Explicit expressions for two optimal control problems

HÅKAN JOHANSSON

Department of Applied Mechanics
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg, Sweden 2014
Explicit expressions for two optimal control problems

Håkan Johansson*

Department of Applied Mechanics, Chalmers University of Technology

SUMMARY
In this report explicit expressions relevant to goal-oriented a posteriori error analysis of two optimal control problems are given. The first problem considers the trajectory of a particle (can be interpreted as a lane change manoeuver in vehicle dynamics) and the second problem is a double pendulum (can be viewed as lifting of an arm). This paper is to be considered as supporting material and is not a self-contained complete paper.

KEY WORDS: particle trajectory, movement planning, supplementary material

1. Purpose

The purpose of this paper is to collect a set of explicit expressions to support an intended publication regarding a posteriori error estimation for optimal control problem, [1]. Hence, motivations and definitions are sometimes omitted, for instance the definition of relevant function spaces. As the solution method used in [1] involves a lot of rather straightforward differentiations, that would be lengthy in a journal paper these are collected here instead. Differentiation of the abstract forms involved are made as Gâteaux-derivatives, and the following notation will be used:

\[
A'(\bullet, u; \delta u) \overset{\text{def}}{=} \lim_{\epsilon \to 0} \frac{A(\bullet, u + \delta u \epsilon) - A(\bullet, u)}{\epsilon}.
\]

2. General format of optimal control problem

We consider the steering of a mechanical system (without feedback) described by the following state equation

\[
M(u)\ddot{u}(t) + J(u(t), t) = f(p(t), t), \quad t \in [0, T]
\]

(1)

where \( u \) is a vector-valued collection of \( N \) state variables, matrix \( M(u) \) and internal force vector \( J(u(t), t) \) define the mechanical system and \( f(p(t), t) \) is forces emanating from external sources.

*Correspondence to: H. Johansson, Department of Applied Mechanics, Chalmers University of Technology, 412 96 Göteborg, Sweden. E-mail: hakan.johansson@chalmers.se
excitation determined by the $K$ control(s) $p(t)$. We here consider mechanical systems consisting of rigid bodies assembled at joints, although the pertinent formulation can be extended to the situation of elastic bodies. The state $u$ must satisfy boundary conditions determined by desired initial and target (end) configuration of the system as follows

$$u(0) = u_0, \quad u(T) = u_T.$$  \tag{2}

The desired control $p(t)$ is found as the minimizer to the scalar performance measure (objective functional)

$$F(p, u).$$  \tag{3}

In addition, $p$ and $u$ must satisfy $M$ inequality constraints on the form

$$g(p, u) \leq 0, \quad t \in [0, T].$$  \tag{4}

In summary, the optimal control problem can be formulated as follows: Determine the (vector-valued) control $p(t)$ that minimizes the objective function (3) while satisfying the state equation (1), the boundary condition (2) and the (nonlinear) inequality constraint (4).

Upon solving the optimal control problem using a Finite Element approximation, we shall consider a set of measures of the solution, so-called goal functions $Q(p, u)$, of engineering interest in which the effect of discretization errors is to be estimated.

2.1. Derivatives of state equation

The state equation is given on weak form as

$$A(u; v) = L(p; v) \quad \forall v \in V \tag{5}$$

$$A(u; v) \overset{\text{def}}{=} \int_0^T v^T [M(u)\dot{u} + J(u, t)] \, dt, \quad L(p; v) \overset{\text{def}}{=} \int_0^T v^T f(p, t) \, dt \tag{6}$$

Differentiation wrt $u$ gives

$$A'_u(u; v, \delta u) = \int_0^T v^T [M(u)\delta \dot{u} + (M'_u\delta u)\dot{u} + J'_u\delta u] \, dt, \quad L'_p(p; v, \delta p) = \int_0^T v^T f'_p\delta p \, dt \tag{7}$$

and

$$A''_{uu}(u; v, \delta u_1, \delta u_2) = \int_0^T (\delta u_2)^T v^T M'_u\delta \dot{u}_1 + (\delta \dot{u}_2)^T v^T M'_u\delta u_1 + (\delta u_2)^T v^T M''_{uu}\delta \dot{u}_1 + (\delta \dot{u}_2)^T v^T J''_{uu}\delta u_1 \, dt \tag{8}$$

$$L''_{pp}(p; v, \delta p, \delta p) = \int_0^T (\delta p)^T v^T f''_{pp}\delta p \, dt \tag{9}$$

where $M'_u, J''_{uu}$ and $f''_{pp}$ are 3-dimensional arrays with suitable indexing, and $M''_{uu}$ as a 4-dimensional array.

2.2. Derivatives of equality constraints

The target condition is enforced weakly as an equality constraint

$$h(u; v^h) = (v^h)^T (u(T) - u_T) = 0 \quad \forall v^h \in \mathbb{R}^N \tag{10}$$

differentiation of $h$ w.r.t $u$ gives

$$h(u; v^h, \delta u) = (v^h)^T \delta u(T) \tag{11}$$

and second derivatives are zero.
2.3. Derivatives of equality constraints

The inequality constraint is enforced weakly as
\[ g(p, u; v^e) = \int_0^T [v^e]^T g(p, u) \, dt \leq 0 \]  
(12)
differentiation of \( g \) w.r.t \( p \) and \( u \) gives
\[ g'_p(p, u; v^e, \delta p) = \int_0^T (v^e)^T g_p(p, u) \delta p \, dt \]
(13)
\[ g''_p(p, u; v^e, \delta p_1, \delta p_2) = \int_0^T (\delta p_2)^T (v^e)^T g_{pp}(p, u) \delta p_1 \, dt \]
(14)
\[ g'_{pu}(p, u; v^e, \delta p, \delta u) = \int_0^T (\delta p)^T (v^e)^T g'_{pu}(p, u) \delta u \, dt \]
(15)
\[ g''_{uu}(p, u; v^e, \delta u_1, \delta u_2) = \int_0^T (\delta u_2)^T (v^e)^T g''_{uu}(p, u) \delta u_1 \, dt \]
(16)
Where \( g_{pp}, g'_{pu}, g''_{uu} \) can be defined as 3-dimensional arrays.

2.3.1. Penalty formulation  
A straightforward manner to treat the inequality constraints is to introduce a suitable penalty formulation, where a violation of the constraints is given a cost that is added to the objective function\(^\dagger\)

\[ F(p, u) := F(p, u) + F_{pen} \]
(17)
where \( a_1, a_2 \) and \( \epsilon \) are problem-dependant "tuning" parameters for the penalty term. This will give the following expressions for additions to \( F \)
\[ (F_{pen})'_p(p, u; \delta p) = a_1 a_2 \sum_i \int_0^T (\max([0, g_i(p, u) - \epsilon])^{(a_2-1)} \delta g'_p(p, u) \delta p \, dt \]
(18)
\[ (F_{pen})'_u(p, u; \delta u) = a_1 a_2 \sum_i \int_0^T (\max([0, g_i(p, u) - \epsilon])^{(a_2-1)} \delta g'_u(p, u) \delta u \, dt \]
(19)
\[ (F_{pen})''_{pp}(p, u; \delta p_1, \delta p_2) = a_1 a_2 \sum_i \int_0^T (\max([0, g_i(p, u) - \epsilon])^{(a_2-1)} \delta g''_{pp}(p, u) \delta p_1 \, dt \]
(20)
\[ (\delta p_2)^T [(g'_p)^2 + (a_2-1)(\max([0, g_i(p, u) - \epsilon])^{(a_2-2)} g''_{pp}) \delta p_1 \, dt \]
(21)

\(^\dagger\)A penalty formulation for the equality condition \( h(u) \) is also possible along the same lines, but is not further considered here.
\[(\mathcal{F}^{\text{pen}})_{u_p} (p, u; \delta p, \delta u) = a_1 a_2 \sum_i^M \int_0^T \max[0, g_i (p, u) - \epsilon])^{(a_2 - 1)} \]

\[(\delta p)^T \left[ \left(g_i \right)_p^u \left(\frac{g_i}{p} \right)_u + (a_2 - 1) \max[0, g_i (p, u) - \epsilon])^{(a_2 - 2)} \right] \delta u \ dt \quad (23)\]

\[(\mathcal{F}^{\text{pen}})_{u_u} (p, u; \delta u_1, \delta u_2) = a_1 a_2 \sum_i^M \int_0^T \max[0, g_i (p, u) - \epsilon])^{(a_2 - 1)} \]

\[\left(\delta u_2\right)^T \left[ \left(\frac{g_i}{u} \right)_u^2 + (a_2 - 1) \max[0, g_i (p, u) - \epsilon])^{(a_2 - 2)} \right] \delta u_1 \ dt \quad (25)\]

3. Trajectory control

As a first application, we shall consider a particle trajectory, inspired by a vehicle dynamics problem described in [2]. A particle of unit mass travels in the 2D-plane with external force acting as control, i.e.

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{v}_x \\
\dot{v}_y
\end{bmatrix} = 
\begin{bmatrix}
v_x \\
v_y \\
0 \\
0
\end{bmatrix} - 
\begin{bmatrix}
0 \\
0 \\
F_x \\
F_y
\end{bmatrix}
\]

\quad (26)

with initial condition \(u_0 = [0, 10, 30, 0]^T\) and target condition \(u_T = [100, 0, 20, 0]^T\) to be reached at final time \(T = 4s\). In addition, we have inequality constraints.

\[g(p, u) = 
\begin{bmatrix}
(F_x^2 + F_y^2) - (\mu g)^2 \\
v_{\text{min}} - v_x \\
1 - ((x - x_{p1})/x_{L1})^{N_1} + ((y - y_{p1})/y_{L1})^{N_1} \\
1 - ((x - x_{p2})/x_{L2})^{N_2} + ((y - y_{p2})/y_{L2})^{N_2} \\
\vdots
\end{bmatrix}
\]

\quad (27)

where in the first constraint \(\mu g\) represents a maximum available force (due to friction between vehicle and ground), \(v_{\text{min}}\) is a prescribed minimum speed in \(x\)-direction (to avoid loops), and a number of hyper-ellipsoid obstacles defined by center points \((x_{p1, y_{p1}})\), size \((x_{L1, y_{L1}})\) and exponent \(N_i\) the shape of obstacles \((N = 2\) for circular obstacles). The first two constraints are convex whereas the latter are not.

3.1. Derivatives of state equation

By defining \(u\) as vector of state variables \(u = [x, y, v_x, v_y]^T\) and extracting from (26) we identify

\[M = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad J = 
\begin{bmatrix}
-u_3 \\
-u_4 \\
0 \\
0
\end{bmatrix}, \quad f = 
\begin{bmatrix}
0 \\
0 \\
F_x \\
F_y
\end{bmatrix}
\]

\quad (28)
we obtain as derivatives

\[ M'\mathbf{u} = 0, \quad J'\mathbf{u} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad f'_p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \] (29)

and second derivatives are zero.

### 3.2. Objective functions

To allow for analysis of trade-off between deviation from desired route \((y = 0\) is a desired route) and forces and forces rates, we express the objective function as

\[ \mathcal{F} = \gamma_1 \int_0^T F_x^2 dt + \gamma_2 \int_0^T F_y^2 dt + \gamma_3 \int_0^T x^2 dt + \gamma_4 \int_0^T y^2 dt + \gamma_5 \int_0^T v_x^2 dt + \gamma_6 \int_0^T v_y^2 dt \] (30)

where \(\gamma_1 - \gamma_6\) are parameters to be studied.

\[ \mathcal{F}_p'(\mathbf{p}, \mathbf{u}; \delta \mathbf{p}) = 2\gamma_2 \int_0^T F_x \delta F_x dt + 2\gamma_3 \int_0^T F_y \delta F_y dt + 2\gamma_4 \int_0^T \dot{F}_x \delta \dot{F}_x dt + 2\gamma_5 \int_0^T \dot{F}_y \delta \dot{F}_y dt \] (31)

\[ \mathcal{F}_u'(\mathbf{p}, \mathbf{u}; \delta \mathbf{u}) = 2\gamma_1 \int_0^T \dot{y} \delta y dt \] (32)

\[ \mathcal{F}_p''(\mathbf{p}, \mathbf{u}; \delta \mathbf{p}, \delta \mathbf{p}) = 2\gamma_2 \int_0^T \delta F_x \delta F_x dt + 2\gamma_3 \int_0^T \delta F_y \delta F_y dt + 2\gamma_4 \int_0^T \delta \dot{F}_x \delta \dot{F}_x dt + 2\gamma_5 \int_0^T \delta \dot{F}_y \delta \dot{F}_y dt \] (33)

\[ \mathcal{F}_{uu}''(\mathbf{p}, \mathbf{u}; \delta \mathbf{u}, \delta \mathbf{u}) = 2\gamma_1 \int_0^T \delta \dot{y} \delta y dt \] (34)

and \(\mathcal{F}_{pu}''(\mathbf{p}, \mathbf{u}; \delta \mathbf{p}, \delta \mathbf{u}) = 0\)

### 3.3. Derivatives of inequality constraints

\[ g(\mathbf{p}, \mathbf{u}) = \begin{bmatrix} (F_x^2 + F_y^2) - (\mu g)^2 \\ v_{\text{min}} - v_x \\ 1 - ((x - x_{p1})/x_{L1})^{N_1} - ((y - y_{p1})/y_{L1})^{N_1} \\ 1 - ((x - x_{p2})/x_{L2})^{N_2} - ((y - y_{p2})/y_{L2})^{N_2} \\ \vdots \end{bmatrix} \] (35)

Derivatives of first constraint

\[ (g_1)'_p = \begin{bmatrix} 2F_x \\ 2F_y \end{bmatrix}, \quad (g_1)'_u = 0 \] (36)

\[ (g_1)''_p = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad (g_1)''_u = 0, \quad (g_1)''_{u,u} = 0 \] (37)
Second constraint (minimum speed in x-direction) derivatives

\[
(g_2)'_p = 0, \quad (g_2)'_u = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]  
(38)

\[
(g_2)''_p = 0, \quad (g_2)''_p, u = 0, \quad (g_2)''_u, u = 0
\]  
(39)

Obstacle constraint derivatives

\[
(g_2)''_p = 0, \quad (g_2)''_p, u = 0, \quad (g_2)''_u, u = \begin{bmatrix} -\frac{N_1(N_1-1)}{x_{L_1}}((x - x_p)/(x_{L_1})^{(N_1-2)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]  
(40)

3.4. Goal functions

For the case of trajectory control, the following goal functions were considered

\[
Q_1 = F, \quad Q_2 = x(t_s), \quad Q_3 = y(t_s), \quad Q_4 = v_x(t_s), \quad Q_5 = v_y(t_s)
\]

\[
Q_6 = x^\text{sim}(T), \quad Q_7 = y^\text{sim}(T), \quad Q_8 = v_x^\text{sim}(T), \quad Q_9 = v_y^\text{sim}(T), \quad Q_{10} = \int_0^T v_x F_x + v_y F_y dt
\]  
(41)

where \(x^\text{sim}\) and \(y^\text{sim}\) refers to a subsequent 'exact' simulation where the discrete solution \(p_h\) is used, i.e. \(u^\text{sim}\) solves

\[
A(u^\text{sim}; v) = L(p_h; v), \quad \forall v \in \mathbb{V}^\text{sim}
\]  
(42)

for a very fine mesh \(\mathbb{V}^\text{sim}\). Differentiation of the goal quantities gives:

\[
(Q_1)'_p(p, u; \delta p) = F'_p(p, u; \delta p), \quad (Q_1)'_u(p, u; \delta u) = F'_u(p, u; \delta u)
\]  
(43)

\[
(Q_2)'_p(p, u; \delta p) = 0, \quad (Q_2)'_u(p, u; \delta u) = \delta x(t_s)
\]  
(44)

\[
(Q_3)'_p(p, u; \delta p) = 0, \quad (Q_3)'_u(p, u; \delta u) = \delta y(t_s)
\]  
(45)

\[
(Q_4)'_p(p, u; \delta p) = 0, \quad (Q_4)'_u(p, u; \delta u) = \delta v_x(t_s)
\]  
(46)

\[
(Q_5)'_p(p, u; \delta p) = 0, \quad (Q_5)'_u(p, u; \delta u) = \delta v_y(t_s)
\]  
(47)

\[
(Q_6)'_p(p, u; \delta p) = d_p x^\text{sim}(T), \quad (Q_6)'_u(p, u; \delta u) = 0
\]  
(48)

\[
(Q_7)'_p(p, u; \delta p) = d_p y^\text{sim}(T), \quad (Q_7)'_u(p, u; \delta u) = 0
\]  
(49)

\[
(Q_8)'_p(p, u; \delta p) = d_p v_x^\text{sim}(T), \quad (Q_8)'_u(p, u; \delta u) = 0
\]  
(50)

\[
(Q_9)'_p(p, u; \delta p) = d_p v_y^\text{sim}(T), \quad (Q_9)'_u(p, u; \delta u) = 0
\]  
(51)

\[
(Q_{10})'_p(p, u; \delta p) = \int_0^T v_x \delta F_x + v_y \delta F_y dt, \quad (Q_{10})'_u(p, u; \delta u) = \int_0^T \delta v_x F_x + \delta v_y F_y dt
\]  
(52)

Note: \(d_p u^\text{sim}(T)\) needed for goal functions 6-9. that can be solved for from the tangent equation

\[
A'_u(u^\text{sim}, v, d_p u^\text{sim}) = L'_p(p_h; v, \delta p), \quad \forall v \in \mathbb{V}^\text{sim}.
\]  
(53)
As a second example we consider the control of a double pendulum (inspired by a bio-mechanic modeling of a human arm, cf. [3]). The configuration of the pendulum is described by the two angles $\theta_1$ and $\theta_2$ and is steered by "applied" bending moment $M_1(t)$, $M_2(t)$ in the joints, defining the controls $p = [p_1, p_2]^T = [M_1, M_2]^T$. The inertia of the pendulum is given by three point masses $m_1$-$m_3$, and the links have lengths $l_1$ and $l_2$.

Initial condition corresponds to a vertical pendulum at rest, $u(0) = [0, 0, 0, 0]^T$, and we consider a target condition that the pendulum should be horizontal and at rest, i.e. that $u(T) = [0, 0, \pi/2, \pi/2]^T$. As constraints we have: The controls has some limited capacity, i.e. the allowable bending moments in the joints are restricted as $-7.5 = p_{1,\text{min}} \leq p_1 \leq p_{1,\text{max}} = 20$ and $-10 = p_{2,\text{min}} \leq p_2 \leq p_{2,\text{max}} = 10$. Moreover, the rotation in the second joint is restricted as $0 = \theta_{\Delta,\text{min}} \leq \theta_1 - \theta_2 \leq \theta_{\Delta,\text{max}} = \frac{3}{4}\pi$. The inequality constraints $g(p, u)$ are thus defined as

$$
g(p, u) = \begin{bmatrix}
-p_1 + p_{1,\text{min}} \\
p_1 - p_{1,\text{max}} \\
-p_2 + p_{2,\text{min}} \\
p_2 - p_{2,\text{max}} \\
\theta_2 - \theta_1 + \theta_{\Delta,\text{min}} \\
\theta_1 - \theta_2 - \theta_{\Delta,\text{max}}
\end{bmatrix}
$$

(54)

Given thses constraints, we seek to find the controls such that the objective function measuring the magnitude of moments and moment velocities exerted in joints is minimized

$$
\mathcal{F}(p) = \frac{1}{2} \int_0^T \left[ |p(t)|^2 + \alpha |\dot{p}(t)|^2 \right] dt
$$

(55)

the parameter $\alpha$ is used the weight between the two components (which in a numerical realization acts as a regularization parameter).
4.1. Derivatives of state equation

The Lagrangian function entailing kinematic and potential energy for the 3-mass double pendulum takes the form

\[
\mathcal{L} = l_1^2 \left( \frac{m_1}{4} + m_3 \right) \frac{\dot{\theta}_1^2}{2} + l_2^2 \left( m_1 + \frac{m_2}{4} + m_3 \right) \frac{\dot{\theta}_2^2}{2} + gl_1 \left( \frac{m_1}{2} + m_3 \right) \cos(\theta_1) + gl_2 \left( m_1 + \frac{m_2}{2} + m_3 \right) \cos(\theta_2) + l_1 l_2 \left( \frac{m_1}{2} + m_3 \right) \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)
\]

from which the Euler-Lagrange equations of motion are obtained as

\[
l_1^2 \left( \frac{m_1}{4} + m_3 \right) \ddot{\theta}_1 + l_1 l_2 \left( \frac{m_1}{2} + m_3 \right) \left( \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \right) + gl_1 \left( \frac{m_1}{2} + m_3 \right) \sin(\theta_1) = M_1(t)
\]

\[
l_2^2 \left( m_1 + \frac{m_2}{4} + m_3 \right) \ddot{\theta}_2 + l_1 l_2 \left( \frac{m_1}{2} + m_3 \right) \left( \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \right) + gl_2 \left( m_1 + \frac{m_2}{2} + m_3 \right) \sin(\theta_2) = M_2(t)
\]

We define \( \mathbf{u} \) as vector of state variables \( \mathbf{u} = [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]^T \) from which we identify

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
l_1 l_2(\frac{m_1}{2} + m_3) & l_1 l_2(\frac{m_1}{2} + m_3) \cos(u_1 - u_2) & l_2^2(\frac{m_1}{2} + m_2 + m_3) & 0 \\
0 & 0 & 0 & 0 \\
l_1 l_2(\frac{m_1}{2} + m_3) \sin(u_1 - u_2) & l_1 l_2(\frac{m_1}{2} + m_3) \cos(u_1 - u_2) & l_2^2(\frac{m_1}{2} + m_2 + m_3) & 0
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
\begin{matrix}
-u_3 \\
-u_4
\end{matrix} \\
l_1 l_2(\frac{m_1}{2} + m_3) u_1^2 \sin(u_1 - u_2) + gl_1 \left( \frac{m_1}{2} + m_3 \right) \sin(u_1) \\
-l_1 l_2(\frac{m_1}{2} + m_3) u_2^2 \sin(u_1 - u_2) + gl_2 \left( m_1 + \frac{m_2}{2} + m_3 \right) \sin(u_2)
\end{bmatrix}
\]

\[
f = \begin{bmatrix}
0 \\
0 \\
M_1 \\
M_2
\end{bmatrix}
\]

The derivatives of \( M \) are obtained as

\[
M'_{u_1} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -l_1 l_2(\frac{m_1}{2} + m_3) \sin(u_1 - u_2) \\
0 & 0 & -l_1 l_2(\frac{m_1}{2} + m_3) \sin(u_1 - u_2) & 0
\end{bmatrix}
\]

\[
M'_{u_2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & l_1 l_2(\frac{m_1}{2} + m_3) \sin(u_1 - u_2) \\
0 & 0 & 0 & 0 \\
0 & 0 & l_1 l_2(\frac{m_1}{2} + m_3) \sin(u_1 - u_2) & 0
\end{bmatrix}
\]

\[M'_{u_3, u_4} = M'_{u_3, u_4} = 0\]

8
with all other components of $M''_{uu} = 0$. Furthermore, for derivatives of $J$, we consider each component individually:

$((J)_1)'_u = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^T$, $((J)_1)''_{uu} = 0$ \hspace{1cm} (67)

$((J)_2)'_u = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}^T$, $((J)_2)''_{uu} = 0$ \hspace{1cm} (68)

$((J)_3)'_u = \begin{bmatrix} l_1l_2(m_1/2 + m_3)u_2^2 \cos(u_1 - u_2) + gl_1(m_1/2 + m_3) \cos(u_1) \\ -l_1l_2(m_1/2 + m_3)u_4^2 \cos(u_1 - u_2) \\ 0 \\ 2l_1l_2(m_1/2 + m_3)u_4 \sin(u_1 - u_2) \end{bmatrix}$ \hspace{1cm} (69)

$((J)_3)''_{u_1} u = \begin{bmatrix} -l_1l_2(m_1/2 + m_3)u_2^2 \sin(u_1 - u_2) - gl_1(m_1/2 + m_3) \sin(u_1) \\ l_1l_2(m_1/2 + m_3)u_4^2 \sin(u_1 - u_2) \\ 0 \\ 2l_1l_2(m_1/2 + m_3)u_4 \cos(u_1 - u_2) \end{bmatrix}$ \hspace{1cm} (70)

$((J)_3)''_{u_2} u = \begin{bmatrix} l_1l_2(m_1/2 + m_3)u_2^2 \sin(u_1 - u_2) \\ -l_1l_2(m_1/2 + m_3)u_4^2 \sin(u_1 - u_2) \\ -2l_1l_2(m_1/2 + m_3)u_4 \cos(u_1 - u_2) \end{bmatrix}$ \hspace{1cm} (71)

$((J)_3)''_{uu} u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ \hspace{1cm} (72)
Finally, \( \frac{d^2J}{du^2} \) can be rewritten as:

\[
((J)_4)'' \quad \frac{u}{u_1} \quad \frac{u}{u_2} \quad \frac{u}{u_3} \quad \frac{u}{u_4} = \begin{bmatrix}
2l_1l_2(m_1/2 + m_3)u_4 \cos(u_1 - u_2) \\
-2l_1l_2(m_1/2 + m_3)u_4 \cos(u_1 - u_2) \\
0 \\
2l_1l_2(m_1/2 + m_3) \sin(u_1 - u_2)
\end{bmatrix}
\]

(73)

\[
((J)_4)' \quad \frac{u}{u_1} \quad \frac{u}{u_2} \quad \frac{u}{u_3} \quad \frac{u}{u_4} = \begin{bmatrix}
l_1l_2(m_1/2 + m_3)u_3^2 \cos(u_1 - u_2) \\
-l_1l_2(m_1/2 + m_3)u_3^2 \cos(u_1 - u_2) + gl_2(m_1 + \frac{m_2}{2} + m_3) \cos(u_2) \\
-2l_1l_2(m_1/2 + m_3)u_3 \sin(u_1 - u_2) \\
0
\end{bmatrix}
\]

(74)

\[
((J)_4)'' \quad \frac{u}{u_1} \quad \frac{u}{u_2} \quad \frac{u}{u_3} \quad \frac{u}{u_4} = \begin{bmatrix}
l_1l_2(m_1/2 + m_3)u_3^2 \sin(u_1 - u_2) \\
l_1l_2(m_1/2 + m_3)u_3^2 \sin(u_1 - u_2) \\
2l_1l_2(m_1/2 + m_3)u_3 \cos(u_1 - u_2) \\
0
\end{bmatrix}
\]

(75)

\[
((J)_4)'' \quad \frac{u}{u_1} \quad \frac{u}{u_2} \quad \frac{u}{u_3} \quad \frac{u}{u_4} = \begin{bmatrix}
l_1l_2(m_1/2 + m_3)u_3^2 \sin(u_1 - u_2) - gl_2(m_1 + \frac{m_2}{2} + m_3) \sin(u_2) \\
-2l_1l_2(m_1/2 + m_3)u_3 \cos(u_1 - u_2) + gl_2(m_1 + \frac{m_2}{2} + m_3) \sin(u_2) \\
-2l_1l_2(m_1/2 + m_3)u_3 \cos(u_1 - u_2) \\
0
\end{bmatrix}
\]

(76)

\[
((J)_4)'' \quad \frac{u}{u_1} \quad \frac{u}{u_2} \quad \frac{u}{u_3} \quad \frac{u}{u_4} = \begin{bmatrix}
-2l_1l_2(m_1/2 + m_3)u_3 \cos(u_1 - u_2) \\
2l_1l_2(m_1/2 + m_3)u_3 \cos(u_1 - u_2) - 2l_1l_2(m_1/2 + m_3) \sin(u_1 - u_2) \\
0 \\
0
\end{bmatrix}
\]

(77)

Finally,\( \frac{d^2J}{du^2} \) can be rewritten as:

\[
f' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad f'' = 0
\]

(79)

4.2. Derivatives of inequality constraints

Recalling:

\[
g(p, u) = \begin{bmatrix}
-p_1 + p_{1,min} \\
p_{1} - p_{1,max} \\
p_2 + p_{2,min} \\
p_{2} - p_{2,max} \\
\theta_2 - \theta_1 + \theta_{\Delta,min} \\
\theta_1 - \theta_2 - \theta_{\Delta,max}
\end{bmatrix}
\]

(80)
we have the following component derivatives

\[
(g_1)'_p = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad (g_1)'_u = 0
\]

\[
(g_2)'_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (g_2)'_u = 0
\]

\[
(g_3)'_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad (g_3)'_u = 0
\]

\[
(g_4)'_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (g_4)'_u = 0
\]

\[
(g_5)'_p = 0, \quad (g_5)'_u = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}
\]

\[
(g_6)'_p = 0, \quad (g_6)'_u = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}
\]

Due to its linear nature of the constraints, all second derivatives of \( g \) are zero.

### 4.3. Goal functions

For the case of movement planning, the following goal functions were considered

\[
Q_1 = \mathcal{F}, \quad Q_2 = \theta_1(t_s), \quad Q_3 = \theta_2(t_s), \quad Q_4 = \dot{\theta}_1(t_s), \quad Q_5 = \dot{\theta}_2(t_s)
\]

\[
Q_6 = \theta^\text{sim}_1(T), \quad Q_7 = \theta^\text{sim}_2(T), \quad Q_8 = \dot{\theta}^\text{sim}_1(T), \quad Q_9 = \dot{\theta}^\text{sim}_2(T), \quad Q_{10} = \int_0^T M_1 \dot{\theta}_1 + M_2 \dot{\theta}_2 dt
\]

which are in direct analogy to the trajectory control example above.

### REFERENCES