THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

Supersymmetric Geometries in Type IIA Supergravity

Classification using the Spinorial Geometry Method

Christian von Schultz

Department of Fundamental Physics
Chalmers University of Technology
Göteborg, Sweden 2014

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Christian von Schultz

Department of Fundamental Physics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden

Telephone +46 (0)31–772 1000

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Abstract

This thesis is about supersymmetric geometries in type IIA supergravity and their classification using spinorial geometry. Appended is our first paper on the topic. There are four types of Killing spinors to treat, distinguished by their isotropy group. The appended paper treats the cases where the isotropy group is $\mathrm{Spin}(7)$ or $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$.

The first part of the thesis introduces the method and some important concepts that should help the reader understand the paper. After an introduction describing supersymmetry and supergravity, we turn to spinors and the $\mathrm{Spin}(p,q)$ groups, and proceed to the Killing spinor equations and the method of spinorial geometry. The method results in a linear system of equations (the exact expression is appended the paper), and in chapter 4 the reader is shown how to go about in simplifying it to arrive at the final expressions found in the paper.

Keywords: Supersymmetry, Spinorial Geometry, Classification, IIA Supergravity

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Chapter 1

Introduction

Physics can be about making some cool new gadget or device, or about saving the environment. But that is applied physics. Fundamental physics is, more than anything, a quest to find the truth — the nature of reality on its most fundamental level, discovering the rules that govern it, unweaving the rainbow. In practice, much theoretical work in fundamental physics is done on theories and models with no known relation to the real world, or even theories that are known not to be phenomenologically viable — toy models exhibiting some interesting mathematical properties which might lead to new insights that could possibly, maybe, be useful in the future development of more realistic models of reality.

The classification of supersymmetric geometries in type IIA supergravity is somewhere in between. Supersymmetric solutions have historically been very important, for instance when going from weak coupling to strong coupling. Strong coupling physics is normally beyond control, signifying the need to rethink the degrees of freedom chosen to describe the problem. Extrapolations between weak and strong coupling are normally impossible — unless you consider quantities protected by some symmetry. Treating objects protected by supersymmetry you can learn a surprising amount, without large quantum corrections getting in your way and spoiling the party. Thus the present work would be useful for studying toy models, but it might also be possible to compactify and do deformations of supersymmetric geometries so that connection is made with phenomenologically viable models. By classifying the geometries we delineate the possibilities within the theoretical framework of type IIA supergravity. That should be of some help if you wish to construct new models in type IIA supergravity — whether realistic or not.

At the centre of fundamental physics stands the concept of *symmetry*, often implemented using the mathematics of group theory and Lie algebras. A symmetry transformation is a mathematical transformation which leaves all measurable quantities intact. It can be an internal symmetry, such as changing the overall phase of a complex wavefunction, or it can be an external symmetry, involving angles and distances of spacetime itself, rather than just the components of the fields living in spacetime.

A symmetry transformation typically involves a number of scalar parameters; e.g., the phase shift α of a U(1) transformation of a complex field: $\psi(x) \mapsto e^{i\,\alpha}\,\psi(x)$. If the symmetry parameters, such as α, do not depend on the position x in spacetime, we call it a *global symmetry* or a *rigid symmetry*. If, on the other hand, the symmetry parameters depend on x, e.g. $\psi(x) \mapsto e^{i\,\alpha(x)}\,\psi(x)$, we talk about a *local symmetry* or *gauge symmetry*. Often, when a theory exhibits some global symmetry, it is useful to consider what would happen if the symmetry were local. (That is called *gauging* the symmetry.) Simply replacing α by $\alpha(x)$ would normally mean that the transformation is no longer a symmetry transformation, because of derivatives making trouble. Symmetry is then restored by replacing all derivatives with *covariant derivatives*, which differ from ordinary derivatives by some *connection* or *gauge potential* that you invent with its own transformation rule, created to restore the symmetry. The symmetry transformation, or *gauge transformation* would then be done both to the original fields of the theory according to the original transformation rule, and simultaneously to the gauge potential according to the rule you made up to restore the symmetry. The commutator of the covariant derivative gives you the field strength associated to the gauge potential. According to the rules of quantum field theory, all renormalisable terms that you can construct (still respecting all the desired symmetries) must be added to the Lagrangian of the theory.

The importance of the gauging procedure to fundamental physics can hardly be overstated. For example, take the Dirac field $\psi(x)$, which may be used to describe electrons. The theory is invariant under the U(1) symmetry mentioned above, $\psi(x) \mapsto \mathrm{e}^{\mathrm{i}\,\alpha}\,\psi(x)$. Gauging the symmetry, i.e. making α a function on spacetime, requires us to have a covariant derivative with a gauge potential — the electromagnetic potential — and a corresponding field strength — the electromagnetic field strength, composed of the ordinary electric and magnetic fields. Then the rules for ordinary field theory gives you Maxwell's equations. The rules for quantum field theory gives you quantum electrodynamics, which is capable of describing all physical phenomena of everyday experience (except gravity and nuclear physics). The gauging procedure takes you from the existence of the electron to the full theory of electromagnetism. The standard model of particle physics does essentially the same, but with a larger symmetry group: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, and describes all physical phenomena of everyday experience (except gravity). The corner stones of the standard model of cosmology — dark energy and dark matter — are still left out however, so the standard model of particle physics is not the end of the story.

One attractive solution to the dark matter problem is *supersymmetry*. As a consequence of supersymmetry, all fermion particles get their own boson superpartner, and all boson particles get their own fermion superpartner. From a mathematical point of view, supersymmetry is essentially the manifestation of the following idea: What if the symmetry parameters don't have to be Lorentz scalars? It turns out that it is possible to have symmetry transformations where the symmetry parameter is not a phase shift or some other such Lorentz scalar, but actually a *spinor*. The matter fields of the fermions are spinors, so when we take the symmetry parameter to be a spinor, the symmetry transformation necessarily relates the bosons to fermions, and vice versa. When talking about supersymmetry, one normally means *rigid* supersymmetry; i.e., a global symmetry, whose symmetry parameters do not depend on the point x in spacetime. As you may guess from the preceding discussion, one natural thing to ask when faced with such a global symmetry is if we can make it local — if we can *gauge* it.

It turns out we can, and moreover the resulting theory contains Einstein's theory of general relativity. For this reason, rather than talking about local supersymmetry or gauged supersymmetry, the established term is *supergravity*. It does not mean that the gravity is super-strong and that we are treating black holes or something (though black holes *are* interesting objects to study in supergravity theories); it simply means that there is local supersymmetry and there is gravity. Supergravity means that we have a symmetry whose symmetry parameter is a spinor that depends on the position x in the spacetime.

If the Lagrangian (or the action) of the theory is invariant under the symmetry transformation we say that the theory has the symmetry. (Otherwise the transformation wouldn't be a symmetry transformation.) If the theory has the symmetry, then the equations of motion (loosely speaking "the laws of physics") have that symmetry. This doesn't necessarily mean that the solution has that symmetry, however. There are a variety of ways to break a symmetry, and I won't go into the details here. Suffice to say, that the solution has a symmetry if it is invariant under the symmetry transformation. We are looking for supersymmetric geometries, and we get them by insisting that the solution is invariant under the supersymmetry transformation. If it is, we call the supersymmetry parameter a *Killing spinor*.

The most promising attempt at a quantum theory of gravity is widely regarded to be string theory, of which there are various types related by certain limits and dualities. In the limit where quantum gravity effects are small, these string theories give rise to different types of supergravity.

The focus of this work has been type IIA supergravity, and the classification of supersymmetric type IIA geometries. Type IIA supergravity is a ten-dimensional theory which can be obtained either by taking a certain limit in type IIA string theory, or by doing a dimensional reduction of eleven-dimensional supergravity, which is the supergravity theory with the highest possible dimensionality.

Why study IIA supergravity?

Because there is no systematic classification of type IIA geometries yet. We start from one Killing spinor (minimal supersymmetry), and make no assumptions. From this we obtain the most general structure that *all* supersymmetric solutions must satisfy, since all supersymmetric solutions will have at least one Killing spinor.

Type IIA supergravity also has a two-form field strength, just like the ordinary electromagnetic field. That means that the intuition physicists have developed for Maxwell's theory applies to (at least part of) solutions of IIA supergravity. It is e.g. possible to have a black hole with some electric charge in this theory.

A systematic classification *has* been done before for eleven-dimensional supergravity [1]. So why bother with IIA supergravity, when we can get everything from eleven dimensions? One reason is the Romans cosmological constant S of massive IIA supergravity: you don't get that from eleven dimensions, only the massless version of IIA supergravity; and some things that may be difficult to do in eleven dimensions, such as the study of black holes, may be easier to do in IIA theory directly, compared to doing the work in eleven dimensions and follow up by a dimensional reduction. The analysis in eleven dimensions would have to deal with all the higher Kaluza–Klein modes in type IIA.

Next, we turn in chapter 2 to spinors and the $\mathrm{Spin}(p, q)$ groups. In chapter 3 we proceed to the Killing spinor equations and the method of spinorial geometry. The method results in a linear system of equations (the exact expression is appended the paper), and in chapter 4 the reader is shown how to go about in simplifying it to arrive at the final expressions found in the paper.

Chapter 2

Spinors

2.1 Introduction

Clifford algebra, also called geometric algebra, is the algebra of physical space. Space is spanned by a number of basis vectors $\boldsymbol{e}_A$, called the generators of the Clifford algebra, since any element in the algebra can be written as a linear combination of the Clifford product, or geometric product, of such generators.

The symmetric part of the geometric product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is the ordinary scalar product:

$$\frac{1}{2}(\boldsymbol{x}\boldsymbol{y} + \boldsymbol{y}\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{y}. \tag{2.1a}$$

This can also be written in terms of the basis vectors $\boldsymbol{e}_A$ and the metric g_{AB}:

$$\boldsymbol{e}_A\,\boldsymbol{e}_B + \boldsymbol{e}_B\,\boldsymbol{e}_A = 2\,g_{AB}. \tag{2.1b}$$

The antisymmetric part of the geometric product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is the wedge product (in three dimensions this is dual to the familiar cross product):

$$\frac{1}{2}(\boldsymbol{x}\boldsymbol{y} - \boldsymbol{y}\boldsymbol{x}) = \boldsymbol{x} \wedge \boldsymbol{y}. \tag{2.2a}$$

In terms of the basis vectors $\boldsymbol{e}_A$:

$$\boldsymbol{e}_A\,\boldsymbol{e}_B - \boldsymbol{e}_B\,\boldsymbol{e}_A = 2\,\boldsymbol{e}_A \wedge \boldsymbol{e}_B. \tag{2.2b}$$

Sometimes, e_{AB} is used to denote $\boldsymbol{e}_A \wedge \boldsymbol{e}_B$.

The Clifford product is associative and distributive. As always, using an orthonormal basis will simplify your life a bit.

The natural thing for a physicist, when faced with an associative algebra, is to consider a matrix representation of the algebra. Each basis vector $\boldsymbol{e}_A$ has an associated *gamma matrix* $\mathit{\Gamma}_A$. The expression (2.1b) is often rendered as

$$\mathit{\Gamma}^A \mathit{\Gamma}^B + \mathit{\Gamma}^B \mathit{\Gamma}^A = 2g^{AB} \tag{2.3}$$

in the matrix representation, where g^{AB} is the matrix inverse of the metric g_{AB}.

Matrices can of course be considered to be linear transformations acting on some sort of multi-component objects. The space that the gamma matrices act upon is called *spinor space.* One way of thinking about spinors is that they are simply the things the gamma matrices act upon, though some people prefer the name "pinor" for this.

The Pin(n) group is the set of all reflections and rotations in n dimensions, and can be constructed from the Clifford algebra $\mathcal{C}\ell_n$. It is the double cover of the O(n) group of orthogonal transformations: for each rotation or reflection in the O(n) group there are two elements in the Pin(n) group. If $s \in \mathrm{Pin}(n) \subset \mathcal{C}\ell_n$, both s and $-s$ correspond to the same O(n) transformation, and affect vectors (and tensors) in the same manner, making the same reflection or rotation. Their action on spinors (or pinors) is different though, corresponding to the concept of orientation entanglement.

The Spin(n) group is a subgroup of Pin(n) that only contain the rotations (and Lorentz boosts, if we are working with a Lorentzian spacetime). If the distinction is made, a spinor is a representation of the Spin(n) group, whereas a pinor is a representation of Pin(n). Thus, when talking about irreducible representations of the Spin(n) group, a spinor may actually have fewer components than the size of the gamma matrices. In even dimensions, the general spinor splits into left-handed and right-handed Weyl spinors, each transforming separately under Spin(n) — though still related by elements of the larger Pin(n) group (a reflection can switch handedness).

The Spin(n) group may be defined in terms of the Clifford algebra $\mathcal{C}\ell_n$ as [2]

$$\mathrm{Spin}(n) \coloneqq \left\{ s \in \mathcal{C}\ell_n^+ \;\middle|\; s\, s^{\mathrm{t}} = 1, \forall \boldsymbol{x} \in \mathbb{R}^n, s\boldsymbol{x}s^{-1} \in \mathbb{R}^n \right\}, \tag{2.4}$$

where $\mathcal{C\ell}_n^+$ are the even grade elements of the Clifford algebra in n dimensions, and s^{t} is the reversal (or transpose) of s. The action of $\mathrm{Spin}(n)$ on a vector is $\boldsymbol{x} \mapsto s\boldsymbol{x}s^{-1}$, and it is a double cover of $\mathrm{SO}(n)$, since s and $-s$ correspond to the same transformation on the vector. Since the elements of the spin group act on a spinor from the left only, s and $-s$ would still be distinct transformations if applied to a spinor.

The generalisation to Lorentzian spacetime and groups like $\mathrm{Spin}(9,1)$ is straightforward: just use a Lorentzian metric in the definition of $\mathcal{C\ell}_{9,1}$. Also, since it is not possible to normalise all invertible elements $s \in \mathcal{C\ell}_{9,1}^+$ to $+1$, we define

$$\mathrm{Spin}(p,q) \coloneqq \left\{ s \in \mathcal{C\ell}_{p,q}^+ \;\middle|\; s\,s^{\mathrm{t}} = \pm 1, \forall \boldsymbol{x} \in \mathbb{R}^{p,q}, s\boldsymbol{x}s^{-1} \in \mathbb{R}^{p,q} \right\}. \tag{2.5}$$

This means that $\mathrm{Spin}(p,q)$ has two disconnected pieces: the proper, orthochronous Lorentz transformations with $s\,s^{\mathrm{t}} = +1$, and the improper, non-orthochronous transformations with $s\,s^{\mathrm{t}} = -1$. The part connected to the identity transformation is sometimes called $\mathrm{Spin}_+(p,q)$:

$$\mathrm{Spin}_+(p,q) \coloneqq \left\{ s \in \mathcal{C\ell}_{p,q}^+ \;\middle|\; s\,s^{\mathrm{t}} = +1, \forall \boldsymbol{x} \in \mathbb{R}^{p,q}, s\boldsymbol{x}s^{-1} \in \mathbb{R}^{p,q} \right\}. \tag{2.6}$$

Going to the matrix representation, an $s \in \mathcal{C\ell}_n^+$ is represented by a linear combination of even powers of gamma matrices, such as $\exp\left(\frac{1}{2}\,\theta\,\mathit{\Gamma}_{12}\right)$ for a rotation by an angle θ in the $\boldsymbol{e}_1\boldsymbol{e}_2$ plane, where $\mathit{\Gamma}_{12} = \frac{1}{2}\left(\mathit{\Gamma}_1\,\mathit{\Gamma}_2 - \mathit{\Gamma}_2\,\mathit{\Gamma}_1\right)$ is the matrix corresponding to $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2$. The vector $\boldsymbol{x}$ would be represented by the matrix $x^A\,\mathit{\Gamma}_A$ — in the matrix representation of the Clifford algebra, the vectors are matrices too.

2.2 Curved space

Handling scalars, vectors and higher tensors in curved space is not too difficult — any course in general relativity will cover the basics. Scalars are particularly simple, and the vectors (and tensors) are essentially handled by replacing ordinary derivatives, $\partial_M V^N$, by covariant derivatives,

$$\nabla_M V^N \coloneqq \partial_M V^N + \mathit{\Gamma}^N_{ML}\,V^L,$$

where the $\mathit{\Gamma}^N_{ML}$ are the Christoffel symbols, or connection coefficients.

Digging a little deeper, the concept of a vector requires a bit more thought compared to the case of flat space, since the common “it’s an arrow” intuition doesn’t really work all that well out of the box if the space is curved. The components V^M of a vector are properly seen to multiply the basis vectors, $\boldsymbol{V} = V^M \boldsymbol{e}_M$, but how do we actually make sense of the basis vectors $\boldsymbol{e}_M$? The answer lies in directional derivatives. Already in flat space, there is a one-to-one correspondence between a vector $\boldsymbol{v}$ and the associated directional derivative $\boldsymbol{v} \cdot \nabla$ at a point. Nothing prevents us from taking the directional derivative to *be* the definition of a vector. Thus $\boldsymbol{V} \equiv V^M \boldsymbol{e}_M \equiv V^M \partial_M$. The directional derivative makes perfect sense even on curved manifolds. The vector space spanned by the partial derivatives ∂_M evaluated at a point p is called the tangent space of the manifold at that point, and may be visualised as a flat infinite space laying tangent to the manifold at the point, like a plane laying tangent on a circle.

This formalism doesn’t work for spinors. To work with a curved spacetime, we want to represent the basis vectors $\boldsymbol{e}_M$ by a derivative ∂_M, but to work with spinors, we want to represent the basis vectors $\boldsymbol{e}_A$ by some gamma matrix Γ_A. Clearly, we cannot do both at the same time. Clearly, we *need* to do both at the same time.

In order to handle spinors on a curved manifold we need vielbeins, which essentially translate back and forth between curved indices as in $\boldsymbol{e}_M$ (which we identify with the derivative ∂_M) and flat indices as in $\boldsymbol{e}_A$ (which we identify with the gamma matrix Γ_A). Since we know how to handle spinors in flat space, and curved space is still locally flat, we assign a local frame with an orthonormal basis $\{\boldsymbol{e}_A\}$ at each point of the spacetime, related to the tangent space of the manifold by $\boldsymbol{e}_A = e_A^M \partial_M$, where $A, B, \ldots$ are the flat indices and $M, N, \ldots$ are the curved indices. They are related by the *vielbein* e_A^M. Instead of the Christoffel symbol, we have the spin connection, $\Omega_{M,AB}$ (which is antisymmetric in A and B). The expression for the covariant derivative of a vector expressed in flat indices is then

$$\nabla_M V^A = \partial_M V^A + \Omega_{M,}{}^A{}_B V^B.$$

But the real advantage is that with $\Omega_{M,AB}$, unlike the Γ^L_{MN}, we can act on a spinor ε:

$$\nabla_M \varepsilon = \partial_M \varepsilon + \frac{1}{4}\, \Omega_{M,AB}\, \Gamma^{AB}\, \varepsilon.$$

This will be necessary when we turn to supergravity and the Killing spinor equations.

Chapter 3

Supergravity

The fields of type IIA supergravity are (bosonic) the graviton g_{MN}, the dilaton Φ, the NSNS 2-form B_{MN}, the RR 1-form C_M, the RR 3-form C_{MNP}; and (fermionic) one Majorana non-chiral gravitino ψ_M, and one Majorana non-chiral dilatino λ. We use H for the NSNS 3-form field strength and $\tilde{S}$, $\tilde{F}$ and $\tilde{G}$ for the RR k-form field strength. The latter all tend to come with the dilaton as $\mathrm{e}^{\Phi}\tilde{S}$, $\mathrm{e}^{\Phi}\tilde{F}$ and $\mathrm{e}^{\Phi}\tilde{G}$, so we will absorb a factor of e^{Φ} into them and drop the tilde.

We are looking for classical supergravity solutions. Classical solutions have vanishing fermion fields.

A supersymmetry transformation relates bosons to fermions and fermions to bosons. The supersymmetry variation of a boson will be given by the fermionic fields and the supersymmetry parameter (which is a spinor), possibly multiplied by some Γ_M and numerical factors. For example

$$\delta\Phi = \tfrac{1}{2}\,\bar{\varepsilon}\,\lambda, \tag{3.1a}$$

$$\delta e_M^A = \bar{\varepsilon}\,\Gamma^A\,\psi_M. \tag{3.1b}$$

These, and the other bosonic supersymmetry variations, can be found e.g. in [3]. The precise expressions do not concern us, only that all of them are built from the fermionic fields: the graviton ψ_M and the dilatino λ. Since classical solutions have $\psi_M = 0$ and $\lambda = 0$, we know that the supersymmetry variations of the bosons vanish automatically. When looking for classical supergravity solutions, we will not get any constraints from the variation of the bosons.

The supersymmetric variations of the graviton ψ_M and the dilatino λ are:

$$\begin{aligned}\delta\psi_M &= \nabla_M\,\varepsilon + \tfrac{1}{8}\,H_{MP_1P_2}\,\Gamma^{P_1P_2}\,\Gamma_{11}\,\varepsilon + \tfrac{1}{8}\,\mathrm{e}^{\Phi}\tilde{S}\,\Gamma_M\,\varepsilon + \\ &\quad + \tfrac{1}{16}\,\mathrm{e}^{\Phi}\,\tilde{F}_{P_1P_2}\,\Gamma^{P_1P_2}\,\Gamma_M\,\Gamma_{11}\,\varepsilon + \tfrac{1}{8\cdot 4!}\mathrm{e}^{\Phi}\,\tilde{G}_{P_1\cdots P_4}\,\Gamma^{P_1\cdots P_4}\,\Gamma_M\,\varepsilon, \qquad (3.2)\\ \delta\lambda &= \partial_P\Phi\,\Gamma^P\,\varepsilon + \tfrac{1}{12}\,H_{P_1P_2P_3}\,\Gamma^{P_1P_2P_3}\,\Gamma_{11}\,\varepsilon + \tfrac{5}{4}\,\mathrm{e}^{\Phi}\tilde{S}\,\varepsilon + \\ &\quad + \tfrac{3}{8}\,\mathrm{e}^{\Phi}\,\tilde{F}_{P_1P_2}\,\Gamma^{P_1P_2}\,\Gamma_{11}\,\varepsilon + \tfrac{1}{4\cdot 4!}\,\mathrm{e}^{\Phi}\tilde{G}_{P_1\cdots P_4}\,\Gamma^{P_1\cdots P_4}\,\varepsilon. \qquad (3.3)\end{aligned}$$

Unlike the bosonic case, these variations will not automatically vanish. We need to set the variations to zero, and these equations need to be solved.

3.1 Killing spinors

The concept of a Killing vector will be familiar to anyone who has studied differential geometry or general relativity. A Killing vector is a coordinate independent way of describing a bosonic symmetry. There is a certain maximal amount of symmetry that the geometry can have in a given number of dimensions, and the number of (linearly independent) Killing vectors tells you what amount of symmetry you have.

Fewer will be familiar with the concept of a *Killing spinor*. Similarly to the bosonic case, there is a maximum amount of possible supersymmetry, and the number of Killing spinors tells you how much of that supersymmetry is realised for a given solution.

Once we have a spinor, we can construct spacetime form bilinears. (This is discussed further in section 3.2.1.) A one-form corresponds to a vector, and so it turns out that we can get a Killing vector from a Killing spinor. A vector constructed from spinors this way is quadratic in the spinors. The Killing vector is in some sense the square of the Killing spinor.

3.1.1 The Killing spinor equations

A spinor ε is a Killing spinor if it fulfils the Killing spinor equations,

$$\mathcal{D}_M\,\varepsilon = 0 \qquad \text{and} \qquad \mathcal{A}\,\varepsilon = 0 \qquad (3.4)$$

where

$$\mathcal{D}_M\,\varepsilon = \nabla_M\,\varepsilon + \tfrac{1}{8}\,H_{MP_1P_2}\,\Gamma^{P_1P_2}\,\Gamma_{11}\,\varepsilon + \tfrac{1}{8}\,\mathrm{e}^{\Phi}\tilde{S}\,\Gamma_M\,\varepsilon + \\ + \tfrac{1}{16}\,\mathrm{e}^{\Phi}\,\tilde{F}_{P_1P_2}\,\Gamma^{P_1P_2}\,\Gamma_M\,\Gamma_{11}\,\varepsilon + \tfrac{1}{8\cdot 4!}\mathrm{e}^{\Phi}\,\tilde{G}_{P_1\cdots P_4}\,\Gamma^{P_1\cdots P_4}\,\Gamma_M\,\varepsilon, \qquad (3.5)$$

$$\mathcal{A}\varepsilon = \partial_P\Phi\,\Gamma^P\,\varepsilon + \tfrac{1}{12}\,H_{P_1P_2P_3}\,\Gamma^{P_1P_2P_3}\,\Gamma_{11}\,\varepsilon + \tfrac{5}{4}\,\mathrm{e}^{\Phi}\tilde{S}\,\varepsilon + \\ + \tfrac{3}{8}\,\mathrm{e}^{\Phi}\,\tilde{F}_{P_1P_2}\,\Gamma^{P_1P_2}\,\Gamma_{11}\,\varepsilon + \tfrac{1}{4\cdot 4!}\,\mathrm{e}^{\Phi}\tilde{G}_{P_1\cdots P_4}\,\Gamma^{P_1\cdots P_4}\,\varepsilon. \qquad (3.6)$$

Note that (3.5) and (3.6) are simply the gravitino equation (3.2) and the dilatino equation (3.3), respectively, with the Killing spinor ε as the supersymmetry parameter. Sometimes we call $\mathcal{D}_M$ the supercovariant derivative.

3.1.2 Handling spinors

There are two methods for handling spinors: using spinor bilinears as e.g. in [4, 5], or taking the spinorial geometry approach. When using spinor bilinears, we don't have explicit expressions for the spinors themselves. It is possible to find algebraic relations between the various bilinears using Fierz identities, and take it from there. The downside of that method, is that it takes a fundamentally linear problem, $\mathcal{D}_M\,\varepsilon = 0$ and $\mathcal{A}\,\varepsilon = 0$, and turns it into a non-linear problem.

The method named spinorial geometry [1], on the other hand, uses explicit spinors, and the problem remains linear. We express the spinors in terms of forms, not to be confused with the spacetime forms. The general spinor is spanned by the forms

$$\mathbb{1}, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4, \mathsf{e}_5, \mathsf{e}_{12}, \mathsf{e}_{13}, \mathsf{e}_{14}, \mathsf{e}_{15}, \mathsf{e}_{23}, \mathsf{e}_{24}, \mathsf{e}_{25}, \mathsf{e}_{34}, \mathsf{e}_{35}, \mathsf{e}_{45}, \\ \mathsf{e}_{123}, \mathsf{e}_{124}, \mathsf{e}_{125}, \mathsf{e}_{134}, \mathsf{e}_{135}, \mathsf{e}_{145}, \mathsf{e}_{234}, \mathsf{e}_{235}, \mathsf{e}_{245}, \mathsf{e}_{345}, \qquad (3.7) \\ \mathsf{e}_{1234}, \mathsf{e}_{1235}, \mathsf{e}_{1245}, \mathsf{e}_{1345}, \mathsf{e}_{2345}, \text{ and } \mathsf{e}_{12345},$$

where $\mathsf{e}_{ab} = \mathsf{e}_a \wedge \mathsf{e}_b$ and so on, with coefficients that are functions on spacetime. This is the exterior algebra of $\mathbb{R}^5$ (with complex coefficents).

The Γ_A matrices are realised in terms of the wedge product and left contraction:

$$\Gamma_0\,\eta = -\mathsf{e}_5 \wedge \eta + \mathsf{e}_5 \lrcorner\, \eta \qquad \Gamma_i\,\eta = \mathsf{e}_i \wedge \eta + \mathsf{e}_i \lrcorner\, \eta \\ \Gamma_5\,\eta = \mathsf{e}_5 \wedge \eta + \mathsf{e}_5 \lrcorner\, \eta \qquad \Gamma_{5+i} = \mathrm{i}\,\mathsf{e}_i \wedge \eta - \mathrm{i}\,\mathsf{e}_i \lrcorner\, \eta \qquad (3.8) \\ i = 1, 2, 3, 4.$$

A change of basis allows us to see gamma matrices as creation and annihilation operators:

$$
\begin{array}{ll}
\underline{\textsf{Annihilation operators}} & \underline{\textsf{Creation operators}} \\
\Gamma_{\bar\alpha} = \frac{1}{\sqrt{2}}\left(\Gamma_a + \mathrm{i}\,\Gamma_{a+5}\right) = \sqrt{2}\,\mathsf{e}_a \lrcorner & \Gamma_{\alpha} = \frac{1}{\sqrt{2}}\left(\Gamma_a - \mathrm{i}\,\Gamma_{a+5}\right) = \sqrt{2}\,\mathsf{e}_a \wedge \\
\Gamma_{+} = \frac{1}{\sqrt{2}}\left(\Gamma_5 + \Gamma_0\right) = \sqrt{2}\,\mathsf{e}_5 \lrcorner & \Gamma_{-} = \frac{1}{\sqrt{2}}\left(\Gamma_5 - \Gamma_0\right) = \sqrt{2}\,\mathsf{e}_5 \wedge
\end{array}
\tag{3.9}
$$

Note that such a change of basis changes the metric as well: In this basis the nonvanishing components are $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$ and $g_{+-} = 1$. This means that $\Gamma^{\bar\alpha}$ is a creation operator. Furthermore,

$$
\Gamma^{\bar\alpha_1\cdots\bar\alpha_k}\,\mathbb{1} \quad \text{and} \quad \Gamma^{\bar\alpha_1\cdots\bar\alpha_k +}\,\mathbb{1} \quad \text{where } k = 0,1,2,3,4 \tag{3.10}
$$

is a basis in the space of spinors, where $\mathbb{1}$ is the vacuum of our creation/annihilation operator algebra, in the sense that it is annihilated by all the annihilation operators. (The basis elements in (3.10) only differ from those of (3.7) by powers of $\sqrt{2}$.)

The Γ_A matrices are 32×32 matrices in ten dimensions, which means that a general spinor has 32 components. (An irriducible spinor has 16 real components, since the general spinor decomposes into two Majorana–Weyl spinors with opposite chirality.) Studying spinors with 32 functions on spacetime would be a formidable task, even if the problem is linear. It turns out that the problem can be simplified further by an expedient gauge choice. A $\mathrm{Spin}(9,1)$ gauge transformation can be used to transform a general spinor into one of four cases.

3.1.3 Four orbits

There are four orbits of spinors under the $\mathrm{Spin}(9,1)$ gauge transformation. Two spinors are in the same orbit if they are related by some element in the symmetry group (up to normalisation, in this context). If two spinors are in different orbits, it is not possible to relate them to each other using the symmetry group.

Some subset of the $\mathrm{Spin}(9,1)$ group will relate a given spinor to all the other spinors in the same orbit. Some other subset of $\mathrm{Spin}(9,1)$ will leave the spinor invariant. That subset is called the *stability subgroup* (also called the *isotropy group*). It turns out that the four orbits have different stability subgroups, and may be characterised by them: $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$, $\mathrm{Spin}(7)$, $\mathrm{SU}(4)$ and $G_2 \ltimes \mathbb{R}^8$. In each case we can choose a representative spinor to use in the Killing spinor equations; instead of treating a generic 32-component spinor, we consider

$$\varepsilon = f\,(\mathbb{1} + \mathsf{e}_{1234}) + g\,(\mathsf{e}_5 + \mathsf{e}_{12345}) \tag{3.11}$$

with $g \neq 0$ in the $\mathrm{Spin}(7)$ case and $g = 0$ in the $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ case,

$$\varepsilon = f\,(\mathbb{1} + \mathsf{e}_{1234}) + g_1\,(\mathsf{e}_5 + \mathsf{e}_{12345}) + \mathrm{i}g_2\,(\mathsf{e}_5 - \mathsf{e}_{12345}) \tag{3.12}$$

with $g_2 \neq 0$ in the $\mathrm{SU}(4)$ case, and

$$\varepsilon = f\,(\mathbb{1} + \mathsf{e}_{1234}) + g\,(\mathsf{e}_1 + \mathsf{e}_{234}) \tag{3.13}$$

with $g \neq 0$ in the $G_2 \ltimes \mathbb{R}^8$ case.

Although some work has been done on the $\mathrm{SU}(4)$ case (and to some extent on the $G_2 \ltimes \mathbb{R}^8$ case), only the $\mathrm{Spin}(7)$ and $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ cases are ready for publication at the time of writing, so we shall be concentrating on those. The $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ case can be considered a special case of the $\mathrm{Spin}(7)$ case, with enhanced symmetry.

3.2 Spin(7)

The $\mathrm{Spin}(n)$ group is known as the double cover of the $\mathrm{SO}(n)$ group, and may be defined in terms of the Clifford algebra $\mathcal{C}\ell_n$ as in (2.4):

$$\mathrm{Spin}(n) \coloneqq \left\{\, s \in \mathcal{C}\ell_n^+ \;\middle|\; s\,s^{\mathrm{t}} = 1, \forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{s}\boldsymbol{x}\boldsymbol{s}^{-1} \in \mathbb{R}^n \,\right\}.$$

Thus $\mathrm{Spin}(7)$ seems closely linked to seven-dimensional space — and yet $\mathrm{Spin}(7)$ often pops up in the study of eight-dimensional manifolds. Indeed, the study of (3.11) yields an eight-dimensional submanifold with $\mathrm{Spin}(7)$ structure, and two orthogonal directions (one space, one time).

This may seem surprising at first. How can $\mathrm{Spin}(7)$ be embedded into an eight-dimensional setting?

The naive answer would be to see Spin(7) as a subgroup of Spin(8), obtained by simply taking the generators of $\mathcal{C}\ell_7$ from a seven-dimensional subspace of $\mathbb{R}^8$. We might simply think of Spin(7) as a subgroup of Spin(8) that leaves a certain vector, say $\boldsymbol{e}_8$, invariant.

But alas, here we talk of a Spin(7) which doesn't leave *any* vector in $\mathbb{R}^8$ invariant. Instead, it leaves a four-form ϕ invariant. In the naive construction, that seems impossible. This is not your naive Spin(7).

An element s of the Spin(n) group acts on a vector $\boldsymbol{x} \in \mathbb{R}^n$ by $s\boldsymbol{x}s^{-1}$, producing some rotation. However, our Spin(7) acts on a vector $\boldsymbol{x} \in \mathbb{R}^8$ as $\boldsymbol{x} \mapsto s\boldsymbol{x}$. Our Spin(7) acts only on the left, treating the vectors as if they were spinors. Our Spin(7) is a subgroup of O(8)! It is a subgroup of O(8) which leaves a four-form ϕ invariant; or in other words, it leaves the ternary cross product in eight dimensions invariant. ϕ is called the Cayley form and was defined in Harvey and Lawson [6],[1] by identifying $\mathbb{R}^8$ with the octonions (Cayley numbers). This invariant tensor of Spin(7) may be defined as

$$\phi = \mathrm{Re}\,\chi - \frac{1}{2}\,\omega \wedge \omega, \tag{3.14}$$

where

$$\begin{aligned}
\omega &= -\boldsymbol{e}^1 \wedge \boldsymbol{e}^6 - \boldsymbol{e}^2 \wedge \boldsymbol{e}^7 - \boldsymbol{e}^3 \wedge \boldsymbol{e}^8 - \boldsymbol{e}^4 \wedge \boldsymbol{e}^9,\\
\chi &= (\boldsymbol{e}^1 + \mathrm{i}\,\boldsymbol{e}^6) \wedge (\boldsymbol{e}^2 + \mathrm{i}\,\boldsymbol{e}^7) \wedge (\boldsymbol{e}^3 + \mathrm{i}\,\boldsymbol{e}^8) \wedge (\boldsymbol{e}^4 + \mathrm{i}\,\boldsymbol{e}^9).
\end{aligned}$$

In an SU(4) context, we would think of ω as the Kähler form, and χ as a holomorphic volume form.

To be precise, the generators of our Spin(7) are the fifteen SU(4) generators $\tilde{M}_{\alpha\bar{\beta}}$, where the tilde is used to denote the traceless part ($\tilde{M}_\alpha{}^\alpha = 0$), and the six generators of the form

$$M^+_{\alpha\beta} \coloneqq M_{\alpha\beta} + \frac{1}{2}\,\epsilon_{\alpha\beta}{}^{\bar{\gamma}\bar{\delta}}\,M_{\bar{\gamma}\bar{\delta}}.$$

M_{AB} is represented by S_{AB} when acting on spinors, and by J_{AB} when acting on vectors and tensors, where

$$S^{AB} = \frac{\mathrm{i}}{4}\,[\Gamma^A, \Gamma^B], \tag{3.15}$$

$$(J^{AB})_{CD} = \mathrm{i}\left(\delta^A_C\delta^B_D - \delta^A_D\delta^B_C\right). \tag{3.16}$$

[1]For an earlier discussion of Spin(7) manifolds, see Bonan [7].

Above, we have introduced an Hermitian basis, denoted by greek indices $\alpha, \beta, \gamma, \ldots$:

$$\begin{cases} \boldsymbol{e}^{\alpha} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}^{a} + \mathrm{i}\boldsymbol{e}^{a+5}\right), \\ \boldsymbol{e}^{\bar{\alpha}} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}^{a} - \mathrm{i}\boldsymbol{e}^{a+5}\right), \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{e}^{a} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}^{\alpha} + \boldsymbol{e}^{\bar{\alpha}}\right), \\ \boldsymbol{e}^{a+5} = -\frac{\mathrm{i}}{\sqrt{2}}\left(\boldsymbol{e}^{\alpha} - \boldsymbol{e}^{\bar{\alpha}}\right), \end{cases} \tag{3.17}$$

for $a = 1, \ldots, 4$. (Note that (3.17) follows from (3.9).) This basis will be especially useful when we solve the Killing spinor equations in terms of irreducible SU(4) representations in section 4.1, but when working with the Spin(7) invariant spinor (3.11) we will want to make the connection to the invariant four-form ϕ of (3.14). In the Hermitian basis, we have

$$\omega_{\alpha\bar{\beta}} = -\mathrm{i}\, g_{\alpha\bar{\beta}}, \tag{3.18}$$

$$\chi_{\alpha_1\alpha_2\alpha_3\alpha_4} = 4\, \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}. \tag{3.19}$$

We will also want to know the contractions of ϕ and the covariant derivative on ϕ (which is constant in the local Lorentz frame):

$$\phi_{i\ell_1\ell_2\ell_3}\, \phi^{j\ell_1\ell_2\ell_3} = 42\, \delta_i^j, \tag{3.20}$$

$$\phi_{i_1i_2\ell_1\ell_2}\, \phi^{j_1j_2\ell_1\ell_2} = -4\, \phi_{i_1i_2}{}^{j_1j_2} + 12\, \delta_{[i_1i_2]}^{j_1j_2}, \tag{3.21}$$

$$\phi_{i_1i_2i_3\ell}\, \phi^{j_1j_2j_3\ell} = -9\, \delta_{[i_1}^{j_1}\, \phi_{i_2i_3]}{}^{j_2j_3]} + 6\, \delta_{[i_1i_2i_3]}^{j_1j_2j_3}, \tag{3.22}$$

$$\nabla_A\, \phi_{B_1B_2B_3B_4} = 4\, \Omega_{A,[B_1}{}^{C}\, \phi_{|C|B_2B_3B_4]}. \tag{3.23}$$

i, j, ℓ denote eight-dimensional indices, and A, B, C denote ten-dimensional indices, taking values 0 and 5 (or $+$ and $-$) in addition to the eight of i, j, ℓ.

What we are most interested in are all the possible contractions of all the possible derivatives on ϕ, since that gives us expressions in terms of the spin connection Ω, and relates them to covariant things. When solving the Killing spinor equations we get everything in terms of the spin connection, but it looks a bit nicer to express the result in terms of covariant quantities. (The exact expressions for all the possible contractions of all the possible derivatives on ϕ are left as an exercise for the reader.)

3.2.1 The Spin(7) case

From a spinor we can form spacetime form bilinears: essentially, take a scalar product of two spinors with some suitable number of Γ^A matrices put in between. The ordinary Dirac inner product $D(\eta, \theta) \coloneqq \langle \mathit{\Gamma}_0 \eta, \theta \rangle$ may be used for this (for conventions, see [8]). The spinors we can use to form our bilinears are the Killing spinor ε as well as $\mathit{\Gamma}_{11}\, \varepsilon$.

When $g = 0$ the symmetry enhances to $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$. We can create the spacetime form bilinears $\boldsymbol{e}^-$ and $\boldsymbol{e}^- \wedge \phi$, where $\boldsymbol{e}^-$ signifies a lightcone direction. From the point of view of 11-dimensional supergravity, whence type IIA supergravity may be obtained by compactification, this class of solutions correspond to not having any momentum along the compact direction.

If $g \neq 0$ on the other hand, we *do* have momentum along the compact direction. This is directly reflected in the spacetime forms we get from ε: $\kappa = f^2\, \boldsymbol{e}^0$, $\omega = f^2\, \boldsymbol{e}^0 \wedge \boldsymbol{e}^5$ and $\tau = f^2\, \boldsymbol{e}^0 \wedge \phi$, where we have used gauge symmetry to set $f = \pm g$ in (3.11). (Taking both ε and $\mathit{\Gamma}_{11}\, \varepsilon$ in the inner product, we get the one-form $f^2\, \boldsymbol{e}^5$, the four-form $f^2\, \phi$, and the five-form $f^2\, \boldsymbol{e}^5 \wedge \phi$.)

As shown in section 4.1, the one-form κ is Killing. Getting a timelike Killing vector means that the geometry is independent of time (if the time direction is taken to be in the direction of the Killing vector). A privileged direction in time is something one normally finds in the study of massive objects, which have timelike velocities. The $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ case, on the other hand, doesn't have any timelike one-form. How do we understand this?

The $\mathrm{Spin}(7)$ and $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ cases are related from the point of view of eleven-dimensional supergravity — if we take the $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ solution and boost it along the compact direction, we obtain the $\mathrm{Spin}(7)$ solution. The generic $\mathrm{Spin}(7)$ solution has some momentum running along the compact direction. From the ten-dimensional point of view this looks like a mass.[2] Thus, the generic $\mathrm{Spin}(7)$ case looks like a massive variant of the $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ case.

[2]Not to be confused with the Romans cosmological constant, a.k.a. Romans mass parameter. Type IIA supergravity with nonzero Romans cosmological constant is sometimes called massive IIA supergravity, but these solutions are massive in a different sense.

3.3 Integrability conditions

The integrability conditions are $[\mathcal{D}_M, \mathcal{A}]\,\varepsilon = 0$ and $[\mathcal{D}_M, \mathcal{D}_N]\,\varepsilon = 0$. Naturally, for a Killing spinor (which satisfies $\mathcal{A}\,\varepsilon = 0$ and $\mathcal{D}_M\,\varepsilon = 0$) these will have to be satisfied.

Construct $\mathcal{I}\,\varepsilon = \Gamma^M\,[\mathcal{D}_M, \mathcal{A}]\,\varepsilon$ and $\mathcal{I}_M\,\varepsilon = \Gamma^N\,[\mathcal{D}_M, \mathcal{D}_N]\,\varepsilon$. Clearly, both $\mathcal{I}\,\varepsilon = 0$ and $\mathcal{I}_M\,\varepsilon = 0$. But they are also a linear system in terms of the field equations and Bianchi identities of the theory. For example $\mathcal{I}\,\varepsilon$ looks like

$$\begin{aligned} \mathcal{I}\,\varepsilon = {} & \Big(\mathrm{F}\Phi - \mathrm{F}G_{(3)}\,\Gamma^{(3)} + \mathrm{B}G_{(5)}\,\Gamma^{(5)}\Big)\,\varepsilon + \\ & + \Big(-3\,\mathrm{F}F_{(1)}\,\Gamma^{(1)} + \mathrm{F}H_{(2)}\,\Gamma^{(2)} + \mathrm{B}F_{(3)}\,\Gamma^{(3)} + 2\,\mathrm{B}H_{(4)}\,\Gamma^{(4)}\Big)\,\Gamma_{11}\varepsilon \end{aligned} \tag{3.24}$$

where F stands for field equation and B stands for Bianchi identity; the precise expressions may be found in the appended paper, as well as the expression for $\mathcal{I}_M\,\varepsilon$, which is slightly more involved.

The Bianchi identities are first order equations, and therefore easier to solve than the field equations. If we can get a field equation expressed in terms of Bianchi identities, that tends to simplify things.

Now, some gamma matrices will annihilate ε, and then the corresponding coefficient (field equation or Bianchi identity) drops out of the expression, and that particular representation will be unconstrained by the integrability conditions of the Killing spinor equations. This is why we talk about classifying *geometries* rather than solutions — this approach may leave some field equations that still have to be solved in order to have the full solution. As for those representations that aren't annihilated, we get some field equations that are automatically satisfied, or that are given in terms of the Bianchi identities.

So while we fix many general aspects of the solution, there is still some unconstrained parameters to play with, which is natural, since among others, all solutions with enhanced supersymmetry would be special cases of the one we present here.

Chapter 4

Classification of Supersymmetric Solutions

4.1 Linear system in $\mathrm{SU}(4)$ indices

The spinors can be written in terms of gamma matrices acting on the $\mathbb{1}$ from (3.7). In the case of the Spin(7) invariant spinor of (3.11), we have

$$\varepsilon = f\left(1 + \frac{1}{4}\,\Gamma^{\bar{1}\bar{2}\bar{3}\bar{4}}\right)\mathbb{1} + g\left(\frac{1}{\sqrt{2}}\Gamma^{+} + \frac{1}{4\sqrt{2}}\Gamma^{\bar{1}\bar{2}\bar{3}\bar{4}+}\right)\mathbb{1} \tag{4.1a}$$

or equivalently

$$\begin{aligned}\varepsilon = f\left(1 + \frac{1}{4}\,\frac{1}{4!}\,\epsilon_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\,\Gamma^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\right)\mathbb{1} + \\ + g\left(\frac{1}{\sqrt{2}}\,\Gamma^{+} + \frac{1}{4\sqrt{2}}\,\frac{1}{4!}\,\epsilon_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4}\,\Gamma^{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{\alpha}_4+}\right)\mathbb{1}.\end{aligned} \tag{4.1b}$$

When we act on the spinor ε with $\mathcal{A}$ and $\mathcal{D}_M$ we get terms with gamma matrices on the form $\Gamma^{(a)}\Gamma_{(b)}\Gamma^{(c)}$, where (a) denotes a ten-dimensional indices; e.g. $\Gamma^{(a)} := \Gamma^{A_1A_2\cdots A_a}$ and $\Gamma^{(0)} \equiv 1$. A product of gamma matrices may be simplified using (2.3) — i.e. using the Clifford algebra — and the expression is brought to the form

$$\mathcal{D}_M\,\varepsilon = \sum_{a=0}^{5} X_{(a)}\Gamma^{(a)}\mathbb{1} \tag{4.2}$$

for some $X_{(a)}$. (Here $X_{(0)}$ would be a scalar, and again $\Gamma^{(0)} = 1$.) We have $\mathcal{D}_M\varepsilon = 0$ if and only if all $X_{(i)} = 0$ in said expression. A similar procedure works for the algebraic equation $\mathcal{A}\varepsilon = 0$.

We get a linear system in the fluxes and the spin connection $\Omega_{A,BC}$. When solving this system we organise it in terms of irreducible SU(4) representations. For example, here is a subset of the equations we get from the Spin(7) invariant spinor in the gauge where $f = g$:

$$
\begin{array}{ccc}
\text{SU(4) expression} & & \text{Spin(7) expression} \\
\hline
\mathrm{d}f_0 = 0 & \Leftrightarrow & \mathrm{d}f_0 = 0 \\
\mathrm{d}f_5 = -\frac{1}{2} f\,\Omega_{0,05} & \Leftrightarrow & \mathrm{d}f_5 = -\frac{1}{2} f\,\Omega_{0,05} \\
\mathrm{d}f_\alpha = -\frac{1}{2} f\,\Omega_{0,0\alpha} & \Leftrightarrow & \mathrm{d}f_i = -\frac{1}{2} f\,\Omega_{0,0i} \\
\Omega_{5,05} = 0 & \Leftrightarrow & \Omega_{5,05} = 0 \\
\Omega_{5,0\alpha} = -\Omega_{\alpha,05} & \Leftrightarrow & \Omega_{5,0i} = -\Omega_{i,05} \\
\left.\begin{array}{r}
\Omega^{\gamma}{}_{,0\gamma} = -\Omega_{\gamma,0}{}^{\gamma} \\
\Omega^{\text{trless}}_{\bar{\beta},0\alpha} = -\Omega^{\text{trless}}_{\alpha,0\bar{\beta}} \\
\Omega_{(\alpha_1,\alpha_2)0} = 0
\end{array}\right\} & \Leftrightarrow & \Omega_{(i_1,i_2)0} = 0
\end{array}
\tag{4.3}
$$

Of course, with the Spin(7) invariant spinor (3.11), we don't really want SU(4) expressions. We want Spin(7) expressions, with eight-dimensional indices $i, j, \ldots$, rather than the four holomorphic and four anti-holomorphic indices of SU(4). As the table (4.3) above shows, rewriting the SU(4) expressions is often very simple, or even trivial. All the same, Spin(7) is a larger group than SU(4) and sometimes you need to piece together the Spin(7) representation using several of the SU(4) expressions, making the end result much more consise.

Then comes where you try to interpet the equations: What do they really say about the geometry and the fluxes? The spin connection is not a covariant quantity, so what the equations in (4.3) say might not be entirely obvious. In this case, all the equations in (4.3) are captured in

$$\nabla_A \kappa_B + \nabla_B \kappa_A = 0, \tag{4.4}$$

where $\kappa = f^2\,\boldsymbol{e}^0$ is the spacetime one-form spinor bilinear mentioned in section 3.2.1. In other words, κ is a Killing one-form, and the associated vector field K is a Killing vector:

$$\mathcal{L}_K\, g = 0. \tag{4.5}$$

That we should get a Killing vector from our Killing spinor is entirely expected, and these (4.3) are the equations confirming that this is so.

Some of the equations we get from the linear system will be purely geometric constraints, like the equations (4.3). Others involve both the fluxes and the spin connection; then the strategy is to express the fluxes in terms of the geometry.

4.2 Rewriting in terms of Spin(7) expressions

Writing an SU(4) scalar or vector expression in terms of Spin(7) is straightforward, but the higher-degree forms require some more thinking. A two-form F decomposes as

$$\begin{aligned} F &= \tfrac{1}{2}\, F_{AB}\, \boldsymbol{e}^A \wedge \boldsymbol{e}^B \\ &= F_{05}\, \boldsymbol{e}^0 \wedge \boldsymbol{e}^5 + F_{0i}\, \boldsymbol{e}^0 \wedge \boldsymbol{e}^i + F_{5i}\, \boldsymbol{e}^5 \wedge \boldsymbol{e}^i + \tfrac{1}{2}\, F_{ij}\, \boldsymbol{e}^i \wedge \boldsymbol{e}^j \end{aligned} \tag{4.6}$$

where F_{ij} can be further decomposed into two distinct Spin(7) representations: $F_{ij} = F_{ij}^{(\mathbf{7})} + F_{ij}^{(\mathbf{21})}$, where the bold number denotes the number of degrees of freedom in the representation. It is possible for some of these parts to be determined by the geometry (i.e. you can solve for them in terms of $\Omega_{A,BC}$), while other parts can be unconstrained by the Killing spinor equations. Indeed, that is the case for F, where all parts except $F^{(\mathbf{21})}$ are given in terms of the geometry and the Romans cosmological constant of the theory. The exact expression may be found in the paper appended to this thesis.

As in the example above, there is more than one irreducible SU(4) representation corresponding to $F_{ij}^{(\mathbf{7})}$, and more than one corresponding to $F_{ij}^{(\mathbf{21})}$. But even without knowing that F_{ij} decomposes as $F_{ij} = F_{ij}^{(\mathbf{7})} + F_{ij}^{(\mathbf{21})}$ under Spin(7), we are lead to guess the right expressions from the corresponding SU(4) expressions. Consider for example the equation

$$F_{\alpha_1\alpha_2} - \tfrac{1}{2}\, \epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2}\, F_{\bar\beta_1\bar\beta_2} = -2\left(\Omega_{0,\alpha_1\alpha_2} - \tfrac{1}{2}\, \epsilon_{\alpha_1\alpha_2}{}^{\bar\beta_1\bar\beta_2}\, \Omega_{0,\bar\beta_1\bar\beta_2}\right). \tag{4.7}$$

The Levi-Civita tensor ϵ is not an invariant Spin(7) tensor, but rather an SU(4) object. The simple expedient of replacing ϵ by the Spin(7) invariant tensor ϕ results in an expression that reproduces (4.7) with an extra factor of two, as well as another SU(4) equation:

$$F_{\gamma}{}^{\gamma} = -2\,\Omega_{0,\gamma}{}^{\gamma}. \tag{4.8}$$

Now, even though $F_{ij} - \frac{1}{2}\,\phi_{ijk\ell}\,F^{k\ell}$ *is* in the **7** representation of Spin(7), it doesn't mean that it is our $F_{ij}^{(\mathbf{7})}$ in the decomposition $F_{ij} = F_{ij}^{(\mathbf{7})} + F_{ij}^{(\mathbf{21})}$. We want

$$F_{ij}^{(\mathbf{7})} = (P^{(\mathbf{7})})_{ij}^{k\ell}\,F_{k\ell} \tag{4.9}$$

for some projector $P^{(\mathbf{7})}$. Being a projector, we want $P^{(\mathbf{7})}$ to satisfy $(P^{(\mathbf{7})})^2 = P^{(\mathbf{7})}$ — we need to fix the normalisation. The result is

$$F_{ij}^{(\mathbf{7})} = \frac{1}{4}\left(F_{ij} - \tfrac{1}{2}\,\phi_{ijk\ell}\,F^{k\ell}\right). \tag{4.10}$$

Similarly, $F_{ij}^{(\mathbf{21})}$ is given by

$$F_{ij}^{(\mathbf{21})} = \frac{1}{4}\left(3\,F_{ij} - \tfrac{1}{2}\,\phi_{ijk\ell}\,F^{k\ell}\right). \tag{4.11}$$

But we don't have to guess what the Spin(7) representations are from the SU(4) expressions. We can also start from the known expressions for the Spin(7) decompositions of two-, three- and four-forms, found in e.g. [9]. For the three-forms, we have $\Lambda^3(\mathbb{R}^8) = \Lambda^3_{\mathbf{8}} \oplus \Lambda^3_{\mathbf{48}}$ where

$$\Lambda^3_{\mathbf{8}} = \left\{ \star(\alpha \wedge \phi) \mid \alpha \in \Lambda^1(\mathbb{R}^8) \right\}, \quad \Lambda^3_{\mathbf{48}} = \left\{ \alpha \in \Lambda^3(\mathbb{R}^8) \mid \alpha \wedge \phi = 0 \right\}.$$

This can then be rewritten in terms of projectors acting on a general three-form, with

$$(P^{(\mathbf{8})})_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \frac{1}{7} \times \frac{1}{3!}\,\phi^{k j_1 j_2 j_3}\,\phi_{k i_1 i_2 i_3}, \tag{4.12}$$

$$(P^{(\mathbf{48})})_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \delta_{[i_1 i_2 i_3]}^{j_1 j_2 j_3} - (P^{(\mathbf{8})})_{i_1 i_2 i_3}^{j_1 j_2 j_3}. \tag{4.13}$$

We can let the projectors act on our forms, and then write the resulting expressions in irreducible SU(4) expressions, and then go hunt for them in the linear system.

4.3 The end result

After having written out the linear system first in terms of irreducible SU(4) representations, and then rewritten it in terms of Spin(7) representations, it remains to put it in the final form: to piece together the various representations and give the resulting expression for the fluxes in terms of the geometry and other fluxes. Naturally, some of these representations will go into the final expressions still undetermined, as the $F^{(\mathbf{21})}$ mentioned above, while others will be completely determined. The undetermined parts are not completely arbitrary, though, as they will still need to satisfy the field equations.

We also rewrite the spin connection $\Omega_{A,BC}$ in terms of derivatives on the spinor bilinear spacetime forms, including the Spin(7) invariant four-form ϕ:

$$\theta_i = -\frac{1}{36}\nabla^{(8)m}\phi_{mk_1k_2k_3}\phi^{k_1k_2k_3}{}_i, \qquad \theta_5 = -\frac{1}{42}\phi^{k_1k_2k_3k_4}\nabla_{k_1}\phi_{5k_2k_3k_4}.$$

The actual expressions for the fluxes and geometry can be found in the appended paper.

Chapter 5

Outlook

The classification of supersymmetric geometries in type IIA supergravity having the structure group of $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ or $\mathrm{Spin}(7)$ is, of course, only a beginning. The natural next step would be to consider the other possible structure groups: SU(4) and G_2. As it turns out, there is actually an interesting special case with the SU(4) structure group, where the coefficient g_1 in the SU(4) invariant spinor (3.12) vanishes. We believe that this case may be T-dual to the pure spinor case found in type IIB supergravity [8].

Then one could start looking for new interesting solutions. There are some known solutions, but they may not be the most general ones. By plugging in known solutions into this classification, we can see where it is possible to deform the solution in various ways (for instance by turning on some new components of the fluxes). It should also be possible to look for new kinds of supersymmetric black holes, by adding the requirement that there is an horizon in the spacetime. (Outside a black hole there should be a timelike Killing vector field, which becomes a null Killing vector at the horizon. For similar work in other supergravities, see e.g. [10] (heterotic), [11] (IIB), [12, 13] (11D).

There may also be applications to flux compactification. Compactification should be a crucial step in going from the ten-dimensional geometry to something that could make contact with some aspect of the measurable real world. Thus it is possible that the present work might indirectly lay the ground for some phenomenologically viable model.

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