THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

# Field Theories with a Twist


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# *Abstract*

This thesis deals with topological field theories derived from maximally supersymmetric gauge theories. The main part of this work is the three appended research papers, each of which investigates some different aspect of these theories. Leading up to these is the first part of the thesis which aims at introducing the above-mentioned concepts in such a fashion that the papers become legible to even those readers who are less familiar with notions such as topological field theory and $\mathcal{N} = 4$ Yang-Mills theory from the start. This introduction begins with the concept of symmetry, and both gauge- and supersymmetry are explained. This leads on to the introduction of maximally supersymmetric gauge theories, and then finally to outlining the possible ways to create topological field theories from these.

The two last chapters of this thesis are dedicated to the theories featured in the appended papers. Papers I and II both deals with the geometric Langlands-twist of $\mathcal{N} = 4$ Yang-Mills theory and the five-dimensional analog of this. In paper I, spherically symmetric solutions to the localisation equations of both the four- and five-dimensional topological field theories are investigated, whereas paper II presents an explicit expression for the action of the five-dimensional theory. The final paper deals with a topological twisting of (2,0) theory, which was conjectured to be relevant in the understanding of the AGT-correspondence. However, this suggested twist is shown to not result in a topological field theory on a general background by showing that the stress-tensor of the theory cannot in general be both $Q$-exact and conserved.

**Keywords:** Topological Field Theories, Supersymmetric Gauge Theory, Extended Supersymmetry

# *Appended papers*

## *Paper I:*

**Tunneling solutions in topological field theory on** $R \times S^3 \times I$

Louise Anderson and Måns Henningson

*Journal of High Energy Physics, vol 1202.*

## *Paper II:*

**Five-dimensional topologically twisted maximally supersymmetric Yang-Mills theory**

Louise Anderson

*Journal of High Energy Physics, vol 1302.*

## *Paper III:*

**The trouble with twisting (2,0) theory**

Louise Anderson and Hampus Linander

*Submitted*

# *Contents*

*The hardest thing to explain*
*is the glaringly evident*
*which everybody has*
*decided not to see.*
*– Ayn Rand*

## *Chapter 1:*

# *Introduction*

If one is to believe Rand, it is fortunate that this thesis revolves around a world rarely considered evident in everyday life; the very smallest constituents of the world as we know it. These are at the present best understood in terms of kinematics and interactions of elementary particles, and the laws governing these appear amazingly simple and concise. More than that, there seem to be clues pointing towards that all forms of interactions known to man today may be described by one theory; a theory of everything. From such a theory, all other interactions and known laws could in principle be derived. The promises such a theory offers are tempting ones indeed but in the quest for it, there are many obstacles to overcome and with every answer, more questions are presented.

The universe is a system of enormous complexity to say the least and to attempt at describing all of it by one consistent theory poses a huge challenge. This is the reason why research in fundamental physics today to a large extent circles around theories that we know to not be "true" in the sense that they do not describe the world in which we live. However, they do provide us with a simpler setting, a playground so to say, which allows physicists to study certain aspects of the more complicated system which is reality. It is such theories which will have the principal role in this thesis; the so-called maximally supersymmetric Yang-Mills theory together with its more mysterious cousin (2,0) theory. More precisely, this thesis will deal with certain topological versions of these theories.

In the following chapters, I will make a whole-hearted attempt to give a comprehensible introduction to these theories and the concept of supersymmetric gauge theory in general. However, to do this in full has proven to be an overwhelming task. The inevitable result of this is that some previous knowledge on the readers part of quantum field theory as well as introductory group theory must be assumed and required to fully appreciate the work of this thesis.

The structure of this work is as follows: The first three chapters will be of a more general character and provide an introduction to the areas which have been studied in the appended papers. The first one of these will briefly introduce the concept of quantum field theory after which focus will be shifted to the different symmetries possible in such theories. In particular, the concepts of supersymmetry and extended supersymmetry are introduced. Chapter 3 is in full devoted to theories with maximal supersymmetry: maximally supersymmetric Yang-Mills theory and (2,0) theory. The final one of these, chapter 4, will provide an overview of the area of topological field theories and the process of topological twisting, thus completing the first part.

In the two last chapters of the thesis, the concepts introduced in the first part are combined into the particular topological twistings of the maximally supersymmetric theories which have been the subjects of PAPERS I-III. Chapter 5 will attempt to provide a motivation, introduction and summary of the first two of the papers presented in this thesis, and chapter 6 will be likewise related to PAPER III.

## *Chapter 2:*

# *The World as We Know it:*
# *– Symmetries and Quantum Field Theory*

Quantum field theory is the main framework used in modern physics to describe the subatomic world. In this framework, particles are thought of as excitations in some underlying, space-time dependent fields and are emergent after quantisation of these. It is thus the fields which are the fundamental constituents of the theory, and a theory can be completely specified by its field content together with the transformation properties and equations of motion of all fields. A more convenient way of describing the kinematics and interactions of the theory is through the Lagrangian formalism, relying on the possibility of writing down the action of the theory and deriving corresponding equations of motion from the principle of least action.

A well-known feature in most theories in physics is the appearance of some forms of symmetries, which may in certain cases reduce problems that at first glance seem virtually impossible to trivial, or at least solvable. This thesis revolves around theories which are highly symmetrical creatures, and without the presence of this amount of symmetry in them, the work presented in the research papers appended herein would not be possible. In this chapter we shall make an attempt at explaining the kinds of symmetries relevant for this work.

## *2.1 Space-Time Symmetries*

The symmetries which may be easiest to grasp are the global symmetries of space-time, and thus this is where we take our starting point.

The translational symmetries of space and time might be some of the most well-known symmetries present in physical theories. These symmetries correspond to the laws of physics being the same at all points in space-time, meaning that the outcome of a certain experiment will be the same regardless of whether if it is carried out here and now, or in Paris tomorrow morning for example. In an equivalent fashion, rotational invariance of space corresponds to the laws of physics not favouring any specific direction.

With the introduction of special relativity, time and space are treated on equal footing, and rotations in space-time is required to be a symmetry of all physical theories hoping to be compatible with the theory special of relativity. The invariance under this symmetry is known as Lorentz invariance. All of these symmetries may from a group theoretical perspective conveniently be described by stating the group formed by the symmetry transformations, the generators of which will form the corresponding Lie algebra. For example, the Lorenz transformations in $d$ space-like- and one time-like dimension form the group $SO(d,1)$ with the corresponding Lie Algebra $\mathfrak{so}(d,1)$.

In 1967, Coleman and Mandula showed that Lorentz symmetry together with translational symmetry in space-time is the largest amount of global symmetry which may be present in any theory with a mass gap. This maximal amount of symmetry is known as Poincaré symmetry. In the special case where all particles are massless, the Poincaré invariance may be extended to so-called conformal invariance, meaning invariance under scalings which preserve angles. This is however as far as this kind of global symmetry reaches [1].

In the Lagrangian formalism, the symmetries of a theory are captured by the invariance of the Lagrangian under the action of the corresponding symmetry group.

## 2.2 Local Symmetries and Gauge Theories

In addition to the global symmetries, where we have now reached our maximal amount according to Coleman and Mandula, the action of the quantum theories relevant for describing our surroundings also admits another kind of continuous symmetry. This is formed by introducing a coordinate dependence of the generators of the symmetry group. Such a local symmetry is known as a *gauge symmetry*, and the group formed by the gauge transformations has received the rather unimaginative name *gauge group*. Every generator of this group will give rise to a vector field, called a gauge potential, which, (or more often whose field strength) will be present in the Lagrangian of the theory. During quantisation of the theory, these gauge fields give rise to the gauge bosons, or force mediating particles, present in the theory.

Amongst the simplest examples of theories like these is the theory of QED, which has a gauge group of $U(1)$. This group has only one generator, and thus there is a single gauge field present in the theory. The field-strength of this is the familiar electromagnetic field strength, which will be present in the Lagrangian of the theory, and after quantisation, this gauge field will give rise to the photon as we expect in a theory describing electromagnetism. These concepts naturally generalise to non-abelian gauge groups. Theories of this kind have been amongst, if not *the* most successful at describing the world in the history of science.The Standard model of particle physics, which describes all fundamental forces of nature with the exception of gravity, is for example a gauge theory with gauge group $G = SU(3) \times SU(2) \times U(1)$.

Non-abelian gauge theories like this one are in general referred to as *Yang-Mills theories* after Yang and Mills who first introduced the concept by using the gauge group of $SU(2) \times U(1)$ in order to satisfactory formulate the theory of weak interactions as a gauge theory [2].

## *2.3 Supersymmetry*

The Coleman-Mandula theorem is, though highly useful, not entirely water-proof. There is a loophole present in the theorem in that it only applies to bosonic symmetry generators, i.e. symmetries which transforms bosons into bosons and fermions into fermions. There is nothing which prevents us from a further generalisation of the concept of symmetry − analogues as done when going from global to local − from bosonic to fermonic. Such a symmetry will relate fermionic- and bosonic fields to one another, and is known under the name of *supersymmetry*.

The history leading up to the formulation of supersymmetry as we know it is long, with aspects in both two-dimensional field theory and string theory in the late 60's and early 70's. It was not until it was formulated in the setting of a four-dimensional quantum field theory that it became widely known. This original supersymmetric quantum field theory is the now-famous Wess-Zumino model [3, 4]. In 1975, a new, more general version of the Coleman-Mandula theorem was presented in [5], which also included fermionic symmetries, and the work on classifying the different possible supersymmetries commenced. Finally, a classification of all supersymmetries possible and of their representations in dimensions larger than $d = 1 + 1$ was presented by Nahm in 1977 [6]. In the next section, we shall occupy ourselves further with the implications of the introduction of these new fermionic symmetry generators.

As in the case of the bosonic symmetries presented in sections 2.1 and 2.2, where the generators form a Lie algebra, the generators of the supersymmetry form the *supersymmetry algebra*. In general, all symmetry generators may be combined to form a super-Lie algebra, where fermionic objects are represented by anti commuting, i.e. Grassman-odd, quantities whereas bosonic objects are represented by Grassman-even ones. Straight from these claims, one may deduce that two supersymmetry generators must combine to a bosonic symmetry generator, which amongst others will have important consequences in the recipe of topological twisting presented in section 4.2.

There is nothing which prevents us from having more than one generator of the superalgebra, and theories which admit these kinds of symmetries are said to have *extended supersymmetry*. The amount of supersymmetry present in the theory is often specified by the number of times the minimal amount of supersymmetry a theory contains, i.e. the number of generators of the superalgebra, and is denoted by $\mathcal{N}$. The group which rotates the different supercharges is called the *R-symmetry group* of the theory.

However, there is a natural upper limit on the amount of supersymmetry possible, which may be understood by considering the action of the supersymmetry charges on the fields. The supersymmetry generators are as previously mentioned fermionic generators with corresponding charges, $Q$, of helicity $-1/2$, thus they lower the helicity of the fields they act on by one half. Similarly, $\bar{Q}$ raises this helicity by the same amount. Thus the number of generators of the supersymmetry algebra will effect the helicity content of the theory.

The maximal amount of supersymmetry possible is in our case decided by the requirements that the theory should not include gravity, that is, there should not be any fields in the theory with helicity larger than one. Such theories are known as having *maximal supersymmetry*. In ten dimensions, this restriction from helicity content means that there is only one unique supersymmetric Yang-Mills theory [7], whereas in four dimensions, there are three different theories, all with different amount of supersymmetry: $\mathcal{N} = 1$, $\mathcal{N} = 2$, and finally the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory.

It shall be noted that, in theories where this symmetry remains unbroken, any superpartners will have identical quantum numbers with the exception of spin (since the supersymmetry generators commute with all bosonic symmetry generators of the theory). Thus mass and charge will remain unchanged by a supersymmetry transformation, and with this simple observation, we may conclude that any supersymmetry present in nature must be broken at the energy levels available for us to study today. This since our world is void of particle such as a bosonic cousin of the electron or the neutrino. The manners in which this symmetry is broken by nature is a bustling field of research, but lies beyond the scope of this work. All theories featured herein will be theoretical models in which at least some amount of supersymmetry remains unbroken.

*Chapter 3:*

# A Brief History of Theories with Maximal Supersymmetry

Theories with maximal supersymmetry provide excellent training grounds for physicists to study “real world”-problems and phenomena in a simpler setting, since more symmetry in general means easier calculations. Two such theories have been the starting point of the papers appended to this thesis and this section is aimed at providing an introduction to these two theories.

## 3.1 $\mathcal{N} = 4$ Super Yang-Mills theory

Four-dimensional $\mathcal{N} = 4$ Yang-Mills theory has been claimed to be the fruit fly of fundamental physics. It is both maximally supersymmetric as well as conformal; thus a much simpler, much more restricted theory than any theory which one may hope would describe the world as we know it. It is this simplicity that allows for calculations to be carried out therein. These calculations can then in turn be shown to provide (at least) parts of the answers to physical problems.

After this brief introduction to the theory, and the attempt to adequately describe the benefits of studying it, it is high time to consider the actual formulation of the theory itself. Whereas the above section was specific to the four-dimensional case, we shall for the moment adopt a more general approach, applicable to maximally supersymmetric Yang-Mills theory in ten dimensions or less. The starting point will be the unique, maximally supersymmetric Yang-Mills theory in ten dimensions. The process where this is used to obtain lower-dimensional theories is known as *dimensional reduction.*

The ten-dimensional theory has an amazingly simple Lagrangian formulation, given by:

$$\mathcal{S} = \int_{M_{10}} \mathrm{d}^{10}x \ \mathrm{Tr} \left( -\frac{1}{4} F_{MN} F^{MN} + \frac{i}{2} \bar{\lambda} D_M \Gamma^M \lambda \right), \tag{3.1}$$

where $M$,$N$ denotes indices in ten-dimensions. $F_{MN}$ is the field strenght of the gauge field $A_M$, and the fermionic degrees of freedom in the theory are contained in the Majorana-Weyl spinor $\lambda$ (when $M_{10}$ is of Minkowski signature). $\Gamma^M$ denotes here the ten-dimensional gamma matrices which satisfy the Clifford algebra. The action in (3.1) is invariant under a set of supersymmetry transformations, which are best represented as the expression for the variations of the individual fields, and may as such be written as:

$$\delta A_M = i\bar{\varepsilon}\Gamma_M\lambda \qquad \delta\lambda = \frac{1}{2}F_{MN}\Gamma^{MN}\varepsilon, \tag{3.2}$$

where $\varepsilon$ is a the supersymmetry parameter.

The fields in the ten-dimensional theory are naturally dependent on all ten coordinates, but by restricting this and only allowing them to depend on the $d$ first coordinates, a lower-dimensional theory is obtained. In such a setting, the group rotating the $(10-d)$ remaining coordinates will be the $R$-symmetry group of the $d$-dimensional theory. Similarly, the gauge field along these $(10-d)$ directions turns into scalars. The fermion content of the lower-dimensional theory may be obtained by studying how the spinor representation decomposes when space-time is taken to be a product of some $d$-dimensional- and some $10-d$ dimensional manifold. In such a way, all maximally supersymmetric Yang-Mills theories in dimensions less than ten may be obtained [7].

Especially, it should be noted that this gives the symmetry group of the four-dimensional $\mathcal{N}=4$ Yang-Mills theory as $Spin(4)\times Spin(6)_R$ (in Euclidean signature), where the latter factor is the $R$-symmetry group. This is seen by letting the ten-dimensional space-time be a product of some four-manifold $M_4$ and some six-manifold $M_6$ and taking all the fields to be independent on the coordinates along the latter. The Lorentz group of the four-dimensional manifold on which the theory now lives will thus simply be given by the Lorentz group of $M_4$, whereas the $R$-symmetry group will rather be given by the Lorentz group of the manifold subject to dimensional reduction, i.e. $Spin(6)$. The corresponding field content may be obtained by studying the decomposition of the spinor- and vector-representations of the ten-dimensional theory under such a restriction of space-time.

In general, such a lower-dimensional theory will contain auxiliary fields in order to make the supersymmetry invariance manifest. In some dimensions however, these may easily be removed from the formulation by altering the requirement on the spinor field $\lambda$ to be either Majorana and/or Weyl. This is the case in four or six dimensions.

## 3.2 The Mysterious (2,0) Theory

There are other theories which do enjoy maximal supersymmetry but cannot be derived from the nice, ten-dimensional super-Yang-Mills theory of equation (3.1). One of the most mysterious of these theories is the six-dimensional self-dual superconformal field theory known as (2,0) theory [8, 9]. This theory is interesting for many reasons, amongst others being that it is the highest-dimensional superconformal field theory possible* [6]. That such a superconformal field theory actually exists in dimensions higher than four was a surprising fact when first presented, and the theory seems to have more surprises in store since it does not seem to behave like any other theory we know of.

*out of theories not involving gravity. In the supergravity case, the maximum number of dimensions is eleven

First and foremost, there appears to be an absence of any Lagrangian formulation of this six-dimensional theory, yet, on a quantum mechanical level, its existence is clear [9]. The theory moreover contains no parameter which in some way may be seen as "small", so calculations using perturbation theory is impossible.

If one chooses to restrict to the non-interacting sector of the theory, that is to the free tensor multiplet, the mysteries fade slightly. For this abelian version of the theory, a classical formulation in terms of equations of motion exists, as well an expression for the component-wise supersymmetry transformations of the fields. It is this restriction of the theory which has been the starting point of PAPER III.

Let us now briefly describe the content of the tensor multiplet of (2,0) theory. The eight real fermionic degrees of freedom are represented by a chiral spinor obeying a symplectic Majorana-Weyl condition, $\Psi$. This resides in the spinor representation of both the Lorentz group as well as the $R$-symmetry group of the theory. The bosonic degrees of freedom are slightly more complicated. In general for a $4k+2$-dimensional space with Minkowski signature it is possible to have a self-dual middle-dimensional, real form. In the (2,0) theory, such a form is present as the three-form field strength, $H$, which is a singlet under the $R$-symmetry. Finally, there are five real bosonic Lorentz scalars, $\Phi_M$, transforming in the vector representation of the $R$-symmetry group which complete the tensor multiplet.

The description of this theory is, as mentioned, existent on the level of equations of motion, and the supersymmetry transformations induce an isomorphism on the space of solutions of these. Let $M, N, P$ denote six-dimensional Lorentz indices, $K$ denote an index in the vector representation of the $R$-symmetry group and $\alpha, \beta$ be spinor indices. The equations of motions for the free abelian tensor multiplet on a flat background are then given by:

$$D^M D_M \Phi = 0 \tag{3.3}$$

$$dH = 0 \tag{3.4}$$

$$\Gamma^M D_M \Psi = 0. \tag{3.5}$$

Similarly, the supersymmetry transformations may in the same situation be written as:

$$\delta H_{MNP} = 3\partial_{[M} \left( \bar{\Psi}_\alpha \Gamma_{NP]} \epsilon^\alpha \right) \tag{3.6}$$

$$\delta \Phi_K = 2(\Gamma^R_K)_{\alpha\beta} \bar{\Psi}^\alpha \epsilon^\beta \tag{3.7}$$

$$\delta \Psi^\alpha = \frac{i}{12} H_{MNP} \Gamma^{MNP} \epsilon^\alpha + i M_{\beta\gamma} \partial_M (\Gamma^R_K)^{\alpha\beta} \Phi^K \Gamma^M \epsilon^\gamma. \tag{3.8}$$

In spite of its many mysteries, there exist clear connections to the well-known $\mathcal{N}=4$ Yang-Mills theory. By compactifying the (2,0) theory on circle, it gives rise to maximally supersymmetric Yang-Mills theory in five dimensions [10], and when instead compactified on a Riemann surface $C$, one obtains just precisely the four-dimensional $\mathcal{N}=4$ Yang-Mills theory [11]. However, one must remember that the (2,0) theory should be considered to be fundamentally different from the ten-dimensional maximal supersymmetric Yang-Mills theory since no known relation between these two maximally supersymmetric theories exist. This six-dimensional theory is of a different, more mysterious kind with many questions still unanswered.

*Chapter 4:*

# *Topological Field Theories*

In the previous chapter, the maximally supersymmetric theories which will be featured in this thesis were introduced. However, PAPER I-III deals with some *topological field theories* originating from these theories presented in chapter 3. Here we will give a brief introduction to the concept of topological field theories in general, and how one may obtain them from supersymmetric field theories.

The characteristic of a topological quantum field theory is, as the name suggests, that all observables (i.e. the correlation functions of the theory) are topological invariants of the manifold on which the theory is placed and thus completely independent of any metric. Naturally, this presents a huge advantage when one wishes to perform calculations. The metric independence of the theory makes several otherwise unavailable techniques possible, one of the most useful being the ability to localise the path integral on classical field configurations, something which is further described in section 4.3.

One of the main accomplishments in the field of topological quantum field theory is the physical interpretations that have been provided to objects which were previously seen as strictly mathematical beings. The two earliest examples of this are the Donaldson polynomial of four-manifolds [12] and the Jones polynomial [13] of knots [14, 15].

The latter one of these accomplishments of finding a physics solution to a problem which was previously considered as purely mathematical; finding an intrinsic three-dimensional definition of the Jones polynomial of a knot, was the starting point of what would be an ongoing 25-year period filled with activity on both the mathematics- and the physics side to unveil the mysteries of these theories and an ongoing search for more interesting areas where they may be applicable. This field has since its infancy held a position in the twilight zone between mathematics and physics, and both areas have greatly benefited from this collaboration.

There are two main classes of topological quantum field theories today, the first one of these being theories of Schwarz type [16], which are topological for the simple reason that the formulation of these theories does not require the introduction of any metric. A famous example from this family of topological field theories is the Chern-Simons theory [15]. In the other class of theories, those of Cohomological type, however, the metric independence is much more subtle. The theories which are featured in the work presented herein belong to this latter class of theories, and we shall thus spend a moment in explaining this further.

## 4.1 Theories of Cohomological Type

Theories of this class of supersymmetrical topological quantum field theories can as previously mentioned not be formulated with a manifest metric independence, but the feature they have in common is that they contain (at least) one nilpotent supersymmetry generator, denoted by $Q$ which satisfies $Q^2 = 0$.

The theory may be restricted to only considering states which are closed under this operator, or equivalently: restricting the observables of the theory to the cohomology classes of the operator. This means that the physical states of the theory will be those that obey $\{Q, \Psi\} = 0$, and any two states $\Psi$ and $\hat{\Psi}$ will be considered equivalent if they differ by something $Q$-exact (meaning that they differ by a quantity on the form $\{Q, \dots\}$).

A common property for theories of this type is that their stress-tensor must be $Q$-exact, which makes the metric variation of any observable, which in this case will be the correlation functions of the theory, vanish in cohomology. In this way, the observables of such theories are indeed topological invariants, even though the formulation of the theory itself contains the metric.

## 4.2 Topological Twisting

Topological field theories of cohomological type, as described above, requires the existence of a nilpotent, scalar supersymmetry charge. Under certain circumstances, a theory containing such a supercharge may be created through a process known as *topological twisting* from an ordinary supersymmetric quantum field theory. This procedure was first introduced in [14], and will, if certain requirements are fulfilled, result in a theory which contains at least one nilpotent supercharge. The cohomology of this will then consistute the physical observables of the theory.

This recipe all revolves around the relatively simple and well-known fact that the supersymmetry charges are spinors and thus transform under both the Lorentz group as well as the $R$-symmetry group of the theory. The aim is to find a suitable diagonal subgroup of these two, such that the spinor representation of the original Lorentz- and $R$-symmetry groups decomposes into at least one scalar representation. If this such a subgroup is found, this will leave us with a theory containing at least one supersymmetry charge which is scalar under this new diagonal group.

The corresponding supersymmetry generator, $Q$, will then be nilpotent directly by the supersymmetry algebra. The argument to why this is may in short be expressed as follows: It is clear that $Q^2$ must be given by a bosonic symmetry operator which is scalar under both the Lorentz- and $R$-symmetry groups, but no such symmetry exist in the theory. Thus the only reasonable possibility is to have $Q^2 = 0$, which is precisely what is required if we wish to obtain a topological field theory of cohomological type. The physical interpretation of the theory may now be changed such that we choose to only consider $Q$-invariant path integrals, observables and states. Furthermore, anything of the form $\{Q, \dots\}$ will be regarded as trivial. In short, we follow the procedure outlined in section 4.1 and take the physical states and observables of the theory to be the cohomology classes of this scalar, nilpotent supersymmetry charge $Q$.

When the background on which the theory is placed has vanishing curvature, the twisting procedure amounts to a convenient relabelling of the field content of the theory, but the situation is more intricate when curvature of the background is introduced. Then corrections to both the equations of motion and supersymmetry transformation rules, as well as to the action itself may be required to maintain the supersymmetry invariance of the theory. These corrections must, to my knowledge, be manually added to the theory and follows no general approach or procedure. In most cases, they are however possible to find using a reasonable amount of effort, (as for example is the case in PAPER II), but this does not appear to always be the case, as exemplified in PAPER III.

The twisting itself is defined by the homomorphism used to embed the Lorentz group in the $R$-symmetry group, and this choice also then naturally defines the topological field theory obtained. Thus, this process may be used to find a variety of topological field theories, and the number of possible topological field theories which may be created from a certain supersymmetric quantum field theory will be given by the number of ways to create a subgroup of the Lorentz- and $R$-symmetry groups such that the spinor representation decomposes into at least one scalar representation.

To illustrate this, we consider four-dimensional maximally supersymmetric Yang-Mills theory. This has been shown to admit three inequivalent topological twistings, [17, 18, 19, 20], one of which has been the focus of PAPERS I and II. This twisting in question is known as the GL-twist in four dimensions, and will, together with its five-dimensional analog, be described in detail in chapter 5, and in even greater detail in PAPERS I and II respectively.

It should also be noted that since the concept of topological twisting amounts to finding an embedding of the Lorentz group in the $R$-symmetry group (fulfilling certain criteria), topological twisting naturally resides in Euclidean signature. This since cases where the background is of Minkowski signature will have a non-compact Lorentz-group and there will thus be no way of embedding this in the compact $R$-symmetry group. However, this issue may be circumvented on some occasions if the time-like direction is somehow singled out from the space-like ones, as for example has been done in the topological twistings presented in PAPERS II and III.

## 4.3 Localisation

One of the more appealing features of topological field theories (which was briefly mentioned in section 4.2) is that the path integral may be localised on field configurations which are invariant under the nilpotent operator $Q$. This concept of localisation in the context of supersymmetric theories was first suggested in the early 1980's [21], and applied shortly thereafter [22]. Under favourable circumstances, which are better specified in [19], it suffices to consider bosonic field configurations which are invariant under $Q$. The variation of the bosonic quantities can be consider to vanish trivially, since they transform into fermionic quantities under the supersymmetry transformations which in some sense anyhow are considered to be infinitesimal. Thus the only requirement we are left with is that the variation of the fermionic fields must vanish. This is equivalent to only considering field configurations which satisfy:

$$\{Q, \lambda\} = 0, \tag{4.1}$$

where $\lambda$ denotes the fermionic fields of the theory. In cases where we have a topological field theory of cohomological type as described in section 4.1, the path integral may be localised on configurations satisfying equation (4.1), as shall be explained below.

A quite straight-forward way of viewing this localisation process is to note that any observable in the theory may, when written in the pant-integral formalism, be expressed on the form:

$$\int \mathcal{D} \ldots e^{-\mathcal{S}}, \tag{4.2}$$

where $\mathcal{S}$ denotes the action. Since we have choosen to restrict ourselves to observables lying in the cohomology of $Q$, this allows for the freedom of adding anything $Q$-exact* to the action of the theory without changing the physical interpretation. By using this freedom, and choosing this "something" in a clever way, one may reduce the infinite-dimensional configuration space of the path integral to a finitely-dimesional one. The magical term which accomplishes this will be

$$-\frac{1}{\varepsilon}\int_V \mathrm{Tr}\left(\sum_{\text{all fermions } \lambda_i} \{Q, \lambda_i\}^2\right), \tag{4.3}$$

where $\lambda_i$ denotes the fermion fields of the theory and the sum runs over all of these. As $\varepsilon \to 0$, this term diverges unless $\{Q, \lambda_i\} = 0$ for all fermionic fields $\lambda_i$. Hence the path integral in equation 4.2 is suppressed for any field configurations that does not fulfil this, and as such indeed localises on configurations which satisfy the above equation (4.1). Because of obvious reasons, these equations are often referred to as the *localisation equations of the theory.*

*anything satisfying $x = \{Q, \ldots\}$.

## *Chapter 5:*

# *The GL-twist in Four- and Five Dimensions*

After all this rather general introduction to the field, we are finally at a point at which a discussion of the theories investigated in the papers appended to this thesis may commence.

PAPER I deals with the four-dimensional topological field theory which was first mentioned in [17], and later shown to have applications to the geometric Langlands program in [19], as well as the five-dimensional analog of this. This five-dimensional theory is also the focus of PAPER II, though the settings differ. The four-dimensional theory is named after its applications as the geometric Langlands-twist, or GL-twist for short, of $\mathcal{N} = 4$ Yang-Mills theory. The five-dimensional has, to my knowledge, no commonly accepted name yet, and will herein simply be referred to as *the five-dimensional analog of GL-twisted Yang-Mills*, since this name is very descriptive of the theory.

## *5.1 The GL-Twist in Four Dimensions*

Starting from the four-dimensional maximally supersymmetric Yang-Mills theory as it is presented in section 3.1, we recall the symmetry group of this theory, (in Euclidean signature) is $Spin(4) \times Spin(6)_R$. The $Spin(6)_R$ will, as the notation indicates, denote the $R$-symmetry group. As mentioned in section 4.2, this theory admits three inequivalent twistings. In this section, the details of the particular twist that has been studied in PAPERS I and II, the so-called *GL-twist* are presented.

To perform the twisting, it is convenient to notice that $Spin(4) \simeq SU(2)_l \times SU(2)_r$ as well as $Spin(6) \simeq SU(4)$. Furthermore, we may now note that $SU(4)$ contains a subgroup of $SU(2)^l_R \times SU(2)^r_R \times U(1)_R$. The twisting procedure now replaces the four $SU(2)$'s by a new group, which shall be denoted by $SU(2)'_l \times SU(2)'_r$. This new $SU(2)'_l \times SU(2)'_r$ is defined such that $SU(2)'_l$ is the diagonal subgroup of both the Lorentz-$SU(2)_l$ and the R-symmery one, $SU(2)^R_l$, and vice versa for $SU(2)'_r$.

We now restrict ourselves to consider only the theory invariant under this new symmetry group. The decomposition of the spinor representation under this identification will result in two scalar representations with opposite charges under the remaining $U(1)$, which means that the recipe of topological twisting has given us a theory with two scalar supersymmetry charges.

A schematical image of the twisting, as well as how the spinor representation transforms under it, can be seen in table 5.1. Therein, the representations under the $SU(2)$'s are denoted by bold face numbers and the charge under the $U(1)$ is denoted by a superscript.

| | $SU(2)_l \times SU(2)_r \times SU(2)_l^R \times SU(2)_r^R \times U(1)_R$ | $\xrightarrow{twist}$ | $SU(2)_l' \times SU(2)_r' \times U(1)$ |
|---|---|---|---|
| Spinor rep. | $(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})^{+1/2} \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})^{-1/2}$ $\oplus(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})^{+1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})^{-1/2}$ | | $(\mathbf{1},\mathbf{1})^{\pm 1/2} \oplus (\mathbf{2},\mathbf{2})^{\pm 1/2}$ $\oplus(\mathbf{3},\mathbf{1})^{+1/2} \oplus (\mathbf{1},\mathbf{3})^{-1/2}$ |

***Table 5.1:*** *Table describing the decomposition of the spinor representation under the four-dimensional GL-twist. Bold face numbers denotes representations under the $SU(2)$'s and superscript the charges under the $U(1)$.*

The original six scalar fields which arose in the process of dimensional reduction will under this twisting split into one one-form as well as one complex scalar field. Moreover, the gauge field decomposes hereunder into a one-form, and in the same manner the fermion fields decompose into two zero-forms, two one-forms as well as one two-form, which is conveniently split into its self-dual- and anti-self-dual parts. This may all be obtained by studying how the corresponding representations of fields change during the twistng procedure.

Likewise, the action and equations of motion for the twisted theory may be obtained by studying how their constituents alter during the twisting procedure. The full Lagrangian for this theory, on a general, curved background, was presented by Kapustin and Witten in the geometric Langlands-paper of 2007 [19]. This shows that this twist indeed results in a good, topological field theory.

## *5.2 The Five-Dimensional Analog of the GL-Twist*

The five-dimensional analog of the topological twisting described above was first introduced by Witten in 2011 [23] in the context of Khovanov homology for knots [24]. The five-dimensional theory has not been as well-studied as its four-dimensional contrapart, though it plays a vital role in the first two papers appended to this thesis. In order to formulate this topological field theory, a convenient starting point is once more the five-dimensional maximally supersymmetric Yang-Mills theory when viewed as a dimensional reduction from the ten-dimensional one.

After this dimensional reduction (again in a Euclidean signature), we find, for a general five-manifold, that both the Lorentz group and the $R$-symmetry group of the theory are $SO(5)$. Thus there exist a unique embedding of the Lorentz group in the $R$-symmetry group and hence such a theory admits a unique topological twisting.

However, one may make things more interesting by restricting the appearance of the five-manifold such that $M_5 = M_4 \times I$, for $M_4$ a four-manifold and $I$ an interval. The requirement of Euclidean signature is now however superfluous, and we find ourselves in one of the rather rare occasions where a topological twisting may be carried out even if

| | $SU(2)_l \times SU(2)_r \times U(1) \times SU(2)_R \xrightarrow{twist}$ | $SU(2)_l \times SU(2)' \times U(1)$ |
|---|---|---|
| Spinor rep. | $(\mathbf{1},\mathbf{2},\mathbf{2})^{+1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})^{-1/2} \oplus(\mathbf{2},\mathbf{1},\mathbf{2})^{+1/2} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})^{-1/2}$ | $(\mathbf{1},\mathbf{1})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{3})^{\pm 1/2} \oplus(\mathbf{2},\mathbf{2})^{\pm 1/2}$ |

***Table 5.2:*** *Table describing the decomposition of the spinor representation during the twisting procedure of the five-dimensional GL-twist. Bold face numbers denotes representations under the $SU(2)$'s and superscript the charges under the $U(1)$.*

the background is of Minkowski signature (as briefly discussed in section 4.2). A time-like direction is now allowed as long as it is taken to lie along the interval. This all results in a reduction of the Lorentz group of the theory to $SO(4)$. There may be several inequivalent ways of embedding $SO(4)$ in $SO(5)$, and thus the theory may now admit several inequivalent topological twistings, one of which will be five-dimensional analog of the four-dimensional $GL$-twist.

This twisting is illustrated in table 5.2, where the Lorentz group is written as $SO(4) \simeq SU(2)_l \times SU(2)_r$ and the $R$-symmetry group as $SO(5) \supset U(1) \times SU(2)_R$. This twisting then amounts to identifying the two factors $SU(2)_r$ and $SU(2)_R$, and replacing them by the diagonal subgroup $SU(2)'$. The notations in the table follows those of table 5.1. Equivalently, this twist may be described by the homomorphism from the Lorentz group $Spin(4)$ to the $R$-symmetry group $Spin(5)$ being chosen such that the spinor representation, $\mathbf{4}$ of $Spin(5)$ decomposes as a direct sum of two chiral spinor representations of $Spin(4)$, $\mathbf{2} \otimes \mathbf{2}$.

As may be seen in table 5.2, this twisting will, just as in the four-dimensional case, result in two scalar spinor representations, and the topological field theory obtained will thus have two scalar supersymmetries. The remaining field content may again be obtained by studying the decomposition of the corresponding representations under the twist (for details on this, see PAPER II).

Even further restrictions on the five-manifold, in terms of restricting the appearance of $M_4$, are sometimes made. This is necessary when one wishes to make contact with applications in knot theory, and is done in the latter part of PAPER I where the five-dimensional analogs of the four-dimensional localisation equations of the GL-twisted $\mathcal{N} = 4$ Yang-Mills are considered. In cases like these, the theory is considered on a five-manifold of the form $M_5 = M_3 \times I \times \mathbb{R}$ instead of the more general $M_5 = M_4 \times I$. The relation of the theory on such a manifold to knot theory will be briefly outlined in section 5.3, but a more detailed explanation may be found in [23]. It should here be noted that the five-dimensional topological field theory on this more restricted background may be obtained straight from its four-dimensional analog through a process of first applying $S$-duality then followed by $T$-duality [23].

In PAPER II, the focus lies solely on this five-dimensional theory and the curvature corrections required when the theory is formulated on a general background. The complete and explicit expression for the action for a general $M_5 = M_4 \times I$ is presented and shown to be supersymmetric.

## 5.3 Relation to Knots

As mentioned on several occasions, one of the twists of the four-dimensional $\mathcal{N} = 4$ Yang-Mills theory described above has been shown to have applications to the geometric Langlands program [19]. However, if the theory is considered on a four-manifold with a product structure like the one in PAPER I, namely $M_4 = M_3 \times I$ for $I$ an interval with some appropriate boundary conditions [23, 25] at its endpoints, the theory will instead have interesting applications in the field of knot theory [23, 26].

A knot, in the mathematical sense of the word, is an embedding of a circle in $\mathbb{R}_3$, and to each knot, a series of knot invariants may be assigned. One such invariant is the so-called Jones polynomial, [13], which were one of the subjects studied at the very birth of topological field theories in 1989. Until a gauge theory description of it was given [15], it lacked any intrinsic three-dimensional formulation, which is desired since it arises from objects embedded just in three-dimensional space. In the gauge theory description [15], the Jones polynomial is computed as expectation values of certain loop operators, known as Wilson loops, in three-dimensional Chern-Simons theory.

It is from its two-dimensional definition of the mathematics literature long known that the Jones polynomial is a Laurent polynomial with integer coefficients, but this is not obvious from the gauge theory definition. Nor is there a clear reason from this point of view why the coefficients must be integers. However, both of the above questions find their answer in the subject of Khovanov homology [24]. Khovanov homology associates to each knot, instead of a polynomial, a finite-dimensional vector space. For our intents and purposes, the details of the relations between Khovanov homology and the Jones polynomial are not of vital importance, but it suffices to notice that one may recover the Jones polynomial from Khovanov homology (the curious reader may find a detailed explanation in [23]). This provides the original motivation for finding a gauge theory description of Khovanov homology as done with the Jones polynomial in the late eighties.

From a quantum field theory perspective, it is quite straight forward to assign a number to a knot via the path integral formalism (i.e. the value of the path integral in the presence of the Wilson loop encoding the knot). However, it is slightly harder to associate a vector space to an object in this formalism. This was accomplished in [23] (and further studied in [26]) by using a four-dimensional topological quantum field theory and surface operators instead of three-dimensional Chern-Simons with line operators, which was used when considering the Jones polynomial. This lift from three to four dimensions can be done by adding what may be viewed as a "time direction" to the three manifold, so that the theory instead is considered on $\mathbb{R} \times M_3$.

However, this is more complicated than it may appear at first glance. In order to do this in a consistent matter, one must first use the fact that the Chern-Simons path integral on $M_3$ may be expressed as the path integral of a four-dimensional topological field theory on $M_3 \times I$, where $I$ is an interval (often taken to be $\mathbb{R}^+$), with suitable boundary conditions at its endpoints [23, 25]. This four-dimensional topological field theory is precisely the GL-twist of $\mathcal{N} = 4$ Yang-Mills theory [27]. When placed on a four-manifold with the described product structure and Wilson operators in the boundary, this four-dimensional theory will thus be able to reproduce the Jones polynomial of a knot corresponding to this Wilson operator in the same manner as the Chern-Simons theory did in [15]. The Khovanov homology will then obtain a gauge theory interpretation by aid of the five-dimensional analog of the GL-twisted $\mathcal{N} = 4$ Yang-Mills on a five-manifold of the form $M_5 = M_4 \times \mathbb{R} = M_3 \times I \times \mathbb{R}$.

Thus both the four-dimensional GL-twist and its five-dimensional analog are theories of great physical interest since they provide a gauge theory description of these knot invariants, descriptions which in some ways seem more natural than the descriptions previously known.

## 5.4 Paper I – *Localisation Equations and Their Solutions*

In paper I, the localisation equations of the GL-twist of $\mathcal{N} = 4$ Yang-Mills, as well as its five-dimensional analog on manifolds with the structure relevant for the applications to knots are studied more closely, and spherically symmetric solutions to these are found.

Since solving the localisation equations of a topological field theory amounts to finding field configurations such that the supersymmetry variation of the fermionic fields vanishes, and since the fermionic field content differ between the four- and five-dimensional theories, so will the localisation equations and their solutions. However, they are not completely unrelated to one another, since the fifth dimension also may be considered as an additional "time direction" which is added to the four-dimensional theory. In such a way, the five-dimensional solutions may be seen as tunnelling solutions between a set of two four-dimensional solutions. Details, as well as the implications of this, may be found in the appended paper I.

## 5.5 Paper II – *The Complete Action of the Five-Dimensional Theory*

In paper II, the focus is shifted to the five-dimensional theory instead. More specifically, since the topological twisting only, in a straight forward manner, gives us a topological field theory on a flat background, the case when the background has non-vanishing curvature still remained an interesting problem. In the paper, an explicit expression of the action is computed for the theory on a general five-manifold of the form $M_4 \times I$. It is shown that some curvature terms are needed additions to the action in order to maintain supersymmetry invariance, but that no corrections to either the supersymmetry transformation laws or the equations of motions are possible. By providing an explicit expression of the action which fulfils all symmetry properties required, the theory is fully specified even on a general background.

The choice of product structure of the manifold, which is slightly more general than the one required to make contact with Khovanov homology, has been used herein since the relation to knot theory is not the main focus of the paper but rather the topological field theory on its own.

*Chapter 6:*

# *Twisting* (2,0) *Theory and Lower-Dimensional Correspondencs*

The last paper presented in this thesis deals with something slightly different, namely a topological field theory originating from another maximally supersymmetric theory: the mysterious (2,0) theory which was briefly introduced in section 3.2. This theory is highly interesting for many reasons, one simply being how little is known about it in comparison to other theories. Another being that it is believed to give rise to lower-dimenisonal correspondences through different compactifications of topological twistings of it. One such topological twisting of the theory which is believed to be related to the so-called *AGT-correspondance* is the subject of PAPER III, and this chapter will attempt to give a brief introduction to this as well as some other lower-dimensional correspondences which may have their origin in (2,0) theory. Such correspondences provide the motivation for pursuing topological twists of this mysterious six-dimensional theory. After the small review of these, we arrive at the particular twist of (2,0) theory which is studied in PAPER III.

## *6.1 The hitchhikers guide to the AGT correspondence and the conjectured relation to* (2,0) *theory*

In recent years, many correspondences between gauge theories in different dimensions have been shown or conjectured. The most famous one of these is probably the AdS/CFT conjecture [28], which relates supergravity theories in the bulk of an Anti de Sitter space-time to conformal field theories living on the boundary of this. This conjecture was presented by Maldacena in 1997, and may be considered one of the greatest advances in fundamental physics of the 1990's. This is for several reasons; one being that it allows for calculations to be carried out in some strongly coupled gauge theories by using the correspondence to move the problem to its weakly-coupled supergravity analog, another one being that it provides a non-perturbative formulation of some string theories with suitable boundary conditions via the boundary theory.* It is thus not surprising that the discovery of a correspondence with so many benefits led to the question whether or not there are more similar ones?

*For a more adequate introduction to AdS/CFT, see for example [29].

The AGT correspondence, which was first presented by Alday, Gaiotto and Tachikawa in 2009 [30] is one of these correspondences which have since been conjectured. This correspondence relates objects in four-dimensional super-Yang-Mills theory to objects in a two-dimensional gauge theory. More specifically, it relates the correlation functions in two-dimensional Liouville theory to the Nekrasov partition function [31, 32] of certain $\mathcal{N} = 2$ superconformal gauge theories in four dimensions. This class of four-dimensional theories are sometimes referred to as *class $\mathcal{S}$ theories* [11, 33], and may be obtained by compactification of the (2,0) theory on a two-dimensional Riemann surface with possible defects [34]. The idea behind the correspondence is as follows:

The four-dimensional theories on one side of the correspondence can be considered to be labelled by the properties of the Riemann surface used in the compactification (more rigoruosly by the genus $g$ and number of punctures $n$). There are in general, however, several ways to “create” such a Riemann surface through a process known by the remarkably descriptive name of “sewing together pairs of pants”. From the point of view of the four-dimensional field theory, two different ways of sewing the same Riemann surface will result in different Lagrangian descriptions of the theory, and the operations relating two different sewings correspond to $S$-duality transformations relating the different Lagrangians. In some way, these class $\mathcal{S}$ theories must thus encode information on the possible ways of sewing the Riemann surface which was used in the compactification of (2,0) theory.

There are however also objects in two-dimensional field theory which may be defined in terms of ways of sewing Riemann surfaces from pairs of pants, one such being the Liouville correlation function in two-dimensional conformal field theory on the Riemann surface in question. (The precise way in which this is done is however beyond the scope of this thesis, but I refer interested readers to [30].) This motivates the search for a correspondence between objects on the four-dimensional- and two-dimensional side, which both may be defined in terms of the different ways of sewing the Riemann surface. This will result in a conjecture between precisely the Nekrasov partition function of the class $\mathcal{S}$ theories on the four-dimensional side and the correlation functions in two-dimensional Liouville theory as stated above.

This conjecture has however not been proven for Riemann surfaces of general genera and number of punctures, even though extensive tests have been carried out. One possible way to prove the conjecture may lie in the observation that the theories on one end of it may be obtained from the mysterious (2,0) theory. Thus, this six-dimensional theory may be the origin of the correspondence. Since 2009 and the suggestion of the original AGT-correspondence, several other conjectured correspondences have been stated between theories in dimensions $d$ and $(6 - d)$, all with a conjectured origin in the (2,0) theory compactified on a $d$-dimensional Riemann manifold. This was for example done with great success for $d = 3$ in [35].

## *6.2* Paper III - *The Trouble with Twisting* (2,0) *Theory*

The third and final paper of this thesis revolves around the concepts presented above, more specifically, it aims at a detailed investigation on how the four-dimensional theory which lies on one side of the AGT-correspondence may arise from (2,0) theory. Once again, the lack of any Lagrangian formulation of the six-dimensional theory presents us with a challenge, and forces us to consider the abelian, non-interacting version of the full theory, which as previously mentioned has a well-defined definition on the level of equations of motion. It is hence on this level we must carry out our calculations.

The belief was that the four-dimensional side of the correspondence would be the result of a compactification of the (2,0) theory on a two-manifold $C$. However, for a general metric, all supersymmetry would in such a case be broken, and thus there would be no possibility that the resulting theory would be superconformal, and hence in no way be related to the correspondence. To preserve some amount of supersymmetry, a topological twisting of the six-dimensional theory must first be performed, after one may compactify on $C$ to obtain a four-dimensional, supersymmetric theory.

The next obstacle which presents itself is that the classical formulation which exists for the free tensor multiplet of (2,0) theory is valid only in Minkowski space, whereas topological twisting in principle only works in Euclidean signature (as outlined in section 4.2). This is a problem discussed in detail in PAPER III, but which is overcome in the same manner as in PAPER II: by singling out the time-like direction and carrying out the twisting along the other directions. In the last paper appended to this thesis, this is achieved by placing the time-like direction along the two-manifold $C$. Thus this setup leaves us with a four-manifold of Euclidean signature on which a topological twisting may be carried out.

In previous work, this twist has been claimed to be analogous to the Donaldson-Witten twist of $\mathcal{N} = 2$ Yang-Mills theory in four dimensions. However, in PAPER III we find that there are some slight differences between the twist relevant for the AGT-correspondence and its well-known cousin. These differences, though small, turn out to have far-reaching consequences for the topological nature of the theory. That they are there is however clear and may schematically be viewed as follows: Since our aim is to shed some light on the origins of the AGT-correspondence, we would like to be able to carry out the twisting on $C$ as well to obtain a topological field theory on both sides of the correspondence. Since this is not possible because of signature issues, the second best thing is done and the theory is considered under the supercharge which would become scalar on the two-manifold as well under such a hypothetical twist. This particular supersymmetry charge will be some linear combination of the charges which are found to be scalars on $M_4$ after the first part of the twisting, but it is not the same linear combination of these two as in the Donaldson-Witten case.

Regardless of the above complications, it is shown that a compactification on $C$ is needed to obtain a theory which may still be topological on the four-manifold.

All of the reasoning above is spelled out in detail in PAPER III, and the topological nature of the theory along the four-manifold $M_4$ is investigated by considering the stress-tensor of the theory along this. A stress-tensor, $T^{\mu\nu}$, with all desired properties required for the theory to be topological (i.e. it being both covariantly conserved and $Q$-exact) is explicitly computed as $\{Q, \lambda^{\mu\nu}\}$ and an explicit expression for $\lambda^{\mu\nu}$ is presented. However, things take a turn for the worse when the four-manifold becomes curved. We argue that there is no way to add curvature corrections to $\lambda^{\mu\nu}$, and only two possible additions to the equations of motion. There is however no way that this would result in a stress-tensor which has vanishing divergence on a general $M_4$. Hence, the theory on a curved background does not seem to be topological in nature. This feature; that a topological field theory exists on a flat manifold but not on the generalisation to a curved one, is to the authors knowledge something which has not been seen before, and which I find deeply perplexing.

## *Chapter 7:*
# *Conclusion*

Herein, the three first papers of my scientific carrier are presented and an attempt is made to put them into context.

As is clearly seen, the red line connecting them all is topological field theories. One may actually be more specific in saying that the theories considered in the three papers of this thesis are all topologically twisted versions of maximally supersymmetric gauge theories. The subject of topological field theories is a very diverse one, and to my knowledge, any classification of all topological field theories similar to the classification of supersymmetric Yang-Mills theories has not been discovered yet. The latest one of the topological twists of $\mathcal{N} = 4$ super Yang-Mills was discovered almost 15 years after the introduction of the concept, and new applications of these theories are continuously discovered.

In this area, there are several problems which may be addressed in future work, the most acute for my peace of mind being the issue of a topological field theory on a flat manifold which does not generalise to a curved background. An attempt at finding a general proof of why, at least in the presence of an action for the topological theory in the flat case, it should be possible to generalise it to generic appearances of the background is something I feel requires some thought. If it is not possible to construct such a proof, then the point where it breaks down may shed some light on some of the more subtle properties of topological field theories and the concept of topological twisting. As for the case studied in PAPER III, the situation is further complicated by the lack of any Lagrangian formulation of the (2,0) theory, but the simpler setting of theories where such a formulation exists should non the less be a good starting point for future work.

Something else which fascinates me at the moment is the possibility of there being a higher-dimensional analog of the interpretation of knot invariants in terms of gauge theory. There might be something similar for higher-dimensional knot-like objects, where the invariants instead may be related to surfaces embedded in four-dimensional space and then these invariants may find a gauge-theory description in terms of a five-dimensional topological field theory. This five-dimensional theory will then, I believe, most likely be closely related, if not the one, which was the subject of the second paper of this thesis. This is a question I believe deserves some effort as well.

Over all, I must admit to there being too many other exciting possible areas for future work than I have had the opportunity to reflect on, nor less mention within in the scope of this thesis, and I can only hope that some of them will be subjects included in my future doctoral thesis.

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