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Explicit expressions for two optimal control problems

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SUMMARY

In this report explicit expressions relevant to goal-oriented a posteriori error analysis of two optimal control problems are given. The first problem considers the trajectory of a particle (can be interpreted as a lane change manoeuver in vehicle dynamics) and the second problem is a double pendulum (can be viewed as lifting of an arm). This paper is to be considered as supporting material and is not a self-contained complete paper.

KEY WORDS: particle trajectory, movement planning, supplementary material

1. Purpose

The purpose of this paper is to collect a set of explicit expressions to support an intended publication regarding a posteriori error estimation for optimal control problem, [1]. Hence, motivations and definitions are sometimes omitted, for instance the definition of relevant function spaces. As the solution method used in [1] involves a lot of rather straightforward differentiations, that would be lengthy in a journal paper these are collected here instead. Differentiation of the abstract forms involved are made as Gâteaux-derivatives, and the following notation will be used:

$$A'_u(\bullet, u; \delta\boldsymbol{u}) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0} \frac{A(\bullet, \boldsymbol{u} + \delta\boldsymbol{u}\epsilon) - A(\bullet, \boldsymbol{u})}{\epsilon}.$$

2. General format of optimal control problem

We consider the steering of a mechanical system (without feedback) described by the following state equation

$$\boldsymbol{M}(\boldsymbol{u})\dot{\boldsymbol{u}}(t) + \boldsymbol{J}(\boldsymbol{u}(t), t) = \boldsymbol{f}(\boldsymbol{p}(t), t), \qquad t \in [0, T] \tag{1}$$

where $\boldsymbol{u}$ is a vector-valued collection of N state variables, matrix $\boldsymbol{M}(\boldsymbol{u})$ and internal force vector $\boldsymbol{J}(\boldsymbol{u}(t), t)$ define the mechanical system and $\boldsymbol{f}(\boldsymbol{p}(t), t)$ is forces emanating from external

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excitation determined by the K control(s) $\boldsymbol{p}(t)$. We here consider mechanical systems consisting of rigid bodies assembled at joints, although the pertinent formulation can be extended to the situation of elastic bodies. The state $\boldsymbol{u}$ must satisfy boundary conditions determined by desired initial and target (end) configuration of the system as follows

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}(T) = \boldsymbol{u}_T. \tag{2}$$

The desired control $\boldsymbol{p}(t)$ is found as the minimizer to the scalar performance measure (objective functional)

$$\mathcal{F}(\boldsymbol{p}, \boldsymbol{u}). \tag{3}$$

In addition, $\boldsymbol{p}$ and $\boldsymbol{u}$ must satisfy M inequality constraints on the form

$$\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{u}) \leq \boldsymbol{0}, \qquad t \in [0, T]. \tag{4}$$

In summary, the optimal control problem can be formulated as follow: Determine the (vector-valued) control $\boldsymbol{p}(t)$ that minimizes the objective function (3) while satisfying the state equation (1), the boundary condition (2) and the (nonlinear) inequality constraint (4).

Upon solving the optimal control problem using a Finite Element approximation, we shall consider a set of measures of the solution, so-called *goal functions* $\mathcal{Q}(\boldsymbol{p}, \boldsymbol{u})$, of engineering interest in which the effect of discretization errors is to be estimated.

2.1. Derivatives of state equation

The state equation is given on weak form as

$$A(\boldsymbol{u}; \boldsymbol{v}) = L(\boldsymbol{p}; \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbb{V} \tag{5}$$

$$A(\boldsymbol{u}; \boldsymbol{v}) \stackrel{\text{def}}{=} \int_0^T \boldsymbol{v}^{\mathrm{T}}[\boldsymbol{M}(\boldsymbol{u})\dot{\boldsymbol{u}} + \boldsymbol{J}(\boldsymbol{u}, t)]\,\mathrm{d}t, \quad L(\boldsymbol{p}; \boldsymbol{v}) \stackrel{\text{def}}{=} \int_0^T \boldsymbol{v}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{p}, t)\,\mathrm{d}t \tag{6}$$

Differentiation wrt $\boldsymbol{u}$ gives

$$A'_u(\boldsymbol{u}; \boldsymbol{v}, \delta\boldsymbol{u}) = \int_0^T \boldsymbol{v}^{\mathrm{T}}[\boldsymbol{M}(\boldsymbol{u})\delta\dot{\boldsymbol{u}} + (\boldsymbol{M}'_{\boldsymbol{u}}\delta\boldsymbol{u})\dot{\boldsymbol{u}} + \boldsymbol{J}'_{\boldsymbol{u}}\delta\boldsymbol{u}]\,\mathrm{d}t, \quad L'_p(\boldsymbol{p}; \boldsymbol{v}, \delta\boldsymbol{p}) = \int_0^T \boldsymbol{v}^{\mathrm{T}}\boldsymbol{f}'_p\delta\boldsymbol{p}\,\mathrm{d}t. \tag{7}$$

and

$$A''_{uu}(\boldsymbol{u}; \boldsymbol{v}, \delta\boldsymbol{u}_1, \delta\boldsymbol{u}_2) = \int_0^T (\delta\boldsymbol{u}_2)^{\mathrm{T}}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{M}'_{\boldsymbol{u}}\delta\dot{\boldsymbol{u}}_1 + (\delta\dot{\boldsymbol{u}}_2)^{\mathrm{T}}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{M}'_{\boldsymbol{u}}\delta\boldsymbol{u}_1 + (\delta\boldsymbol{u}_2)^{\mathrm{T}}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{M}''_{\boldsymbol{uu}}\dot{\boldsymbol{u}}\delta\boldsymbol{u}_1 + (\delta\boldsymbol{u}_2)^{\mathrm{T}}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{J}''_{\boldsymbol{uu}}\delta\boldsymbol{u}_1\,\mathrm{d}t \tag{8}$$

$$L''_{pp}(\boldsymbol{p}; \boldsymbol{v}, \delta\boldsymbol{p}, \delta\boldsymbol{p}) = \int_0^T (\delta\boldsymbol{p})^{\mathrm{T}}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{f}''_{\boldsymbol{pp}}\delta\boldsymbol{p}\,\mathrm{d}t \tag{9}$$

where $\boldsymbol{M}'_{\boldsymbol{u}}$, $\boldsymbol{J}''_{\boldsymbol{uu}}$ and $\boldsymbol{f}''_{\boldsymbol{pp}}$ are 3-dimensional arrays with suitable indexing, and $\boldsymbol{M}''_{uu}$ as a 4-dimensional array.

2.2. Derivatives of equality constraints

The target condition is enforced weakly as an equality constraint

$$h(\boldsymbol{u}; \boldsymbol{v}^{\mathrm{h}}) = (\boldsymbol{v}^{\mathrm{h}})^{\mathrm{T}}(\boldsymbol{u}(T) - \boldsymbol{u}_T) = 0 \qquad \forall \boldsymbol{v}^{\mathrm{h}} \in \mathbb{R}^N \tag{10}$$

differentiation of h w.r.t $\boldsymbol{u}$ gives

$$h(\boldsymbol{u}; \boldsymbol{v}^{\mathrm{h}}, \delta\boldsymbol{u}) = (\boldsymbol{v}^{\mathrm{h}})^{\mathrm{T}}\delta\boldsymbol{u}(T) \tag{11}$$

and second derivatives are zero.

2.3. Derivatives of equality constraints

The inequality constraint is enforced weakly as

$$g(\boldsymbol{p}, \boldsymbol{u}; \boldsymbol{v}^{\mathrm{g}}) = \int_0^T [\boldsymbol{v}^{\mathrm{g}}]^{\mathrm{T}} \boldsymbol{g}(\boldsymbol{p}, \boldsymbol{u}) \, \mathrm{d}t \le 0 \tag{12}$$

differentiation of g w.r.t $\boldsymbol{p}$ and $\boldsymbol{u}$ gives

$$g'_p(\boldsymbol{p}, \boldsymbol{u}; \boldsymbol{v}^{\mathrm{g}}, \delta\boldsymbol{p}) = \int_0^T (\boldsymbol{v}^{\mathrm{g}})^{\mathrm{T}} \boldsymbol{g}'_{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{u}) \delta\boldsymbol{p} \, \mathrm{d}t, \quad g'_u(\boldsymbol{p}, \boldsymbol{u}; \boldsymbol{v}^{\mathrm{g}}, \delta\boldsymbol{u}) = \int_0^T (\boldsymbol{v}^{\mathrm{g}})^{\mathrm{T}} \boldsymbol{g}'_{\boldsymbol{u}}(\boldsymbol{p}, \boldsymbol{u}) \delta\boldsymbol{u} \, \mathrm{d}t \tag{13}$$

$$g''_{pp}(\boldsymbol{p}, \boldsymbol{u}; \boldsymbol{v}^{\mathrm{g}}, \delta\boldsymbol{p}_1, \delta\boldsymbol{p}_2) = \int_0^T (\delta\boldsymbol{p}_2)^{\mathrm{T}} (\boldsymbol{v}^{\mathrm{g}})^{\mathrm{T}} \boldsymbol{g}''_{\boldsymbol{pp}}(\boldsymbol{p}, \boldsymbol{u}) \delta\boldsymbol{p}_1 \, \mathrm{d}t \tag{14}$$

$$g''_{pu}(\boldsymbol{p}, \boldsymbol{u}; \boldsymbol{v}^{\mathrm{g}}, \delta\boldsymbol{p}, \delta\boldsymbol{u}) = \int_0^T (\delta\boldsymbol{p})^{\mathrm{T}} (\boldsymbol{v}^{\mathrm{g}})^{\mathrm{T}} \boldsymbol{g}''_{\boldsymbol{pu}}(\boldsymbol{p}, \boldsymbol{u}) \delta\boldsymbol{u} \, \mathrm{d}t \tag{15}$$

$$g''_{uu}(\boldsymbol{p}, \boldsymbol{u}; \boldsymbol{v}^{\mathrm{g}}, \delta\boldsymbol{u}_1, \delta\boldsymbol{u}_2) = \int_0^T (\delta\boldsymbol{u}_2)^{\mathrm{T}} (\boldsymbol{v}^{\mathrm{g}})^{\mathrm{T}} \boldsymbol{g}''_{\boldsymbol{uu}}(\boldsymbol{p}, \boldsymbol{u}) \delta\boldsymbol{u}_1 \, \mathrm{d}t \tag{16}$$

Where $\boldsymbol{g}''_{\boldsymbol{pp}}$, $\boldsymbol{g}''_{\boldsymbol{pu}}$, $\boldsymbol{g}''_{\boldsymbol{uu}}$ can be defined as 3-dimensional arrays.

2.3.1. Penalty formulation A straightforward manner to treat the inequality constraints is to introduce a suitable penalty formulation, where a violation of the constraints is given a cost that is added to the objective function†

$$\mathcal{F}(\boldsymbol{p}, \boldsymbol{u}) := \mathcal{F}(\boldsymbol{p}, \boldsymbol{u}) + \mathcal{F}^{\mathrm{pen}} \qquad \mathcal{F}^{\mathrm{pen}} \overset{\mathrm{def}}{=} a_1 \sum_i^M \int_0^T (\max[0, g_i(\boldsymbol{p}, \boldsymbol{u}) - \epsilon])^{a_2} \mathrm{d}t \tag{17}$$

where a_1, a_2 and ϵ are problem-dependant "tuning" parameters for the penalty term. This will give the following expressions for additions to

$$(\mathcal{F}^{\mathrm{pen}})'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = a_1 a_2 \sum_i^M \int_0^T (\max[0, g_i(\boldsymbol{p}, \boldsymbol{u}) - \epsilon])^{(a_2-1)} (g_i)'_{\boldsymbol{p}} \delta\boldsymbol{p} \, \mathrm{d}t \tag{18}$$

$$(\mathcal{F}^{\mathrm{pen}})'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = a_1 a_2 \sum_i^M \int_0^T (\max[0, g_i(\boldsymbol{p}, \boldsymbol{u}) - \epsilon])^{(a_2-1)} (g_i)'_{\boldsymbol{u}} \delta\boldsymbol{u} \, \mathrm{d}t \tag{19}$$

$$(\mathcal{F}^{\mathrm{pen}})''_{pp}(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}_1, \delta\boldsymbol{p}_2) = a_1 a_2 \sum_i^M \int_0^T (\max[0, g_i(\boldsymbol{p}, \boldsymbol{u}) - \epsilon])^{(a_2-1)} \tag{20}$$

$$(\delta\boldsymbol{p}_2)^{\mathrm{T}} \left[((g_i)'_p)^2 + (a_2 - 1)(\max[0, g_i(\boldsymbol{p}, \boldsymbol{u}) - \epsilon])^{(a_2-2)} (g_i)''_{\boldsymbol{pp}} \right] \delta\boldsymbol{p}_1 \, \mathrm{d}t \tag{21}$$

†A penalty formulation for the equality condition $h(\boldsymbol{u})$ is also possible along the same lines, but is not further considered here.

$$(\mathcal{F}^{\text{pen}})''_{pu}(\boldsymbol{p},\boldsymbol{u};\delta\boldsymbol{p},\delta\boldsymbol{u}) = a_1 a_2 \sum_i^M \int_0^T \max[0, g_i(\boldsymbol{p},\boldsymbol{u}) - \epsilon])^{(a_2-1)} \tag{22}$$

$$(\delta\boldsymbol{p})^{\mathrm{T}} \left[(g_i)'_p (g_i)'_u + (a_2 - 1) \max[0, g_i(\boldsymbol{p},\boldsymbol{u}) - \epsilon])^{(a_2-2)} (g_i)''_{\boldsymbol{pu}} \right] \delta\boldsymbol{u}\, \mathrm{d}t \tag{23}$$

$$(\mathcal{F}^{\text{pen}})''_{uu}(\boldsymbol{p},\boldsymbol{u};\delta\boldsymbol{u}_1,\delta\boldsymbol{u}_2) = a_1 a_2 \sum_i^M \int_0^T \max[0, g_i(\boldsymbol{p},\boldsymbol{u}) - \epsilon])^{(a_2-1)} \tag{24}$$

$$(\delta\boldsymbol{u}_2)^{\mathrm{T}} \left[((g_i)'_u)^2 + (a_2 - 1) \max[0, g_i(\boldsymbol{p},\boldsymbol{u}) - \epsilon])^{(a_2-2)} (g_i)''_{\boldsymbol{uu}} \right] \delta\boldsymbol{u}_1\, \mathrm{d}t \tag{25}$$

3. Trajectory control

As a first application, we shall consider a particle trajectory, inspired by a vehicle dynamics problem described in [2]. A particle of unit mass travels in the 2D-plane with external force acting as control, i.e.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} - \begin{bmatrix} v_x \\ v_y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_x \\ F_y \end{bmatrix} \tag{26}$$

with initial condition $\boldsymbol{u}_0 = [0,\ 10,\ 30,\ 0]^{\mathrm{T}}$ and target condition $\boldsymbol{u}_T = [100,\ 0,\ 20,\ 0]^{\mathrm{T}}$ to be reached at final time $T = 4s$. In addition, we have inequality constraints.

$$\boldsymbol{g}(\boldsymbol{p},\boldsymbol{u}) = \begin{bmatrix} (F_x^2 + F_y^2) - (\mu g)^2 \\ v_{\min} - v_x \\ 1 - ((x - x_{p1})/x_{L1})^{N_1} + ((y - y_{p1})/y_{L1})^{N_1} \\ 1 - ((x - x_{p2})/x_{L2})^{N_2} + ((y - y_{p2})/y_{L2})^{N_2} \\ \vdots \end{bmatrix} \tag{27}$$

where in the first constraint μg represents a maximum available force (due to friction between vehicle and ground), $v_{\min}$ is a prescribed minimum speed in x-direction (to avoid loops), and a number of hyper-ellipsoid obstacles defined by center points (x_{pi}, y_{pi}), size (x_{L1}, y_{Li}) and exponent N_i the shape of obstacles ($N = 2$ for circular obstacles). The first two constraints are convex whereas the latter are not.

3.1. Derivatives of state equation

By defining $\boldsymbol{u}$ as vector of state variables $\boldsymbol{u} = [x,\ y,\ v_x,\ v_y]^{\mathrm{T}}$ and extracting from (26) we identify

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{J} = \begin{bmatrix} -u_3 \\ -u_4 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{f} = \begin{bmatrix} 0 \\ 0 \\ F_x \\ F_y \end{bmatrix} \tag{28}$$

we obtain as derivatives

$$\boldsymbol{M}'_{\boldsymbol{u}} = \boldsymbol{0}, \qquad \boldsymbol{J}'_{\boldsymbol{u}} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{f}'_{\boldsymbol{p}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

and second derivatives are zero.

3.2. Objective functions

To allow for analysis of trade-off between deviation from desired route ($y = 0$ is a desired route) and forces and forces rates, we extress the objective function as

$$\mathcal{F} = \gamma_1 \int_0^T F_x^2 \mathrm{d}t + \gamma_2 \int_0^T F_y^2 \mathrm{d}t + \gamma_3 \int_0^T x^2 \mathrm{d}t + \gamma_4 \int_0^T y^2 \mathrm{d}t + \gamma_5 \int_0^T v_x^2 \mathrm{d}t + \gamma_6 \int_0^T v_y^2 \mathrm{d}t \tag{30}$$

where $\gamma_1 - \gamma_6$ are parameters to be studied.

$$\mathcal{F}'_p(\boldsymbol{p}, \boldsymbol{u}; \delta \boldsymbol{p}) = 2\gamma_2 \int_0^T F_x \delta F_x \mathrm{d}t + 2\gamma_3 \int_0^T F_y \delta F_y \mathrm{d}t + 2\gamma_4 \int_0^T \dot{F}_x \delta \dot{F}_x \mathrm{d}t + 2\gamma_5 \int_0^T \dot{F}_y \delta \dot{F}_y \mathrm{d}t \tag{31}$$

$$\mathcal{F}'_u(\boldsymbol{p}, \boldsymbol{u}; \delta \boldsymbol{u}) = 2\gamma_1 \int_0^T y \delta y \mathrm{d}t \tag{32}$$

$$\mathcal{F}''_p(\boldsymbol{p}, \boldsymbol{u}; \delta \boldsymbol{p}, \delta \boldsymbol{p}) = 2\gamma_2 \int_0^T \delta F_x \delta F_x \mathrm{d}t + 2\gamma_3 \int_0^T \delta F_y \delta F_y \mathrm{d}t + 2\gamma_4 \int_0^T \delta \dot{F}_x \delta \dot{F}_x \mathrm{d}t + 2\gamma_5 \int_0^T \delta \dot{F}_y \delta \dot{F}_y \mathrm{d}t \tag{33}$$

$$\mathcal{F}''_{uu}(\boldsymbol{p}, \boldsymbol{u}; \delta \boldsymbol{u}, \delta \boldsymbol{u}) = 2\gamma_1 \int_0^T \delta y \delta y \mathrm{d}t \tag{34}$$

and $\mathcal{F}''_{pu}(\boldsymbol{p}, \boldsymbol{u}; \delta \boldsymbol{p}, \delta \boldsymbol{u}) = 0$

3.3. Derivatives of inequality constraints

$$\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{u}) = \begin{bmatrix} (F_x^2 + F_y^2) - (\mu g)^2 \\ v_{\min} - v_x \\ 1 - ((x - x_{p1})/x_{L1})^{N_1} - ((y - y_{p1})/y_{L1})^{N_1} \\ 1 - ((x - x_{p2})/x_{L2})^{N_2} - ((y - y_{p2})/y_{L2})^{N_2} \\ \vdots \end{bmatrix} \tag{35}$$

Derivatives of first constraint

$$(g_1)'_{\boldsymbol{p}} = \begin{bmatrix} 2F_x \\ 2F_y \end{bmatrix}, \quad (g_1)'_{\boldsymbol{u}} = \boldsymbol{0} \tag{36}$$

$$(g_1)''_{\boldsymbol{p},\boldsymbol{p}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad (g_1)''_{\boldsymbol{p},\boldsymbol{u}} = \boldsymbol{0}, \quad (g_1)''_{\boldsymbol{u},\boldsymbol{u}} = \boldsymbol{0} \tag{37}$$

Second constraint (minimum speed in x-direction) derivatives

$$(g_2)'_{\boldsymbol{p}} = \mathbf{0}, \quad (g_2)'_{\boldsymbol{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{38}$$

$$(g_2)''_{\boldsymbol{p},\boldsymbol{p}} = \mathbf{0}, \quad (g_2)''_{\boldsymbol{p},\boldsymbol{u}} = \mathbf{0}, \quad (g_2)''_{\boldsymbol{u},\boldsymbol{u}} = \mathbf{0} \tag{39}$$

Obstacle constraint derivatives

$$(g_2)''_{\boldsymbol{p},\boldsymbol{p}} = \mathbf{0}, \quad (g_2)''_{\boldsymbol{p},\boldsymbol{u}} = \mathbf{0}, \quad (g_2)''_{\boldsymbol{u},\boldsymbol{u}} = \begin{bmatrix} -\frac{N_1(N_1-1)}{x_{L1}^2}((x-x_{p1})/x_{L1})^{(N_1-2)} & 0 & 0 & 0 \\ 0 & -\frac{N_1(N_1-1)}{y_{L1}^2}((y-y_{p1})/y_{L1})^{(N_1-2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

3.4. Goal functions

For the case of trajectory control, the following goal functions were considered

$$\begin{aligned} &\mathcal{Q}_1 = \mathcal{F}, && \mathcal{Q}_2 = x(t_s), && \mathcal{Q}_3 = y(t_s), && \mathcal{Q}_4 = v_x(t_s), && \mathcal{Q}_5 = v_y(t_s) \\ &\mathcal{Q}_6 = x^{\text{sim}}(T), && \mathcal{Q}_7 = y^{\text{sim}}(T), && \mathcal{Q}_8 = v_x^{\text{sim}}(T), && \mathcal{Q}_9 = v_y^{\text{sim}}(T), && \mathcal{Q}_{10} = \textstyle\int_0^T v_x F_x + v_y F_y \mathrm{d}t \end{aligned} \tag{41}$$

where x^{sim} and y^{sim} refers to a subsequent 'exact' simulation where the discrete solution $\boldsymbol{p}_h$ is used, i.e. $\boldsymbol{u}^{\text{sim}}$ solves

$$A(\boldsymbol{u}^{\text{sim}}; \boldsymbol{v}) = L(\boldsymbol{p}_h; \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbb{V}^{\text{sim}} \tag{42}$$

for a very fine mesh $\mathbb{V}^{\text{sim}}$. Differentiation of the goal quantities gives:

$$(\mathcal{Q}_1)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = \mathcal{F}'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}), \quad (\mathcal{Q}_1)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = \mathcal{F}'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) \tag{43}$$

$$(\mathcal{Q}_2)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = 0, \quad (\mathcal{Q}_2)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = \delta x(t_s) \tag{44}$$

$$(\mathcal{Q}_3)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = 0, \quad (\mathcal{Q}_3)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = \delta y(t_s) \tag{45}$$

$$(\mathcal{Q}_4)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = 0, \quad (\mathcal{Q}_4)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = \delta v_x(t_s) \tag{46}$$

$$(\mathcal{Q}_5)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = 0, \quad (\mathcal{Q}_5)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = \delta v_y(t_s) \tag{47}$$

$$(\mathcal{Q}_6)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = d_p x^{\text{sim}}(T), \quad (\mathcal{Q}_6)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = 0 \tag{48}$$

$$(\mathcal{Q}_7)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = d_p y^{\text{sim}}(T), \quad (\mathcal{Q}_7)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = 0 \tag{49}$$

$$(\mathcal{Q}_8)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = d_p v_x^{\text{sim}}(T), \quad (\mathcal{Q}_8)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = 0 \tag{50}$$

$$(\mathcal{Q}_9)'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = d_p v_y^{\text{sim}}(T), \quad (\mathcal{Q}_9)'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = 0 \tag{51}$$

$$(\mathcal{Q}_{10})'_p(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{p}) = \int_0^T v_x \delta F_x + v_y \delta F_y \mathrm{d}t, \quad (\mathcal{Q}_{10})'_u(\boldsymbol{p}, \boldsymbol{u}; \delta\boldsymbol{u}) = \int_0^T \delta v_x F_x + \delta v_y F_y \mathrm{d}t \tag{52}$$

Note: $d_p\boldsymbol{u}^{\text{sim}}(T)$ needed for goal functions 6-9. that can be solved for from the tangent equation

$$A'_u(\boldsymbol{u}^{\text{sim}}; \boldsymbol{v}, d_p\boldsymbol{u}^{\text{sim}}) = L'_p(\boldsymbol{p}_h; \boldsymbol{v}, \delta\boldsymbol{p}), \qquad \forall \boldsymbol{v} \in \mathbb{V}^{\text{sim}}. \tag{53}$$

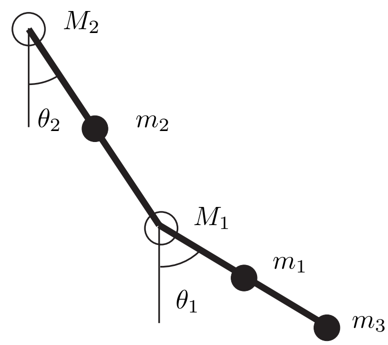

Figure 1. Double pendulum-Lifting

4. Movement planning

As a second example we consider the control of a double pendulum (inspired by a bio-mechanic modeling of a human arm, cf. [3]). The configuration of the pendulum is described by the two angles θ_1 and θ_2 and is steered by "applied" bending moment $M_1(t)$, $M_2(t)$ in the joints, defining the controls $\boldsymbol{p} = [p_1, p_2]^{\mathrm{T}} = [M_1, M_2]^{\mathrm{T}}$. The inertia of the pendulum is given by three point masses m_1-m_3, and the links have lengths l_1 and l_2.

Initial condition corresponds to a vertical pendulum at rest, $\boldsymbol{u}(0) = [0, 0, 0, 0]^{\mathrm{T}}$, and we consider a target condition that the pendulum should be horizontal and at rest, i.e. that $\boldsymbol{u}(T) = [0, 0, \pi/2, \pi/2]^{\mathrm{T}}$. As constraints we have: The controls has some limited capacity, i.e. the allowable bending moments in the joints are restricted as $-7.5 = p_{1,min} \leq p_1 \leq p_{1,max} = 20$ and $-10 = p_{2,min} \leq p_2 \leq p_{2,max} = 10$. Moreover, the rotation in the second joint is restricted as $0 = \theta_{\Delta,min} \leq \theta_1 - \theta_2 \leq \theta_{\Delta,max} = \frac{3}{4}\pi$. The inequality constraints $\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{u})$ are thus defined as

$$\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{u}) = \begin{bmatrix} -p_1 + p_{1,min} \\ p_1 - p_{1,max} \\ -p_2 + p_{2,min} \\ p_2 - p_{2,max} \\ \theta_2 - \theta_1 + \theta_{\Delta,min} \\ \theta_1 - \theta_2 - \theta_{\Delta,max} \end{bmatrix} \tag{54}$$

Given theses constraints, we seek to find the controls such that the objective function measuring the magnitude of moments and moment velocities exerted in joints is minimized

$$\mathcal{F}(\boldsymbol{p}) = \frac{1}{2} \int_0^T \left[|\boldsymbol{p}(t)|^2 + \alpha |\dot{\boldsymbol{p}}(t)|^2 \right] \mathrm{d}t \tag{55}$$

the parameter α is used the weight between the two components (which in a numerical realization acts as a regularization parameter).

4.1. Derivatives of state equation

The Lagrangian function entailing kinematic and potential energy for the 3-mass double pendulum takes the form

$$\begin{aligned}\mathcal{L} =& l_1^2\left(\frac{m_1}{4}+m_3\right)\frac{\dot\theta_1^2}{2}+l_2^2\left(m_1+\frac{m_2}{4}+m_3\right)\frac{\dot\theta_2^2}{2}\\ &+gl_1\left(\frac{m_1}{2}+m_3\right)\cos(\theta_1)+gl_2\left(m_1+\frac{m_2}{2}+m_3\right)\cos(\theta_2)\\ &+l_1l_2\left(\frac{m_1}{2}+m_3\right)\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\end{aligned} \tag{56}$$

from which the Euler-Lagrange equations of motion are obtained as

$$\begin{aligned}l_1^2\left(\frac{m_1}{4}+m_3\right)\ddot\theta_1+l_1l_2\left(\frac{m_1}{2}+m_3\right)\left(\ddot\theta_2\cos(\theta_1-\theta_2)+\dot\theta_2^2\sin(\theta_1-\theta_2)\right)\\ +gl_1\left(\frac{m_1}{2}+m_3\right)\sin(\theta_1)=M_1(t)\end{aligned} \tag{57}$$

$$\begin{aligned}l_2^2\left(m_1+\frac{m_2}{4}+m_3\right)\ddot\theta_2+l_1l_2\left(\frac{m_1}{2}+m_3\right)\left(\ddot\theta_1\cos(\theta_1-\theta_2)-\dot\theta_1^2\sin(\theta_1-\theta_2)\right)\\ +gl_2\left(m_1+\frac{m_2}{2}+m_3\right)\sin(\theta_2)=M_2(t)\end{aligned} \tag{58}$$

We define $\boldsymbol{u}$ as vector of state variables $\boldsymbol{u}=[\theta_1,\ \theta_2,\ \dot\theta_1,\dot\theta_2]^{\mathrm{T}}$ from which we identify

$$\boldsymbol{M}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & l_1^2(m_1/4+m_3) & l_1l_2(m_1/2+m_3)\cos(u_1-u_2)\\ 0 & 0 & l_1l_2(m_1/2+m_3)\cos(u_1-u_2) & l_2^2(m_1+m_2/4+m_3)\end{bmatrix} \tag{59}$$

$$\boldsymbol{J}=\begin{bmatrix}-u_3\\ -u_4\\ l_1l_2(m_1/2+m_3)u_4^2\sin(u_1-u_2)+gl_1\left(\frac{m_1}{2}+m_3\right)\sin(u_1)\\ -l_1l_2(m_1/2+m_3)u_3^2\sin(u_1-u_2)+gl_2\left(m_1+\frac{m_2}{2}+m_3\right)\sin(u_2)\end{bmatrix}\qquad \boldsymbol{f}=\begin{bmatrix}0\\0\\M_1\\M_2\end{bmatrix} \tag{60}$$

The derivatives of $\boldsymbol{M}$ are obtained as

$$\boldsymbol{M}'_{u_1}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -l_1l_2(m_1/2+m_3)\sin(u_1-u_2)\\ 0 & 0 & -l_1l_2(m_1/2+m_3)\sin(u_1-u_2) & 0\end{bmatrix} \tag{61}$$

$$\boldsymbol{M}'_{u_2}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & l_1l_2(m_1/2+m_3)\sin(u_1-u_2)\\ 0 & 0 & l_1l_2(m_1/2+m_3)\sin(u_1-u_2) & 0\end{bmatrix},\qquad \boldsymbol{M}'_{u,u_3}=\boldsymbol{M}'_{u,u_4}=\boldsymbol{0} \tag{62}$$

$$\boldsymbol{M}''_{u_1,u_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) \\ 0 & 0 & -l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) & 0 \end{bmatrix} \tag{63}$$

$$\boldsymbol{M}''_{u_1,u_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) \\ 0 & 0 & l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) & 0 \end{bmatrix} \tag{64}$$

$$\boldsymbol{M}''_{u_2,u_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) \\ 0 & 0 & l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) & 0 \end{bmatrix} \tag{65}$$

$$\boldsymbol{M}''_{u_2,u_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) \\ 0 & 0 & -l_1 l_2 (m_1/2 + m_3) \cos(u_1 - u_2) & 0 \end{bmatrix} \tag{66}$$

with all other components of $\boldsymbol{M}''_{\boldsymbol{uu}} = \mathbf{0}$. Furthermore, for derivatives of $\boldsymbol{J}$, we consider each component individually:

$$((\boldsymbol{J})_1)'_{\boldsymbol{u}} = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^{\mathrm{T}}, \qquad ((\boldsymbol{J})_1)''_{uu} = \mathbf{0} \tag{67}$$

$$((\boldsymbol{J})_2)'_{\boldsymbol{u}} = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}^{\mathrm{T}}, \qquad ((\boldsymbol{J})_2)''_{uu} = \mathbf{0} \tag{68}$$

$$((\boldsymbol{J})_3)'_{\boldsymbol{u}} = \begin{bmatrix} l_1 l_2 (m_1/2 + m_3) u_4^2 \cos(u_1 - u_2) + g l_1 \left(m_1/2 + m_3\right) \cos(u_1) \\ -l_1 l_2 (m_1/2 + m_3) u_4^2 \cos(u_1 - u_2) \\ 0 \\ 2 l_1 l_2 (m_1/2 + m_3) u_4 \sin(u_1 - u_2) \end{bmatrix} \tag{69}$$

$$((\boldsymbol{J})_3)''_{u_1\,\boldsymbol{u}} = \begin{bmatrix} -l_1 l_2 (m_1/2 + m_3) u_4^2 \sin(u_1 - u_2) - g l_1 \left(m_1/2 + m_3\right) \sin(u_1) \\ l_1 l_2 (m_1/2 + m_3) u_4^2 \sin(u_1 - u_2) \\ 0 \\ 2 l_1 l_2 (m_1/2 + m_3) u_4 \cos(u_1 - u_2) \end{bmatrix} \tag{70}$$

$$((\boldsymbol{J})_3)''_{u_2\,\boldsymbol{u}} = \begin{bmatrix} l_1 l_2 (m_1/2 + m_3) u_4^2 \sin(u_1 - u_2) \\ -l_1 l_2 (m_1/2 + m_3) u_4^2 \sin(u_1 - u_2) \\ 0 \\ -2 l_1 l_2 (m_1/2 + m_3) u_4 \cos(u_1 - u_2) \end{bmatrix} \tag{71}$$

$$((\boldsymbol{J})_3)''_{u_3\,\boldsymbol{u}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

$$((\boldsymbol{J})_3)''_{u_4}\boldsymbol{u} = \begin{bmatrix} 2l_1l_2(m_1/2+m_3)u_4\cos(u_1-u_2) \\ -2l_1l_2(m_1/2+m_3)u_4\cos(u_1-u_2) \\ 0 \\ 2l_1l_2(m_1/2+m_3)\sin(u_1-u_2) \end{bmatrix} \tag{73}$$

$$((\boldsymbol{J})_4)'_{\boldsymbol{u}} = \begin{bmatrix} l_1l_2(m_1/2+m_3)u_3^2\cos(u_1-u_2) \\ -l_1l_2(m_1/2+m_3)u_3^2\cos(u_1-u_2) + gl_2\left(m_1+\frac{m_2}{2}+m_3\right)\cos(u_2) \\ -2l_1l_2(m_1/2+m_3)u_3\sin(u_1-u_2) \\ 0 \end{bmatrix} \tag{74}$$

$$((\boldsymbol{J})_4)''_{u_1}\boldsymbol{u} = \begin{bmatrix} -l_1l_2(m_1/2+m_3)u_3^2\sin(u_1-u_2) \\ l_1l_2(m_1/2+m_3)u_3^2\sin(u_1-u_2) \\ 2l_1l_2(m_1/2+m_3)u_3\cos(u_1-u_2) \\ 0 \end{bmatrix} \tag{75}$$

$$((\boldsymbol{J})_4)''_{u_2}\boldsymbol{u} = \begin{bmatrix} l_1l_2(m_1/2+m_3)u_3^2\sin(u_1-u_2) \\ -l_1l_2(m_1/2+m_3)u_3^2\sin(u_1-u_2) - gl_2\left(m_1+\frac{m_2}{2}+m_3\right)\sin(u_2) \\ -2l_1l_2(m_1/2+m_3)u_3\cos(u_1-u_2) \\ 0 \end{bmatrix} \tag{76}$$

$$((\boldsymbol{J})_4)''_{u_3}\boldsymbol{u} = \begin{bmatrix} -2l_1l_2(m_1/2+m_3)u_3\cos(u_1-u_2) \\ 2l_1l_2(m_1/2+m_3)u_3\cos(u_1-u_2) \\ -2l_1l_2(m_1/2+m_3)\sin(u_1-u_2) \\ 0 \end{bmatrix} \tag{77}$$

$$((\boldsymbol{J})_4)''_{u_4}\boldsymbol{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{78}$$

Finally,

$$\boldsymbol{f}'_{\boldsymbol{p}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{f}''_{\boldsymbol{pp}} = \boldsymbol{0} \tag{79}$$

4.2. Derivatives of inequality constraints

Recalling

$$\boldsymbol{g}(\boldsymbol{p},\boldsymbol{u}) = \begin{bmatrix} -p_1 + p_{1,min} \\ p_1 - p_{1,max} \\ -p_2 + p_{2,min} \\ p_2 - p_{2,max} \\ \theta_2 - \theta_1 + \theta_{\Delta,min} \\ \theta_1 - \theta_2 - \theta_{\Delta,max} \end{bmatrix} \tag{80}$$

we have the following component derivatives

$$(g_1)'_{\boldsymbol{p}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad (g_1)'_{\boldsymbol{u}} = \mathbf{0} \tag{81}$$

$$(g_2)'_{\boldsymbol{p}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (g_2)'_{\boldsymbol{u}} = \mathbf{0} \tag{82}$$

$$(g_3)'_{\boldsymbol{p}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad (g_3)'_{\boldsymbol{u}} = \mathbf{0} \tag{83}$$

$$(g_4)'_{\boldsymbol{p}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (g_4)'_{\boldsymbol{u}} = \mathbf{0} \tag{84}$$

$$(g_5)'_{\boldsymbol{p}} = \mathbf{0}, \quad (g_5)'_{\boldsymbol{u}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{85}$$

$$(g_6)'_{\boldsymbol{p}} = \mathbf{0}, \quad (g_6)'_{\boldsymbol{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \tag{86}$$

Due to its linear nature of the constraints, all second derivatives of $\boldsymbol{g}$ are zero.

4.3. Goal functions

For the case of movement planning, the following goal functions were considered

$$\begin{array}{lllll} \mathcal{Q}_1 = \mathcal{F}, & \mathcal{Q}_2 = \theta_1(t_s), & \mathcal{Q}_3 = \theta_2(t_s), & \mathcal{Q}_4 = \dot{\theta}_1(t_s), & \mathcal{Q}_5 = \dot{\theta}_2(t_s) \\ \mathcal{Q}_6 = \theta_1^{\text{sim}}(T), & \mathcal{Q}_7 = \theta_2^{\text{sim}}(T), & \mathcal{Q}_8 = \dot{\theta}_1^{\text{sim}}(T), & \mathcal{Q}_9 = \dot{\theta}_2^{\text{sim}}(T), & \mathcal{Q}_{10} = \int_0^T M_1\dot{\theta}_1 + M_2\dot{\theta}_2 \mathrm{d}t \end{array} \tag{87}$$

which are in direct analogy to the trajectory control example above.

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