U_\alpha \cap U_\beta \neq \emptyset$, the so called transition function

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{2.1}$$

(which hence is a mapping between two open sets in $\mathbb{R}^k$), is smooth.

On smooth manifolds it is possible to give meaning to and develop the concepts and tools of calculus. If M and N are smooth manifolds, one can define what it means for functions $f : M \to \mathbb{R}$ and $F : M \to N$ to be smooth, and also how to differentiate and integrate on smooth manifolds. As simple as this may sound, this is not an easy task at all, and in fact most (good) textbooks on the subject (like e.g. [L2]) have to spend several hundred pages on motivating and developing these tools. The main difficulty stems from the fact that manifolds only locally look like $\mathbb{R}^k$, where these concepts are well-known. Hence, one has to find constructions that, (to a large extent), resemble say differentiation or integration, yet are invariant under changes of coordinates.

For integration this leads to the concept of differential forms, which are objects that can be integrated in a coordinate invariant way. These play a very central part in modern differential geometry, (as well as in algebraic topology and algebraic geometry, see e.g. [BT] and [GH]), and will be a very useful tool for us as well. However, we will not spend any time developing this theory here, but will assume that the reader is familiar with the basic parts, like e.g. exterior differentiation and wedge products. We recommend [L2] for a good introduction to differential forms, as well as the basic concepts and theorems of differential geometry; (see also [M] for a nice exposition of the central position occupied by differential forms in modern geometry).

We will in fact be interested in a more general type of differential forms called currents. Integration of differential forms on smooth, (orientable), manifolds, corresponds to integration of functions in ordinary calculus. In the same way, the concept of currents are the manifold counterpart of distributions, or generalised functions. Thus, a current is a linear functional on differential forms, (together with a weak continuity condition), which we intuitively should think of as a "singular differential form". Currents will be of great importance to us in Paper I.

Several natural counterparts for differentiation on smooth manifolds exist as well. For us, the most important concept will be that of connections, which we will return to in section 7 below.

Now from the definition of manifolds it is clear that tools which make it possible to patch together local constructions into global objects are of great value. One such technical tool which is of utmost importance in the theory of smooth manifolds are so called *partitions of unity*, the existence and properties of which are given in the following theorem, (see e.g. [L2], Theorem 2.25 for a proof).

Theorem 2.1. *Suppose that M is a smooth manifold and $U = \{U_\alpha\}_{\alpha \in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to U, i.e. there exists an indexed family $\{\psi_\alpha\}_{\alpha \in A}$ of smooth functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:*

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.

(ii) supp $\psi_\alpha \subset U_\alpha$ for each $\alpha \in A$.

(iii) The family of supports, $\{\text{supp } \psi_\alpha\}_{\alpha \in A}$ is locally finite.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Here, condition (iii) means that each point in M has a neighborhood that intersects supp ψ_α for only finitely many values of α. This in turn implies that the sum in (iv) only has finitely many non-zero terms in a neighborhood of each point, so there will not be any problems related to convergence.

Thus, a partition of unity is a family of smooth functions of compact support, that are used to form global object from local ones. For example this is precisely how one defines integrals of differential forms. Unfortunately, due to analytic continuation this passage from local to global is not available in the holomorphic setting. We will soon discuss this in detail, but first we will introduce complex manifolds, which are our main geometric objects of interest.

3. Complex manifolds and the additive Cousin problem

The subject of this thesis is complex geometry. Hence, we are not really interested in smooth manifolds, but rather a more restrictive class of manifolds called *complex manifolds*. These are defined in exactly the same way as smooth manifolds, (Definition 4), except that one requires the charts ϕ_α to take values in $\mathbb{C}^k$ instead of $\mathbb{R}^k$. One also requires the transition functions (2.1) to be holomorphic, and not just smooth.

Since complex manifolds are geometric objects that locally look like $\mathbb{C}^k$, they are always even-dimensional. Furthermore, one can prove that they always are orientable, i.e. have a well-defined inside and outside. For complex manifolds of dimension one, (i.e. "genuine" surfaces as they locally look like $\mathbb{C} \simeq \mathbb{R}^2$), one can show that orientability is sufficient. This means that all orientable, smooth manifolds of (real) dimension two, can be given a complex structure.

In higher dimensions, however, this is no longer true. In fact, despite the similar looking definitions, the world of smooth and complex manifolds are very different. For example, as we have already mentioned, for smooth manifolds we have the Whitney embedding theorem, which states that *any* smooth manifold M, can be smoothly embedded in $\mathbb{R}^n$, for some $n \in \mathbb{N}$. In stark contrast to this, it is not very difficult to show that the only compact complex manifolds that can be holomorphically embedded into some $\mathbb{C}^n$, are points.

At the end of the previous section, we introduced partitions of unity and described them as an important tool in going from local to global. We also mentioned that, due to analytic continuity, this tool unfortunately is missing in the complex analytic setting. It is important to point out that this does not just relate to complex manifolds; patching together local objects into global ones in a holomorphic way is a highly non-trivial problem already for domains in $\mathbb{C}^n$. We will now describe this in greater detail and in order to keep things as simple as possible, we will only treat $\mathbb{C}^n$ for quite some time.

Two famous local to global results from one variable complex analysis are the Weierstrass product theorem, and the Mittag-Leffler theorem, (see e.g. [A]). Attempts to generalize these, (especially the latter), to several complex variables, historically turned out to be very important for the development of the field. The natural several variable generalization of both these theorems is the following decomposition problem.

ADDITIVE COUSIN PROBLEM: Let Ω be an open set in $\mathbb{C}^n$. Suppose that $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω, and that for any $j, k \geq 1$, functions $g_{jk} \in \mathcal{O}(U_j \cap U_k)$ are given, with

$$g_{jk} + g_{kl} + g_{lj} = 0 \qquad \text{on} \quad U_j \cap U_k \cap U_l, \tag{3.1}$$

whenever $U_j \cap U_k \cap U_l \neq \emptyset$.

Find functions $g_j \in \mathcal{O}(U_j)$, such that $g_j - g_k = g_{jk}$ on $U_j \cap U_k$.

To see how this decomposition problem is related to local to global problems, let us see how we can use it to obtain a several complex variable version of the Mittag-Leffler problem. The formulation of this problem is the following.

MITTAG-LEFFLER PROBLEM: Let Ω be an open set in $\mathbb{C}^n$. Suppose that $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω. For each $j \geq 1$, let m_j denote a meromorphic function on U_j, (i.e. a quotient of holomorphic functions), and assume that these "match up", in the sense that $m_j - m_k =: g_{jk}$ is holomorphic on $U_j \cap U_k$, whenever this set is non-empty.

Find a global meromorphic function m on Ω, such that $m - m_j \in \mathcal{O}(U_j)$ for all $j \geq 1$, (hence the "singularities", or the principal parts of m and m_j are the same).

The Mittag-Leffler problem is an immediate consequence of the Cousin problem. Namely, as $g_{jk} := m_j - m_k \in \mathcal{O}(U_j \cap U_k)$ clearly satisfy (3.1), the Cousin problem yields holomorphic functions $g_j \in \mathcal{O}(U_j)$, such that $m_j - m_k = g_j - g_k$, or equivalently,

$$m_j - g_j = m_k - g_k \qquad \text{on} \qquad U_j \cap U_k.$$

Thus, we get a globally well-defined meromorphic function m on Ω, with $m - m_j \in \mathcal{O}(U_j)$, by setting $m := m_j - g_j$ on U_j.

The several variable generalization of the Weierstrass product theorem also follows from the (multiplicative) Cousin problem, but the argument is more involved, (see e.g. [R]).

Historically, the Cousin problem was solved on certain domains $\Omega \subset \mathbb{C}^n$, called domains of holomorphy, in a spectacular way by the Japanese mathematician Kiyoshi Oka, in 1936, (see [R] for a nice survey of the early developments of several variable complex analysis). This important and difficult achievement was later simplified and expanded by Henri Cartan during the 1940's and 1950's, using methods of sheaf cohomology theory. Beginning in the 1960's, other analytical ways to form global holomorphic objects out of local ones were discovered. Before we can start describing these however, we first need to introduce the key differential operator of complex analysis.

4. The $\bar{\partial}$-operator

In the first section, we defined holomorphicity of a single variable function $f : \mathbb{C} \to \mathbb{C}$, as being complex differentiable. If we regard f as a real-valued mapping instead, i.e.

$$f(x, y) = u(x, y) + iv(x, y),$$

for some real-valued functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, one can show that the holomorphicity of f is equivalent to the following system of partial differential equations, know as the *Cauchy-Riemann equations*,

$$\begin{cases} u'_x = v'_y, \\ u'_y = -v'_x. \end{cases}$$

The idea behind the $\bar{\partial}$-operator comes from yet another reformulation of the holomorphicity condition.

It is a consequence of the Cauchy integral formula that holomorphic functions always can be expanded in power-series. Hence, we can expand f in a Taylor-series about a point $z_0 = x_0 + iy_0$,

$$f(z) = f(z_0) + f'_x(z_0)(x - x_0) + f'_y(z_0)(y - y_0) + o(|z - z_0|).$$

If we replace x and y by

$$\begin{cases} x = \frac{z+\bar{z}}{2}, \\ y = \frac{z-\bar{z}}{2i}, \end{cases} \tag{4.1}$$

i.e. make the change of variables

$$\begin{cases} z = x + iy, \\ \bar{z} = x - iy, \end{cases}$$

then this becomes,

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)(\bar{z} - \bar{z}_0) + o(|z - z_0|),$$

where,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Writing $f = u + iv$, we get that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Hence, using the Cauchy-Riemann equations, we see that in this formulation, f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

i.e. the power-series expansion of f does not contain any powers of $\bar{z}$.

In the several variable setting, we can in the same way set,

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

for $j = 1, \ldots, n$. Then a several variable function $f : \mathbb{C}^n \to \mathbb{C}$ will be holomorphic if and only if,

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \qquad \text{for all } j = 1, \ldots, n. \tag{4.2}$$

Using the language of differential forms, it is in fact possible to express these conditions in an even more compressed form.

The exterior differentiation operator, d, applied to a function $f : \mathbb{C}^n \simeq \mathbb{R}^{2n} \to \mathbb{C}$, yields

$$df = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}dx_j + \frac{\partial f}{\partial y_j}dy_j\right).$$

Making the change of variables (4.1) again, and setting

$$\begin{cases} dz_j = dx_j + idy_j, \\ d\bar{z}_j = dx_j - idy_j, \end{cases}$$

this transforms into,

$$df = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}dz_j + \frac{\partial f}{\partial \bar{z}_j}d\bar{z}_j\right).$$

Hence, for functions on $\mathbb{C}^n$, we can define two operators, ∂ and $\bar{\partial}$, by setting,

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}dz_j \quad \text{and} \quad \bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}d\bar{z}_j,$$

thereby splitting exterior differentiation into two parts, $d = \partial + \bar{\partial}$. Comparing the defintion of $\bar{\partial}$ with (4.2), we see that a several variable function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic, if and only if,

$$\bar{\partial} f = 0.$$

Just as with exterior differentiation, it is possible to extend ∂ and $\bar{\partial}$ to act on differential forms of higher degree. A differential form α, of degree $p + q$, which is of the form,

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{I,J} dz_I \wedge d\bar{z}_J,$$

where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ are multiindices with integer components, $|I|$, $|J|$ stand for the number of components, and

$$dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \quad \text{and} \quad d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q},$$

is said to be of *bidegree* (p, q). For these, we set

$$\partial \alpha := \sum_{j=1}^{n} \sum_{|I|=p,|J|=q} \frac{\partial \alpha_{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J,$$

and,

$$\bar{\partial} \alpha := \sum_{j=1}^{n} \sum_{|I|=p,|J|=q} \frac{\partial \alpha_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J.$$

Since $d^2 = 0$, it then follows that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.

Now for local to global problems, such as the additive Cousin problem of the previous section, it turns our that the inhomogenous $\bar{\partial}$-problem is of great interest.

INHOMOGENOUS $\bar{\partial}$-PROBLEM: Let Ω be a domain in $\mathbb{C}^n$, and let f be a (0,1)-form on Ω, with $\bar{\partial} f = 0$. Find a function $u : \Omega \to \mathbb{C}$, such that

$$\bar{\partial} u = f. \tag{4.3}$$

Since $\bar{\partial}^2 = 0$, the condition $\bar{\partial} f = 0$ is necessary for the solvability of this equation. If

$$f = \sum_{j=1}^{n} f_j dz_j,$$

then $\bar{\partial} f = 0$ means that,

$$\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j} \quad \text{for all } j, k = 1, \ldots, n. \tag{4.4}$$

Hence, (4.3) is a compressed way of saying: Given an n-tuple of functions $\{f_j\}_{j=1}^n$ on Ω such that (4.4) holds, find a function $u : \Omega \to \mathbb{C}$, such that

$$\frac{\partial u}{\partial \bar{z}_j} = f_j \quad \text{for } j = 1, \ldots, n.$$

And so for $n \geq 2$, the inhomogenous $\bar{\partial}$-equation is an over-determined system of first order, linear, partial differential equations.

It turns out that the necessary condition, $\bar{\partial} f = 0$, is sufficient for the existence of *local* solutions to (4.3). This is the famous Grothendieck-Dolbeault lemma, (see e.g. [H4], Proposition 1.3.8). The existence of *global* solutions is, however, a much more difficult problem, closely related to the domain Ω. In fact, we have the following theorem.

Theorem 4.1. *The existence of global solutions to the inhomogenous $\bar{\partial}$-equation on a domain $\Omega \subset \mathbb{C}^n$, is equivalent to the solvability of the additive Cousin problem on Ω.*

Proof. The easy direction here is if we assume known that the Cousin problem is solvable on Ω. Then, as $\bar{\partial} u = f$ is known to have local solutions, there exists an open covering $\{U_j\}_{j=1}^\infty$ of Ω, and corresponding functions u_j, such that

$$\bar{\partial} u_j = f \quad \text{on } U_j \text{ for any } j.$$

Now let $g_{jk} := u_j - u_k \in \mathcal{O}(U_j \cap U_k)$. Then, by the solvability of the Cousin problem, there exists functions $g_j \in \mathcal{O}(U_j)$, such that, $g_{jk} = g_j - g_k$, or equivalently

$$u_j - g_j = u_k - g_k \quad \text{whenever } U_j \cap U_k \neq \emptyset.$$

Thus, setting $u := u_j - g_j$ on U_j, yields a well-defined global function on Ω, with

$$\bar{\partial} u = \bar{\partial} u_j - \bar{\partial} g_j = f.$$

Conversely, assume that the inhomogenous $\bar{\partial}$-equation is (globally) solvable on Ω. Assume that $\{U_j\}_{j=1}^\infty$ is an open covering of Ω, and that functions $g_{jk} \in \mathcal{O}(U_j \cap U_k)$ are given, which satisfy the cocycle condition,

$$g_{jk} + g_{kl} + g_{lj} = 0 \quad \text{on } U_j \cap U_k \cap U_l \neq \emptyset.$$

We want to find $g_j \in \mathcal{O}(U_j)$, such that $g_j - g_k = g_{jk}$ on $U_j \cap U_k$.

For this, we start by constructing smooth solutions to the Cousin problem. Let $\{\phi_j\}_{j=1}^\infty$ be a partition of unity subordinate to $\{U_j\}_{j=1}^\infty$, (Theorem 2.1). Set

$$h_j := \sum_{k=1}^{\infty} \phi_k g_{jk} \quad \text{on } U_j.$$

By the properties of partitions of unity, for any $z \in U_j$, this sum contains only a finite number of terms, so we do not have any convergence problems. Then h_j is well-defined and smooth on U_j. Furthermore, using the cocycle condition, we get that on $U_j \cap U_k$,

$$h_j - h_k = \sum_{l=1}^{\infty} \phi_l (g_{lj} - g_{lk}) = g_{jk} \sum_{l=1}^{\infty} \phi_l = g_{jk}.$$

Thus, $\{h_j\}_{j=1}^{\infty}$ yield a smooth solution to the Cousin problem.

Since every g_{jk} is holomorphic, we have that

$$\bar{\partial} h_j - \bar{\partial} h_k = \bar{\partial} g_{jk} = 0 \quad \text{ on } U_j \cap U_k.$$

Hence, we get a well-defined (0,1)-form, f, on Ω, by setting $f := \bar{\partial} h_j$ on U_j. By construction, $\bar{\partial} f = 0$. The global solvability of the $\bar{\partial}$-equation now implies that there exists a global function, $u : \Omega \to \mathbb{C}$, such that $\bar{\partial} u = f$ on Ω. Set $g_j := h_j - u$ on U_j. Then,

$$\bar{\partial} g_j = \bar{\partial} h_j - \bar{\partial} u = \bar{\partial} h_j - f = 0,$$

so that $g_j \in \mathcal{O}(U_j)$ for all j, and furthermore,

$$g_j - g_k = (h_j - u) - (h_k - u) = h_j - h_k = g_{jk} \quad \text{ on } U_j \cap U_k.$$

Thus $\{g_j\}_{j=1}^{\infty}$ solves the Cousin problem. □

Hence, returning to the discussion of the previous section, instead of attacking the Cousin problem with the methods of sheaf cohomology theory, one can study the existence of global solutions to the inhomogenous $\bar{\partial}$-equation, using methods from the theory of partial differential equations. This approach, which became popular in the 1960's, turned out to be very fruitful, in particular after the work of Lars Hörmander. We will now spend quite some time explaining some of the basic components of this theory, which plays a very central part in this thesis.

5. L^2-theory for the $\bar{\partial}$-equation

Using Hilbert space methods, Hörmander ([H1],[H2]), in 1965 showed that not only is it possible to solve $\bar{\partial} u = f$ on certain domains $\Omega \subset \mathbb{C}^n$, but also provided very useful estimates for the solutions. These estimates have since then become an indispensible tool for the construction of global holomorphic functions with specified properties. To illustrate the main ideas more clearly, we begin by studying the one-variable version of this theorem.

Let Ω be any domain in $\mathbb{C}$, and let $f : \Omega \to \mathbb{C}$ be any locally integrable function, (as we are in $\mathbb{C}$, we can without loss of generality interchange

(0,1)-forms with functions). In this setting, the inhomogenous $\bar{\partial}$-equation translates into finding a function $u : \Omega \to \mathbb{C}$, such that

$$\frac{\partial u}{\partial \bar{z}} = f \quad \text{on } \Omega \tag{5.1}$$

in the sense of distributions.

The "usual" Hilbert space approach to linear partial differential equations now is to make a weak reformulation of this equation, make some suitable estimates and finally, after having made some assumption about the regularity of f, deduce that a solution exists by invoking the Riesz representation theorem.

In our case, this would translate into multiplying our equation with the complex conjugate of a test function $\alpha \in C_c^\infty(\Omega)$, and apply integration by parts to arrive at,

$$-\int_\Omega u\overline{\frac{\partial \alpha}{\partial z}} = \int_\Omega f\bar{\alpha}.$$

Then we would try to estimate the right hand side.

Hörmander showed that this approach can be made successful, and also produce nice estimates for the solution, if we introduce *weighted* L^2 inner products instead.

Let $\phi \in C^2(\Omega)$ be a real-valued function and introduce the weighted scalar product

$$\langle f, g\rangle_\phi := \int_\Omega f\bar{g}e^{-\phi}.$$

With respect to this scalar product, the weak formulation of (5.1) becomes,

$$\int_\Omega u\overline{\bar{\partial}_\phi^* \alpha}e^{-\phi} = \int_\Omega f\bar{\alpha}e^{-\phi}, \tag{5.2}$$

where

$$\bar{\partial}_\phi^* \alpha := -e^\phi \frac{\partial}{\partial z}\big(\alpha e^{-\phi}\big)$$

is the formal adjoint of the $\bar{\partial}$-operator. We then have the following proposition.

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ be any domain, and let*

$$L^2(e^{-\phi}) := \{g \in L^2_{loc}(\Omega)\ ;\ \int_\Omega |g|^2 e^{-\phi} < \infty\}.$$

Then, for any $f \in L^2(e^{-\phi})$, there exists a solution $u : \Omega \to \mathbb{C}$ to the inhomogenous $\bar{\partial}$-equation (5.1) satisfying,

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega |f|^2 e^{-\phi}, \tag{5.3}$$

if and only if

$$\Big| \int_\Omega f \bar{\alpha} e^{-\phi} \Big|^2 \leq \int_\Omega |f|^2 e^{-\phi} \int_\Omega |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi} \tag{5.4}$$

holds for all $\alpha \in C_c^2(\Omega)$.

Proof. One direction is immediate: If u is a solution to (5.1) satisfying (5.3), then (5.4) follows at once by applying the Cauchy-Schwarz inequality to (5.2).

Now suppose that (5.4) holds for all $\alpha \in C_c^2(\Omega)$, and let $E \subset L^2(e^{-\phi})$ be the subspace,

$$E := \{\bar{\partial}_\phi^* \alpha; \alpha \in C_c^2(\Omega)\}.$$

Define the anti-linear functional $L : E \to \mathbb{C}$ as,

$$L(\bar{\partial}_\phi^* \alpha) := \int_\Omega f \bar{\alpha} e^{-\phi}.$$

Then, (5.4) says that L is well-defined and of norm not exceeding,

$$\int_\Omega |f|^2 e^{-\phi}. \tag{5.5}$$

The Hahn-Banach extension theorem can now be used to extend L to an anti-linear functional on all of $L^2(e^{-\phi})$, with the same norm. By the Riesz representation theorem, there exists some element $u \in L^2(e^{-\phi})$, with norm less than or equal to (5.5), such that

$$L(g) = \int_\Omega u \bar{g} e^{-\phi},$$

for all $g \in L^2(e^{-\phi})$. Choosing $g = \bar{\partial}_\phi^* \alpha$ yields,

$$\int_\Omega u \overline{\bar{\partial}_\phi^* \alpha} e^{-\phi} = \int_\Omega f \bar{\alpha} e^{-\phi},$$

so u solves the inhomogenous $\bar{\partial}$-equation (5.1). □

Hence, we have reduced the existence of solutions to (5.1), to proving the inequality (5.4). The next step is to rewrite this inequality a bit further. Assume that $\psi : \Omega \to \mathbb{R}$ is a strictly positive function. Then, with the same reasoning we get that there exists a solution to (5.1) with

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega \frac{|f|^2}{\psi} e^{-\phi},$$

if and only if,

$$\int_\Omega \psi|\alpha|^2 e^{-\phi} \leq \int_\Omega |\bar\partial_\phi^* \alpha|^2 e^{-\phi}, \tag{5.6}$$

for all $\alpha \in C_c^2(\Omega)$.

The reason for this reformulation is that there exists a special choice of ψ, for which (5.6) always holds, for any domain in $\mathbb{C}$. Namely,

$$\psi = \frac{\partial^2 \phi}{\partial z \partial \bar z} =: \Delta\phi.$$

(This is the reason for requiring that $\phi \in C^2(\Omega)$.)

Proposition 5.2. *Let $\Omega \subset \mathbb{C}$ be any domain, and let $\phi \in C^2(\Omega)$. Then, for any $\alpha \in C_c^2(\Omega)$,*

$$\int_\Omega \Delta\phi|\alpha|^2 e^{-\phi} + \int_\Omega |\frac{\partial\alpha}{\partial\bar z}|^2 e^{-\phi} = \int_\Omega |\bar\partial_\phi^*\alpha|^2 e^{-\phi}.$$

Proof. By integration by parts

$$\int_\Omega |\bar\partial_\phi^*\alpha|^2 e^{-\phi} = -\int_\Omega \bar\partial_\phi^*\alpha \frac{\partial}{\partial \bar z}(\bar\alpha e^{-\phi}) = \int_\Omega (\frac{\partial}{\partial\bar z}\bar\partial_\phi^*\alpha)\bar\alpha e^{-\phi}.$$

Also, by definition,

$$\bar\partial_\phi^*\alpha = -e^{\phi}\frac{\partial}{\partial z}\big(\alpha e^{-\phi}\big) = -\frac{\partial\alpha}{\partial z} + \alpha\frac{\partial\phi}{\partial z},$$

and so,

$$\frac{\partial}{\partial\bar z}\bar\partial_\phi^*\alpha = -\frac{\partial^2\alpha}{\partial\bar z\partial z} + \frac{\partial\alpha}{\partial\bar z}\frac{\partial\phi}{\partial z} + \frac{\partial^2\phi}{\partial\bar z\partial z}\alpha = \bar\partial_\phi^*(\frac{\partial\alpha}{\partial\bar z}) + (\Delta\phi)\alpha.$$

Hence,

$$\begin{aligned}\int_\Omega |\bar\partial_\phi^*\alpha|^2 e^{-\phi} &= \int_\Omega \bar\partial_\phi^*(\frac{\partial\alpha}{\partial\bar z})\bar\alpha e^{-\phi} + \int_\Omega \Delta\phi|\alpha|^2 e^{-\phi} = \\ &= \int_\Omega \Big|\frac{\partial\alpha}{\partial\bar z}\Big|^2 e^{-\phi} + \int_\Omega \Delta\phi|\alpha|^2 e^{-\phi},\end{aligned}$$

where we have just used that $\bar\partial_\phi^*$ is the adjoint of $\frac{\partial}{\partial\bar z}$ with respect to the scalar product defined by ϕ. $\square$

Altogether, we have proved the following one-dimensional version of Hörmander's theorem.

Theorem 5.3. *Let $\Omega \subset \mathbb{C}$ be any domain, and let $\phi \in C^2(\Omega)$ be any function with $\Delta\phi > 0$. Then, for any $f \in L^2_{loc}(\Omega)$, there exists a solution u to the inhomogenous $\bar{\partial}$-equation,*

$$\frac{\partial u}{\partial \bar{z}} = f \quad \text{on } \Omega,$$

with,

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega \frac{|f|^2}{\Delta\phi} e^{-\phi}.$$

Let us quickly recapitulate what we have done so far. We started off with the additive Cousin problem for domains Ω in $\mathbb{C}^n$, i.e. the problem of patching together local holomorphic functions in a holomorphic way. We then reformulated this problem into the problem of finding global solutions to the inhomogenous $\bar{\partial}$-equation on Ω. And now, we have just shown that for domains in $\mathbb{C}$, this equation is always globally solvable, and there exists good estimates for the solutions as well.

The natural question to pose now is whether it is possible to proceed with this analytic approach in higher dimensions? Before we can start answering this, we need to introduce some new notation, since we can no longer interchange functions and (0,1)-forms.

Let f and α be (0,1)-forms on $\Omega \subset \mathbb{C}^n$,

$$f = \sum_{j=1}^n f_j d\bar{z}_j \quad , \quad \alpha = \sum_{j=1}^n \alpha_j d\bar{z}_j,$$

let $\phi \in C^2(\Omega)$, and define a scalar product with respect to ϕ, through

$$\langle f, \alpha \rangle_\phi := \sum_{j=1}^n \int_\Omega f_j \alpha_j e^{-\phi}.$$

One can check that the formal adjoint of the $\bar{\partial}$-operator, (which now takes (0,1)-forms to functions), with respect to this scalar product becomes,

$$\bar{\partial}^*_\phi \alpha := -e^\phi \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(\alpha_j e^{-\phi} \right).$$

With this notation, the weighted dual formulation of the inhomogenous $\bar{\partial}$-equation,

$$\bar{\partial} u = f \quad \text{on } \Omega,$$

becomes: Find $u : \Omega \to \mathbb{C}$, such that

$$\int_\Omega u\overline{\bar{\partial}^*_\phi \alpha} e^{-\phi} = \sum_{j=1}^n \int_\Omega f_j \bar{\alpha}_j e^{-\phi},$$

for all $\alpha \in C^2_{(0,1)}(\Omega)$, where

$$C^2_{(0,1)}(\Omega) := \Big\{\alpha = \sum_{j=1}^n \alpha_j d\bar{z}_j \ ; \ \alpha_j \in C^2_c(\Omega)\Big\}.$$

If we replace $L^2(e^{-\phi})$ with,

$$L^2_{(0,1)}(e^{-\phi}) := \Big\{f = \sum_{j=1}^n f_j d\bar{z}_j \ ; \ \sum_{j=1}^n \int_\Omega |f_j|^2 e^{-\phi} < \infty\Big\},$$

in Proposition 5.1, then a careful study of the proof reveals that this result holds, basically unchanged, in this several variable setting as well. However, this is not as good news as one might think at first. In fact, in several variables, it is impossible to prove that

$$\big|\langle f, \alpha\rangle_\phi\big|^2 \leq \Big(\sum_{j=1}^n \int_\Omega |f_j|^2 e^{-\phi}\Big) \int_\Omega |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi}, \tag{5.7}$$

for all $\alpha \in C^2_{(0,1)}(\Omega)$. If we could prove this, then we would have shown that there exists a solution to $\bar{\partial} u = f$, without using the compatibility condition $\bar{\partial} f = 0$. Hence, we need some way of taking this extra information into consideration as well. For this, we first need to introduce some notation from the theory of Hilbert spaces.

Let T denote the operator $\bar{\partial}$ but with the specified domain,

$$Dom(T) := \{u \in L^2(e^{-\phi}) \ ; \ \bar{\partial} u \in L^2_{(0,1)}(e^{-\phi})\}.$$

Furthermore, let T^* denote the adjoint operator, which is $\bar{\partial}^*_\phi$ but with the specified domain,

$$\begin{aligned} Dom(T^*) : = \{v \in L^2_{(0,1)}(e^{-\phi}) \ ; \ u \in Dom(T) \mapsto \langle \bar{\partial} u, v\rangle_\phi \text{ is a} \\ \text{bounded linear functional}\}. \end{aligned}$$

The following proposition is the several variable analogue of, (the rewritten form of), Proposition 5.1, (see [B3], Proposition 1.3.2 for a proof).

Proposition 5.4. *Let Ω be a domain in $\mathbb{C}^n$, and let $\psi : \Omega \to \mathbb{C}^{r\times r}$ be a continuous function whose value at any point $z \in \Omega$ is a strictly positive definite, and uniformly bounded hermitian matrix $\psi(z) = (\psi_{jk}(z))_{1\leq j,k\leq n}$.*

If, for any $\alpha \in Dom(T^*) \cap Ker(\bar{\partial})$ *it holds that,*

$$\sum_{j,k=1}^{n} \int_\Omega \psi_{jk}\alpha_j\bar{\alpha}_k e^{-\phi} \le \int_\Omega |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi},$$

then, for any $f \in L^2_{(0,1)}(e^{-\phi})$ *with* $\bar{\partial} f = 0$*, there exists a solution to* $\bar{\partial} u = f$ *satisfying,*

$$\int_\Omega |u|^2 e^{-\phi} \le \sum_{j,k=1}^{n} \int_\Omega \psi^{-1}_{jk} f_j \bar{f}_k e^{-\phi}.$$

Comparing this with Propostion 5.1, we see that the main difference, which reflects the necessary condition $\bar{\partial} f = 0$ in several variables, is the class of test functions.

Although we have chosen not to include it, the proof of this proposition is not overly difficult. Instead, the main difficulty lies in finding the appropriate several variable analogue of Proposition 5.2. We now turn to studying this in greater detail.

First off, consider the matrix-valued function ψ. It turns out that the appropriate choice is,

$$\psi_{jk} = \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}, \tag{5.8}$$

which clearly is consistent with Proposition 5.1. Furthermore, the several variable counterpart of $\Delta\phi > 0$, is requiring the Hessian (5.8) to be strictly positive definite.

Functions defined on domains in $\mathbb{C}^n$, with positive definite complex Hessians of this type, are called *plurisubharmonic functions.* These functions are the complex-variable analogues of convex functions in real analysis, and just as for convex functions, there exists several alternative definitions of plurisubharmonicity that do not require the functions to be twice differentiable. One definition that bears a striking resemblance to the definition of convex functions, as well as explains the name, is the following.

Definition 5. (i) Let $\Omega \subset \mathbb{C}$ be a domain, and let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function. Then u is called *subharmonic* if, for every compact subset $K \subset \Omega$, and every continuous function $h : K \to \mathbb{R}$ which is harmonic on the interior of K, the inequality $u \le h$ is valid in K, if it holds on ∂K.

(ii) Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function. Then u is called *plurisubharmonic* if its restriction to every complex line in Ω is subharmonic, i.e. for arbitrary $z, w \in \mathbb{C}^n$, the function

$$\tau \mapsto u(z + \tau w),$$

is subharmonic in the open subset of $\mathbb{C}$ where it is defined.

The complex-analytic counterpart of convex sets in $\mathbb{R}^n$, is the notion of pseudo-convexity.

Definition 6. A domain Ω in $\mathbb{C}^n$ is called *pseudo-convex* if there exists a continuous, plurisubharmonic function ψ defined in Ω, which tends to infinity at the boundary; (such a ψ is called an *exhaustion function* for Ω).

The study of plurisubharmonic functions and pseudo-convexity are extremely important and key parts of several complex variable theory. We will, however, not treat this theory in any great detail. For the interested reader we recommend the nice survey [K1], as well as the standard treatises [H3] and [K2].

One intuitive way of regarding pseudo-convex sets is that they are the domains in $\mathbb{C}^n$ that behave most similar to domains in $\mathbb{C}$; ("things work as usual"). The classical example of this is the Hartogs' extension theorem and domains of holomorphy, (once again, see [R]). Another example is that pseudo-convexity is a sufficient condition for the several variable version of Proposition 5.2 to hold.

Proposition 5.5. *Let Ω be a pseudo-convex domain in $\mathbb{C}^n$, and let $\phi \in C^2(\Omega)$. Then, for any $\alpha \in C^2_{(0,1)}(\overline{\Omega}) \cap Dom(T^*)$,*

$$\sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-\phi} \leq \int_\Omega |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi} + \frac{1}{2} \int_\Omega |\bar{\partial} \alpha|^2 e^{-\phi},$$

where,

$$|\bar{\partial}\alpha|^2 = \sum_{j<k} \Big| \frac{\partial \alpha_j}{\partial \bar{z}_k} - \frac{\partial \alpha_k}{\partial \bar{z}_j} \Big|^2.$$

See e.g. [B3], Theorem 1.4.2 for a proof.

Hence, we get that for a pseudo-convex domain Ω in $\mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \alpha_j \bar{\alpha}_k e^{-\phi} \leq \int_\Omega |\bar{\partial}^*_\phi \alpha|^2 e^{-\phi},$$

for any $\alpha \in C^2_{(0,1)}(\overline{\Omega}) \cap Dom(T^*) \cap Ker(\bar{\partial})$. Assume that $\phi \in C^2(\Omega)$ is strictly plurisubharmonic, i.e. the complex Hessian of ϕ is strictly positive definite. If we compare this with the setting of Proposition 5.4, we see that the only difference is the extra requirement of smoothness up to the boundary, $\alpha \in C^2_{(0,1)}(\overline{\Omega})$. This extra requirement can be removed through approximations, but doing so is highly non-trivial.

Once this, far from obvious step, has been taken we arrive at the following theorem, ([H1, H2, H3]).

Theorem 5.6. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, let $\phi \in C^2(\Omega)$ be a strictly plurisubharmonic function, and let $f \in L^2_{(0,1)}(e^{-\phi})$. If $\bar{\partial} f = 0$, (in the sense of distributions), then there exists a global solution, u, to the inhomogenous $\bar{\partial}$-equation $\bar{\partial} u = f$. Furthermore, this solution satisfies the estimate,*

$$\int_\Omega |u|^2 e^{-\phi} \leq \sum_{j,k=1}^{n} \int_\Omega \phi^{jk} f_j \bar{f}_k e^{-\phi}, \tag{5.9}$$

provided that the left hand side is finite, where ϕ^{jk} denotes the inverse matrix of the complex Hessian of ϕ.

This is (one version of) the Hörmander theorem for the inhomogenous $\bar{\partial}$-equation in $\mathbb{C}^n$. Apart from the relation to local to global phenomena discussed so far, this theorem is an extremely useful tool for constructing holomorphic functions with specified properties. In the words of Feodor Nazarov ([N3])

This amazing theorem has become the main tool for constructing analytic functions in $\mathbb{C}^n$ with good growth/decay estimates. It has essentially wiped out all previous ad hoc procedures based on power series, Cauchy integrals and such.

The philosophy behind these constructions is that through the fundamental estimate (5.9), the sought after properties of a holomorphic function can be reduced to the construction of a specific plurisubharmonic weight function ϕ, which is a much less rigid object than holomorphic functions. The analytic proof of the Kodaira embedding theorem at the end of section 8 below is a nice illustration of this.

Now as we have tried to point out throughout this section, the proof of the Hörmander theorem has two main parts:

(i) Reducing the existence of solutions to an inequality.

(ii) Finding conditions which ensure that this inequality holds.

We will now turn to the study of inhomogenous $\bar{\partial}$-equations on complex manifolds, where the Hörmander estimates are an important tool in the analytic study of complex geometry. For complex manifolds, just as for domains in $\mathbb{C}^n$, the difficult part is step (ii). The first part of Paper II is devoted to illustrating a new method, which we believe is simpler than the traditional methods, for the establishment of this step.

6. Holomorphic vector bundles - Motivation

A general strategy in order to understand the geometry of a complex manifold X, is to study:

(i) Holomorphic maps from other complex manifolds into X.

(ii) Holomorphic maps from X into other, (easier), complex manifolds.

(iii) Holomorphic vector bundles on X.

(We will explain what (iii) means shortly.) As we will soon see, these are all closely related.

In (i), one is particularly interested in investigating the complex submanifolds, or more generally the analytic subvarieties, of X. These are defined in the following way.

Definition 7. Let X be a complex manifold. An *analytic subvariety* of X is a closed subset $Y \subset X$, such that for any point $x \in X$, there exists an open neighborhood $U \subset X$ containing x, such that $Y \cap U$ is the zero set of finitely many holomorphic functions on U, i.e. there exists $f_1, \ldots, f_k \in \mathcal{O}(U)$ with,

$$Y \cap U = \{f_1 = \ldots = f_k = 0\} \cap U.$$

An analytic subvariety is not always a submanifold, since it can have singularities. For example, the union of the two coordinate axes in $\mathbb{C}^2$ can be written as $\{z_1 z_2 = 0\}$, so it is an analytic subvariety with a singularity at the origin. However, one can show that for any analytic subvariety Y, the set of regular points $Y_{reg} := Y \setminus Y_{sing}$ is a non-empty complex submanifold of X.

We will need the following standard notions from the theory of analytic subvarieties.

Definition 8. Let X be a complex manifold, and let $Y \subset X$ be an analytic subvariety.

• We say that Y is *irreducible* if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties $Y_i \subset Y$.

• If Y is irreducible, then the *dimension* of Y is defined as the dimension of the manifold of its regular points, $\dim(Y) :=\dim(Y_{reg})$.

• Y is said to be an analytic $hypersurface$ of X, if it has codimension one, i.e. $\dim(Y) = \dim(X) - 1$.

Varieties and hypersurfaces, (not necessarily analytic ones), are the main objects of investigation in algebraic geometry. Many deep and ingenious tools have been developed for their study, (see e.g. [SKKT] for a nice introduction). From our point of view, however, analytic subvarieties are mainly interesting because of their close connection to (iii) above.

The basic definition of vector bundles is the following.

Definition 9. Suppose X is a complex manifold. A *holomorphic vector bundle of rank r over X* is a complex manifold E, together with a surjective holomorphic map, $\pi : E \to X$, satisfying the following conditions:

I. For each $x \in X$, the set $E_x := \pi^{-1}(x)$, called the *fiber over x*, has the structure of an r-dimensional complex vector space.

II. For each $x \in X$, there exists a neighborhood $U \subset X$ of x, and a biholomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^r$, such that $\pi|_U = \pi_1 \circ \Phi$, where π_1 is projection on the first factor. Furthermore, for any $y \in U$, the restriction $\Phi|_{E_y} : E_y \to \{y\} \times \mathbb{C}^r$ is a $\mathbb{C}$-linear isomorphism. The pair (U, Φ) is called a *local trivialization of E over U.*

A holomorphic vector bundle of rank 1 is called a *holomorphic line bundle.*

The main example, and motivation, for vector bundles is the so called *tangent bundle* of a manifold. The intuition behind this construction is to think of the manifold as a submanifold of $\mathbb{C}^n$. At each point of the manifold we then have a tangent space, and the idea is to glue all of these together so that the resulting geometric object also is a manifold. Generalising this construction, making it independent of the ambient space and coordinates, has led to the abstract definition above; for each $x \in X$, the fiber E_x represents the tangent space at x.

As vector bundles are, possibly twisted, disjoint unions of vector spaces, there is a meta-theorem to the effect that any canonical construction in linear algebra gives rise to a geometric version for vector bundles. Thus, it is possible to construct the dual, tensor product, direct sum, etc. of vector bundles.

Now vector bundles are one of the main building blocks of modern geometry. Considering that the tangent bundle of a manifold, by construction, contains a lot of interesting geometric information about the manifold, this is perhaps not so surprising. For us, holomorphic vector bundles, and their rich theory, are *the* main objects of study. Unfortunately, this richness also makes it very difficult to give a self-contained introduction. In this section we will give a rather detailed motivation for their study by describing their relation to analytic varieties. After this, we will recapitulate some basic concepts and theory in the next section, but this will be done hastily and mainly to establish the notation and our conventions. We recommend [W1], [H4], and [K3] among many others for the uninitiated reader.

We have the following simple consequence of Definition 9, ([L2], Lemma 5.4).

Lemma 6.1. *Let $E \to X$ be a rank r holomorphic vector bundle over a complex manifold X. Whenever (U_α, Φ_α) and (U_β, Φ_β) are two local trivializations of E that overlap, the composite map $\Phi_\beta \circ \Phi_\alpha^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{C}^r$ to itself, will be of the form,*

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, z) = (x, \tau_{\alpha\beta}(x)z),$$

where $\tau_{\alpha\beta}$ is a holomorphic map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$, called the transition function from Φ_α to Φ_β.

It follows immediately from the definition of transition functions that they have the following cocycle property: Whenever there are three overlapping local trivializations, we have that

$$\tau_{\alpha\beta}(x)\tau_{\beta\gamma}(x) = \tau_{\alpha\gamma}(x) \quad \text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma, \tag{6.1}$$

where the juxtaposition on the left hand side denotes matrix multiplication.

It turns out, rather surprisingly, that the transition functions contain all the information about the vector bundle. This remarkable feature is the key property for the link between holomorphic vector bundles and analytic subvarieties alluded to above.

Proposition 6.2. *Let X be a complex manifold. Suppose we are given an open cover $\{U_\alpha\}_{\alpha \in A}$ of X, and for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a holomorphic map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$, satisfying the cocycle condition (6.1) above. Then there is a rank r holomorphic vector bundle $E \to X$, with local trivializations (U_α, Φ_α), whose transition functions are the given maps $\tau_{\alpha\beta}$.*

See e.g. [W1], Chapter 1.2 for a proof.

Hence, a set of transition functions is all we need to get a vector bundle. This can be used to create a line bundle from a hypersurface $Y \subset X$ in the following way: By definition, there exists open sets $U_\alpha \subset X$ covering Y, and holomorphic functions $f_\alpha \in \mathcal{O}(U_\alpha)$ with $Y \cap U_\alpha = \{f_\alpha = 0\} \cap U_\alpha$. The idea is to define the transition functions for a line bundle as

$$\tau_{\alpha\beta} := \frac{f_\alpha}{f_\beta}, \tag{6.2}$$

whenever U_α and U_β overlap. These will certainly satisfy the cocycle condition (6.1), but unfortunately, this is not enough. The problem is that we do not know the vanishing order of the local holomorphic functions, and so the $\tau_{\alpha\beta}$:s could vanish or blow up on Y. This leads to the notion of *effective divisors*. Intuitively, these are hypersurfaces, every branch, (i.e. irreducible component), of which are endowed with a non-negative multiplicity. Locally then, an effective divisor D is the zero locus of holomorphic functions $f \in \mathcal{O}(U_\alpha)$ which vanish to the given multiplicity on every branch of D.

If we form the transition functions (6.2) for effective divisors, they will now be holomorphic and non-vanishing on $U_\alpha \cap U_\beta$. By Proposition 6.2 they form the transition functions of a line bundle. Thus, instead of studying the hypersurfaces of a complex manifold X, we can study line bundles on X. From an analytical viewpoint, line bundles turn out to be easier to work with.

For example, if X is a *compact* complex manifold, then by the maximum principle any holomorphic function $f : X \to \mathbb{C}$ is constant. Now suppose that $\pi : L \to X$ is a holomorphic line bundle and study holomorphic maps from subsets $U \subset X$ to L satisfying $\pi \circ f = Id_U$, called *local holomorphic sections* to L. If L is trivial, i.e. $L = X \times \mathbb{C}$, the set of global holomorphic sections to L are in one-to-one correspondence with the set of holomorphic functions on X. The point here is of course that if X is compact, there might exist non-trivial holomorphic line bundles on X, having non-constant global holomorphic sections, which one can apply the tools of analysis to.

7. Holomorphic vector bundles - The setting

Just as in the line bundle setting, sections of a holomorphic vector bundle, $\pi : E \to X$ are defined as maps from subsets $U \subset X$ to E, $f : U \to E$, satisfying $\pi \circ f = Id_U$. We denote the set of smooth sections of E by $C^\infty(X, E)$, and the set of holomorphic sections by $\mathcal{O}(X, E)$.

As sections of vector bundles correspond to functions on the manifold, we want to be able to apply the tools of analysis to their study. We will now quickly recall the most important of these tools and notions from the theory of vector bundles.

Assume that $\text{rank}_{\mathbb{C}} E = r$. A *local frame* for E is an ordered r-tuple of local sections $(e_i) = (e_1, \ldots, e_r)$ over U, with the property that for each $x \in U$, the r-tuple $(e_1(x), \ldots, e_r(x))$ forms a basis for the fiber E_x.

We have already introduced differential forms, which are nothing but sections to exterior powers of the cotangent bundle, the dual of the tangent bundle. For a vector bundle $E \to X$, we can form sections of the bundle $\Lambda T^*X \otimes E$. We call these *differential forms on X with values in E.* Locally, such a section is just a linear combination of tensor products of differential forms and sections of E. For $p \geq 1$ we let $C_p^\infty(X, E)$ denote the set of smooth differential p-forms with values in E.

A *hermitian metric*, h, on a vector bundle $E \to X$ is a positive definite hermitian inner product, $(\cdot, \cdot)_{h(x)}$, on each fiber E_x that varies smoothly with x. More explicitly, given two sections s, t the function $(s, t)_h$ on X is smooth. Locally in a neighborhood U of a point $x \in X$ we can identify s and t with vectors of functions on U and h with a matrix-valued function on U so that

$$(s, t)_h = t^* hs$$

where t^* is the transpose conjugate of t and juxtaposition denotes matrix multiplication.

As soon as we have a metric on a vector bundle, we also get a well-defined bilinear map, $\{\cdot, \cdot\}$, for differential forms on X with values in E. Namely by letting $\{\alpha \otimes s, \beta \otimes t\} := \alpha \wedge \bar{\beta}\ (s, t)_h$ for forms α, β and sections s, t, and then extend to arbitrary forms with values in E by linearity.

A *connection* D on E is a $\mathbb{C}$-linear mapping $D : C^\infty(X, E) \to C_1^\infty(X, E)$ satisfying the Leibniz rule

$$D(fs) = df \otimes s + fDs$$

for any smooth map $f \in C^\infty(X)$ and section $s \in C^\infty(X, E)$. Hence a connection is a first order differential operator that allows us to take directional derivatives of sections. Locally, if we regard a section s as a vector of functions we have that

$$Ds = ds + \theta s$$

where θ is a matrix of one-forms.

On a holomorphic vector bundle with a hermitian metric h, there exists a special connection D called the *Chern connection* which reflects the geometric and holomorphic structure of the vector bundle. This connection is characterized by:

(i) Compatibility with h

$$d(s, t)_h = \{Ds, t\}_h + \{s, Dt\}_h.$$

(ii) In the decomposition of D into (1,0) and (0,1) parts, $D = D' + D''$, the (0,1) part D'' equals $\bar{\partial}$.

One can show that the Chern connection is unique and that locally the connection matrix θ is a matrix of (1,0)-forms given by $\theta = h^{-1}\partial h$.

Given a connection D we can extend it to $C^\infty_p(X,E)$ for $p \geq 1$ by the Leibniz rule

$$D(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge Ds$$

where α is a k-form on X and s is a smooth section of E. We use this to define the *curvature* Θ associated with a connection D through $\Theta s := D^2 s$. We then have

$$\Theta(fs) = D(df \otimes s + fDs) = d^2 f \otimes s - df \otimes Ds + df \otimes Ds + f\Theta s = f\Theta s$$

for $f \in C^\infty(X)$, $s \in C^\infty(X,E)$ so Θ is a form-valued endomorphism of E, i.e. locally just a matrix of two-forms.

For a hermitian holomorphic vector bundle (E,h), we call the curvature associated with the Chern connection the curvature of E and one can show that in this case

$$\Theta = \bar\partial\theta = \bar\partial(h^{-1}\partial h).$$

Hence, if we let $\{dz_j\}_{j=1}^n$ be a basis for the holomorphic cotangent space of X, the curvature can locally be represented as a matrix of (1,1)-forms which we can write either as

$$\Theta = \sum_{j,k=1}^n \Theta_{jk} dz_j \wedge d\bar z_k,$$

where Θ_{jk} are local $r \times r$ matrix-valued functions, or as

$$\Theta = \sum_{\substack{1\leq j,k\leq n\\ 1\leq \mu,\lambda\leq r}} c_{jk\mu\lambda} dz_j \wedge d\bar z_k \otimes e^*_\mu \otimes e_\lambda,$$

where $c_{jk\mu\lambda}$ are local functions, $\{e_\mu\}_{\mu=1}^r$ is a local frame for E, and $\{e^*_\mu\}_{\mu=1}^r$ is the corresponding dual coframe.

In Riemannian geometry one studies the geometry of a manifold M, through the geometry of its tangent bundle, TM. A Riemannian metric on M is a scalar product on each tangent space that varies smoothly with respect to the base point. Just as for holomorphic vector bundles, given a Riemannian metric there exists a canonical choice of connection, the *Levi-Civita connection*, ∇, that reflects the geometry of M.

Suppose now that $M = X$ is a complex manifold with a hermitian metric g. By definition, this means that for each $x \in X$, $g(x)$ is a hermitian scalar product on the complex tangent space $T^{1,0}_x X$. This means that if $z = (z_1, \ldots, z_n)$ are local coordinates near x, g can be written as

$$g = \sum_{j,k=1}^n g_{jk} dz_j \otimes d\bar z_k. \tag{7.1}$$

If v is a holomorphic vector field on X,

$$v = \sum_{j=1}^{n} v_j \frac{\partial}{\partial z_j},$$

we can hence define the norm of v as,

$$\|v\|_g^2 = \sum_{j,k=1}^{n} g_{jk} v_j \bar{v}_k.$$

It turns out, however, to be much more convenient to work with the imaginary part of g, instead of the metric itself. This two-form, ω, is called the *Kähler form* of g, and if g is given in local coordinates as in (7.1), then it is not difficult to show that

$$\omega = i \sum_{j,k=1}^{n} g_{jk} dz_j \wedge d\bar{z}_k. \tag{7.2}$$

One example of the way things are simplified by working with ω instead of g, is the relation between ω and the volume form. By definition, a *volume form* is a differential form of maximal degree that can be written as,

$$dV_\omega = i^n \xi_1 \wedge \bar{\xi}_1 \wedge \ldots \wedge \xi_n \wedge \bar{\xi}_n$$

whenever $\xi = (\xi_1, \ldots, \xi_n)$ is an orthonormal basis for the holomorphic cotangent space. This is a well-defined global form, (any other orthonormal basis is related to ξ via a unitary linear transformation), and it can also be expressed as

$$dV_\omega = \omega^n / n!,$$

in terms of the Kähler form

$$\omega = i \sum_{j,k=1}^{n} g_{jk} dz_j \wedge d\bar{z}_k = i \sum_{j=1}^{n} \xi_j \wedge \bar{\xi}_j.$$

This, in turn, can be utilized to introduce a convenient formalism for computing the norms of forms. We will do this in the setting of differential forms with values in a vector bundle over a compact complex manifold.

Thus, let $(E, h) \to (X, \omega)$ be a holomorphic vector bundle over a compact complex manifold. Let α be an E-valued form of bidegree $(p, 0)$. We define the norm of α with respect to the metrics h and ω through

$$\|\alpha\|^2 dV_\omega = c_p \{\alpha, \alpha\}_h \wedge \omega_{n-p} \tag{7.3}$$

where $\omega_{n-p} := \omega^{n-p}/(n-p)!$, and $c_p := i^{p^2}$ is a unimodular constant chosen so that the right hand side is positive. One can show that if $\{\xi_j\}$ are orthonormal coordinates at a point and

$$\alpha = \sum_{|I|=p} \alpha_I \xi_I$$

where $\{\alpha_I\}$ are sections of E, then

$$\|\alpha\|^2 = \sum_{|I|=p} \|\alpha_I\|_h^2.$$

We also use (7.3) to define the norm of E-valued forms of bidegree $(0,q)$. In particular then $\|\alpha\| = \|\bar{\alpha}\|$.

Using this definition one can now proceed to show that it is possible to define the norm of an E-valued form η of arbitrary bidegree in such a way that if

$$\eta = \sum \eta_{IJ} \xi_I \wedge \bar{\xi}_J$$

in terms of an orthonormal basis at a point, then

$$\|\eta\|^2 = \sum \|\eta_{IJ}\|_h^2.$$

This norm can then be polarized yielding an inner product for E-valued forms of arbitrary bidegree. Hence if μ is another form with values in E, which is of the same bidegree as η, then

$$(\eta, \mu) = \sum (\eta_{IJ}, \mu_{IJ})_h$$

if we express η and μ in terms of an orthonormal basis as above.

Integrating these norms over X with respect to the volume form, one can hence extend the Hilbert space formalism of section 5 to vector bundle valued differential forms on complex manifolds. But before we can turn to the study of the inhomogenous $\bar{\partial}$-equation on complex manifolds, we need to introduce the concept of Kähler manifolds.

We say that g is a *Kähler metric*, and that X is a *Kähler manifold*, if the two-form ω is closed, $d\omega = 0$.

Kähler manifolds are very important in complex geometry. There are conceptual and computational reasons for this.

First off, given a metric g, we have seen that there are two canonical connections on the holomorphic tangent bundle associated with g:

(i) The Chern connection, D_g, induced by the holomorphic structure on X.

(ii) The Levi-Civita connection, ∇_g, induced by the Riemannian structure on X.

We then have the following result, (see e.g. [B3], Proposition 3.4.2):

Proposition 7.1. *Let g be a hermitian metric on a complex manifold X. Let D_g be the Chern connection, and ∇_g the Levi-Civita connection, induced by g. If (X, g) is a Kähler manifold, then $D_g = \nabla_g$.*

Secondly, computations in local coordinates are usually greatly simplified if one is able to use *normal coordinates.* These are local coordinates at a point $x \in X$ in which the metric g, resembles the euclidean metric to first order. More precisely,

$$g_{jk} = \delta_{jk},$$

and,

$$dg_{jk} = 0$$

at x.

It is an immediate consequence of (7.2) that if we can choose normal coordinates at any point, then $d\omega = 0$, so the metric must be Kähler. To a large extent, the computational advantage of Kähler manifolds stems from the fact that the converse of this also holds, (see e.g. [H4], Proposition 1.3.12).

Proposition 7.2. *Let g be a Kähler metric. For each $x \in X$, there are local holomorphic coordinates near x that are normal in x.*

8. L^2-methods for the $\bar{\partial}$-equation on compact Kähler manifolds

With the concepts and formalism of the previous section at our disposal, we can now formulate the compact Kähler generalisation of Theorem 5.6 on the global solvability of the inhomogenous $\bar{\partial}$-equation together with Hörmander's L^2-estimates.

Theorem 8.1. *Let (X, ω) be a compact Kähler manifold, and let (L, h) be a holomorphic line bundle over X. Assume that the metric h has strictly positive curvature,*

$$i\Theta \geq \varepsilon\omega, \tag{8.1}$$

for some $\varepsilon > 0$.

For any $\bar{\partial}$-closed (n, q)-form f with values in L and $q \geq 1$, there exists an $(n, q-1)$-form u with values in L, such that

$$\bar{\partial}u = f,$$

and

$$\int_X \|u\|^2 dV_\omega \leq \frac{1}{\varepsilon q} \int_X \|f\|^2 dV_\omega. \tag{8.2}$$

(Apart from the L^2-estimates (8.2), this is the famous *Kodaira vanishing theorem*, [K4].)

To see that this theorem is the compact Kähler version of Theorem 5.6, note that on a line bundle $L \to X$, a metric h is locally just a strictly positive function. Hence we can write it as $h = e^{-\varphi}$ for some smooth, real-valued, function φ, (strictly speaking, φ is not a function but rather a collection of smooth real-valued local functions, related in a consistent way on overlaps; we will return to this shortly), so norms of L-valued differential forms correspond to weighted L^2-norms in $\mathbb{C}^n$. We will emphasize this by writing the norm of a section as $\|u\|^2 = |u|^2 e^{-\varphi}$, from now on.

Furthermore, in $\mathbb{C}^n$ a sufficient condition for the global solvability of the inhomogenous $\bar{\partial}$-equation was for the domain to be pseoduconvex. This meant that there existed a plurisubharmonic exhaustion or weight function φ. Formally, in our setting, this corresponds to,

$$0 < i\partial\bar{\partial}\varphi = i\bar{\partial}\partial \log e^{-\varphi} = i\bar{\partial}(h^{-1}\partial h) = i\Theta,$$

i.e. the existence of a strictly positively curved metric on L.

We mentioned earlier that by the Whitney embedding theorem, any smooth manifold can be smoothly embedded in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. We also noted that the situation is very different in the complex setting, as the only compact complex manifolds that can be holomorphically embedded into some $\mathbb{C}^m$, are points. It is then quite natural to ask: Is there any similar embedding theorem for compact complex manifolds?

The answer is provided by the celebrated *Kodaira embedding theorem*, [K5].

Theorem 8.2. *Let X be a compact complex manifold. There exists a positive line bundle over X, (i.e. a line bundle carrying a strictly positively curved metric), if and only if X can be holomorphically embedded into complex projective space $\mathbb{P}^m$, for some $m \in \mathbb{N}$.*

This deep and important theorem nicely illustrates how closely intertwined the three general strategies to understand the geometry of manifolds, mentioned at the beginning of section 6, are. In order to study the subvarieties of a compact complex manifold X, one was led to introduce holomorphic line bundles, and using these in turn, one can decide if X in fact is a submanifold of complex projective space, the simplest and most fundamental compact complex manifold.

Kodaira's original proof for the theorem relies heavily on, and was the main motivation for, his vanishing theorem. The argument is quite involved and algebraic, using blow-ups and sheaf cohomology theory, ([K5];

see also [W1] Chapter 6). Using a more general form of the L^2-estimates, (which of course did not exist at the time Kodaira proved the theorem), an analytic proof of the theorem can be given. In order to formulate these, we first need to introduce singular metrics on line bundles.

Up until now, a hermitian metric $h = e^{-\varphi}$ on a line bundle $L \to X$, has been a *smooth* mapping from the base manifold to the space of strictly positive hermitian norms on the fiber. This means that given an open cover $\{U_j\}$ of X, φ is given by a collection of smooth, real-valued functions $\{\varphi_j\}$, subordinate to $\{U_j\}$, such that

$$\varphi_j - \varphi_k = \log|\tau_{jk}|^2 \quad \text{on } U_j \cap U_k \neq \emptyset,$$

where $\{\tau_{jk}\}$ are the transition functions of L. (An immediate consequence of this is that if φ is a metric and χ is a global function on X, then $\varphi + \chi$ is also a metric.)

As the τ_{jk}:s are holomorphic and non-vanishing, $\log|\tau_{jk}|^2$ will be pluriharmonic, and so

$$\partial\bar{\partial}\varphi_j = \partial\bar{\partial}\varphi_k \quad \text{on } U_j \cap U_k \neq \emptyset.$$

Hence, for smooth local representatives, φ_j, the curvature form

$$\Theta = \partial\bar{\partial}\varphi := \partial\bar{\partial}\varphi_j,$$

is a globally defined $(1,1)$-form, although φ_j is just locally defined.

A *singular metric* φ on L is defined in the same way, but without requiring the φ_j:s to be smooth. Instead one requires $\varphi_j \in L^1_{loc}(X)$ and so the curvature form, $\partial\bar{\partial}\varphi$, is well-defined in the sense of currents.

Singular hermitian metrics on holomorphic line bundles were introduced by Demailly in [D3], and ever since then they have been a fundamental tool in interpreting notions of complex algebraic geometry analytically.

It is possible to prove the existence of solutions to the inhomogenous $\bar{\partial}$-equation with L^2-estimates, in the setting of singular metrics. This is the content of the Demailly-Nadel vanishing theorem, [D2, N1]. Demailly showed this in a very general context, (namely for complex manifolds carrying some complete Kähler metric), but we will just need it for *projective* manifolds. These are compact Kähler manifolds on which it is known to exist some line bundle carrying a positively curved, smooth metric. (The terminology is explained by the Kodaira embedding theorem which implies that these manifolds can be seen as submanifolds of projective space.) We then have the following theorem.

Theorem 8.3. *Let (X, ω) be a projective manifold. Let L be a holomorphic line bundle over X having a, possibly singular, metric $h = e^{-\varphi}$ whose* *curvature satisfies,*

$$i\partial\bar{\partial}\varphi \geq \varepsilon\omega, \tag{8.3}$$

for some $\varepsilon > 0$.

For any $\bar{\partial}$*-closed* (n, q)*-form* f *with values in* L *and* $q \geq 1$*, there exists an* $(n, q-1)$*-form* u*, with values in* L*, such that*

$$\bar{\partial}u = f,$$

and

$$\int_X |u|^2 e^{-\varphi} dV_\omega \leq \frac{1}{\varepsilon q} \int_X |f|^2 e^{-\varphi} dV_\omega, \tag{8.4}$$

provided that the right hand side is finite.

Here, the curvature assumption (8.3) is in the sense of currents, (it basically just means that the local representatives of φ can be chosen to be strictly plurisubharmonic).

An important difference compared to Theorem 8.1, is the very last proviso. The finiteness of the L^2-norm with respect to a singular metric, implies that f must vanish on the non-integrability locus of $e^{-\varphi}$. In particular, if $\varphi = \log |s|^2$, for some holomorphic function s, this means that f must vanish on the zero locus of s. Through the L^2-estimates (8.4), this in turn implies that u must vanish on the zero locus of s too. This observation is originally due to Bombieri, [B6], and is extremely useful for constructing global holomorphic functions with specific zero sets.

A nice illustration of this technique is the analytic proof of the Kodaira embedding theorem mentioned previously. We end this section with a rough sketch of the difficult direction of the proof.

Given a projective manifold X, we want to construct a holomorphic mapping $\mathcal{K} : X \to \mathbb{P}^N$ for some $N \in \mathbb{N}$, which is an embedding. This means that $\mathcal{K}$ is injective, and has an injective differential $d\mathcal{K}$. We will only discuss the injectivity of $\mathcal{K}$.

Let $L \to X$ be a line bundle and let E denote the space of global holomorphic sections of L, (which we will assume to be non-empty). Furthermore, let $s_0, \ldots, s_N$ be a basis for E, (one can show that E is finite-dimensional by using the Montel theorem).

We claim that there exists a line bundle L such that

$$\mathcal{K}(x) := [s_0(x), \ldots, s_N(x)],$$

is the sought for embedding. Although it might look strange at first sight, the right hand side is well-defined. What we mean is simply the values of the s_j:s with respect to some local trivialisation of L. If we change to another trivialisation, all the sections get multiplied with the same quantity, so we will still get the same point in $\mathbb{P}^N$. Hence $\mathcal{K}$ is well-defined and clearly holomorphic.

The main step in proving the injectivity of $\mathcal{K}$, is to show that for any two points $a, b \in X$, $a \neq b$, we can construct a global holomorphic section s to L, such that $s(a) \neq 0$ and $s(b) = 0$. We will now outline how Theorem 8.3 can be used to achieve this.

As X is projective, there exists a holomorphic line bundle $F \to X$ with a smooth metric ϕ, which is strictly positively curved,

$$i\partial\bar{\partial}\phi \geq \varepsilon\omega,$$

for some $\varepsilon > 0$. Assume that $\dim_{\mathbb{C}} X = n$, identify a neighborhood of a with $\mathbb{C}^n$, and set

$$\psi_a(z) := \chi_a(z) \log |z - a|^{2n},$$

where χ_a is a cut-off function with $\chi_a \equiv 1$ close to a. ψ_a is a locally integrable function on X, and one can show that,

$$i\partial\bar{\partial}\psi_a \geq -C\omega,$$

for some constant C, which is independent of a if X is compact.

For $k \in \mathbb{N}$,

$$k\phi + \psi_a$$

will then define a singular metric on $kF := F^{\otimes k}$, (the k times tensor product of F), and

$$i\partial\bar{\partial}(k\phi + \psi_a) \geq (k\varepsilon - C)\omega \geq \omega,$$

for large enough k.

As a first step we will use this metric in Theorem 8.3 to construct a global, $\bar{\partial}$-closed $(n, 0)$-form, u with values in kF, and $u(a) \neq 0$.

Choose local coordinates $(z_1, \ldots, z_n)$ in a neighborhood of a, let χ be a cut-off function with $\chi \equiv 1$ near the origin, and let $u_{loc} := dz_1 \wedge \ldots \wedge dz_n$. Define a smooth $(n, 0)$-form $\tilde{s}$ with values in kF locally through, $\tilde{s} := \chi u_{loc}$. Then,

$$f = \bar{\partial}\tilde{s} = \bar{\partial}\chi \wedge u_{loc},$$

will be a smooth, $\bar{\partial}$-closed $(n, 1)$-form with values in kF. Also, the L^2-norm of f with respect to the metric $k\phi + \psi_a$ will be finite, as f vanishes on the non-integrability locus of $e^{-\psi_a}$. Hence, Theorem 8.3 can be applied to produce a kF-valued $(n, 0)$-form v, with $\bar{\partial}v = f$ and

$$\int_X |v|^2 e^{-(k\phi + \psi_a)} dV_\omega \leq \int_X |f|^2 e^{-(k\phi + \psi_a)} dV_\omega.$$

Since f is smooth, it follows from regularity theory for the $\bar{\partial}$-operator that v is smooth as well. Furthermore, as the L^2-norm of v is finite with respect to $k\phi + \psi_a$, this implies that $v(a) = 0$.

Thus,

$$u := \tilde{s} - v$$

is a $\bar{\partial}$-closed, $(n, 0)$-form with values in kF and $u(a) \neq 0$.

Differential forms of bidegree $(n, 0)$ with values in kF, are just sections of $kF \otimes K_X$, where K_X is the canonical bundle of X. If we repeat the above argument but with kF replaced by $kF \otimes K_X^*$, (i.e. take the tensor product with the dual bundle of K_X), we get a global holomorphic section u to kF with $u(a) \neq 0$.

Finally, we can make sure that $u(a) \neq 0$, and $u(b) = 0$, by repeating this last argument once again, but this time with,

$$f = \tilde{\chi} u_{loc},$$

where $\tilde{\chi} \equiv 1$ in a neighborhood of a, but $\tilde{\chi} \equiv 0$ in a neighborhood of b. Also, we add the function

$$\psi_b(z) := \chi \log |z - b|^{2n},$$

to the metric.

Returning to the beginning of the proof and the mapping $\mathcal{K}$, we see that by choosing the line bundle L as kF, and using the technique sketched above for large enough k, we can construct a global holomorphic section s with $s(a) \neq 0$ and $s(b) = 0$. As $\{s_j\}_{j=0}^N$ form a basis for the set of global holomorphic sections, s can be written as

$$s = \sum_{j=0}^{N} c_j s_j,$$

for some functions $\{c_j\}_{j=0}^N$. Hence,

$$\sum_{j=0}^{N} c_j s_j(a) \neq 0, \quad \text{and} \quad \sum_{j=0}^{N} c_j s_j(b) = 0,$$

and so,

$$\mathcal{K}(a) = [s_0(a) : \ldots : s_N(a)] \neq [s_0(b) : \ldots : s_N(b)] = \mathcal{K}(b).$$

9. Positivity concepts and L^2-theory for vector bundles

In the previous section we introduced L^2-theory for the inhomogenous $\bar\partial$-equation for line bundle valued differential forms. We now turn to the extension of these results to vector bundles.

Given a hermitian, holomorphic vector bundle (E, h) over a complex manifold X, we have already defined the curvature tensor Θ of h in section 7. For Theorem 8.1 hold, the existence of a strictly positively curved metric on the line bundle is crucial. Hence the first thing we need to consider in the vector bundle setting is what it should mean for Θ to be positive.

Let $\{dz_j\}_{j=1}^n$ denote an orthonormal basis for the holomorphic cotangent bundle of X, at some fixed point. Then, if $\mathrm{rank}_{\mathbb{C}} E = r$, the curvature tensor is of the form,

$$\Theta = \sum_{j,k=1}^{n} \Theta_{jk} dz_j \wedge d\bar{z}_k,$$

where $\{\Theta_{jk}\}_{j,k=1}^n$ are $r \times r$ matrix-valued functions on X.

In the line bundle setting, $r = 1$, and so the Θ_{jk}:s are just scalar-valued functions. Hence the natural definition of positivity for Θ is to define it as the positivity of the real $(1,1)$-form $i\Theta$; (the i is needed since $dz \wedge d\bar{z} = -2i dx \wedge dy$).

For matrix-valued coefficients, however, there exists no similar 'canonical' way of defining the positivity of Θ. Over time, two different, but equally important notions of positivity for vector bundles have evolved: Positivity in the sense of Griffiths, and positivity in the sense of Nakano.

Definition 10. Let $(E, h) \to X$, and Θ be as above.

(i) We say that Θ is *strictly positively curved in the sense of Griffiths*, if for any (smooth) section s of E, and any n-tuple of complex numbers $\{v_j\}_{j=1}^n$

$$\sum_{j,k=1}^{n} \big(\Theta_{jk} s, s\big)_h v_j \bar{v}_k \geq \varepsilon \|s\|_h^2 \sum_{j=1}^{n} |v_j|^2, \tag{9.1}$$

for some $\varepsilon > 0$.

(ii) We say that Θ is *strictly positively curved in the sense of Nakano*, if for any n-tuple of sections $\{s_j\}_{j=1}^n$ of E,

$$\sum_{j,k=1}^{n} \big(\Theta_{jk} s_j, s_k\big)_h \geq \varepsilon \sum_{j=1}^{n} \|s_j\|_h^2, \tag{9.2}$$

for some $\varepsilon > 0$.

Semi-positivity, semi-negativity and strict negativity are defined similarly.

It follows immediately from the definitions that when $r = 1$, both these notions coincide and reduce to the positivity condition in the line bundle setting. Other immediate consequences of the definitions are that Griffiths and Nakano positivity also coincide when $n = 1$, i.e. for vector bundles over Riemann surfaces, and that being positively curved in the sense of Nakano implies being positively curved in the sense of Griffiths; just choose $s_j = sv_j$ in (9.2). The converse property, however, does not hold in general (see e.g. [D1] Chapter VII, Example 6.8, or Example 4 in Paper IV).

Positivity in the sense of Griffiths is closely connected to, and stems from, a common algebro-geometric procedure of reducing vector bundles to line bundles. For a vector bundle $E \to X$ this is achieved through the study of the so called *Serre line bundle* $\mathcal{O}_{\mathbb{P}(E)}(1)$ over the *projective bundle* $\mathbb{P}(E) \to X$, associated to E.

These concepts can be constructed in an abstract, global way, (see e.g. [L1], Appendix A), but in order to present the ideas as, (in our view), concretely as possible, we will only discuss the local setting here. Hence let E denote an arbitrary complex vector bundle over X with $\operatorname{rank}_{\mathbb{C}} E = r$, and let E^* denote the dual bundle. We can define a fiber bundle $\pi : \mathbb{P}(E) \to X$ by defining each fiber through $\mathbb{P}(E)_x := \mathbb{P}(E^*_x)$, the projectivization of an r-dimensional vector space. Locally, for an open set $U \subset X$, we have that $E^*\big|_U \simeq U \times \mathbb{C}^r$ and then $\mathbb{P}(E)\big|_U \simeq U \times \mathbb{P}^{r-1}$. Furthermore the pullback bundle $\pi^* E^* \to \mathbb{P}(E)$ will then locally be given by

$$\pi^* E^*\big|_U \simeq U \times \mathbb{P}^{r-1} \times \mathbb{C}^r$$

and so we can define the tautological line subbundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ of $\pi^* E^*$ as

$$\mathcal{O}_{\mathbb{P}(E)}(-1)\big|_U := \{(x, [w], z) \ ; \ z \in [w]\}.$$

The Serre line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is then defined as the dual of $\mathcal{O}_{\mathbb{P}(E)}(-1)$. The notation is justified by the fact that fiberwise this is nothing but the usual line bundle $\mathcal{O}(1)$ over $\mathbb{P}^r$. Thus we have that the global holomorphic sections of $\mathcal{O}_{\mathbb{P}(E)}(1)$ over any fiber are in one-to-one correspondence with the linear forms on E^*_x, i.e. with the elements of E_x; (this is the reason for projectivizing E^* instead of E).

Suppose now that the vector bundle E is equipped with a hermitian metric h. Then the Serre line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ will inherit a corresponding metric $\pi^* h$. One can show that positivity in the sense of Griffiths is equivalent to $\pi^* h$ being positively curved in the line bundle sense, ([G]).

Remark 1. It might very well happen that $\mathcal{O}_{\mathbb{P}(E)}(1)$ can be equipped with a positively curved metric that does not stem from a metric on E to begin with. The Griffiths conjecture, ([G]), says that E then can be equipped with a Griffiths positive metric. This has been shown to be true on Riemann surfaces by Umemura [U], and Campana and Flenner [CF], but the general case is still unresolved.

The following useful and important properties hold in the Griffiths context.

Proposition 9.1. *Let $(E, h) \to X$ be a hermitian, holomorphic vector bundle over a complex manifold X.*

(i) Θ^h is Griffiths positive if and only if the dual metric h^{-1} on E^* is negatively curved in the sense of Griffiths.

(ii) Θ^h is Griffiths negative if and only if for any holomorphic section u to E,

$$\log \|u\|_h^2,$$

is plurisubharmonic.

We will prove (i) in the setting of real metrics in Paper IV, (Proposition 2.1; see e.g. [B2] section 2 for a complex proof), and (ii) will be discussed in Paper I. Both these properties are of fundamental importance in the latter paper.

Now, in contrast to the algebro-geometric origins of Griffiths positivity, curvature in the sense of Nakano is an analytic concept directly connected to the existence of solutions to the inhomogenous $\bar{\partial}$-equation, for form-valued sections of vector bundles. Namely, if one reworks the proof of Theorem 8.1 in the vector bundle setting, strict Nakano positivity will be the necessary replacement of the positivity condition (8.1). In fact, we have the following vector-bundle version of Theorem 8.1.

Theorem 9.2. *Let (X, ω) be a compact Kähler manifold, and let (E, h) be a hermitian, holomorphic vector bundle over X. Assume that the metric h is strictly positively curved in the sense of Nakano,*

$$i\Theta \geq_{Nak.} \varepsilon\omega \otimes I, \tag{9.3}$$

for some $\varepsilon > 0$.

For any $\bar{\partial}$-closed (n, q)-form f with values in E and $q \geq 1$, there exists an $(n, q-1)$-form u with values in E, such that

$$\bar{\partial}u = f,$$

and

$$\int_X \|u\|^2 dV_\omega \leq \frac{1}{\varepsilon q}\int_X \|f\|^2 dV_\omega. \tag{9.4}$$

Apart from the L^2-estimates, Theorem 9.2 is known as the *Nakano vanishing theorem*, [N2]. In the first part of Paper II we show that this theorem can be proven using the $\partial\bar{\partial}$-Bochner-Kodaira method, introduced by Siu in [S], which we believe to be much simpler than the traditional proofs.

Being so closely related to the inhomogenous $\bar{\partial}$-equation, Nakano positivity is of fundamental importance for the analytical study of holomorphic vector bundles. Unfortunately, it is difficult to obtain an intuitive understanding of this concept, and in contrast to the Griffiths setting, there are not many nice functorial properties. For example, the dual of a Nakano positive vector bundle, in general, is not Nakano negative.

We have already noted that Nakano positivity implies Griffiths positivity. In the other direction we have the following important theorem due to Demailly and Skoda, ([DS]; see also Paper IV, Theorem 2.2).

Theorem 9.3. *Let $(E, h) \to X$ be a hermitian, holomorphic vector bundle over a complex manifold X. If h is positively curved in the sense of Griffiths, then $h \det h$ is positively curved in the sense of Nakano.*

10. Summary of papers

We end the introduction with a brief summary of the papers.

10.1. Paper I: Singular hermitian metrics on holomorphic vector bundles. In section 8 we introduced singular metrics on holomorphic line bundles and the Demailly-Nadel vanishing theorem, and identified these as fundamental tools in the analytic study of complex geometry. In Paper I, we wanted to investigate whether anything similar existed for holomorphic vector bundles.

Up until now, a hermitian metric $\tilde{h}$ on a holomorphic vector bundle $E \to X$ has been a positive definite hermitian inner product $(\cdot,\cdot)_{\tilde{h}(x)}$ on each fiber E_x, that varies smoothly with x. Assume now that we drop the smoothness assumption and introduce *singular hermitian metrics* on holomorphic vector bundles, as just measurable maps from the base space to the space of positive definite hermitian forms on the fiber. What can be said about these?

Let h denote such a singular metric. In the line bundle setting, h is locally just a function, and so the connection matrix and the curvature can be written as $\theta = h^{-1}\partial h = \partial \log h$ and $\Theta = \bar{\partial}\theta = \bar{\partial}\partial \log h$. Hence, all that is needed for these concepts to be well-defined in the sense of currents is that $\log h \in L^1_{loc}$. In the vector bundle setting, h is matrix-valued, which makes the situation much more complicated.

Now although we can not define the curvature tensor of h immediately, we can nevertheless still define what it means for a singular hermitian metric to be positively and negatively curved in the sense of Griffiths. This is due to the equivalent characterisation of Griffiths negativity of Proposition 9.1 (ii), which does not require any regularity from h, and property (i). It turns out that this definition of Griffiths curvature rules out most of the possible pathological examples of singular hermitian metrics, (Paper I, Proposition 1.1).

The main question that we wanted to investigate in Paper I was:

Given a singular hermitian metric h on a holomorphic vector bundle $E \to X$ with $\mathrm{rank}_{\mathbb{C}} E \geq 2$, where h is curved in the sense of Griffiths as in Proposition 9.1, is it possible to define θ, and in particular Θ, in a meaningful way; for example as currents with measure coefficients?

In Paper I, Proposition 1.2, the current ∂h is shown to be locally L^2-valued and $\theta := h^{-1}\partial h$ an a.e. welldefined matrix of $(1,0)$-forms.

For the curvature, however, the situation turns out to be more involved. In Paper I, Theorem 1.3, we give a simple example which shows that $\Theta := \bar{\partial}(h^{-1}\partial h)$ can not be defined everywhere as a current with measure coefficients. Thus Griffiths curvature in the sense of Proposition 9.1, is not enough to define the curvature in general.

In the example of Theorem 1.3, the set of points that cause problems is the singular locus, $\{\det h = 0\}$, of the metric. Hence, the natural thing to investigate next is if it is possible to define the curvature outside of this set; (it is an immediate consequence of Paper I, Proposition 1.2, (ii) that the singular locus has Lebesgue measure zero). In Paper I, Theorem 1.4, we show that it is indeed possible to define the curvature as a current with measure coefficients outside of the singular locus.

In particular, it is now possible to define what it means for a singular hermitian metric to be *strictly* positively curved in the sense of Griffiths and Nakano, outside of the singular locus. Since the curvature assumptions needed in order to solve the inhomogenous $\bar{\partial}$-equation with Hörmander's L^2-estimates, only depend on the absolutely continuous part of the curvature, the first ingredient needed in order to prove Demailly-Nadel type of vanishing theorems on vector bundles, can hence be achieved.

The second ingredient that is needed are regularisation results; it is of utmost importance to be able to approximate the strictly positively curved singular metric with a sequence of smooth metrics, while keeping the strict positivity. We end Paper I with showing that such regularisations are possible for strictly Griffiths positive and negative, and Nakano negative singular hermitian metrics. We discuss these approximation results and the resulting vanishing theorem in the summary of Paper II.

10.2. Paper II: The Nakano vanishing theorem and a vanishing theorem of Demailly-Nadel type for holomorphic vector bundles. As mentioned in the introduction, the first part of Paper II is devoted to proving the Nakano vanishing theorem with Hörmader type L^2-estimates, (Theorem 9.2), using Siu's so called $\partial\bar{\partial}$-Bochner-Kodaira method, ([S]).

In Riemannian geometry the basic idea behind the Bochner method is (very vaguely) to calculate the Laplacian of the norm of forms. Then one can draw conclusions about the geometry by carefully analyzing the resulting expression and putting restrictions on the curvature of the metric. The straightforward adaptation of this method in our complex setting would then be to calculate and analyze

$$\Delta\|\alpha\|^2 \tag{10.1}$$

where α is an E-valued, (n,p)-form. However, it turns out that this approach will not work out well and so the historical approach to the vanishing theorem has been through the Kähler identities.

What Siu demonstrates in [S], (among other things), is that if the metric is dually, negatively curved in the sense of Nakano, an approach that is very similar to the classical Bochner method can be applied. The main idea is to let the E-valued $(0,q)$-form α remain form-valued, replace Δ by $i\partial\bar{\partial}$ and calculate

$$i\partial\bar{\partial}c_q\{\alpha,\alpha\}\wedge\omega^{n-q-1}/(n-q-1)!$$

instead of (10.1).

In [B1] Berndtsson shows that in the line bundle case, this method can be applied directly, without resorting to dual bundles, and he also derives the Hörmander L^2-estimates. Here the situation is slightly more involved. Let (L,ϕ) be a positively curved line bundle over X and let α be an (n,p)-form with values in L. It turns out that the appropriate counterpart of (10.1) in this case is

$$i\partial\bar{\partial}c_{n-p}\gamma_\alpha\wedge\gamma_\alpha\wedge\omega^{p-1}e^{-\phi}/(p-1)!$$

where γ_α, (up to a constant), is the Hodge-$*$ of α, i.e. an L-valued $(n-p,0)$-form such that

$$\alpha = \gamma_\alpha \wedge \omega^p/p!.$$

The first aim of Paper II is to show that this latter approach works almost without change for forms with values in a vector bundle, thereby proving Theorem 9.2.

In the second part of Paper II, we return to the singular hermitian metrics of Paper I. As mentioned previously, in order to extend the Demailly-Nadel vanishing theorem to vector bundles, we need to define what it means for the singular hermitian metric to be strictly positively curved in the sense of Nakano, (at least a.e. with respect to Lebesgue measure), and we must also be able to approximate the metric with a sequence of smooth metrics, while keeping the strict positivity.

In Paper I we show that it is possible to obtain a regularising sequence when the metric is strictly negatively curved in the sense of Griffiths, (Paper I, Proposition 6.1). Through duality, (Proposition 9.1, (i)), it then follows that this can be obtained in the strictly Griffiths positive case as well. In Paper I we also prove a similar approximation result for singular hermitian metrics that are strictly negatively curved in the sense of Nakano, (Paper I, Proposition 1.6). However, as the dual of a Nakano negative metric, in general is not Nakano positive, the same trick can not be applied here.

Both regularisation results of Paper I are based on alternative characterisations of Griffiths and Nakano negativity in terms of some plurisubharmonic function. For Nakano positive metrics, such an alternative characterisation does not exist and so some other approach to regularisation is needed. Unfortunately, despite many efforts, we have so far failed to find any succeful way to achieve this.

When $n = 1$, i.e. for vector bundles over Riemann surfaces, the concepts of Griffiths and Nakano positivity coincide. Hence, using the regularisation result of Paper I, in the second part of Paper II we prove a Demailly-Nadel type of vanishing theorem for holomorphic vector bundles over Riemann surfaces, (Paper II, Theorem 1.2).

10.3. Paper III: Extensions of Ohsawa-Takegoshi type for sections of a vector bundle. The extension theorem of Ohsawa and Takegoshi, which first appeared in [OT], is a very useful tool in complex analysis, with a lot of applications. This theorem has many different variants, one of the most basic being the so called *adjunction version*. This version states the following.

Let X be a compact Kähler manifold and let S be a smooth hypersurface in X. S then defines a line bundle on X, which we will denote by (S) and which has a global holomorphic section s such that $S = s^{-1}(0)$. Also let L be a complex line bundle over all of X. Assume that the line bundles L and (S) have smooth metrics ϕ and ψ respectively, satisfying the curvature assumptions

$$i\partial\bar{\partial}\phi \geq 0, \tag{10.2}$$

and

$$i\partial\bar{\partial}\phi \geq \delta i\partial\bar{\partial}\psi, \tag{10.3}$$

for some $\delta > 0$. Assume furthermore that s is normalized so that

$$|s|^2 e^{-\psi} \leq e^{-1/\delta}.$$

Finally let u be a global holomorphic section of $K_S + L|_S$.

Then there exists a global holomorphic section U of $K_X + (S) + L$ such that

$$U = ds \wedge u$$

on S and such that U satsfies the estimate

$$\int_X c_n U \wedge \bar{U} e^{-\phi-\psi} \leq C \int_S c_{n-1} u \wedge \bar{u} e^{-\phi}$$

for some constant C, where we use the shorthand notation $c_p := i^{p^2}$.

Hence, we see that just as in Hörmander's theorem on the solvability of the inhomogenous $\bar{\partial}$-equation, (Theorem 8.1), the Ohsawa-Takegoshi extension theorem consists of two parts: One which states that an extension is possible, and a second part which gives an L^2-estimate for the extension. Just as with Hörmander's theorem it is mainly this estimate, (with a completely universal constant), that makes the theorem so useful; (see e.g. [D1], Chapter VIII for some applications).

There are many different ways of proving this extension theorem, but basically all of them are rather involved. The approach that we are interested in is the one introduced by Berndtsson in [B4], where he shows that finding an extension with L^2-estimates is equivalent to solving the inhomogenous $\bar{\partial}$-equation,

$$\bar{\partial} v = u \wedge [S],$$

where $[S]$ is the current of integration on S. What makes the analysis involved in this proof, is that the right hand side no longer is an L^2-valued differential form, but a current. In [B1], Lecture 6, it is shown that a modified version of the $\partial\bar{\partial}$-Bochner-Kodaira method can be applied.

Having studied the $\partial\bar{\partial}$-Bochner-Kodaira method extensively in Paper II, our main goal in Paper III is to use it to prove a vector bundle version of the extension theorem. This is achieved in Paper III, Theorem 1.1 and 1.2, and the proof of these theorems constitute the main part of the paper.

After the publication of [OT], Ohsawa extended the theorem in different directions in a long series of papers. In one of these papers, [O2], he obtains a result which shares some similarities to our extension theorems, although the formulation is quite different from ours, ([O2] Theorem 4). We believe that our compact Kähler setting is slightly more general, as [O2] Theorem 4 only treats complex manifolds that become Stein after removing a closed subset. The main difference, however, lies in our methods of proof. We consider our adaptation of the $\partial\bar{\partial}$-Bochner-Kodaira method to the vector bundle setting to be our main originality. Furthermore, Guan and Zhou have recently proven a much more general version of the extension theorem, and also managed to determine the optimal constant in the L^2-estimate, ([GZ], Theorem 2.1).

Now in section 9 we introduced the so called Serre line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over the projective bundle $\mathbb{P}(E)$, associated with a given vector bundle $E \to X$. We also showed that this line bundle, in some sense, contains all the information about E. Hence, a common method when one wants to generalize a result that is already known for line bundles to vector bundles, is to study $\mathcal{O}_{\mathbb{P}(E)}(1) \to \mathbb{P}(E)$, instead of $E \to X$. This can be done for the Ohsawa-Takegoshi extension theorem as well.

This approach is interesting in our case since it turns out that the curvature assumptions needed for our vector bundle versions of the extension theorem, (i.e. the vector bundle replacements of (10.2)-(10.3)), require positivity in the sense of Nakano, which is a very strong requirement. Thus, it is natural to inquire about the relation between our vector bundle assumptions, and the curvature assumptions in the Serre line bundle setting.

In the last part of Paper III, we show that the curvature assumptions in the Serre line bundle setting, imply the vector bundle conditions. Hence, although being curved in the sense of Nakano is a strong condition to impose on a metric, the conditions that arise when one reduces the problem to line bundles are in fact even stronger. A key ingredient in proving these implications is Theorem 9.3 by Demailly and Skoda.

10.4. Paper IV: Log concavity for matrix-valued functions and a matrix-valued Prékopa theorem. When we introduced plurisubharmonic functions in section 5, we mentioned that they are the complex-analytic counterparts of convex functions in real analysis. Using this analogy, it is not too difficult to show that a positively curved metric on a trivial line bundle, is nothing but the complex version of a log concave function.

In Paper IV we turn this analogy around and extend it to trivial vector bundles, i.e. we introduce two new 'convexity' notions for real, matrix-valued functions, corresponding to Griffiths and Nakano positivity in the complex-analytic setting. We call these being log concave in the sense of Griffiths and Nakano. In the first part of the paper we study some examples and investigate the fundamental properties of these new concepts; (these turn out to be very similar to the basic complex properties introduced in section 9).

For log concave functions an important result that is closely related to the Brunn-Minkowski inequality is the following theorem due to Prékopa, ([P]).

Theorem 10.1. *Let $\varphi : \mathbb{R}^m_t \times \mathbb{R}^n_y \to \mathbb{R}$ be convex and define $\tilde{\varphi} : \mathbb{R}^m \to \mathbb{R}$ through*

$$e^{-\tilde{\varphi}(t)} = \int_{\mathbb{R}^n} e^{-\varphi(t,y)} dV(y).$$

Then $\tilde{\varphi}$ is convex.

Just as for the Brunn-Minkowski inequality, Prékopa's theorem can be proven in many different ways, each pointing towards various directions of generalisations. One of these proofs, due to Brascamp and Lieb [BL], is based on a weighted Poincaré inequality, which in fact turns out to be a real variable version of the Hörmander L^2-estimates for the inhomogenous $\bar{\partial}$-equation, (see e.g. [B1], section 1.3). Hence, it is quite natural to ask if there exist any corresponding complex variants of the Prékopa theorem.

This question has been extensively studied in recent years, mainly by Berndtsson, who in a series of papers has obtained complex analytic counterparts of the Prékopa theorem, with gradually increasing generality. In their most general form, these are theorems on the curvature properties of certain infinite rank holomorphic vector bundles associated with holomorphic fibrations, ([B2], Theorem 1.1 and 1.2). We will not describe these results and their relation to the Brunn-Minkowski and Prékopa theorem here, but refer the reader to [B5], sections 2 and 3.

Now after we introduce the notions of Griffiths and Nakano log concavity for matrix-valued functions in the first part of Paper IV, we proceed to show a matrix-valued Prékopa theorem in the second part, (Paper IV, Theorem 1.2). The main idea behind the proof of this theorem is to generalize one of the above mentioned complex-analytic Prékopa theorems of Berndtsson, (Paper IV, Theorem 1.5), and then recast this theorem in the real variable setting. This latter reformulation is achieved through a weighted, vector-valued Paley-Wiener type of theorem, (Paper IV, Theorem 1.4), and the proof of this theorem and Paper IV, Theorem 1.5, constitutes the main bulk of the second part of Paper IV.

REFERENCES

[A] M. Andersson, *Topics in complex analysis*, Universitext, Springer-Verlag, New York, 1997.

[B1] B. Berndtsson, *An introduction to things $\overline{\partial}$*, Analytic and algebraic geometry, IAS/Park City Math. Ser., vol. 17, Amer. Math. Soc., Providence, RI, 2010, pp. 7–76.

[B2] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2) **169** (2009), no. 2, 531–560.

[B3] B. Berndtsson, *L^2-methods for the $\bar{\partial}$-equation*, Kass University Press, CTH (1995).

[B4] B. Berndtsson, *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 4, 1083–1094.

[B5] B. Berndtsson, *The openness conjecture and complex Brunn-Minkowski inequalities* (2014), Preprint, available at `arXiv:1405.0989[math.CV]`.

[B6] E. Bombieri, *Algebraic values of meromorphic maps*, Invent. Math. **10** (1970), 267–287.

[BL] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), no. 4, 366–389.

[BT] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982.

[CF] F. Campana and H. Flenner, *A characterization of ample vector bundles on a curve*, Math. Ann. **287** (1990), no. 4, 571–575.

[D1] J-P. Demailly, *Complex Analytic and Differential Geometry*, Monograph, available at `http://www-fourier.ujf-grenoble.fr/~demailly`.

[D2] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 457–511 (French).

[D3] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992, pp. 87–104.

[DS] J.-P. Demailly and H. Skoda, *Relations entre les notions de positivités de P. A. Griffiths et de S. Nakano pour les fibrés vectoriels*, Séminaire Pierre Lelong-Henri Skoda (Analyse). Années 1978/79 (French), Lecture Notes in Math., vol. 822, Springer, Berlin, 1980, pp. 304–309 (French).

[G] P. A. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 185–251.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[GZ] Q. Guan and X. Zhou, *A solution of an L2 extension problem with optimal estimate and applications* (2013), Preprint, available at `arXiv:1310.7169[math.CV]`.

[H1] L. Hörmander, *L^2 estimates and existence theorems for the $\bar\partial$ operator*, Acta Math. **113** (1965), 89–152.

[H2] L. Hörmander, *An introduction to complex analysis in several variables*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[H3] L. Hörmander, *Notions of convexity*, Progress in Mathematics, vol. 127, Birkhäuser Boston, Inc., Boston, MA, 1994.

[H4] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005. An introduction.

[K1] C. O. Kiselman, *Plurisubharmonic functions and potential theory in several complex variables*, Development of mathematics 1950–2000, Birkhäuser, Basel, 2000, pp. 655–714 (English, with English and Esperanto summaries).

[K2] M. Klimek, *Pluripotential theory*, London Mathematical Society Monographs. New Series, vol. 6, The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.

[K3] K. Kodaira, *Complex manifolds and deformation of complex structures*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 283, Springer-Verlag, New York, 1986.

[K4] K. Kodaira, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 1268–1273.

[K5] K. Kodaira, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Ann. of Math. (2) **60** (1954), 28–48.

[L1] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Vol. 48, Springer-Verlag, Berlin, 2004.

[L2] J. M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003.

[M] S. Morita, *Geometry of differential forms*, Translations of Mathematical Monographs, vol. 201, American Mathematical Society, Providence, RI, 2001.

[N1] A. M. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596.

[N2] S. Nakano, *On complex analytic vector bundles*, J. Math. Soc. Japan **7** (1955), 1–12.

[N3] F. Nazarov, *The Hörmander proof of the Bourgain-Milman theorem*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2050, Springer, Heidelberg, 2012, pp. 335–343.

[O1] T. Ohsawa, *Analysis of several complex variables*, Translations of Mathematical Monographs, vol. 211, American Mathematical Society, Providence, RI, 2002.

[O2] T. Ohsawa, *On the extension of L^2 holomorphic functions. V. Effects of generalization*, Nagoya Math. J. **161** (2001), 1–21.

[OT] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), no. 2, 197–204.

[P] A. Prékopa, *On logarithmic concave measures and functions*, Acta Sci. Math. (Szeged) **34** (1973), 335–343.

[R] R. M. Range, *Complex analysis: a brief tour into higher dimensions*, Amer. Math. Monthly **110** (2003), no. 2, 89–108.

[S] Y. T. Siu, *Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems*, J. Differential Geom. **17** (1982), no. 1, 55–138.

[SKKT] K. E. Smith, L. Kahanpää, P. Kekäläinen, and W. Traves, *An invitation to algebraic geometry*, Universitext, Springer-Verlag, New York, 2000.

[SS] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, II, Princeton University Press, Princeton, NJ, 2003.

[U] H. Umemura, *Moduli spaces of the stable vector bundles over abelian surfaces*, Nagoya Math. J. **77** (1980), 47–60.

[W1] R. O. Wells Jr., *Differential analysis on complex manifolds*, 3rd ed., Graduate Texts in Mathematics, vol. 65, Springer, New York, 2008.

[W2] H. Whitney, *Differentiable manifolds*, Ann. of Math. (2) **37** (1936), no. 3, 645–680.