THESIS FOR THE DEGREE OF LICENTIATE OF FUNDAMENTAL PHYSICS

# Twisting and Turning in Six Dimensions

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# *Abstract*

This thesis investigates certain aspects of a six-dimensional quantum theory known as (2,0) theory. This theory is maximally supersymmetric and conformal, making it the most symmetric higher dimensional quantum theory known. It has resisted an explicit construction as a quantum field theory yet its existence can be inferred from string theory. These properties suggests that an understanding of the theory will create a deeper understanding of the foundations of both.

In the first part of the thesis an explicit formulation of the non-interacting version of the theory is investigated on space-time manifolds that are circle fibrations. The circle fibration geometry enables a compactification to a five dimensional supersymmetric Yang-Mills theory. A unique extension to an interacting theory is found and conjectured to be the compactification of the interacting theory in six dimensions.

The second part of the thesis concerns the topological twisting of the free theory in six dimensions. A space-time manifold which is a product of a four-dimensional and a two-dimensional part is considered. This setup has recently been proposed as an explaination for the conjectured correspondence between four dimensional gauge theory and two-dimensional conformal field theory known as the AGT correspondence. We perform the twisting and subsequent compactification on the two-dimensional manifold of the free tensor multiplet in Minkowski signature to avoid the problems associated with the definition of (2,0) theory on Euclidean manifolds. With the same choice of supercharge as in the usually preferred Euclidean scenario we conclude that there is no stress tensor which exhibits the topological properties previously found in similar theories.

**Keywords:** Supersymmetry, Yang-Mills theory, Topological field theory, Topological twisting, (2,0) theory, Compactification, Circle fibrations.

# *Appended papers*

This thesis is based on the following papers henceforth referred to as Paper I and Paper II.

I. Hampus Linander and Fredrik Ohlsson, *(2,0) theory on circle fibrations*, JHEP **01** (2012) 159, arXiv:1111.6045 [hep-th]

II. Louise Anderson and Hampus Linander, *The trouble with twisting (2,0) theory*, arXiv:1311:3300 [hep-th]

# *Contents*

*Chapter 1:*

# *Introduction*

In the search for a unified theory of gravitation and quantum mechanics we have been led into the world of higher dimensions. This thesis investigates an enigmatic six-dimensional theory that seems to hold many keys to the understanding of a large class of interesting phenomena.

As a warm-up to the more technical investigations of PAPER I and II, I would like to give an overview of why this six-dimensional theory is important and how it rather recently came into the spotlight. Let us recall a few facts about the world of theoretical physics as it stands today. We have a beautiful theory of the elementary particles and forces known as the standard model. It correctly predicts the behaviour of our world at the smallest scales to a stupendous degree of accuracy [1]. The standard model is a quantum field theory and is based on the marriage between quantum mechanics and the theory of special relativity. The one thing it does not describe however is gravity. Einstein's theory of general relavitity describes gravity at length scales that are large in a certain very precise sense. It shows us how mass and energy curves space-time so that planets orbit their suns and light deflects when travelling past a cluster of galaxies. When it comes to details of a gravitational collapse however, or the early events of our universe, it falls short. In these very energetic processes gravity is not well described by the classical theory of general relativity but rather is expected to have a quantum mechanical formulation. The problem of quantum gravity is a difficult one, and to find a solution we are forced to give up the notion of four-dimensional universe. Even though a full answer to this question is still out of reach we have a good candidate: string theory.

String theory is higher dimensional in two ways. One in that it is formulated in more than four dimensions and another in that the fundamental objects are not zero-dimensional but rather one-dimensional: they are strings. Why is string theory interesting? It is a theory that describes both the fundamental particles and their interactions, and gives a quantum mechanical description of gravity. Why is string theory not the answer? Maybe it is, but even though much progress has been made in understanding the theory we still know too little about it to say for sure.

It is in the cross roads of string theory and quantum field theory that we find the theory that is the topic of this thesis. The rather dull name of (2,0) theory does not convey the importance that it warrants. As will be covered in the next chapter it is a very special theory that enjoys a host of interesting properties, the most peculiar of which is that as of yet there is no framework where it can be explicitly defined. String theory provides evidence for its existence yet quantum field theory cannot accomodate its formulation. This is the precarious situation that the theory finds itself in still today, about 20 years after its inception.

This thesis aims to add a small piece of the puzzle that hopefully increases the understanding of the theory and in the end will result in its explicit formulation.

## 1.1 Outline

In chapter 2 an introduction to (2,0) theory, the main theme of this thesis, is given. The theory is described in terms of its symmetries and the problems of its explicit formulation are explained. The simpler non-interacting version of the theory is introduced with the field content of the tensor multiplet together with classical equations of motion.

Chapter 3 begins with an introduction to the concept of compactification. A simple example of a vector field in a circle geometry is worked through. Fibre bundles are then introduced to facilitate the generalisation to circle fibrations. The last section in this chapter summarises Paper I where (2,0) theory is placed on a circle fibration.

The last chapter concerns topological twisting, the topic of Paper II. Rigid supersymmetry on curved manifolds is used as a motivating problem and the technique is then introduced through a concrete example of $N = 2$, $D = 4$ SYM. This section ends with some comments on the general features of topologically twisted theories. The final section gives an overview of Paper II where (2,0) theory is topologically twisted and compactified to a four-dimensional theory.

# *Chapter 2:*
# *(2,0) theory*

In both Paper I and Paper II the main subject is a six-dimensional superconformal theory known as (2,0) theory. It is the purpose of this chapter to try to give an overview of this theory and its place in theoretical physics.

## *2.1 The hard life of a quantum field theory*

In the early parts of the 20th century theoretical physics produced two wonderful theories that both completely changed the way we view the world around us: special relativity and quantum mechanics. They describe a strange reality where light travels at constant velocity independent of the reference frame and where things at a small scale become quantised and uncertain. Both have now been shown to describe our world extremely well. It was therefore very annoying that quantum mechanics did not seem to be on friendly terms with special relativity. It took an enormous effort and the greater part of the 20th century to find the correct way of joining these two theories into what is now called quantum field theory. The efforts were not without reward because the result was the standard model, a quantum field theory that describes almost everything we see around us and more in terms of a handful of elementary particles and forces.

The standard model is a four-dimensional quantum field theory. There are now many examples of well defined quantum field theories in four dimensions and lower, but above four dimensions there is as of yet no known well defined quantum field theory *. One of the developments that have taken place, beginning in the 90s, is that there is now strong indications that there should exist a theory in six dimensions. This theory is intriguing for many reasons. It has strong connections to string theory from which its existence was first derived [2]. Here it has a privileged position that can be described as somewhere in the middle between a quantum field theory and string theory. Understanding the theory will hopefully help us to understand them both better. From another perspective the theory is very interesting in its own right. It is one of the most symmetric theories we know of, having in a certain sense the maximal possible amount of symmetry a theory can have†. In physics, symmetry almost always provide a tool for understanding a theory better, in many cases solving a theory completely. This makes the case of (2,0) theory baffling because as of yet there is no explicit formulation of the theory. Here we have a theory whose existence is almost a certainty but for which no consistent equations are known. This makes the problem of an explicit formulation a very intriguing problem, but also a very

*Well defined here means, among other things, the UV finiteness of the theory.

†This will be expanded on in greater detail in the coming sections.

hard one.

## 2.2 Shadows

How symmetric can a theory be? It turns out that this question has a definite answer in the context of quantum field theories. The answer gives surprising restrictions on possible theories and at the very extreme end we find a special theory in six dimensions. A non-interacting version of the theory can be explicitly defined in terms of a six-dimensional field theory. The difficulties that a formulation of the interacting theory present might indicate that the free field theory is only a shadow, but a shadow non the less represents some features of its creator and this is the approach that motivates this thesis.

### 2.2.1 Symmetries

The space-time symmetry group of any physically viable theory must contain the Lorentz group to abide by the laws of special relativity. It can furthermore be invariant under scaling transformations as the theory of classical electromagnetism is. The smallest group containing both Lorentz transformations and scaling transformations is the conformal group which in addition to the aforementioned transformations also contain what is called special conformal transformations. By a theorem due to Coleman and Mandula [3] this is the largest possible space-time symmetry group for a consistent quantum field theory under some very natural conditions on the theory and on the form of the symmetry group. There can also be internal continuous symmetries of the theory that have to commute with the space-time group. It turns out however that there is a loop hole to this argument [4] that makes it possible to extend the spacetime symmetry of a theory. The loop hole is to regard symmetries that are not generated by ordinary numbers but rather by anticommuting numbers. This extends the possible symmetries to include what is called supersymmetry. This is a symmetry that looks very peculiar, it exchanges fermions and bosons. If we combine the conformal group with the supersymmetry transformations we get what is called the superconformal group. This constitutes the largest possible space-time symmetry group for a reasonable theory.

Armed with the knowledge of the possible symmetries a theory can possess we can now look for its representations. In the 70s Nahm [5] classified the superconformal algebras and showed among other things that they exist only in space-time dimensions less than or equal to six.

Thus if we are looking for superconformal theories we need only look in dimension six and below. The existence of a superconformal algebra is a necessary condition but certainly not a sufficient one. Actually it was not at all clear that there should exist any well defined superconformal theories above space-time dimension four[‡]. It was therefore a very interesting development when evidence for a theory in six dimensions was put forward [2] in the mid 90s.

[‡]There are now many examples of superconformal theories in dimension four and below, the most prominent one being $N = 4$ SYM in $D = 4$.

Since its inception the theory has remained very mysterious. Still today, 20 years later, there is no mathematical formulation of the interacting theory. Some of the reasons for this ghostly existence will be reviewed in the following section, and they can be seen quite easily even from the very basic ingredients of the theory.

There is a by now rather large literature on (2,0) theory. For a review on its relation to M-theory and string theory see [6, 7]. There have been much work on finding a consistent formulation of the theory, a non-exhaustive list of relevant references is [8, 9, 10, 11, 12].

### 2.2.2 Tensor multiplet

From the work of Nahm [5] we know that there is one possible representation of the (2,0) superconformal algebra which we call the tensor multiplet. This representation contains scalars, fermions and a three-form. Thus if we would like to try to write down a field theory description of the theory our field contents is fixed [§].

The tensor multiplet contains the fields summarised in table 2.1 where the bold face numbers indicate the dimensionality of the representations and a subscript c indicates that the spinor has positive chirality.

| Field | Spin(5,1) | × | $\mathrm{Spin}(5)_{\mathrm{R}}$ |
|---|---|---|---|
| $\Phi$ | **1** | | **5** |
| $\Psi$ | $\mathbf{4}_{\mathrm{c}}$ | | **4** |
| $H$ | **10** | | **1** |

***Tab. 2.1:*** *Field content of free (2,0) theory, the tensor multiplet.*

A few general comments on this field content is in order. Firstly the fermions $\Psi$ are symplectic Majorana-Weyl, where the word symplectic stems from the fact that they transform[¶] under $\mathrm{Spin}(5)_{\mathrm{R}} \cong \mathrm{Sp}(4)$. The observant reader will have noticed that the theory contains no gauge field in the usual sense of a two-form field strength. Instead we have a three-form that, as can be seen from the dimensionality of its representation, must be self-dual. This is the source of the main mystery surrounding the theory. It is known that the theory is interacting and that it is classified by a choice of Lie group in the ADE-series, however the natural way to implement such an interaction would be through a gauge field which from just representation considerations is not present. Furthermore a dynamical theory of a self-dual three-form in six dimensions is notoriously difficult as we will shortly experience.

For the moment we can sidestep the problems of interactions and regard just the free theory. In this case the three-form poses no immediate conceptual difficulties, apart from the interesting features we will look closer at in a moment.

The next step in constructing a candidate theory would be to write down an action for the fields. In the case of the scalars and fermions the answer is essentially

[§]There is of course the possibility of having multiple tensor multiplets which would be the case for the general theory.

[¶]This is in fact crucial for the Majorana reality condition which in six dimensions requires an interplay between the R-symmetry and complex conjugation.

unique and is given by the standard expressions

$$S_\Phi = \int_{M_6} \mathrm{d}^6x\sqrt{-G}\left(\nabla_M\Phi\nabla^M\Phi + \frac{1}{5}R\Phi^2\right) \tag{2.1}$$

and

$$S_\Psi = \int_{M_6} \mathrm{d}^6x\sqrt{-G}\bar{\Psi}\Gamma^M\nabla_M\Psi. \tag{2.2}$$

Here $R$ is the scalar curvature of the manifold $M_6$ and $\Gamma^M$ are six-dimensional gamma matrices. The curvature term in the scalar action is required for the theory to be conformally invariant. These actions give rise to the local equations of motion

$$\nabla^2\Phi - \frac{1}{5}R\Phi = 0 \tag{2.3}$$

$$\Gamma^M\nabla_M\Psi = 0. \tag{2.4}$$

When it comes to the three-form the situation is more precarious. The natural action for an $n$-form can be generalised from the action of electromagnetism $S_{\mathrm{EM}} = \int F \wedge \star F$, where $F$ is the U(1) field strength. This form of the action carries over to the general case immediately and we are led to consider

$$S_H = \int_{M_6} H \wedge \star H. \tag{2.5}$$

Here we face a problem. Since $H$ is self-dual we have that $\star H = H$ and substituting this into the action we find $H \wedge H = 0$. This is the second mystery, there is no known six-dimensional covariant action for a self-dual three-form.

If we restrict attention to the equations of motion we are for the moment saved. The equation of motion that follow from the action for a general, non-self-dual, $H$ is given by

$$\mathrm{d} \star H = 0. \tag{2.6}$$

This equation works perfectly well also for a self-dual field and reduces in this case to

$$\mathrm{d}H = 0, \tag{2.7}$$

the condition that $H$ is a closed three-form.

This means that a consistent set of equations for the three-form is

$$\begin{aligned} H &= \star H \\ \mathrm{d}H &= 0. \end{aligned} \tag{2.8}$$

The theory should be invariant under the superconformal algebra and hence there should exist suitable supersymmetry variations transforming solutions to these equations of motion into each other. Indeed one finds that the transformations (2.9)-(2.11) transforms solutions to the equation of motion into each other provided the supersymmetry parameter satisfies condition (2.12). A few words on notation is here warranted. Apart from the six-dimensional indices $M$,$N$,.. these expressions contain lower case Greek indices $\alpha, \beta, \cdots$. These are indices for the four-dimensional spinor representation of the $\mathrm{Spin}(5)_{\mathrm{R}}$ symmetry. The bilinear forms $M_{\alpha\beta}$ and $T^{\alpha\beta}$

lowers and raises indices in this representations. It is now very convenient to regard the five-dimensional vector representation of $\mathrm{Spin}(5)_{\mathrm{R}}$ as the symmetric traceless tensor product of two spinor representations instead. This gives the scalar $\Phi$ a representation as a bispinor, enabling a compact and computationally convenient form of the supersymmetry transformations.

$$\delta H_{MNP} = 3\nabla_{[M}\left(\overline{\Psi}_{\alpha}\Gamma_{NP]}\varepsilon^{\alpha}\right) \tag{2.9}$$

$$\delta\Phi^{\alpha\beta} = 2\overline{\Psi}^{[\alpha}\varepsilon^{\beta]} - \frac{1}{2}T^{\alpha\beta}\overline{\Psi}_{\gamma}\varepsilon^{\gamma} \tag{2.10}$$

$$\delta\Psi^{\alpha} = \frac{i}{12}H_{MNP}\Gamma^{MNP}\varepsilon^{\alpha} + 2iM_{\beta\gamma}\nabla_M\Phi^{\alpha\beta}\Gamma^M\varepsilon^{\gamma} + \frac{4i}{3}M_{\beta\gamma}\Phi^{\alpha\beta}\Gamma^M\nabla_M\varepsilon^{\gamma} \tag{2.11}$$

$$\nabla_M\varepsilon^{\alpha} - \frac{1}{6}\Gamma_M\Gamma^N\nabla_N\varepsilon^{\alpha} = 0 \tag{2.12}$$

Equation (2.12) is the conformal Killing spinor equation. For rigid supersymmetry the natural condition that comes to mind is for the parameter to be covariantly constant, this is however not a conformally invariant equation. The operator in (2.12) is, together with the Dirac operator, the only natural conformally invariant operators available [13]. On a manifold that admits two independent solutions to (2.12) the theory is maximally supersymmetric with 16 supercharges.

Conformal invariance of abelian (2,0) theory manifests itself in the fact that the equations of motion and supersymmetry transformations depend only on the conformal class of the metric. This means that the theory is invariant under a change of the metric of the form

$$G \to e^{-2\sigma(x)}G, \tag{2.13}$$

where $\sigma$ is a function on $M_6$.

At last we find ourselves with a starting point for explicit investigations, the tensor multiplet together with its classical equations of motion. This might seem like a poor substitute for the full interacting quantum theory but this is the only explicit formulation available at the moment. It seems reasonable that some general features of the theory should also be present in the free, classical version. In fact there are some quantities of the full theory that can be calculated in terms of only the free theory, see [14] for such an example.

Before moving on to the description of Paper I let us recall the counting of dimensionality of the representations in table 2.1. The scalar $\Phi$ is a vector under $\mathrm{Spin}(5)_{\mathrm{R}}$. The fermionic field $\Psi$ is a chiral spinor in six dimensions, hence it transforms in a four-dimensional representation of Spin(5,1). It also transforms in the spinor representation of $\mathrm{Spin}(5)_{\mathrm{R}}$, enabling the symplectic Majorana condition to be imposed. The self-dual three-form $H$ has three antisymmetric indices with $\binom{6}{3} = 20$ components that gets halved by the condition that $H$ is self-dual.

## *Chapter 3:*

# *Circle fibrations*

The topic of the first paper is compactification of (2,0) theory on a space-time with the special property of being a circle fibration. It is the purpose of this chapter to introduce the concept of fibered spaces and how they are used in physics. There is a beautiful but rather large theoretical basis behind the methods used in this chapter which will not be covered. This chapter will instead contain rather informal discussions and where details are given the reader is expected to be familiar with Riemannian geometry and the basics of field theory. For an excellent exposition of these topics in the context of (2,0) theory see [7].

## *3.1 Compactification*

Starting with a theory defined in $D$ space-time dimensions there is a way to create a whole class of $d < D$-dimensional theories. This process is called compactification and as the name implies it involves the use of compact manifolds. The concept dates back to the early 20th century when a unified theory of electromagnetism and gravity was sought. It was found that general relativity defined on a five-dimensional space-time, where one of the directions is periodic, describes gravity and electromagnetism in four dimensions [15].

The unifying theory of electromagnetism and gravity did not work out in the end but the concept of building lower dimensional theories from higher dimensional ones became a widely used method. The process of compactification can be readily described by an example and usually the simplest possible example is the theory of a scalar field compactified on a circle. In the next section a slightly more involved example of a vector field compactified on a circle is described. The purpose of this is two-fold: firstly it provides an example where there arises new fields in the compactification of a different type than the original fields, secondly the specific example lies closer to the computations carried out in PAPER I and the reader may therefore find it elucidating to compare the results.

### *3.1.1 Theories on a circle*

The canonical example of a compactification is when the space-time is taken to be of the form *

$$M_D = \mathbb{R}_{D-1} \times \mathrm{S}_1. \tag{3.1}$$

Where $\mathrm{S}_1$ denotes a circle. Let us investigate what happens to the theory of a single gauge field on this manifold. Given a one-form potential $A = A_M \mathrm{d}x^M$ and its field

*For simplicity the space-time is taken to be flat but in general it can be curved, as will be the case for the later parts of this chapter.

strength $F = \mathrm{d}A$, the canonical action is given by

$$S = \int_{M_D} F \wedge \star F \tag{3.2}$$

where $\star$ denotes the Hodge dual†. Let us denote the index corresponding to the circle by $\varphi$ and the $D-1$ other directions by lower case Greek letters. Let us also, by a slight abuse of notation, use $\varphi$ to denote the coordinate on the circle. The fact that the circle is periodic enables us to make a Fourier expansion in the coordinate $\varphi$.

$$A = \sum_{n=0}^{\infty} A_n(x_\mu) e^{in\varphi} \tag{3.3}$$

Here $A_n$ are one forms on $M_D$ and can be split into the parts that lie in the direction of $\mathbb{R}_{D-1}$ and in the direction of the circle

$$A_n = A_n^{D-1} + A_n^{\varphi} \mathrm{d}\varphi. \tag{3.4}$$

Substituting this into the expansion and taking the exterior derivative, recalling that $\mathrm{d}\varphi \wedge \mathrm{d}\varphi = 0$, gives us three terms

$$\mathrm{d}A = \sum_{n=0}^{\infty} \mathrm{d}A_n^{D-1} e^{in\varphi} + \sum_{n=0}^{\infty} \mathrm{d}A_n^{\varphi} \wedge \mathrm{d}\varphi e^{in\varphi} + \sum_{n=0}^{\infty} A_n^{D-1} in e^{in\varphi} \wedge \mathrm{d}\varphi. \tag{3.5}$$

Substituting this back into the action might not seem to give us anything particularly nice but with a few observations we arrive at a very elegant answer. The first observation is that the Hodge dual induces an inner product between forms and that the action in (3.2) is nothing but the inner product $\langle \mathrm{d}A, \mathrm{d}A \rangle$. The basis forms $\mathrm{dx}^M$ are orthogonal with respect to this inner product which has as a consequence that ‡

$$\begin{aligned} \langle \mathrm{d}A, \mathrm{d}A \rangle = & \left\langle \sum_{n=0}^{\infty} \mathrm{d}A_n^{D-1} e^{in\varphi}, \sum_{m=0}^{\infty} \mathrm{d}A_m e^{im\varphi} \right\rangle \\ & + \left\langle \sum_{n=0}^{\infty} \mathrm{d}A_n^{\varphi} \wedge \mathrm{d}\varphi e^{in\varphi}, \sum_{m=0}^{\infty} \mathrm{d}A_m^{\varphi} \wedge \mathrm{d}\varphi e^{im\varphi} \right\rangle \\ & + \left\langle \sum_{n=0}^{\infty} A_n^{D-1} in e^{in\varphi} \wedge \mathrm{d}\varphi, \sum_{m=0}^{\infty} A_m^{D-1} im e^{im\varphi} \wedge \mathrm{d}\varphi \right\rangle. \end{aligned} \tag{3.6}$$

The second observation is that $e^{in\varphi}$ are orthogonal functions on $S_1$. This means that if we perform the integration over $\varphi$ we end up with

$$\int_0^{2\pi} \langle \mathrm{d}A, \mathrm{d}A \rangle = \sum_{n=0}^{\infty} \left\langle \mathrm{d}A_n^{D-1}, \mathrm{d}A_n^{D-1} \right\rangle - \sum_{n=0}^{\infty} n^2 \left\langle A_n^{D-1}, A_n^{D-1} \right\rangle + \sum_{n=0}^{\infty} \langle \mathrm{d}A_n^{\varphi}, \mathrm{d}A_n^{\varphi} \rangle. \tag{3.7}$$

The first and second term describe an infinite tower of massive vector fields in $D-1$ dimensions as well as a single massless vector field (the $n=0$ term). This is accompanied by the third term which is a kinetic term for a massless scalar in

†Here taken to be defined by $\star 1 = \mathrm{vol}(M)$.

‡There are also additional cross terms that are gauge equivalent to zero.

$D-1$ dimensions. These are quite general features when compactifying a theory, the appearance of a tower of new massive fields as well as massless ones. It might seem distressful that we now have an infinite number of fields in our theory. On the other hand if we are only interested in what happens below a certain energy scale the increasingly heavy fields in the tower will not play a role. In the low energy limit we are only left with the massless fields which as we have seen are finite in number.

For a very readable introduction to the concept of compactification in the context of field theory see [16].

### 3.1.2 Generalisation

In the previous section we saw an example of the basic features of compactification: a space-time with a compact direction gives rise to a theory in one dimension lower containing new fields. This method is tremendously useful and has been used to derive a great deal of information about various theories, as well as constructing new ones. There are some very natural generalisations to this scheme. Firstly we can let the space-time be curved. This complicates the above discussion but does not change the basic results of compactification. A more interesting direction is to let the compact space vary in its geometry as we change position in the lower dimensional part.

In the above example the whole of space-time was a product of a lower dimensional part and a circle. This means that wherever we are positioned on $\mathbb{R}_{D-1}$ the circle looks the same. If we relax this property and let the circle vary in its orientation and size we get what is called a fibration. One would then expect that the geometric data of the circle will somehow be reflected in the compactified, lower dimensional, theory. This is the topic of the next section where the concept of a fibre bundle is introduced.

## 3.2 Fibre bundles

Intuition for fibre bundles comes naturally with a simple example. Consider an idealised hedgehog. It consists of a sphere with spines. The sphere is a two-dimensional space and the spines are one-dimensional. Mathematically we can describe the hedgehog by saying that it consists of a sphere with a line segment attached at every point on the surface. This evokes an interpretation of a hedgehog as a bundle of fibres.

More generally we start with a manifold $M$ called the base and a manifold $F$ called the fibre. The intuitive statement that we attach a copy of the fibre to every point of the base can be made precise with the following definition.

**Definition 3.2.1.** A fibre bundle $E$ over a manifold $M$ with fibre $F$ is a space that is locally diffeomorphic to the direct product of an open neighbourhood of $M$ with $F$.

In other words if we take an open neighbourhood $V \subset E$ there exists an open neighbourhood $U \in M$ such that

$$V = U \times F. \tag{3.8}$$

This is illustrated schematically in figure 3.1, where over each point in the neighbourhood $U \subset M$ we find a copy of the fibre $F$ which in this case is indicated by a line.

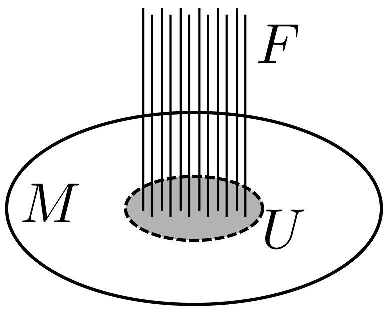


***Fig. 3.1:*** *A fibre bundle. Over each point in a neighbourhood $U$ we find a copy of the fibre $F$.*

### *3.2.1 Circle fibration*

In the special case when the fibre $F$ is a circle, the resulting bundle is called a circle bundle. This means that at each point in the base $M$ we find a circle. In this work a manifold that can be described as a circle bundle will be called a circle fibration. A simple example of a circle fibration is a torus. Here both the base and the fibre is a circle. The manifolds involved can be summarised neatly with the diagram in figure 3.2. Here the manifold $M_6$ is a circle fibration with fibre $\mathrm{S}_1$. The arrow from $\mathrm{S}_1$ to $M_6$ indicates the embedding of the fibre in $M_6$. The base of the fibration is $M_5$ and the arrow from $M_6$ to $M_5$ indicates that to each point in $M_6$ there is an associated point in the base given by a projection $\pi$. The last arrow indicates that locally we can find an isometry to the product space $M_5 \times \mathrm{S}_1$.

$$
\begin{array}{ccccc}
\mathrm{S}_1 & \longrightarrow & M_6 & \xrightarrow{\text{locally}} & M_5 \times \mathrm{S}_1 \\
 & & \downarrow{\scriptstyle \pi} & & \\
 & & M_5 & &
\end{array}
$$

***Fig. 3.2:*** *A diagramatic description of a circle fibration $M_6$.*

### *3.2.2 Geometry of circle fibrations*

So far we have only been talking about the smooth structure of the manifolds in question. This section will give a brief overview of how the metric information about a manifold is represented in the special situation of a circle fibration.

On a general six-dimensional Lorentzian manifold $M_6$ there is a semi-definite metric tensor $G_{MN}$. If $M_6$ is a circle fibration we have locally that

$$M_6|_V \cong M_5|_U \times S_1. \tag{3.9}$$

That is an open neighbourhood of $M_6$ is isometric to a product of an open neighbourhood in $M_5$ and a circle. The metric on the product space on the right is not necessarily a product metric, i.e. it will in general not have a block diagonal structure.

Let $\varphi$ be the coordinate in the direction of the circle, and as before we also let $\varphi$ be the value of the index for this direction. The metric $G_{MN}$ will then have the structure of equation (3.10).

$$G = G_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + 2G_{\mu\varphi}\mathrm{d}x^{\mu}\mathrm{d}\varphi + G_{\varphi\varphi}\mathrm{d}\varphi\mathrm{d}\varphi \tag{3.10}$$

We are however free to parametrise our metric in any way we see fit and the above choice is not the most convenient one. A better way to keep track of the reparametrisation invariance in six dimensions is

$$G = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + r^2\left(\mathrm{d}\varphi + \theta_{\mu}\mathrm{d}x^{\mu}\right)^2 . \tag{3.11}$$

This is just a renaming of the components in (3.10). It keeps track of reparametrisation invariance since

$$\varphi \to \varphi + \lambda(x^{\mu}) \qquad \Leftrightarrow \qquad \theta_{\mu} \to \theta_{\mu} + \partial_{\mu}\lambda. \tag{3.12}$$

The vector $\theta$ transforms as a U(1) gauge field under reparametrisations and the fact that the six-dimensional theory is reparametrisation invariant means in essence that the compactified theory can only depend on the gauge invariant field strength $\mathcal{F}_{\mu\nu} = \partial_{\mu}\theta_{\nu} - \partial_{\nu}\theta_{\mu}$.

The scalar $r$ also naturally corresponds to the radius of the fibre which can be seen if we take $\theta_{\mu} = 0$ thereby making the metric block diagonal and it is clear that $r$ corresponds to the circle radius.

## *3.3 (2,0) theory on circle fibrations*

We have now reached the point where we can describe the goal of Paper I. Here the free tensor multiplet is compactified on a six-dimensional space-time that is a circle fibration. Let us start by reviewing the situation for circle compactifications. Already in the original paper [2] proposing the existence of (2,0) theory some aspects of its compactification on $\mathbb{R}^5 \times S_1$ were discussed. Actually compactification arguments played an essential role in deducing its existence through the web of string theory dualities. By various consideration one can make it very plausible that the theory compactified on a circle will give rise to five-dimensional super Yang-Mills theory [17]. The lack of an explicit construction of the theory of course means that there are no proofs for such claims but only indications. Lately there have been efforts to show that (2,0) theory can perhaps be completely described by a five-dimensional theory [18, 19].

The purpose of Paper I is to continue these efforts in the more general setup that circle fibrations provide.

### *3.3.1 Overview of paper I*

The plan is summarised in figure 3.3. The starting point is to regard one free tensor multiplet of (2,0) theory on a circle fibration. A low energy limit is taken and we find a five-dimensional supersymmetric theory. This theory is shown to have a unique extension to an interacting theory which we derive. That the extension is unique makes it plausible that this is the low energy limit of the interacting (2,0) theory in six dimensions.

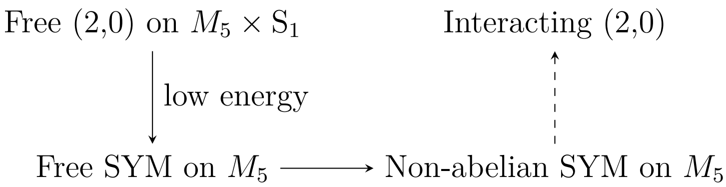


***Fig. 3.3:*** *Method of* Paper I. *Starting from the abelian theory in six dimensions we take the low energy limit on a circle fibration and extend to an interacting theory.*

### *3.3.2 Abelian compactification*

Let $M_6$ be a circle fibration over $M_5$ as described in section 3.2.2. Recall that the fibration metric is given by

$$G = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + r^2\left(\mathrm{d}\varphi + \theta_{\mu}\mathrm{d}x^{\mu}\right)^2 . \tag{3.13}$$

It is now a straight forward calculation to show that the scalar equation of motion§

$$\hat{\nabla}_M\hat{\nabla}^M\Phi - \frac{1}{5}\hat{R}\Phi = 0, \tag{3.14}$$

reduce in five dimensions to an equation that can be integrated to the action in (3.15).

$$S_{\phi} = \int \mathrm{d}^5x\sqrt{-g}\left(-\frac{1}{r}\nabla_{\mu}\phi_{\alpha\beta}\nabla^{\mu}\phi^{\alpha\beta} - \frac{1}{5}\frac{1}{5}R\phi_{\alpha\beta}\phi^{\alpha\beta} + K(g,r,\theta)\phi_{\alpha\beta}\phi^{\alpha\beta}\right) \tag{3.15}$$

where

$$K(g,r,\theta) = \frac{1}{r^3}\nabla_{\mu}r\nabla^{\mu}r - \frac{3}{5}\frac{1}{r^2}\nabla_{\mu}\nabla^{\mu}r + \frac{1}{20}r\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \tag{3.16}$$

contains the geometric information about the circle fibration, i.e. the fibration radius $r$ and the non dynamical field strength $\mathcal{F}_{\mu\nu} = \partial_{\mu}\theta_{\nu} - \partial_{\nu}\theta_{\mu}$. Note that in the general case the geometric data $r$ and $\mathcal{F}$ are functions on $M_5$ and so can vary over the manifold.

For the fermions the computation is also in principle straight forward but is complicated by the somewhat more heavy machinery of spinors in curved backgrounds.

§Here the six-dimensional covariant derivative and scalar curvature are indicated with a hat to distinguish them from their five-dimensional counterparts.

In PAPER I we provide a self contained description of this process in full detail. The result is that the fermion equation of motion

$$\Gamma^M \hat{\nabla}_M \Psi = 0\,, \tag{3.17}$$

reduce in five dimensions to an equation that integrates to an action of the form (3.18).

$$S_\psi = \int \mathrm{d}^5x\sqrt{-g}\left(\frac{1}{r}i\bar{\psi}\gamma^\mu\nabla_\mu\psi - \frac{1}{8}\mathcal{F}_{\mu\nu}\bar{\psi}\gamma^{\mu\nu}\psi\right) \tag{3.18}$$

Both (3.15) and (3.18) have the appearance of five-dimensional super Yang-Mills but with additional geometric terms stemming from the fibration.

Now we come to the self-dual three-form where the story is more interesting. Let us delve a bit deeper into the calculations to elucidate some of the features of its compactification.

On the fibration geometry the three-form $H$ can be written as

$$H = E + F \wedge \mathrm{d}\varphi\,, \tag{3.19}$$

with $E$ a three-form on $M_5$ and $F$ a two-form on $M_5$. A self-dual three-form in six dimensions has 10 independent components, which is the same number as for a three-form and a two-form in five dimensions. One would therefore expect that the components of $F$ and $E$ are identified and this is precisely what happens. Writing out the self-duality condition $H = \star H$ we find that

$$E = -\frac{1}{r} \star F + \theta \wedge F. \tag{3.20}$$

Thus in the end we have a two-form field strength in five dimensions, precisely what is needed for a standard gauge theory¶.

It is now immediate that the equation of motion $\mathrm{d}H = 0$ implies first that $\mathrm{d}F = 0$ from (3.19) and also that $\mathrm{d}E = 0$ which with the identification in (3.20) gives an equation of motion for $F$ that can be integrated to the action in (3.21).

$$S_F = \int_{M_5}\left(-\frac{1}{r}F \wedge \star F + \theta \wedge F \wedge F\right) \tag{3.21}$$

Here we can observe a feature of the compactification of (2,0) theory that is very unusual but which has been known from its inception. Even though the above theory is non interacting we can anticipate the form of the coupling constant from the factors in the first term. It would seem that we have a coupling constant $\sqrt{r}$ which is the inverse of what would be expected from a standard dimensional reduction where we integrate out the circle and pick up a factor of $r$ in the nominator, giving rise to a coupling constant $\frac{1}{\sqrt{r}}$. In the context of compactification on circle fibrations this comes about very naturally from the geometry of the fibration.

In a similar fashion to the reduction of the equation of motions we find the five-dimensional supersymmetry transformations in (3.22), (3.23) and (3.25).

¶This can be compared to the example in section 3.1.1 where the potential gives rise to two new fields.

$$\delta\phi^{\alpha\beta} = 2\overline{\psi}^{[\alpha}\varepsilon^{\beta]} - \frac{1}{2}T^{\alpha\beta}\bar{\psi}_\gamma\varepsilon^\gamma\,, \tag{3.22}$$

$$\delta F_{\mu\nu} = -2i\nabla_{[\mu}\overline{\psi}_\alpha\gamma_{\nu]}\varepsilon^\alpha + i\frac{1}{r}\nabla^\rho r\overline{\psi}_\alpha\gamma_{\mu\nu\rho}\varepsilon^\alpha - 2i\frac{1}{r}\nabla_{[\mu}r\overline{\psi}_\alpha\gamma_{\nu]}\varepsilon^\alpha \tag{3.23}$$

$$+ r\mathcal{F}_{\mu\nu}\overline{\psi}_\alpha\varepsilon^\alpha + \frac{3}{2}r\mathcal{F}_{[\mu}{}^\rho\overline{\psi}_\alpha\gamma_{\nu]\rho}\varepsilon^\alpha - \frac{1}{4}r\mathcal{F}^{\rho\sigma}\overline{\psi}_\alpha\gamma_{\mu\nu\rho\sigma}\varepsilon^\alpha \tag{3.24}$$

and

$$\delta\psi^\alpha = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\alpha + 2iM_{\beta\gamma}\nabla_\mu\phi^{\alpha\beta}\gamma^\mu\varepsilon^\gamma \tag{3.25}$$

$$+\ 2i\frac{1}{r}M_{\beta\gamma}\phi^{\alpha\beta}\nabla_\mu r\gamma^\mu\varepsilon^\gamma - rM_{\beta\gamma}\phi^{\alpha\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\gamma\,. \tag{3.26}$$

The five-dimensional theory is invariant under these transformations provided the supersymmetry parameter satisfies the reduced version of the conformal Killing spinor equation

$$\nabla_\mu\varepsilon^\alpha = \frac{1}{2}\frac{1}{r}\nabla^\nu r\gamma_\mu\gamma_\nu\varepsilon^\alpha + \frac{i}{8}r\mathcal{F}^{\rho\sigma}\gamma_\mu\gamma_{\rho\sigma}\varepsilon^\alpha + \frac{i}{4}r\mathcal{F}_\mu{}^\nu\gamma_\nu\varepsilon^\alpha. \tag{3.27}$$

### *3.3.3 Interacting generalisation*

In the second part of Paper I we extend the abelian theory in five dimensions to include interactions. This process is highly constrained by the symmetries of the theory. It turns out that there is only one possible interacting extension.

Let us list the symmetries of the five-dimensional theory. Apart from five-dimensional Lorentz symmetry and the R-symmetry we have, if the background geometry permits, maximal supersymmetry. The introduction of the length scale $r$ has broken the conformal symmetry but there is still a remnant of it left. To see this, note that the theory in six dimensions only depend on the conformal class of the metric. This means that we end up with the same theory in five dimensions if we instead regard the metric

$$G' = e^{-2\sigma}G \tag{3.28}$$

where $\sigma$ is a smooth function on $M_5$. In terms of the fibration geometry this means that the theory is invariant under

$$g_{\mu\nu} \to e^{-2\sigma}g_{\mu\nu} \tag{3.29}$$

$$r \to e^{-\sigma}r, \tag{3.30}$$

which can be deduced from the form of the metric in (3.13).

Any modifications to the theory must respect these symmetries. The plan is now straight forward, we promote $F$ to be the field strength of a connection of a non abelian gauge group. We let the scalars and fermions transform in the adjoint representation of this gauge group. We then proceed to promote all the covariant derivatives to gauge covariant derivatives. At this point we have a gauge invariant but no longer supersymmetric theory. To continue we begin by satisfying the condition that in the case of a trivial fibration the theory should reduce to supersymmetric Yang-Mills. To this end we add the standard Yukawa and $\phi^4$ terms.

$$S = S_\phi + S_\psi + S_F + \int_{M_5} \mathrm{d}^5x\sqrt{-g}\Big(2\frac{1}{r}f^{abc}\phi_a\bar{\psi}_b\psi_c + \frac{1}{r}f^{ab}{}_e f^{cde}\phi_a\phi_b\phi_c\phi_d\Big) \tag{3.31}$$

To preserve the supersymmetry of the action we also modify the supersymmetry variations according to

$$\delta\psi = \cdots + 2f_a{}^{bc}\phi_b\phi_c\epsilon. \tag{3.32}$$

With these modifications the theory reduces to five-dimensional super Yang-Mills in the case of a trivial fibration with product metric.

The main result of Paper I is that these modifications also in the case of a general fibration geometry constitutes a supersymmetric theory. We also argue that the above modifications constitute the only possible extension to the abelian theory respecting all the symmetries present.

### *3.3.4 Outlook*

There have been many interesting developments regarding the compactification of (2,0) theory in the last few years. In [20] the co-author of Paper I investigated a particular example of a singular circle fibration and solved the equations of motion for the gauge field. It was shown that the theory couples to additional degrees of freedom living on the singularity in the form of a Wess-Zumino-Witten model. The partition function for (2,0) theory on a circle fibration was computed in [21]. In [22] the action for five-dimensional super Yang-Mills in general supergravity backgrounds was computed using similar methods. For a recent discussion on the relation between five-dimensional super Yang-Mills and (2,0) theory see [23].

Another very interesting direction has been the construction of a large class of four-dimensional theories through compactification of (2,0) theory on a Riemann surface [24]. Through their common origin in six dimension there is a whole web of dualities between these theories extending S-duality of $N = 4$ SYM in four dimensions. This also brings us to the second part of this thesis which is concerned with the compactification and subsequent twisting of (2,0) theory on precisely a two manifold.

*Chapter 4:*

# Topological twisting

This chapter concerns the main technique used in Paper II, topological twisting. Briefly, it is a method to create topological field theories out of supersymmetric theories. From another perspective it can be viewed as a method to create supersymmetric theories on general curved manifolds from theories on flat manifolds. The technique dates back to the 80s when it was introduced [25].

## 4.1 Supersymmetry on curved manifolds

Let us start from the perspective of trying to create a globally supersymmetric theory on a curved manifold. A supersymmetry transformation is parametrised by a constant spinor $\varepsilon$. On a flat manifold there is no ambiguity in what we mean by a constant parameter, it simply has no space-time dependence. However when the theory lives on a curved manifold the situation becomes more tricky. The proper generalisation to being constant is to be covariantly constant. So to have a good parameter for supersymmetry we need to find a covariantly constant spinor. This is in general impossible and imposes severe constraints on the geometry of the manifold as can be easily seen. Suppose we have a spinor $\varepsilon$ that is covariantly constant:

$$D_\mu \epsilon = 0. \tag{4.1}$$

The above condition trivially implies $[D_\mu, D_\nu]\,\varepsilon = 0$ and using the fact that the covariant derivatives commutes to the Riemann tensor we find

$$R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon = 0. \tag{4.2}$$

This is an integrability condition for the curvature on the manifold which in general is not satisfied.

## 4.2 The twist

Topological twisting solves this problem in a very elegant fashion using the tools of group theory. The idea is to replace the space-time group by a new one, combining the space-time symmetries with the R-symmetry. The spinor representation of the original theory will now be reducible and will, under certain circumstances, contain a part that does not transform at all under the new space-time group. This means in particular that this part of the spinor also transforms trivially under the new space-time holonomy group and thereby can be considered as a rigid supersymmetry parameter.

### 4.2.1 Example: $N = 2$, $D = 4$ SYM

The details of the technique is best explained by an example. A very instructive example is that of the original paper introducing the concept, namely the twisting of $N = 2$ super Yang-Mills theory on four-dimensional Euclidean space-time. [25].

This theory has an SU(2) R-symmetry and the space-time symmetry group is Spin(4). The space-time group is isomorphic to SU(2)×SU(2), where the two groups are often referred to as the left and right part. The supersymmetry parameters $\varepsilon$ are chiral spinors transforming in the two-dimensional representation of $\mathrm{SU}(2)_\mathrm{R}$. Let us look at one of these parameters, the one transforming under $\mathrm{SU}(2)_r$. Its transformation properties are summarised in (4.3).

$$\begin{array}{cccccc} & \mathrm{SU}(2)_\mathrm{l} & \times & \mathrm{SU}(2)_\mathrm{r} & \times & \mathrm{SU}(2)_\mathrm{R} \\ \varepsilon \in & \mathbf{1} & & \mathbf{2} & & \mathbf{2} \end{array} \tag{4.3}$$

Let us now define a new SU(2). For concreteness let us denote the generators of $\mathrm{SU}(2)_\mathrm{l}$ by $\{T^i_\mathrm{l}\}$ and the generators of $\mathrm{SU}(2)_\mathrm{R}$ by $\{T^i_\mathrm{R}\}$ with $i \in 1,2,3$. We define a new set of generators $T^i_\mathrm{twist}$ generating a new SU(2) as follows.

$$T^i_\mathrm{twist} = T^i_\mathrm{r} + T^i_\mathrm{R} \tag{4.4}$$

The new group is thus what is called the diagonal of $\mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R}$, we rotate in both factors at the same time.

How will $\epsilon$ transform under this new group? What we have been doing simply amounts to taking the tensor product of the two representations and the answer is that

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}. \tag{4.5}$$

The representation $(\mathbf{1},\mathbf{2},\mathbf{2})$ under $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R}$ therefore splits into $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ under $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{twist}$.

$$\begin{array}{ccc} \mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R} & \xrightarrow{\text{twist}} & \mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{twist} \\ (\mathbf{1},\mathbf{2},\mathbf{2}) & & (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \end{array} \tag{4.6}$$

Note that the first term is a singlet under both factors. Thus we find that from the original 8 supercharges we have constructed one scalar supercharge under the new Lorentz group. Let this scalar supercharge be called simply $Q$.

$$Q \in (\mathbf{1},\mathbf{1}) \tag{4.7}$$

This supercharge have many interesting properties. From the supersymmetry algebra

$$\left\{Q^\alpha, Q^\beta\right\} = (\gamma_\mu)^{\alpha\beta} P^\mu , \tag{4.8}$$

we see that after the twisting we have a Lorentz scalar in the left hand side but there is no Lorentz scalar operator available for the right hand side so we must have

$$Q^2 = 0. \tag{4.9}$$

Here we have an operator that squares to zero. This invokes a strong urge to immediately look for $Q$-closed and $Q$-exact quantities and what their cohomology looks like. In words the cohomology of this operator means that we are looking at supersymmetric quantities but we don't care if they differ by a quantity that is the supersymmetry transformation of something else.

A first observation is that the expectation value of a $Q$-exact operator vanishes.

$$\langle \delta_Q \mathcal{O} \rangle = \int \mathcal{D}\Phi \, \delta_Q \mathcal{O} \, e^{-S[\Phi]} \tag{4.10}$$

$$= \int \mathcal{D}\Phi \, \delta_Q \left( \mathcal{O} \, e^{-S[\Phi]} \right) \tag{4.11}$$

$$= 0 \tag{4.12}$$

In the first step we use that the action is supersymmetric. The second step assumes that the path integral measure is supersymmetric so that the total supersymmetry variation can be absorbed by a change of variables in field space.

It also turns out that in this theory the stress tensor is a $Q$-exact quantity.

$$T^{\mu\nu} = \delta_Q \lambda^{\mu\nu} \tag{4.13}$$

This has a very interesting consequence. Lets regard the expectation value of a supersymmetric and metric independent operator $O$ and lets see how it behaves under a metric pertubation.

$$\delta_g \langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \mathcal{O} \delta_g e^{-S[\Phi]} \tag{4.14}$$

By the definition of the stress tensor and the fact that it is $Q$-exact we now have

$$\delta_g \langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \mathcal{O} \delta g_{\mu\nu} T^{\mu\nu} e^{-S[\Phi]} \tag{4.15}$$

$$= \int \mathcal{D}\Phi \mathcal{O} \delta g_{\mu\nu} \delta_Q \lambda^{\mu\nu} e^{-S[\Phi]} \tag{4.16}$$

$$= \int \mathcal{D}\Phi \delta_Q \left( \mathcal{O} \delta g_{\mu\nu} \lambda^{\mu\nu} e^{-S[\Phi]} \right) \tag{4.17}$$

$$= 0 \tag{4.18}$$

The upshot of all this is that if we restrict our attention to $Q$-cohomology then the theory is in fact topological.

## 4.3 Twisting (2,0)

The existence of the interacting (2,0) theory in six dimensions has, as we have seen in the previous chapters, provided an explanation of many properties of lower dimensional theories. One beautiful example of this is the construction due to Gaiotto [24]. Here a whole class of four-dimensional supersymmetric gauge theories* are constructed by compactifying (2,0) on a Riemann surface with possible defects. Their common origin in the six-dimensional theory induces a web of dualities between these theories, a kind of S-duality. Closely related to this construction is a recent

*Usually referred to as class $\mathcal{S}$ [26, 27].

conjecture that there is a correspondence between four-dimensional $N = 2$ gauge theories and two-dimensional conformal field theories. This conjecture[†] is referred to as the AGT[‡] correspondence [28] and states, among other things, that correlation functions in Liouville theory can be computed by the Nekrasov [29] partition function of a four-dimensional $N = 2$ super Yang-Mills theory.

One very natural explanation of this was put forward in [30] and also indicated in [31]. The observation is that if we could somehow compactify (2,0) theory on the four-dimensional manifold instead we would end up with a conformal theory on the Riemann surface. The idea would then be to look for quantities that are protected under both compactifications and this would then hopefully explain the correspondence.

Again the fact that there is no explicit formulation of (2,0) theory means that these ideas rests on the assumption of the existence of the theory. Furthermore in [30] it is assumed that when the compactification on the Riemann surface is performed and the holonomy on the four manifold is twisted we end up with a topological field theory on the four manifold. In Paper II we investigate these claims by explicit calculations in the setting of the abelian theory version of (2,0) theory.

One of the main differences between our treatment and that of [30] is that we work in Lorentzian signature. There are certain conceptual difficulties when formulating the theory in Euclidean signature. One of them is that the Hodge dual does not square to one but rather to minus one which implies that we cannot regard a real self-dual three-form but rather a complex three-form. The situation for the spinors is also different in the two signatures. To be as explicit as possible and to avoid any pitfalls with the choice of Euclidean signature we choose to carry out our investigation in Lorentzian signature.

### *4.3.1 Overview of paper II*

The three main steps of Paper II is summarised in figure 4.1. The starting point is to regard abelian (2,0) theory on a Lorentzian manifold $M_{1,5} = C \times M_4$, where $C$ is a compact two manifold. We choose to work in Lorentzian signature to avoid the problems associated to formulating (2,0) theory in Euclidean signature. This choice of signature is not ideal from the perspective of twisting as will be shown shortly, however we try to stay as close as possible to the proposed construction in Euclidean signature and see where it leads us.

We start by twisting the holonomy on a flat $M_4$ and identify the scalar supercharge $Q$ corresponding to the one used in the Euclidean construction. A $Q$-exact stress tensor is found implying the possibility of a topological theory. However we show that it is not possible to find such a $Q$-exact stress tensor when the theory is considered on a general curved $M_4$. This seems to imply that the specific twist considered in the Euclidean case does not produce the claimed topological theory in four dimensions.

The choice of which groups to twist and how this affects the relevant representations lies at the heart of Paper II. It will therefore be covered in greater detail

[†]There is as of yet no formal proof of this correspondence.

[‡]Alday, Gaiotto and Tachikawa.

Abelian (2,0) theory on $M_{1,5} = C \times M_4$

↓

Twist $M_4$ holonomy

↓

Compactify on $C$

***Fig. 4.1:*** *Method of* Paper II. *Abelian (2,0) theory is twisted and compactified on a Lorentzian manifold of the form $C \times M_4$.*

whereas the more straightforward computations of twisted field content and their dynamics will only be briefly discussed. Finally the situation for the stress tensor of the twisted theory is regarded and the conclusion that there is no $Q$-exact stress tensor is discussed.

### *4.3.2 Lorentzian twist*

In chapter 2 we saw that (2,0) theory has an $R$-symmetry group Spin(5), in this section indicated by a subscript R. Thus on a Lorentzian manifold $M_{1,5}$ the bosonic part of the symmetry group for the theory is given by

$$\mathrm{Spin}(1{,}5) \times \mathrm{Spin}(5)_{\mathrm{R}}. \tag{4.19}$$

We now take the six manifold to be of the form

$$M_{1,5} = C \times M_4, \tag{4.20}$$

where $C$ is a compact two manifold with Minkowski signature and $M_4$ is a four manifold. The space-time symmetry group is broken into two parts and we now have

$$\underbrace{\mathrm{Spin}(1{,}1)}_{C} \times \underbrace{\mathrm{Spin}(4)}_{M_4} \times \mathrm{Spin}(5)_{\mathrm{R}}. \tag{4.21}$$

It should here be pointed out that the space-time symmetry group contains a non-compact part when working in Minkowski signature. Normally to be able to find a scalar supercharge the whole holonomy group needs to be twisted away which means that the Lorentz group must be embeddable into the R-symmetry group of the theory. Since it is not possible to embed a non-compact group into a compact group we will only perform a partial twist.

Now we make a few observations regarding the structure of the symmetry groups. The Lorentz group on $M_4$ is Spin(4) which is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. As before we let a subscript l and r denote the left and the right factor respectively. For the R-symmetry group we have that $\mathrm{Spin}(3) \times \mathrm{Spin}(2) \subset \mathrm{Spin}(5)_R$. Using that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ we thus have that

$$\mathrm{SU}(2)_{\mathrm{l}} \times \mathrm{SU}(2)_{\mathrm{r}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{R}} \subset \mathrm{Spin}(4) \times \mathrm{Spin}(5)_{\mathrm{R}}. \tag{4.22}$$

The supersymmetry parameter $\varepsilon$ is a symplectic Majorana-Weyl spinor of negative chirality. In terms of the Lorentz group on $M_4$ it therefore transforms as $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ under $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r}$. It transforms in the $\mathbf{4}$ of $\mathrm{Spin}(5)_\mathrm{R}$ which under the subgroup $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}$ transforms as $\mathbf{2}^{\frac{1}{2}} \oplus \mathbf{2}^{-\frac{1}{2}}$.

Combining this information we have that the supersymmetry parameter transforms under the subgroups in (4.22) as

$$\varepsilon \in (\mathbf{1},\mathbf{2},\mathbf{2})^{\pm\frac{1}{2}} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})^{\pm\frac{1}{2}}. \tag{4.23}$$

From the results in section 4.2.1 we now see what needs to be done to find a scalar supercharge. There are two possibilities that are equivalent. We choose to twist $\mathrm{SU}(2)_\mathrm{r}$ and $\mathrm{SU}(2)_\mathrm{R}$. Let $\mathrm{SU}(2)' = \mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R}$, then the first term in 4.23 will be twisted according to table 4.1.

| $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)_\mathrm{r} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}$ | $\xrightarrow{\text{twist}}$ | $\mathrm{SU}(2)_\mathrm{l} \times \mathrm{SU}(2)' \times \mathrm{U}(1)_\mathrm{R}$ |
|---|---|---|
| $(\mathbf{1},\mathbf{2},\mathbf{2})^{\pm\frac{1}{2}}$ | | $(\mathbf{1},\mathbf{1})^{\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{3})^{\pm\frac{1}{2}}$ |

***Tab. 4.1:*** *Twisting of the supersymmetry parameter representation.*

After twisting there are two scalar components on $M_4$ with positive and negative $\mathrm{U}(1)_\mathrm{R}$ charge. There is now a definite choice of which supercharge to use. To see this we need to consider the transformation properties under $\mathrm{Spin}(1{,}1)_\mathrm{C}$. Let $\pm$ indicate the two one-dimensional representations of $\mathrm{Spin}(1{,}1) \cong \mathbb{R}$. Here we let the $+$ correspond to the representation that in Euclidean signature would have positive charge under $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Then the full representation of the part of $\varepsilon$ in table 4.1 is given by

$$^{+}(\mathbf{1},\mathbf{2},\mathbf{2})^{\pm\frac{1}{2}} \quad \xrightarrow{\text{twist}} \quad ^{+}(\mathbf{1},\mathbf{1})^{\pm\frac{1}{2}} \oplus {}^{+}(\mathbf{1},\mathbf{3})^{\pm\frac{1}{2}}. \tag{4.24}$$

The supersymmetry parameter is a spinor of negative chirality which fixes the choice of Spin(1,1) representation to be + in (4.24).

If the twisting was carried out in Euclidean signature then we would also have to twist away the dependence on Spin(1,1) by taking the diagonal embedding of $\mathrm{U}(1)'$ in $\mathrm{Spin}(2)_C \times \mathrm{Spin}(2)_\mathrm{R}$. The twisted U(1) charge is then simply the sum of the individual charges. It is then clear that the component of the $M_4$ scalar representation in (4.24) that would have zero charge under $\mathrm{U}(1)'$ is the one with negative $\mathrm{U}(1)_\mathrm{R}$ charge.

Therefore we choose to regard the $M_4$ scalar supercharge in $(\mathbf{1},\mathbf{1})^{-\frac{1}{2}}$ when performing the calculations in Paper II.

### *4.3.3 Twisted tensor multiplet*

The fields of the tensor multiplet gives rise to a number of fields when the twisting is performed. With similar arguments as in the previous section one arrives at the field content in (4.25). Here $E_{\mu\nu}$ is a real self-dual two form, $\sigma$ a complex scalar, $\{\psi,\, \tilde{\psi}\}$ fermionic one forms, $\{\chi,\, \tilde{\chi}\}$ fermionic self-dual two forms, $\{\eta,\, \tilde{\eta}\}$ fermionic scalars, $\{F^-,\, F^+\}$ anti self-dual and self-dual real two forms and finally $A_\mu$ a real one-form.

$$
\begin{aligned}
\Phi &\xrightarrow{\text{twist}} E_{\mu\nu}, \bar{\sigma}, \sigma \\
\Psi &\longrightarrow \psi_\mu, \tilde{\psi}_\mu, \chi_{\mu\nu}, \tilde{\chi}_{\mu\nu}, \eta, \tilde{\eta} \\
H &\longrightarrow F^-_{\mu\nu}, F^+_{\mu\nu}, A_\mu
\end{aligned}
\tag{4.25}
$$

In Paper II the derivation of the twisted field content is performed in detail. Let us here only confirm the counting. The scalar in six dimensions sits in a five-dimensional representation corresponding to the three components of the self-dual two-form and the two components in the complex scalar. The fermionic fields, counting from left to right, contain $4+4+3+3+1+1=16$ real components corresponding to the 16 real components of the symplectic Majorana-Weyl spinor $\Psi$. Similarly the fields arising from the self-dual three-form sums to 10 real components.

The six-dimensional equations of motion and supersymmetry transformations give rise to corresponding equations and transformations for the twisted fields. This is also derived in detail in Paper II from the explicit relations between the twisted and untwisted fields of the tensor multiplet.

### 4.3.4 Stress tensor

After a compactification on $C$ the equations for the twisted fields will be purely four-dimensional. We then proceed to determine the stress tensor for the theory defined on a flat manifold. This is done in two steps. First an ansatz for the stress tensor for the fields arising from $\Phi$ and $\Psi$ is made from the metric variation of the action that do exists for these fields. For the fields arising from $H$ an ansatz is made starting from the stress tensor for a general three-form. It is found that this ansatz needs to be modified for the stress tensor to be supersymmetric. The modified stress tensor is then shown to be $Q$-exact. For completeness let us here give the full expression as it is rather compact.

$$
T^{\mu\nu} = \delta_Q \lambda^{\mu\nu}, \tag{4.26}
$$

where

$$
\begin{aligned}
\lambda^{\mu\nu} = \frac{1}{2}\Big(&\sqrt{2} i \psi^{(\mu}\partial^{\nu)}\sigma + \tilde{\psi}^{(\mu}\partial^\rho E^{\nu)}{}_\rho + \partial_\rho \tilde{\psi}^{(\mu} E^{\nu)}{}_\rho - \partial^{(\mu}\tilde{\psi}^\rho E^{\nu)}{}_\rho \\
&+ i\tilde{\psi}^{(\mu} A^{\nu)} - \frac{i}{2}\tilde{\chi}^{(\mu}{}_\rho F^{-\nu)\rho} - \frac{i}{\sqrt{2}} g^{\mu\nu}\psi_\rho \partial^\rho \sigma - \frac{1}{2} g^{\mu\nu}\tilde{\psi}_\rho \partial_\sigma E_{\rho\sigma} - \frac{i}{2} g^{\mu\nu}\tilde{\psi}_\rho A^\rho\Big).
\end{aligned}
\tag{4.27}
$$

Here $Q$ denotes the scalar supercharge transforming as $(\mathbf{1},\mathbf{1})^{-\frac{1}{2}}$ under the twisted Lorentz group, described in the previous section.

In the final step of Paper II we show that there is no possible extension of this stress tensor to a general curved $M_4$ that is both $Q$-exact and $Q$-closed. When trying to covariantise the above expression for $\lambda^{\mu\nu}$ one meets curvature terms that are not cancelled when checking supersymmetry. These would need to be cancelled by new curvature terms in $\lambda^{\mu\nu}$, however from purely dimensional and symmetry reasons there cannot be any such terms.

### *4.3.5 Outlook*

The results of Paper II indicate that there are some subtleties in twisting (2,0) theory on a Lorentzian manifold in a way that would be natural from the Euclidean perspective. It should be pointed out that our twisting is, from the four-dimensional perspective, very similar to the Donaldson-Witten twist considered in [25]. The difference is in the slightly different field content of the theories and a different choice of scalar supercharge. Here we are motivated by trying to perform the twisting as close as possible to the one which would be performed in the Euclidean scenario. This difference might be the source of the non-topological nature of our result but nevertheless indicates that some care is needed when considering the twisted versions of (2,0) theory.

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# Paper I

# $(2,0)$ theory on circle fibrations


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**Abstract:**

We consider $(2,0)$ theory on a manifold $M_6$ that is a fibration of a spatial $S^1$ over some five-dimensional base manifold $M_5$. Initially, we study the free $(2,0)$ tensor multiplet which can be described in terms of classical equations of motion in six dimensions. Given a metric on $M_6$ the low energy effective theory obtained through dimensional reduction on the circle is a Maxwell theory on $M_5$. The parameters describing the local geometry of the fibration are interpreted respectively as the metric on $M_5$, a non-dynamical U(1) gauge field and the coupling strength of the resulting low energy Maxwell theory. We derive the general form of the action of the Maxwell theory by integrating the reduced equations of motion, and consider the symmetries of this theory originating from the superconformal symmetry in six dimensions. Subsequently, we consider a non-abelian generalization of the Maxwell theory on $M_5$. Completing the theory with Yukawa and $\phi^4$ terms, and suitably modifying the supersymmetry transformations, we obtain a supersymmetric Yang-Mills theory which includes terms related to the geometry of the fibration.



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## Contents



# 1 Introduction

The six-dimensional $(2,0)$ theory [1, 2] has been a subject of much interest recently, in particular through its interpretation as the world volume theory of multiple $M5$-branes and the progress towards a better understanding of various brane configurations in $M$-theory. In [3] the connection, through dimensional reduction on a circle, with supersymmetric Yang-Mills theory in five dimensions is discussed for the case of a six-dimensional manifold that is a direct product of $S^1$ and five-dimensional Minkowski space-time. Recent work exploring this connection between $(2,0)$ theory and supersymmetric Yang-Mills theory in five dimensions includes [4, 5, 6, 7, 8, 9, 10]. In the present paper we consider, along the lines of [11], the generalization of the above geometry to a general circle fibration $M_6 \to M_5$ for the free theory of the $(2,0)$ tensor multiplet [12]. We also consider a non-abelian generalization of the Maxwell theory and construct a candidate for the interacting $(2,0)$ theory on $M_5$, which cannot be directly obtained by dimensional reduction.

The self-duality of the tensor three-form field strength makes a Lagrangian description problematic, but the tensor multiplet possesses a classical description in terms of equations of motion. The low energy effective theory obtained in the dimensional reduction on the $S^1$ fibre is a Maxwell theory on $M_5$ describing an abelian gauge field, five scalars and four spinors satisfying appropriate reality conditions. The coupling strength of the gauge theory, given by the square root of the radius of the $S^1$, is a function on the base manifold of the fibration. Furthermore, the U(1) subgroup of diffeomorphisms of $M_6$, corresponding to reparametrizations along the circle, gives rise to an additional non-dynamical abelian gauge field (the connection on the U(1) bundle $M_6$ over $M_5$) coupled to the gauge theory. In five dimensions it is possible to integrate the equations of motion to obtain an action describing the complete gauge theory, which we derive for a generic metric on $M_6$. In particular, the action contains terms including the U(1) field strength and the gradient of the radius in addition to the topological term for the gauge fields discussed in [11].

The theory of the $(2, 0)$ tensor multiplet depends only on the conformal structure of $M_6$, i.e. the equations of motion are covariant under a Weyl rescaling of the metric and simultaneous rescalings of the fields according to their conformal weight. A consequence of this conformal symmetry in six dimensions is that the gauge theory on $M_5$ obtained by the reduction is invariant under the corresponding simultaneous conformal rescalings of the five-dimensional metric, the dynamical fields and the (varying) coupling strength parameter.

When the manifold $M_6$ admits conformal Killing spinors the theory in six dimensions is also supersymmetric at the level of the equations of motion. (That is, the set of solutions to the equations of motion is closed under supersymmetry transformations.) In this case the same is true also for the five-dimensional theory. Furthermore, the action on $M_5$ is invariant under the same supersymmetry transformations as the equations of motion. In principle (if not in practice) this is a non-trivial feature since it extends the supersymmetry from its stationary points to the full action functional. Of course, the conformal symmetry of the tensor multiplet theory persists regardless of the existence of non-trivial conformal Killing spinors on $M_6$. We will, as previously mentioned, consider the reduction on arbitrary circle fibration, which implies that generically the six-dimensional theory will not be supersymmetric.

It is possible to generalize the abelian theory on $M_5$ to a non-abelian gauge theory by covariantizing the action and adding interaction terms. Including the ordinary Yukawa coupling and quartic scalar self-interaction of ordinary supersymmetric Yang-Mills theory produces a theory on $M_5$ that is invariant under the same generalized conformal symmetry as in the abelian case. Furthermore, when the ordinary non-linear term in the fermionic supersymmetry variation is included, the non-abelian gauge theory is supersymmetric whenever $M_5$ admits non-trivial solutions to the dimensionally reduced conformal Killing spinor equation, providing a non-trivial check of the construction.

The paper is organized as follows: In section 2 we consider the properties, in particular the superconformal symmetry, of the $(2, 0)$ tensor multiplet theory in six dimensions. As is well known, the self-dual three form field does not admit a Lagrangian description (without the introduction of auxiliary fields [13, 14]) and we will thus only consider $H$ at the level of equations of motion. The scalar and spinor of the tensor multiplet, however, can be described using action functionals. In section 3 we discuss the details of the spatial circle fibration and give explicit expressions for various geometrical quantities. We also describe how the Clifford algebra and spinors in six dimensions are decomposed with respect to the fibration. Subsequently, in section 4 we perform the dimensional reduction and obtain Maxwell theory as the low energy effective theory in five dimensions. We also construct an action corresponding to the equations of motion of this theory and consider its properties under conformal rescalings and supersymmetry transformations in five dimensions. Finally, in section 5, we consider extending the supersymmetric Maxwell theory to a non-abelian supersymmetric Yang-Mills theory by adding interaction terms.

## 2 Abelian $(2,0)$ theory in six dimensions

In this section we review some aspects of the starting point for our consideration: The free tensor multiplet of $(2,0)$ theory on a six-dimensional manifold $M_6$ with local coordinates $y^M$, where $M = 0, 1 \ldots, 5$. The abelian $(2,0)$ theory depends only on the conformal structure on $M_6$, i.e. on the conformal class of the metric $G_{MN}$ defined by the equivalence relation

$$G_{MN}(y) \sim e^{-2\sigma(y)} G_{MN}(y) \tag{2.1}$$

for some arbitrary function $\sigma(y)$, the coordinate dependence of which will be left implicit below. The invariance of the theory under $G_{MN} \to e^{-2\sigma} G_{MN}$ require suitable rescalings of the fields of the abelian multiplet which are discussed below. In addition to the conformal rescaling symmetry, there is a global R-symmetry group[1]

$$\mathcal{R} \cong \mathrm{USp}(4) \cong \mathrm{Spin}(5)\,. \tag{2.2}$$

Associated to the fundamental representation **4** of this group there is a symplectic metric $M_{\alpha\beta}$, where $\alpha$ and $\beta$ are spinor indices in the **4** representation. Details on this symplectic structure are provided in the appendix.

The manifold $M_6$ will generically have non-vanishing curvature and in order to describe spinor fields we will therefore need to introduce the vielbein $E_M^A$, defining locally an orthonormal frame. Here, $A = 0, 1, \ldots, 5$ are flat indices, raised and lowered using the Minkowski metric $\eta_{AB}$, on which local Lorentz transformations act. The vielbein and its inverse $E_A^M$ satisfy the relations

$$E_M^A E_{NA} = G_{MN} \quad , \quad E_A^M E_{MB} = \eta_{AB}\,, \tag{2.3}$$

which imply that under conformal transformations of the metric, the vielbein transforms as $E_M^A \to e^{-\sigma} E_M^A$.

[1]Here, USp(4) is the compact real form of the Symplectic group $\mathrm{Sp}(4, \mathbb{C})$ with Lie algebra $C_2$.

### 2.1 The tensor multiplet

The field content of the abelian theory is a real three-form tensor field $H$, a scalar $\Phi^{\alpha\beta}$ and a positive chirality space-time spinor field $\Psi^\alpha$. The spinor and scalar fields transform respectively in the fundamental **4** representation and the **5** vector representation of the USp(4) R-symmetry. We will use the description of the fundamental representation as a spinor of Spin(5) and the vector representation as an antisymmetric bispinor which is traceless with respect to $M_{\alpha\beta}$. Further details regarding the symplectic transformation and reality properties of the fields are given in the appendix, where we also describe our conventions for Lorentz spinors in both $4+1$ and $5+1$ dimensions.

The tensor field transforms trivially under R-symmetry and is closed and self-dual, i.e. it satisfies the equations of motion

$$dH = 0 \quad , \quad H = *_G H \,, \tag{2.4}$$

where the subscript $G$ on the Hodge dual indicates the six-dimensional metric. Both equations are conformally invariant by virtue of the metric independence of $dH = 0$ and the determinant factor $\sqrt{-G}$ in the definition of the Hodge dual. The condition that $H$ be closed can be viewed as a consequence of it being the field strength of a two-form abelian gauge field $B$. However, for the purpose of the considerations of the present paper it is sufficient to consider only the three-form $H$, in which case $dH = 0$ is considered as an equation of motion[2]. As mentioned in the introduction there is no Lagrangian description of the self-dual tensor field in six dimensions.

The scalar $\Phi^{\alpha\beta}$ satisfies the symplectic reality condition $(\Phi^{\alpha\beta})^* = \Phi_{\alpha\beta}$ and the equation of motion

$$G^{MN}\hat{\nabla}_M\hat{\nabla}_N\Phi^{\alpha\beta} + c\hat{R}\Phi^{\alpha\beta} = 0 \,, \tag{2.5}$$

where $\hat{R}$ is the curvature scalar of the metric $G_{MN}$ and $\hat{\nabla}_M$ is the covariant derivative on $M_6$. In $d$ dimensions, this equation transforms covariantly under a conformal rescaling of the metric and a simultaneous transformation $\Phi^{\alpha\beta} \to e^{2\sigma}\Phi^{\alpha\beta}$ of the scalar field, provided that

$$c = -\frac{1}{4}\frac{(d-2)}{(d-1)} = -\frac{1}{5} \,, \tag{2.6}$$

where in the last step we have inserted $d = 6$. In contrast to the three-form, there is a Lagrangian description for the scalar; the equations of motion follow from the action

$$S_\Phi = \int d^6y\sqrt{-G}\,\left(-\hat{\nabla}_M\Phi_{\alpha\beta}\hat{\nabla}^M\Phi^{\alpha\beta} + c\hat{R}\Phi_{\alpha\beta}\Phi^{\alpha\beta}\right) \,, \tag{2.7}$$

which is real and invariant under Lorentz transformations, symplectic transformations and conformal rescalings.

Finally, as mentioned above, the fermionic degrees of freedom of the tensor multiplet are contained in a positive chirality spinor $\Psi^\alpha$ of the local Lorentz group. The spinor also satisfies a symplectic Majorana condition

$$(\Psi^\alpha)^* = M_{\alpha\beta}B_{(6)}\Psi^\beta \,, \tag{2.8}$$

[2] Of course, $dH = 0$ implies that $H = dB$ at least locally.

where $B_{(6)}$ is related to the charge conjugation matrix in six dimensions. The equation of motion for the spinor is the ordinary Dirac equation

$$\Gamma^M \hat{\nabla}_M \Psi^\alpha = 0\,, \tag{2.9}$$

where the curved space-time index Γ-matrices $\Gamma^M = \Gamma^A E_A^M$ are obtained using the vielbein as usual and the covariant derivative acting on the spinor $\Psi^\alpha$ is given by

$$\hat{\nabla}_M \Psi^\alpha = \partial_M \Psi^\alpha + \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}\Psi^\alpha\,. \tag{2.10}$$

Here, $\Omega_M^{AB}$ is the spin connection, i.e. the gauge field of local Lorentz transformations, of the six-dimensional manifold $M_6$. The Dirac equation is conformally covariant requiring that the spinor is rescaled as $\Psi^\alpha \to e^{\frac{5}{2}\sigma}\Psi^\alpha$, and may be obtained as the stationary point of the action

$$S_\Psi = \int d^6y \sqrt{-G}\, i\overline{\Psi}_\alpha \Gamma^M \hat{\nabla}_M \Psi^\alpha\,. \tag{2.11}$$

Using the reality properties of symplectic Majorana bilinears given in the appendix one verifies the hermiticity of this conformally invariant action functional, which is also a Lorentz and USp(4) scalar. Finally, we note that $\Psi^\alpha$ describes eight fermionic on-shell degrees of freedom, matching the number of bosonic ones (five for the scalar $\Phi^{\alpha\beta}$ and three for the tensor field $H$) as required for supersymmetry.

It should be emphasized that the absence of a Lagrangian formulation implies that our treatment of the tensor multiplet through its equations of motion is strictly classical. (It is possible to construct a Lagrangian for the self-dual tensor field through the introduction of an auxiliary scalar field [13, 14]. However, a path integral quantization using such a Lagrangian appears to be problematic and we will refrain from considering the quantum theory in this paper.)

### 2.2 Supersymmetry of the tensor multiplet

We now turn our attention to the supersymmetry variations of the $(2,0)$ tensor multiplet fields, given by

$$\delta H_{MNP} = 3\hat{\nabla}_{[M}\left(\overline{\Psi}_\alpha \Gamma_{NP]}\mathcal{E}^\alpha\right)\,, \tag{2.12}$$

$$\delta\Phi^{\alpha\beta} = 2\overline{\Psi}^{[\alpha}\mathcal{E}^{\beta]} - \frac{1}{2}T^{\alpha\beta}\overline{\Psi}_\gamma \mathcal{E}^\gamma \tag{2.13}$$

and

$$\delta\Psi^\alpha = \frac{i}{12}H_{MNP}\Gamma^{MNP}\mathcal{E}^\alpha + 2iM_{\beta\gamma}\hat{\nabla}_M\Phi^{\alpha\beta}\Gamma^M\mathcal{E}^\gamma + \frac{4i}{3}M_{\beta\gamma}\Phi^{\alpha\beta}\Gamma^M\hat{\nabla}_M\mathcal{E}^\gamma\,, \tag{2.14}$$

where the parameter $\mathcal{E}^\alpha$ is a symplectic Majorana spinor of negative chirality in the **4** of USp(4). The variations $\delta H_{MNP}$, $\delta\Phi^{\alpha\beta}$ and $\delta\Psi^\alpha$ satisfy the same equations of motion as the original fields if (2.4), (2.5) and (2.9) are imposed and the parameter $\mathcal{E}^\alpha$ satisfies the conformal Killing spinor equation

$$P_M\mathcal{E}^\alpha = \hat{\nabla}_M\mathcal{E}^\alpha - \frac{1}{d}\Gamma_M\Gamma^N\hat{\nabla}_N\mathcal{E}^\alpha = 0\,, \tag{2.15}$$

which is conformally covariant in any dimension $d$ provided a rescaling of the parameter according to $\mathcal{E} \to e^{-\frac{1}{2}\sigma}\mathcal{E}$. The existence of non-vanishing $\mathcal{E}$ satisfying the (2.15) imposes a non-trivial condition on the geometry of the manifold $M_6$. In order to have $(2,0)$ supersymmetry we must consequently restrict our attention to manifolds for which the kernel of the operator $P_M$ is non-trivial.

# 3 Spatial circle fibrations

We now proceed to consider the case where $M_6$ is a fibration of $S^1$ over some five-dimensional base manifold $M_5$ of Lorentzian signature. Thus the curved vector index in six dimensions is split according to $M = (\mu, \varphi)$, where $\mu = 0, \ldots, 4$. We will allow ourselves to abuse the notation slightly by introducing $x^\mu = y^\mu$ and $\varphi = y^\varphi$. Here, $\varphi$ is the local coordinate along the $S^1$ fibre, while $x^\mu$ parametrize the base manifold $M_5$. We will adopt the convention that the range of the periodic $S^1$ coordinate is $0 \leq \varphi < 2\pi$. In the following section we will consider the dimensional reduction of the theory of the $(2,0)$ tensor multiplet on the $S^1$ to a (supersymmetric) Maxwell theory on $M_5$. The present section sets the stage for this reduction by investigating the various consequences of the specialization to a geometry in six dimensions that is a circle fibration.

## 3.1 Geometry of the fibration

The most general form of the metric on $M_6$ with the above decomposition is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + r^2\left(d\varphi + \theta_\mu dx^\mu\right)^2 . \tag{3.1}$$

The fact that $M_6$ can be described as a U(1)-bundle over $M_5$ implies the existence of an isometry along the $S^1$ and consequently the coefficient functions of (3.1) are all independent of the coordinate $\varphi$. Thus, $g_{\mu\nu}(x)$ can be interpreted as the metric on $M_5$, $r(x)$ as the radius of the $S^1$ fibre and the vector $\theta_\mu(x)$ as an angular parameter. The special case when $\partial_\mu r = 0$ and $\theta_\mu = 0$ is referred to as the product metric. For generic $r(x)$ and $\theta_\mu(x)$ we can read of the component expressions for the decomposition of the metric

$$G_{\mu\nu} = g_{\mu\nu} + r^2\theta_\mu\theta_\nu \;\;,\;\; G_{\mu\varphi} = r^2\theta_\mu \;\;,\;\; G_{\varphi\varphi} = r^2 \tag{3.2}$$

and its inverse

$$G^{\mu\nu} = g^{\mu\nu} \;\;,\;\; G^{\mu\varphi} = -\theta^\mu \;\;,\;\; G^{\varphi\varphi} = \frac{1}{r^2} + g^{\mu\nu}\theta_\mu\theta_\nu\,. \tag{3.3}$$

In analogy with the curved index $M$, the flat vector index is split according to $A = (a, 5)$ with $a = 0, \ldots, 4$. By a local Lorentz transformation the components of the vielbein $E_M^A$ and its inverse can be cast in the form

$$E_\mu^a = e_\mu^a \;\;,\;\; E_\mu^5 = r\theta_\mu \;\;,\;\; E_\varphi^a = 0 \;\;,\;\; E_\varphi^5 = r \tag{3.4}$$

and

$$E_a^\mu = e_a^\mu \;\;,\;\; E_5^\mu = 0 \;\;,\;\; E_a^\varphi = -\theta_\mu e_a^\mu \;\;,\;\; E_5^\varphi = \frac{1}{r}\,, \tag{3.5}$$

where $e^a_\mu$ is the vielbein on $M_5$, which satisfies the conditions (2.3). From their definition in terms of the vielbein it is now straightforward to compute the expressions for the non-vanishing components of the Levi-Civita connection $\hat{\Gamma}^P_{MN}$ and spin connection $\Omega^{AB}_M$ and obtain

$$\begin{aligned}
\hat{\Gamma}^\rho_{\mu\nu} &= \Gamma^\rho_{\mu\nu} - r^2\mathcal{F}^\rho{}_{(\mu}\theta_{\nu)} - r\nabla^\rho r\theta_\mu\theta_\nu \\
\hat{\Gamma}^\varphi_{\mu\nu} &= \nabla_{(\mu}\theta_{\nu)} + r^2\theta_\rho\mathcal{F}^\rho{}_{(\mu}\theta_{\nu)} + r\theta_\rho\nabla^\rho r\theta_\mu\theta_\nu + 2\frac{1}{r}\nabla_{(\mu}r\theta_{\nu)} \\
\hat{\Gamma}^\rho_{\mu\varphi} &= \frac{1}{2}r^2\mathcal{F}_\mu{}^\rho - r\theta_\mu\nabla^\rho r \\
\hat{\Gamma}^\varphi_{\mu\varphi} &= -\frac{1}{2}r^2\mathcal{F}_\mu{}^\rho\theta_\rho + r\theta_\mu\theta_\rho\nabla^\rho r + \frac{1}{r}\nabla_\mu r \\
\hat{\Gamma}^\rho_{\varphi\varphi} &= -r\nabla^\rho r \\
\hat{\Gamma}^\varphi_{\varphi\varphi} &= r\theta_\rho\nabla^\rho r
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\Omega^{ab}_\mu = \omega^{ab}_\mu - \frac{1}{2}r^2\theta_\mu e^a_\rho e^b_\sigma\mathcal{F}^{\rho\sigma} \quad &, \quad \Omega^{a5}_\mu = \frac{1}{2}re^a_\nu\mathcal{F}_\mu{}^\nu - \theta_\mu e^a_\nu\nabla^\nu r \\
\Omega^{ab}_\varphi = -\frac{1}{2}r^2 e^a_\rho e^b_\sigma\mathcal{F}^{\rho\sigma} \quad &, \quad \Omega^{a5}_\varphi = -e^a_\nu\nabla^\nu r\,,
\end{aligned} \tag{3.7}$$

where $\Gamma^\rho_{\mu\nu}$, $\omega^{ab}_\mu$ and $\nabla_\mu$ are respectively the Levi-Civita connection, the spin connection and the covariant derivative on the five-dimensional base manifold $M_5$. Finally, we can also compute an expression for the curvature scalar $\hat{R}$ appearing in the action (and equation of motion) for the scalar field in six dimensions. Using the expressions in (3.6) we obtain

$$\hat{R} = R - \frac{1}{4}r^2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - 2\frac{1}{r}\nabla_\mu\nabla^\mu r\,, \tag{3.8}$$

where $R$ denotes the curvature scalar of the metric $g_{\mu\nu}$ on $M_5$.

In the expressions above we have introduced the quantity

$$\mathcal{F}_{\mu\nu} = \partial_\mu\theta_\nu - \partial_\nu\theta_\mu\,, \tag{3.9}$$

which from the point of view of the dimensionally reduced theory on $M_5$ can be interpreted as the field strength of the non-dynamical U(1) gauge field $\theta_\mu(x)$ corresponding to reparametrization invariance along the $S^1$ of the six-dimensional theory. Consequently, all physical five-dimensional quantities must be invariant under a U(1) gauge transformation $\theta_\mu \to \theta_\mu + \partial_\mu\lambda$, corresponding to coordinate transformation $\varphi \to \varphi + \lambda(x)$ in six dimensions, which generically implies that they can only depend on the gauge invariant field strength $\mathcal{F}_{\mu\nu}$.

### 3.2 Decomposition of spinors

The dimensional reduction of the $(2,0)$ theory will involve the decomposition of spinors and Γ-matrices in six dimensions in terms their five-dimensional counterparts. Since the dimension $2^{[d/2]}$ of a Dirac spinor is different in five and six dimensions this decomposition involves, in addition to the split of vector indices described above, a corresponding split of the Lorentz spinor index. We choose a representation of the six-dimensional Clifford algebra in terms of the tensor products involving the five-dimensional $\gamma$-matrices as

$$\left\{ \begin{array}{lcl} \Gamma^a & = & \gamma^a \otimes \rho_1 \\ \Gamma^5 & = & \mathbb{1} \otimes \rho_2 \end{array} \right. , \tag{3.10}$$

where $\rho_1$ and $\rho_2$ are the first two Pauli matrices. These satisfy $\rho_i^2 = \mathbb{1}$, $\rho_i^\dagger = \rho_i$ and $\{\rho_i, \rho_j\} = 2\delta_{ij}$, for $i, j = 1, 2$, and consequently furnish a representation of the two-dimensional Euclidean Clifford algebra. We define the chirality operator in two Euclidean dimensions to be

$$\rho = -i\rho_1\rho_2 \,, \tag{3.11}$$

where the overall sign is a matter of convention and the particular choice above will prove convenient in what follows. As a basis of two-dimensional spinors we may take the two eigenvectors $\eta_\pm$ of $\rho$, satisfying $\rho\eta_\pm = \pm\eta_\pm$, which can be chosen to be real and orthonormal. The action of the $\rho_i$ on these basis vectors is given by

$$\begin{array}{lcl} \rho_1\eta_+ = \eta_- & , & \rho_2\eta_+ = i\eta_- \\ \rho_1\eta_- = \eta_+ & , & \rho_2\eta_- = -i\eta_+ \,. \end{array} \tag{3.12}$$

The charge conjugation matrix $C_{(6)}$ and the $B_{(6)}$ matrix are also decomposed as

$$B_{(6)} = B_{(5)} \otimes \mathbb{1} \;\; , \;\; C_{(6)} = C_{(5)} \otimes \rho_1 \,, \tag{3.13}$$

which together with the decomposition (3.10) is consistent with the conventions for $\Gamma$-matrices in $5+1$ and $4+1$ dimensions.

In the dimensional reduction from six to five dimensions the spinors decompose into tensor products in the same way as the $\Gamma$-matrices. Since the spinors $\Psi^\alpha$ and $\mathcal{E}^\alpha$ relevant for the tensor multiplet and its supersymmetry are symplectic Majorana-Weyl, we will restrict considerations to a spinor $\Lambda^\alpha$ in the **4** representation of USp(4) which satisfies the symplectic Majorana condition (2.8) and has a definite chirality $\Gamma\Lambda^\alpha = \pm\Lambda^\alpha$. With the conventions described in the appendix the chirality operator in six dimensions is $\Gamma = \mathbb{1} \otimes \rho$. Consequently, the decomposition of $\Lambda^\alpha$ is given by

$$\Lambda^\alpha = \lambda^\alpha \otimes \eta_\pm \tag{3.14}$$

according to its chirality. Note that we assume the symplectic spinor index $\alpha$ to be carried by the (Lorentz) spinor $\lambda^\alpha$, which is consistent with the fact that the R-symmetry is unchanged by the dimensional reduction, so that $\lambda^\alpha$ is also in the **4** of USp(4). Furthermore, it is consistent with the five-dimensional spinors satisfying the symplectic Majorana condition

$$(\lambda^\alpha)^* = M_{\alpha\beta} B_{(5)} \lambda^\beta \,, \tag{3.15}$$

analogous to (2.8), with the above decomposition of the charge conjugation matrix. Thus, (3.14) produces five-dimensional Lorentz spinors with the correct properties under symplectic transformations and complex conjugation.

## 4 Maxwell theory in five dimensions

We are now ready to consider the procedure at the heart of the present paper; the dimensional reduction on the $S^1$ fibre. In this section we consider the reduction of the theory of the free tensor multiplet. It is well known (see e.g. [3]) that for a direct product of $S^1$ with five-dimensional Minkowski space, equipped with the product metric, this produces the ordinary maximally supersymmetric $N = 4$ Maxwell theory in five dimensions at energies that are small compared to the fibre radius. In particular, the coupling is related to the (constant) radius of the $S^1$ fibre as $\tilde{g} = \sqrt{r}$. The R-symmetry of the Maxwell theory is the same as for the $(2,0)$ theory and the field content is a gauge field $A_\mu$ with field strength $F_{\mu\nu}$, a scalar $\phi^{\alpha\beta}$ and a symplectic Majorana spinor $\psi^\alpha$, the latter two transforming in the **5** and **4** representations of USp(4) respectively. The generalization to an arbitrary circle fibration should therefore in the low energy limit produce Maxwell theory with varying coupling strength, additional couplings to the non-dynamical U(1) gauge field $\theta_\mu$ and terms depending on the gradient of the radius $r(x)$. In the case when $M_6$ allows non-trivial solutions to $P_M \mathcal{E}^\alpha = 0$ there are unbroken supersymmetries of the $(2,0)$ theory and the five-dimensional theory should therefore be supersymmetric as well.

Before deriving the complete action of the dimensionally reduced theory we review a consequence of the fact that $M_6$ is a fibration of $S^1$ over $M_5$ and the existence of local coordinates $(x^\mu, \varphi)$ where $\varphi$ is a periodic coordinate along $S^1$. Collectively denoting the dynamical fields of the $(2,0)$ theory by $\Xi$, we can perform a Fourier expansion in $\varphi$

$$\Xi(x, \varphi) = \sum_{p \in \mathbb{Z}} \Xi_p(x) e^{ip\varphi} \,, \tag{4.1}$$

where $p$ is the momentum along $S^1$. The different Fourier modes constitute the Kaluza-Klein tower obtained in the reduction. All modes in the tower except the zero mode $\Xi_0$ acquire a mass, corresponding to the momentum along $S^1$. As we will see explicitly below, the curvature of $M_6$ will in fact introduce mass terms[3] also for the zero momentum Fourier modes. However, we will assume that the zero mode masses are negligible compared to the ones generated by non-zero momentum. Consequently, at sufficiently low energies in the reduced theory, the $p \neq 0$ modes cannot be excited and therefore do not contribute to the low energy effective theory on $M_5$. The only remaining mode is thus the zero mode and the Fourier series is truncated $\Xi(x, \varphi) = \Xi_0(x)$. In particular, the fields are therefore independent of the fibre coordinate in the low energy limit that we are concerned with here. In what follows the dependence on the coordinates $x^\mu$ on $M_5$ is left implicit.

The condition $\partial_\varphi \Xi = 0$ is not covariant in six dimensions, which is not surprising since the Fourier expansion assumes explicitly the specific choice of local coordinates $y^M = (x^\mu, \varphi)$. In order to obtain fields on $M_5$ that are suitably normalized it is also possible to rescale the Fourier modes with an arbitrary function of $x^\mu$. We will use this freedom below when we consider the reduction of the $(2,0)$ multiplet in the low energy limit.

[3]The masses will be functions on $M_5$ rather than constants. However, they are uniquely determined by the conformal class of the metric on $M_6$.

### 4.1 The Maxwell action on $M_5$

We consider first the scalar field $\Phi^{\alpha\beta}$ of the tensor multiplet. In this case there is an action in six dimensions which can be dimensionally reduced directly to produce the action in five dimensions. Using the freedom to introduce a relative scaling between the fields in five and six dimensions we let

$$\Phi^{\alpha\beta} = \frac{1}{r\sqrt{2\pi}}\phi^{\alpha\beta}\,, \tag{4.2}$$

which implies that $\phi^{\alpha\beta}$ satisfies the same symplectic reality condition $(\phi^{\alpha\beta})^* = \phi_{\alpha\beta}$ as the six-dimensional scalar. Upon insertion in (2.7) and integration along the fibre coordinate (4.2) yields

$$S_\phi = \int d^5x\sqrt{-g}\left(-\frac{1}{r}\nabla_\mu\phi_{\alpha\beta}\nabla^\mu\phi^{\alpha\beta} - \frac{1}{5}\frac{1}{r}R\phi_{\alpha\beta}\phi^{\alpha\beta} + K(g,r,\theta)\phi_{\alpha\beta}\phi^{\alpha\beta}\right) \tag{4.3}$$

where we have introduced the quantity

$$K(g,r,\theta) = \frac{1}{r^3}\nabla_\mu r\nabla^\mu r - \frac{3}{5}\frac{1}{r^2}\nabla_\mu\nabla^\mu r + \frac{1}{20}r\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,, \tag{4.4}$$

which contains information about the geometry, and in particular the curvature, of the manifold $M_6$. The equation of motion for $\phi^{\alpha\beta}$ that follow from the action (4.3) by construction agrees with the one obtained from dimensional reduction of the equations of motion (2.5) in six dimensions as required.

Moving on to the spinors of the $(2,0)$ tensor multiplet we use the decomposition discussed in the previous section to write

$$\Psi^\alpha = \frac{1}{r\sqrt{2\pi}}\psi^\alpha \otimes \eta_+\,, \tag{4.5}$$

which implies that $\psi^\alpha$ satisfies the symplectic Majorana reality condition

$$(\psi^\alpha)^* = M_{\alpha\beta}B_{(5)}\psi^\beta\,. \tag{4.6}$$

Once again we have introduced a rescaling to get canonically normalized spinors in five dimensions. The action (2.11) then yields

$$S_\psi = \int d^5x\sqrt{-g}\left(\frac{1}{r}i\overline{\psi}_\alpha\gamma^\mu\nabla_\mu\psi^\alpha - \frac{1}{8}\mathcal{F}_{\mu\nu}\overline{\psi}_\alpha\gamma^{\mu\nu}\psi^\alpha\right) \tag{4.7}$$

when integration over $S^1$ is performed, which entails the same equations of motion as obtained by dimensional reduction of the corresponding equations (2.9) in six dimensions.

In the case of the tensor $H_{MNP}$ the absence of an action implies that we must consider the equations of motion directly. The three-form $H$ can be decomposed as

$$H = E + F\wedge d\varphi = \frac{1}{3!}E_{\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho + \frac{1}{2!}F_{\mu\nu}dx^\mu\wedge dx^\nu\wedge d\varphi\,. \tag{4.8}$$

In the low energy limit we have $\partial_\varphi H_{MNP} = 0$ which in particular implies that the coefficients of $E_{\mu\nu\rho}$ and $F_{\mu\nu}$ are independent of $\varphi$, so that $E \in \Omega^3(M_5)$ and $F \in \Omega^2(M_5)$. Dimensional reduction of the equations of motion $dH = 0$ and $H = *_G H$ in six dimensions then yields

$$dE = 0 \ , \ \ dF = 0 \tag{4.9}$$

and

$$E = -\frac{1}{r} *_g F + \theta \wedge F \, , \tag{4.10}$$

where $*_g$ denotes the Hodge dual in five dimensions with respect to the metric $g$ and we have taken the liberty to denote the exterior derivative on $\Omega^*(M_5)$ by the same symbol as its six-dimensional counterpart. Using (4.10) we can eliminate[4] $E$ from the theory on $M_5$, in which case $dE = 0$ gives an equation of motion for $F$. The dimensional reduction of $H$ thus amounts to a two-form field strength $F$ on $M_5$ satisfying $dF = 0$ and the equation of motion

$$d\left(\frac{1}{r} *_g F\right) - \mathcal{F} \wedge F = 0 \, . \tag{4.11}$$

This equation of motion can, in contrast to that of $H$, be integrated to an action functional for the vector potential $A$ of which $F = dA$ is the field strength:

$$S_{\mathrm{F}} = \int \left( -\frac{1}{r} F \wedge *_g F + \theta \wedge F \wedge F \right) \, . \tag{4.12}$$

The complete Maxwell theory on $M_5$ obtained by dimensional reduction is thus described by the action

$$S = S_{\mathrm{F}} + S_\psi + S_\phi \, . \tag{4.13}$$

Introducing $\tilde{g} = \sqrt{r(x)}$, in analogy with the case of a direct product manifold $M_6$, we see that (4.13) describes a Maxwell theory with a coupling strength that is a function on $M_5$ as expected. The second part of the $S_F$ action is equivalent to the topological term given in equation (5.2) of [11] in the sense that their variations are identical up to boundary terms. Furthermore, the complete action in five dimensions contains mass terms of geometrical origin, as mentioned in the beginning of this section. We see that requiring $R$, $\partial_\mu r$ and $\mathcal{F}_{\mu\nu}$ to be sufficiently small ensures the consistency of the truncation of the Kaluza-Klein modes. Although the generic features of $S$ were previously known, the precise form of the action has to the best of our knowledge not been computed before.

### 4.2 Conformal invariance

In conventional Maxwell theory with a constant coupling in five dimensions, the fact that $\tilde{g}$ is dimensionful implies that the theory is not conformally invariant. From the

[4]This elimination is possible since the number of independent components are equal for a two-form $F$ and a three-form $E$ in five dimensions, and the same as the number of independent components of the self-dual three-form $H = *_G H$ in six dimensions.

point of view of the reduction on $S^1$ the constant radius of the circle introduces a length scale that explicitly breaks the scale invariance of the six-dimensional theory. However, in the case of a general circle fibration we are currently considering, the coupling parameter is not restricted to be constant and consequently the conformal symmetry of the $(2,0)$ theory survives the reduction. From the decomposition (3.1) of the metric we find that the geometric quantities scale according to

$$g_{\mu\nu} \to e^{-2\sigma} g_{\mu\nu} \ , \ \ r \to e^{-\sigma} r \ , \ \ \theta_\mu \to \theta_\mu \tag{4.14}$$

under a conformal transformation in six dimensions. Here, we restrict considerations to a parameter $\sigma$ that depends only on the coordinates on $M_5$ in order to obtain a conformal rescaling of the five-dimensional metric. This rescaling constitutes a generalized conformal symmetry of the Maxwell theory provided that the scalar and spinor fields are correspondingly rescaled according to (4.2) and (4.5) as

$$\phi^{\alpha\beta} \to e^{\sigma} \phi^{\alpha\beta} \ , \ \ \psi^\alpha \to e^{\frac{3}{2}\sigma} \psi^\alpha \, . \tag{4.15}$$

From the point of view of the gauge theory on $M_5$ we must thus treat $r$ and $g_{\mu\nu}$ on equal footing, and consequently rescale not only the dynamical fields of the theory but also the coupling strength parameter $\tilde{g}$.

The Maxwell theory obtained in the reduction of the tensor multiplet is of course uniquely determined by the theory in six dimensions, but for the purpose of the considerations in the final section of this paper it is nevertheless interesting to consider the restrictions on an arbitrary gauge theory imposed by requiring the existence of generalized conformal invariance on $M_5$. In particular, given the canonically normalized kinetic term for the scalar $\phi^{\alpha\beta}$ it restricts the terms involving the gradient of the fibre radius and the five-dimensional curvature scalar, since these transform inhomogeneously under rescalings. (The inhomogeneous term produced by the kinetic term for the spinors $\psi^\alpha$ is proportional to $\overline{\psi}_\alpha \gamma^\mu \psi^\alpha$ which vanishes by symmetry.) However, terms involving $F_{\mu\nu}$ or $\mathcal{F}_{\mu\nu}$ are invariant under conformal rescalings and therefore not restricted by this symmetry.

### 4.3 Supersymmetry of the action

We can now restrict our attention to the case when the theory of the $(2,0)$ tensor multiplet in six dimensions is supersymmetric. As we saw above this amounts to requiring that the manifold $M_6$ admits non-trivial conformal Killing spinors $\mathcal{E}^\alpha$ satisfying (2.15). Just as the dynamical fields of the Maxwell theory can be expanded in the periodic $\varphi$ coordinate we can expand $\mathcal{E}^\alpha$ in a Fourier series as

$$\mathcal{E}^\alpha(x^\mu, \varphi) = \sum_{p \in \mathbb{Z}} \mathcal{E}^\alpha_p(x) e^{ip\varphi} \, . \tag{4.16}$$

There is however a significant difference: Being the parameter of supersymmetry transformations $\mathcal{E}^\alpha$ is not a dynamical field and we can not simply integrate out the modes with non-zero momentum along $S^1$. However, acting on a dynamical field with a supersymmetry transformation involving any mode other than the zero mode

$\mathcal{E}_0^\alpha$ changes its mode number. In order to restrict considerations to supersymmetry transformations of the low energy effective theory we must therefore truncate the Fourier series of $\mathcal{E}^\alpha$ and consider only the zero-mode $\mathcal{E}_0^\alpha$. In this way we obtain the spinor parameter of supersymmetry of the low energy effective theory, which satisfies $\partial_\varphi \mathcal{E}^\alpha = 0$. Using the decomposition of spinors described in the previous section we then have

$$\mathcal{E}^\alpha = \varepsilon^\alpha \otimes \eta_- \,, \tag{4.17}$$

where $\partial_\varphi \varepsilon^\alpha = 0$ and according to (3.15)

$$(\varepsilon^\alpha)^* = M_{\alpha\beta} B_{(5)} \varepsilon^\beta \,. \tag{4.18}$$

Dimensional reduction of the conformal Killing spinor equation (2.15) in addition yields the condition

$$\nabla_\mu \varepsilon^\alpha = \frac{1}{2}\frac{1}{r}\nabla^\nu r \gamma_\mu \gamma_\nu \varepsilon^\alpha + \frac{i}{8} r \mathcal{F}^{\rho\sigma} \gamma_\mu \gamma_{\rho\sigma} \varepsilon^\alpha + \frac{i}{4} r \mathcal{F}_\mu{}^\nu \gamma_\nu \varepsilon^\alpha \tag{4.19}$$

on the five-dimensional spinor parameter $\varepsilon^\alpha$.

The supersymmetry transformation of the dynamical fields of the Maxwell theory on $M_5$ are obtained by dimensional reduction of the transformations (2.12), (2.13) and (2.14) in six dimensions, yielding

$$\delta\phi^{\alpha\beta} = 2\overline{\psi}^{[\alpha}\varepsilon^{\beta]} - \frac{1}{2} T^{\alpha\beta} \bar{\psi}_\gamma \varepsilon^\gamma \,, \tag{4.20}$$

$$\begin{aligned} \delta F_{\mu\nu} \;=\;& -2i\nabla_{[\mu}\overline{\psi}_\alpha \gamma_{\nu]}\varepsilon^\alpha + i\frac{1}{r}\nabla^\rho r \overline{\psi}_\alpha \gamma_{\mu\nu\rho}\varepsilon^\alpha - 2i\frac{1}{r}\nabla_{[\mu} r \overline{\psi}_\alpha \gamma_{\nu]}\varepsilon^\alpha \\ & + r\mathcal{F}_{\mu\nu}\overline{\psi}_\alpha \varepsilon^\alpha + \frac{3}{2} r \mathcal{F}_{[\mu}{}^\rho \overline{\psi}_\alpha \gamma_{\nu]\rho}\varepsilon^\alpha - \frac{1}{4} r \mathcal{F}^{\rho\sigma}\overline{\psi}_\alpha \gamma_{\mu\nu\rho\sigma}\varepsilon^\alpha \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \delta\psi^\alpha \;=\;& \frac{1}{2} F_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\alpha + 2i M_{\beta\gamma}\nabla_\mu \phi^{\alpha\beta}\gamma^\mu \varepsilon^\gamma \\ & + 2i\frac{1}{r} M_{\beta\gamma}\phi^{\alpha\beta}\nabla_\mu r \gamma^\mu \varepsilon^\gamma - r M_{\beta\gamma}\phi^{\alpha\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\gamma \,. \end{aligned} \tag{4.22}$$

We note that $\mathcal{F}$ and $\nabla_\mu r$ enter in the supersymmetry variation of $F_{\mu\nu}$ and $\psi^\alpha$ through the covariant derivative of $\varepsilon^\alpha$ and the relation (4.19). It is a straightforward but somewhat laborious task to verify that the complete action (4.13) is invariant under the transformations (4.20), (4.21) and (4.22) provided that the supersymmetry parameter $\varepsilon^\alpha$ satisfies (4.19). (Supersymmetry at the level of the equations of motion in five dimensions is an immediate consequence of supersymmetry in six dimensions.) In analogy to the case for the $(2,0)$ tensor multiplet, supersymmetry of the action thus imposes a non-trivial geometrical condition on the manifold $M_5$, namely the existence of non-trivial solutions to (4.19).

### 4.4 The product metric

In order to verify that the results derived in the present section reproduces the known result for $M_6 = M_5 \times S^1$ with the product metric, we will now consider the case $\theta_\mu(x) = 0$ and $\partial_\mu r(x) = 0$. In this case we expect to recover ordinary Maxwell theory on $M_5$, which we still allow to be arbitrary. From (4.3), (4.7) and (4.12) we find that the action for the case of the product metric reduces to

$$S = \frac{1}{\tilde{g}^2} \int d^5x \sqrt{-g} \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}_\alpha \gamma^\mu \nabla_\mu \psi^\alpha - \nabla_\mu \phi_{\alpha\beta} \nabla^\mu \phi^{\alpha\beta} + cR\phi_{\alpha\beta}\phi^{\alpha\beta} \right) \tag{4.23}$$

where we have identified the Maxwell coupling constant as $\tilde{g} = \sqrt{r}$, which has the appropriate dimension in five dimensions (and in this case is a proper constant). The equations of motion obtained from this action are

$$\nabla_\mu F^{\mu\nu} = 0 \ , \ \ \gamma^\mu \nabla_\mu \psi^\alpha = 0 \ , \ \ \nabla_\mu \nabla^\mu \phi^{\alpha\beta} + cR\phi^{\alpha\beta} = 0 \, . \tag{4.24}$$

The appearance of terms proportional to $R$ is a consequence of the conformal symmetry in six dimensions. Note, however, that as discussed above, the radius $r$ explicitly breaks the scale invariance of the theory on $M_5$ as long as we consider it to be constant, since this condition eliminates the possibility of rescaling the coupling to compensate for the inhomogeneous transformation of $S$ under simultaneous rescalings (4.14) and (4.15).

As a next step we consider requiring the existence of supersymmetry in the $(2,0)$ theory with product metric on $M_6 \to M_5$. The expressions (4.20), (4.21) and (4.22) reduce to the familiar variations of supersymmetric Maxwell theory. Furthermore, the supersymmetry parameter must satisfy (4.19), which for the product metric reduces to

$$\nabla_\mu \varepsilon^\alpha = 0 \, . \tag{4.25}$$

Taking another covariant derivative and antisymmetrizing one obtains $\nabla_{[\mu}\nabla_{\nu]}\varepsilon^\alpha = 0$ which implies $R = 0$, so that the action (and the corresponding equations of motion) reduces to that of ordinary supersymmetric Maxwell theory on $M_5$.

## 5 The non-abelian generalization

In the previous sections of this paper we have considered exclusively the free tensor multiplet of $(2,0)$ theory and the low energy Maxwell theory obtained by its reduction on the $S^1$ fibre of $M_6$. We would now like to extend our scope to consider also the $A_r$,$D_r$ and $E_r$ series of $(2,0)$ [1] in the circle fibration geometry. However, a direct derivation of the low energy theory on $M_5$ by dimensional reduction is not possible in this case because, unlike the free tensor multiplet, the ADE type $(2,0)$ theories have no classical field theory description in terms of equations of motion[5].

However, we have some information regarding the low energy theory on $M_5$ obtained by reduction of the $(2,0)$ theory associated to a simply laced group $G$.

[5]In [11] this is explained in terms of the absence of a classical notion of a gerbe with non-abelian structure group of which $H$ is the curvature.

Conformal invariance in six dimensions entails generalized conformal invariance, discussed above for the free tensor multiplet, and if $M_6$ admits non-trivial conformal Killing spinors parametrizing supersymmetry transformations the theory on $M_5$ will also be supersymmetric. Furthermore, the theory on $M_5$ should be a theory of gauge fields with gauge group $G$. In particular, for the case $M_6 = M_5 \times S^1$ with a product metric (i.e. $\theta_\mu = 0$ and $\partial_\mu r = 0$) the theory on $M_5$ is supersymmetric Yang-Mills theory with gauge group $G$[6]. For a generic metric on $M_6$ it should be coupled to the background U(1) gauge field on $M_5$, corresponding to reparametrization invariance of the fibre.

The generalization of the theory described by the action (4.13) thus involves promoting $A$ to the connection of a principal $G$-bundle over $M_5$ and $\phi^{\alpha\beta}$ and $\psi^\alpha$ to sections of associated adjoint bundles. The dynamical fields of the theory are consequently $A^a_\mu$, $\phi^{\alpha\beta}_a$ and $\psi^\alpha_a$ where we denote by $a$ the index in the adjoint representation of the Lie algebra $\mathfrak{g}$ of $G$. (Since we will not use local Lorentz vector indices on $M_5$ explicitly in this section this notational overlap will hopefully not cause any confusion.) With anti-hermitian Lie algebra generators we have the standard expressions for the gauge field strength and the covariant derivative of a field $\chi^a$ in the adjoint representation, given by

$$F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + f^a{}_{bc} A^b_\mu A^c_\nu \tag{5.1}$$

and

$$D_\mu \chi^a = \nabla_\mu \chi^a + f^a{}_{bc} A^b_\mu \chi^c \,, \tag{5.2}$$

where $f^{abc}$ are the structure constants of $\mathfrak{g}$. As in the previous sections, the derivative $\nabla_\mu$ is covariant w.r.t. both general coordinate transformations and local Lorentz transformations. We can then make the action (4.13) gauge invariant by letting all fields transform in the adjoint representation of $\mathfrak{g}$, replacing the field strength and derivatives with gauge covariant ones and taking the trace in the adjoint representation, giving

$$S_\phi = \int d^5x \sqrt{-g} \left( -\frac{1}{r} D_\mu \phi^a_{\alpha\beta} D^\mu \phi^{\alpha\beta}_a - \frac{1}{5}\frac{1}{r} R \phi^a_{\alpha\beta}\phi^{\alpha\beta}_a + K(g,r,\theta)\phi^a_{\alpha\beta}\phi^{\alpha\beta}_a \right) \,, \tag{5.3}$$

$$S_\psi = \int d^5x \sqrt{-g} \left( \frac{1}{r} i \overline{\psi}^a_\alpha \gamma^\mu D_\mu \psi^\alpha_a - \frac{1}{8} \mathcal{F}_{\mu\nu} \overline{\psi}^a_\alpha \gamma^{\mu\nu} \psi^\alpha_a \right) \tag{5.4}$$

and

$$S_{\rm F} = \int \mathrm{tr} \left( -\frac{1}{r} F \wedge *_g F + \theta \wedge F \wedge F \right) \,. \tag{5.5}$$

In the same way we also obtain gauge covariant supersymmetry variations

$$\delta \phi^{\alpha\beta}_a = 2\overline{\psi}^{[\alpha}_a \varepsilon^{\beta]} - \frac{1}{2} T^{\alpha\beta} \bar{\psi}_{\gamma a} \varepsilon^\gamma \,, \tag{5.6}$$

$$\delta F^a_{\mu\nu} \;\; = \;\; -2i D_{[\mu} \overline{\psi}^a_\alpha \gamma_{\nu]} \varepsilon^\alpha + i\frac{1}{r} D^\rho r \overline{\psi}^a_\alpha \gamma_{\mu\nu\rho} \varepsilon^\alpha - 2i \frac{1}{r} D_{[\mu} r \overline{\psi}^a_\alpha \gamma_{\nu]} \varepsilon^\alpha$$

[6] In the special case where $M_5$ is Minkowski the Yang-Mills theory is maximally supersymmetric.

$$+r\mathcal{F}_{\mu\nu}\overline{\psi}^a_\alpha\varepsilon^\alpha+\frac{3}{2}r\mathcal{F}_{[\mu}{}^\rho\overline{\psi}^a_\alpha\gamma_{\nu]\rho}\varepsilon^\alpha-\frac{1}{4}r\mathcal{F}^{\rho\sigma}\overline{\psi}^a_\alpha\gamma_{\mu\nu\rho\sigma}\varepsilon^\alpha \tag{5.7}$$

and

$$\begin{aligned}\delta\psi^\alpha_a &= \frac{1}{2}F_{a\mu\nu}\gamma^{\mu\nu}\varepsilon^\alpha+2iM_{\beta\gamma}D_\mu\phi^{\alpha\beta}_a\gamma^\mu\varepsilon^\gamma \\ &\quad +2i\frac{1}{r}M_{\beta\gamma}\phi^{\alpha\beta}_aD_\mu r\gamma^\mu\varepsilon^\gamma-rM_{\beta\gamma}\phi^{\alpha\beta}_a\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^\gamma\,.\end{aligned} \tag{5.8}$$

The condition (4.19) receives no modification since the conformal Killing spinor equation on $M_6$ is satisfied by the supersymmetry parameter also for non-abelian $(2,0)$ theory. This is consistent, since $\varepsilon^\alpha$ and the parameters $r$ and $\theta_\mu$ are all invariant under $G$ gauge transformations.

In order to recover ordinary supersymmetric Yang-Mills theory in the case where $M_6=M_5\times S^1$ with product metric we must add a Yukawa term and a $\phi^4$ term to the action to obtain

$$\begin{aligned}S_{\rm YM} &= \ldots+\int d^5x\sqrt{-g}\left(2\frac{1}{r}f^{abc}M_{\alpha\gamma}M_{\beta\delta}\phi^{\alpha\beta}_a\overline{\psi}^\gamma_b\psi^\delta_c\right.\\ &\quad\left.+\frac{1}{r}f^{ab}{}_ef^{cde}M_{\sigma\alpha}M_{\beta\gamma}M_{\delta\lambda}M_{\tau\rho}\phi^{\alpha\beta}_a\phi^{\gamma\delta}_b\phi^{\lambda\tau}_c\phi^{\rho\sigma}_d\right)\,,\end{aligned} \tag{5.9}$$

and modify the supersymmetry variation of the fermionic field with a non-linear term according to

$$(\delta\psi^\alpha_a)_{\rm YM}=\ldots+2f_a{}^{bc}M_{\beta\gamma}M_{\delta\lambda}\phi^{\alpha\beta}_b\phi^{\gamma\delta}_c\varepsilon^\lambda\,. \tag{5.10}$$

The above action and supersymmetry variations transform correctly under generalized conformal rescalings and satisfy the appropriate reality conditions. By a straightforward computation (involving some rather lengthy R-symmetry manipulations) one verifies that when $M_5$ admits non-trivial solutions to (4.19) the action (5.9) is invariant under the modified supersymmetry transformations. Thus, the model described by (5.9) and (5.10) constitutes a generalization of the Maxwell theory, obtained for the free tensor multiplet in the case of a general fibration of $S^1$ over $M_5$, to a non-abelian Yang-Mills theory with varying coupling strength, coupled to a background U(1) gauge field.

Just as in the case of the Maxwell theory, the non-vanishing right hand side of (4.19), which from a five-dimensional point of view depends on the gradient of the coupling strength and the non-dynamical background gauge field, implies that the presence of the terms in (5.10) that depend on $\theta_\mu$ and $\partial_\mu r$ is required for supersymmetry of the Yang-Mills theory. For $\phi^{\alpha\beta}$ and $\psi^\alpha$, the terms in the action depending on $\partial_\mu r$ and $\mathcal{F}_{\mu\nu}$, required for invariance under generalized conformal rescalings, are quadratic and consequently introduce no novel interactions. (The topological $\theta$-term, however, is quadratic in the non-linear field strength (5.1) and does represent an interaction related to the fibration geometry.) In this sense, the model constitutes the minimal non-abelian extension of the Maxwell theory obtained for the tensor multiplet.

Since we have no field theory description of $(2,0)$ theory of type ADE it is not possible to verify that (5.9) and (5.10) gives the correct theory on $M_5$ by explicit computation of the reduction. However, it appears to be difficult to construct other non-abelian gauge theories with all the required properties due to the strong restrictions imposed by generalized conformal symmetry and supersymmetry on $M_5$.

## 6 Summary and conclusion

In this paper we first considered the dimensional reduction of the theory of a free $(2,0)$ tensor multiplet on a circle fibration $M_6 \to M_5$ in detail. The low energy effective theory obtained on $M_5$ is a Maxwell theory describing an abelian gauge field $A_\mu$ with field strength $F_{\mu\nu}$, a scalar $\phi^{\alpha\beta}$ and a spinor $\psi^\alpha$, where the latter fields transform respectively in the **5** and **4** of the USp(4) R-symmetry. For a generic metric on $M_6$ the coupling strength of the Maxwell theory is a function on $M_5$ given by the square root of the fibre radius $r(x)$. Furthermore, the Lagrangian contains quadratic terms for the scalar and spinor fields and a topological $\theta$-term for the gauge field, related to the local geometry of the fibration $M_6$. (In a path integral quantization of the gauge theory on $M_5$, the overall normalization of the action is determined by the requirement that the factor in the integrand containing the exponentiation of the topological $\theta$-term be well defined.) The terms are explicitly derived and the result given in (4.3), (4.7) and (4.12).

The equations of motion of the full theory on $M_5$ can (in contrast to those of the $(2,0)$ theory on $M_6$) be integrated to an action functional. The action is invariant under generalized conformal rescalings of the metric, dynamical fields and the coupling strength. Furthermore, it is invariant under the supersymmetry transformations (4.20), (4.21) and (4.22), obtained by reduction of the corresponding variations in six dimensions, when $M_5$ admits non-trivial solutions to (4.19).

We also considered a non-abelian generalization of the Maxwell gauge theory in order to find the description of the dimensional reduction of ADE type $(2,0)$ theory on the $S^1$ fibre of $M_6$. We find that gauge covariantizing the abelian theory and including Yukawa and $\phi^4$ interaction terms produces a theory with the required invariance under generalized conformal rescalings on $M_5$. As a further consistency check we find that with a quadratic modification of the fermionic supersymmetry transformations the theory is supersymmetric if $M_5$ admits solutions to (4.19). Finally, in the case of a product metric on $M_6$, corresponding to $\theta_\mu = 0$ and $\partial_\mu r = 0$ for the coupling strength and background gauge field in the five dimensional perspective, the generalization reduces to ordinary supersymmetric Yang-Mills theory. We emphasize that the gauge theory on $M_5$ is not directly derived from $(2,0)$ theory on $M_6$ but constitutes the minimal (in the sense described above) candidate for its reduction on $S^1$.

As discussed above, supersymmetry of the $(2,0)$ theory requires the existence of conformal Killing spinors, i.e. non-trivial solutions to (2.15) which in the special case of a circle fibration reduces to (4.19) on the base $M_5$. The classification of manifolds of Lorentzian signature admitting conformal Killing spinors has been extensively studied (see e.g. [15] and references therein). It would be interesting to investigate which of these classes contain circle fibrations.

An interesting example of a manifold that does admit conformal Killing spinors (in particular covariantly constant spinors) is discussed in [11]: Let $M_6 = \mathbb{R}^{1,1} \times TN$, where $TN$ is the Taub-NUT hyper-Kähler space which admits a U(1) action that preserves its hyper-Kähler structure. However, the U(1) action has a fix-point at the origin of the $\mathbb{R}^3$ underlying the $TN$ and consequently the description of $M_6$ as a U(1)-fibration becomes singular on $W = \mathbb{R}^{1,1} \times \{0\}$. Over $M_5 \backslash W$ the description of $M_6$ as a U(1) bundle is valid and the results of the present paper are applicable, but on $W$ the curvature $\mathcal{F}$ has a singularity. In particular, this implies that the topological term, which can equivalently be expressed in terms of the Chern-Simons form, transforms anomalously under gauge transformations requiring the introduction of a WZW model localized on $W$ to cancel the anomaly. A natural extension of the present work would be to consider manifolds $M_6$ with codimension 4 singularities as in the example above and investigate the coupling of the WZW model to the gauge theory on $M_5$. We intend to pursue this direction in future work.

During the final preparation of this manuscript [16] appeared, which treats in detail the case of a single $M5$ brane on $M_6 = \mathbb{R}^{1,2} \times S^3$ and the reduction on the Hopf fibration. Related results concerning instantons in the five-dimensional gauge theory are presented in [17].

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# A Conventions

## A.1 Symplectic transformation properties

We first consider the symplectic transformation properties of the scalar and spinor fields, which fall in non-trivial representations of the $\mathcal{R}$-symmetry group USp(4). We let $\alpha = 1, 2, 3, 4$ be the spinor index of the fundamental **4** representation of USp(4) and denote the symplectic structure by $M_{\alpha\beta}$ (we refrain from using the conventional notation $\Omega$ for the symplectic structure to avoid confusion with the spin connection).

We further denote by $V_{\mathbf{4}}$ and $V_{\bar{\mathbf{4}}}$ the dual modules of the **4** representation and its conjugate representation $\bar{\mathbf{4}}$, and let the vertical position of the index indicate the representation according to $v^\alpha \in V_{\mathbf{4}}$ and $w_\alpha \in V_{\bar{\mathbf{4}}}$. The fundamental representation and its conjugate are related under complex conjugation so we can infer that

$$(v^a)^* \in V_{\bar{\mathbf{4}}} \quad , \quad (w_a)^* \in V_{\mathbf{4}} \, . \tag{A.1}$$

While the two representations **4** and $\bar{\mathbf{4}}$ are unitarily equivalent, it is convenient to distinguish between them when considering reality conditions for the various fields of the $(2, 0)$ theory. We will therefore distinguish upper and lower indices and utilize the fact that complex conjugation interchanges these two types of indices according to the relation above.

The symplectic form is non-degenerate and antisymmetric, $M_{\alpha\beta} = -M_{\beta\alpha}$, and thus constitutes a metric on $V_{\mathbf{4}}$ providing an isomorphism $M : V_{\mathbf{4}} \to V_{\bar{\mathbf{4}}}$ between the vector space $V_{\mathbf{4}}$ and its dual. Similarly, its inverse $T^{\alpha\beta}$ defines an isomorphism $T : V_{\bar{\mathbf{4}}} \to V_{\mathbf{4}}$. The isomorphisms are given by

$$v_\alpha = M_{\alpha\beta} v^\beta \quad , \quad w^\alpha = T^{\alpha\beta} w_\beta \, , \tag{A.2}$$

and because $T = M^{-1}$ they satisfy the relations

$$T^{\alpha\beta} M_{\beta\gamma} = \delta^\alpha{}_\gamma \quad , \quad M_{\alpha\beta} T^{\beta\gamma} = \bar{\delta}_\alpha{}^\gamma \, , \tag{A.3}$$

where $\delta^\alpha{}_\beta$ and $\bar{\delta}_\alpha{}^\beta$ are the identity operators on $V_{\mathbf{4}}$ and $V_{\bar{\mathbf{4}}}$ respectively. Using the metric and its inverse we can thus raise and lower USp(4) spinor indices. Finally, the complex conjugate of the symplectic metric is given by $(M_{\alpha\beta})^* = M^{\alpha\beta} = -T^{\alpha\beta}$ and similarly for $T^{\alpha\beta}$.

The spinor field of the tensor multiplet transforms in the fundamental representation of USp(4) and consequently has a single USp(4) spinor index $\Psi^\alpha$. The scalar field of the multiplet, on the other hand, transforms in the vector representation **5**, which is obtained from the antisymmetric part of the tensor product

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{5} \oplus \mathbf{10} \tag{A.4}$$

by imposing the vanishing of the antisymmetric trace constituting the singlet. The scalar thus is an antisymmetric bispinor $\Phi^{\alpha\beta} = -\Phi^{\beta\alpha}$ satisfying the tracelessness condition $M_{\alpha\beta}\Phi^{\alpha\beta} = 0$. Furthermore, the properties of $M_{\alpha\beta}$ allows us to impose a consistent symplectic reality condition on $\Phi^{\alpha\beta}$, given by

$$(\Phi^{\alpha\beta})^* = \Phi_{\alpha\beta} = M_{\alpha\gamma} M_{\beta\delta} \Phi^{\gamma\delta} \, . \tag{A.5}$$

We also note that this condition is consistent with complex conjugation relating the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations. In the next subsection we will consider a symplectic reality condition for the spinor field $\Psi^\alpha$ as well.

### A.2 Spinors in $4+1$ and $5+1$ dimensions

Next, we consider the spinor representations of the Lorentz group in the dimensions relevant for the considerations of the present paper. We work in Lorentzian signature and use conventions where the flat Minkowski metric is $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. The Clifford algebra in 5+1 and 4+1 dimensions is $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ respectively[7]. The hermiticity properties of the Γ-matrices are given by

$$\begin{aligned} (\gamma^a)^\dagger &= \gamma^0\gamma^a\gamma^0 \\ (\Gamma^A)^\dagger &= \Gamma^0\Gamma^A\Gamma^0 \,. \end{aligned} \tag{A.6}$$

The charge conjugation matrix in the respective dimensions is uniquely determined (up to a complex phase) by the relations

$$\begin{aligned} C_{(5)}^T = -C_{(5)} \quad &, \quad (\gamma^a)^T = C_{(5)}\gamma^a C_{(5)}^{-1} \\ C_{(6)}^T = -C_{(6)} \quad &, \quad (\Gamma^A)^T = C_{(6)}\Gamma^A C_{(6)}^{-1} \,, \end{aligned} \tag{A.7}$$

giving the symmetry properties of the Γ-matrices. Similarly, complex conjugation of the Γ-matrices is given by

$$\begin{aligned} (\gamma^a)^* &= -B_{(5)}\gamma^a B_{(5)}^{-1} \\ (\Gamma^A)^* &= -B_{(6)}\Gamma^A B_{(6)}^{-1} \,, \end{aligned} \tag{A.8}$$

where we define the matrices $B_{(5)} = C_{(5)}\gamma^0$ and $B_{(6)} = C_{(6)}\Gamma^0$, satisfying $BB^* = -\mathbb{1}$.

We will next consider spinors $\Lambda^\alpha$, in either five or six dimensions, that carry an additional USp(4) spinor index in agreement with the application to (2,0) theory. The conjugate spinor is defined as

$$\overline{\Lambda}^\alpha = (\Lambda^\alpha)^T C \tag{A.9}$$

and the charge conjugate spinor as

$$(\Lambda^\alpha)^C = B^{-1}(\Lambda^\alpha)^* \,. \tag{A.10}$$

(With our conventions the Dirac conjugate is given by $\overline{(\Lambda^\alpha)^C}$.) It is not possible in neither 4+1 nor 5+1 dimensions to define ordinary Majorana spinors due to the fact that $BB^* = -\mathbb{1}$. It is, however, possible to make use of the symplectic structure $M_{\alpha\beta}$ (described in detail in the previous subsection) to impose a consistent symplectic Majorana reality condition according to

$$(\Lambda^\alpha)^* = M_{\alpha\beta} B \Lambda^\beta \,. \tag{A.11}$$

[7]Here, as before, $a$ and $A$ are flat vector indices in five and six dimensions respectively.

All the spinors we consider will satisfy this condition which implies that we have

$$(\Lambda^\alpha)^C = M_{\alpha\beta}\Lambda^\beta = \Lambda_\alpha \quad , \quad \overline{(\Lambda^\alpha)^C} = M_{\alpha\beta}\overline{\Lambda}^\beta = \overline{\Lambda}_\alpha \tag{A.12}$$

for the charge conjugate spinors.

In the action functionals the spinors appear exclusively as USp(4) invariant bilinears. The symmetry and reality properties of such bilinears can be derived from the defining relations for the charge conjugation matrix and the properties of the symplectic metric $M_{\alpha\beta}$. For our purposes the relevant relations are

$$\overline{\psi}_\alpha\lambda^\alpha = -\overline{\lambda}_\alpha\psi^\alpha \ , \ \overline{\psi}_\alpha\gamma^a\lambda^\alpha = -\overline{\lambda}_\alpha\gamma^a\psi^\alpha \ , \ \overline{\psi}_\alpha\gamma^{ab}\lambda^\alpha = \overline{\lambda}_\alpha\gamma^{ab}\psi^\alpha \tag{A.13}$$

$$(\overline{\psi}_\alpha\lambda^\alpha)^* = \overline{\psi}_\alpha\lambda^\alpha \ , \ (\overline{\psi}_\alpha\gamma^a\lambda^\alpha)^* = -\overline{\psi}_\alpha\gamma^a\lambda^\alpha \ , \ (\overline{\psi}_\alpha\gamma^{ab}\lambda^\alpha)^* = \overline{\psi}_\alpha\gamma^{ab}\lambda^\alpha \tag{A.14}$$

for spinors $\lambda^\alpha$ and $\psi^\alpha$ in $4+1$ dimensions and similarly

$$\overline{\Psi}_\alpha\Lambda^\alpha = -\overline{\Lambda}_\alpha\Psi^\alpha \ , \ \overline{\Psi}_\alpha\Gamma^A\Lambda^\alpha = -\overline{\Lambda}_\alpha\Gamma^A\Psi^\alpha \ , \ \overline{\Psi}_\alpha\Gamma^{AB}\Lambda^\alpha = \overline{\Lambda}_\alpha\Gamma^{AB}\Psi^\alpha \tag{A.15}$$

$$(\overline{\Psi}_\alpha\Lambda^\alpha)^* = \overline{\Psi}_\alpha\Lambda^\alpha \ , \ (\overline{\Psi}_\alpha\Gamma^A\Lambda^\alpha)^* = -\overline{\Psi}_\alpha\Gamma^A\Lambda^\alpha \ , \ (\overline{\Psi}_\alpha\Gamma^{AB}\Lambda^\alpha)^* = \overline{\Psi}_\alpha\Gamma^{AB}\Lambda^\alpha \tag{A.16}$$

for spinors $\Lambda^\alpha$ and $\Psi^\alpha$ in $5+1$ dimensions.

In six dimensions the Dirac spinor representation is decomposed according to the eigenvalue of the chirality operator $\Gamma$, which we define to be

$$\Gamma = \Gamma^0\Gamma^1\ldots\Gamma^5 \,. \tag{A.17}$$

so that we can consider Weyl spinors of definite chirality

$$\Gamma\Lambda^\alpha = \pm\Lambda^\alpha \,. \tag{A.18}$$

The chirality condition is compatible with (A.11) in six dimensions, admitting symplectic Majorana-Weyl spinors. Finally, we consider the matrices $\gamma^0,\ldots,\gamma^3$ which generate the Clifford algebra in $3+1$ dimensions. Here we can define a chirality operator similar to the one in six dimensions, which provides the final generator

$$\gamma^4 = \gamma = i\gamma^0\gamma^1\ldots\gamma^3 \tag{A.19}$$

of the Clifford algebra in $4+1$ dimensions.

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# Paper II

# The trouble with twisting (2,0) theory

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**Abstract**

We consider a twisted version of the abelian (2,0) theory placed upon a Lorenzian six-manifold with a product structure, $M_6 = C \times M_4$. This is done by an investigation of the free tensor multiplet on the level of equations of motion, where the problem of its formulation in Euclidean signature is circumvented by letting the time-like direction lie in the two-manifold $C$ and performing a topological twist along $M_4$ alone. A compactification on $C$ is shown to be necessary to enable the possibility of finding a topological field theory. The hypothetical twist along a Euclidean $C$ is argued to amount to the correct choice of linear combination of the two supercharges scalar on $M_4$. It may be slightly surprising that this is not the same linear combination as in the well known Donaldson-Witten twist. A more surprising fact however, is that this twisted theory contains no $Q$-exact and covariantly conserved stress tensor unless $M_4$ has vanishing curvature. This is to our knowledge a phenomenon which has not been observed before in topological field theories. In the literature, the setup of the twisting used here has been suggested as the origin of the conjectured AGT-correspondence, and our hope is that this work may somehow contribute to the understanding of it.

# Contents

# 1 Introduction

This work is an investigation of the topological twisting of the (2,0) theory which has been suggested to be relevant in the explanation of the origin of the AGT-conjecture. Herein, the simpler model of the free tensor multiplet is considered, and we find that the resulting twisted theory exhibits some curious, undesirable properties. The most severe of these is the lack of any satisfactory formulation of a stress tensor. This surprising result will be clear eventually, but let us first start at the very beginning.

The theory known as (2,0) theory [1, 2] is a six-dimensional superconformal theory that continue to resist attempts at unraveling its mysteries. One way to obtain information about the theory is to look at its different compactifications. For example, when compactified on a circle it gives rise to five-dimensional maximally supersymmetric Yang-Mills theory [3]. Recently a whole class of four-dimensional gauge theories have been constructed in this way by compactifying (2,0) theory on a two-dimensional Riemann surface with possible defects [4–6]. This class of theories is sometimes referred to as "class $\mathcal{S}$" in the literature [7, 8]. The way these theories are obtained through compactification has led to a conjecture about the relation of certain objects in four-dimensional- and two-dimensional theories, the so-called AGT correspondence [9].

More specifically, this correspondence states that the correlation functions in two-dimensional Liouville theory are related to the Nekrasov partition function [10, 11] of certain $\mathcal{N} = 2$ superconformal gauge theories in four dimensions. One natural way to derive it [9, 12, 13] would be to link it to a certain geometric setup in (2,0) theory, where the spacetime is taken to be a product of a two-dimensional- and a four-dimensional manifold. In such a setting, compactifications could either be carried out on the two- or on the four-manifold, after which one could search for protected quantities which have survived the compactification. A relation should then exist between the protected quantities of both compactifications.

However, one is here faced with the great challenge of a lack of any satisfactory definition of (2,0) theory that would permit such detailed calculations. While this is indeed true for the full, interacting (2,0) theory, this is not the whole story for the abelian version. Here, a classical formulation in terms of equations of motion exists.

Moreover, it is important to notice that for a general background all supersymmetry will be broken and such a situation cannot be expected to shed any light on the AGT-correspondence. In order to preserve some supersymmetry, one must first perform a topological twisting [14]. In a case where the six-manifold has the product structure mentioned previously, i.e. $M_6 = C \times M_4$, and $M_6$ is of Euclidean signature, (thus the holonomy groups of both $C$ and $M_4$ are compact), such a theory admits a unique twisting, which has been claimed [12] to be analogous to the Donaldson-Witten twist of four-dimensional $\mathcal{N} = 2$ Yang-Mills theory [14]. We however find that there are some small differences, which will be further explained throughout this work. In the literature (see for example [12, 15]), it has been stated that the twisting described above would result in a theory which would be topological along $M_4$ and holomorphic along $C$ [16]. Herein, the behaviour of the Lorenzian theory (especially along the four-manifold), is investigated explicitly by computing a stress tensor.

However, the elusive side of (2,0) theory once again comes back to bite us here, since not even the abelian version of this theory has a satisfactory description on a Euclidean six-manifold, but rather only on a six-manifold with Minkowski signature. In such a situation, the holonomy group would be non-compact, and a topological twisting that results in a scalar supercharge cannot be performed. If the light-like direction is taken to lie in $C$, one may still obtain supercharges that are scalars on $M_4$ by a twisting procedure. One of these charges has properties that would make it scalar along $C$ as well, were we in the Euclidean scenario. In this work, this is the supercharge we will consider, and the behaviour of the theory under it is the subject of investigation. The final conclusion is that, on a general $M_4$, the stress tensor of the theory cannot be both $Q$-exact and conserved, and the theory is thus not topological in the traditional sense.

The outline of this work is as follows: In section 2 we describe the twisting procedure giving rise to the supercharge that is scalar on $M_4$ and give a detailed description of the field content in this new, twisted theory. Section 3 deals with the equations of motion as well as the supersymmetry transformations of the twisted theory. In section 4, a stress tensor is computed in the flat case which is shown to have all desired properties. An attempt at generalising this to a general $M_4$ is made, and any $Q$-exact stress tensor is shown to not be covariantly conserved. It is also shown that no modifications to either equations of motion or supersymmetry variations may be done which would rectify these obstructions when $M_4$ is curved.

## 2 The Twisting

We consider the free tensor multiplet of the (2,0) theory on a flat six-manifold $M_6$ , endowed with a product metric such that $M_6 = C \times M_4$, with $C$ some two-manifold and $M_4$ some four-manifold. Throughout this work, light-cone coordinates $\{+,-\}$ on $C$, and indices $\mu, \nu \in \{1,2,3,4\}$ denoting directions along $M_4$, will be used. When needed, $M, N \in \{0,1,2,3,4,5\}$ will denote indices in six dimensions.

The tensor multiplet [17] contains a symplectic Majorana-Weyl spinor $\Psi$ transforming in the **4** of the $R$-symmetry group $Spin(5)_R$, a scalar $\Phi$ in the **5** of $Spin(5)_R$ and a self-dual three-form $H_{MNP}$. This section will deal with the decomposition of these representations under the twist, and the next section will provide a detailed dictionary for reinterpreting these in terms of the field content of the twisted theory.

If $M_6$ is of Euclidean signature, as previously mentioned, the theory admits a unique topological twisting. This since the $R$-symmetry group $Spin(5)$ contains a subgroup $SU(2)_R \times U(1)_R$, which also may be found as subgroups of the Lorentz group of $C \times M_4$: $U(1) \times SU(2)_l \times SU(2)_r$. The twisting procedure is carried out by defining $SU(2)'$ to be the diagonal subgroup of $SU(2)_r \times SU(2)_R$ and $U(1)'$ as the same in $U(1) \times U(1)_R$. By considering the theory under the group $U(1)' \times SU(2)_l \times SU(2)'$, one finds a single supercharge which is scalar hereunder, and thus the possibility of a topological field theory exists.

However, the lack of a satisfactory formulation of the free tensor multiplet of (2,0) theory in Euclidean signature forces us to work in a situation where $C$ is of Minkowski signature instead, with the correspondingly non-compact Lorentz group $Spin(1{,}1)$. There will thus be no way to embed this into $U(1)_R$, and hence it is not possible to perform a twisting along the two-manifold $C$ as in the above case. $M_4$ is however still of Euclidean signature, hence the twisting along these directions will not have been affected. This will be described in greater detail below.

In table 1 and 2, the representations of the fields and supersymmetry parameters before and after twisting along $M_4$ are shown. A more detailed explanation on how the six-dimensional field content should be translated to the fields of the twisted theory will as mentioned follow in the next section.

The superscripts indicate the charge under $U(1)_R$. For clarity, it should here be pointed out that the representations for the fermions and for the

| | $SU(2)_l \times SU(2)_r \times SU(2)_R \times U(1)_R$ |
|---|---|
| $\Phi$ | $(\mathbf{1},\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})^{\pm 1}$ |
| $\Psi$ | $(\mathbf{2},\mathbf{1},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})^{\pm 1/2}$ |
| $H$ | $(\mathbf{3},\mathbf{1},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})^0 \oplus (\mathbf{2},\mathbf{2},\mathbf{1})^0$ |
| $\varepsilon$ | $(\mathbf{2},\mathbf{1},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})^{\pm 1/2}$ |

Table 1: Representations before twisting.

| | $SU(2)_l \times SU(2)' \times U(1)_R$ | Twisted fields |
|---|---|---|
| $\Phi$ | $(\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{1},\mathbf{1})^{\pm 1}$ | $E_{\mu\nu}, \bar{\sigma}, \sigma$ |
| $\Psi$ | $(\mathbf{2},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{3},)^{\pm 1/2} \oplus (\mathbf{1},\mathbf{1})^{\pm 1/2}$ | $\psi_\mu, \tilde{\psi}_\mu, \chi_{\mu\nu}, \tilde{\chi}_{\mu\nu}, \eta, \tilde{\eta}$ |
| $H$ | $(\mathbf{3},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{3})^0 \oplus (\mathbf{2},\mathbf{2})^0$ | $F^-_{\mu\nu}, F^+_{\mu\nu}, A_\mu$ |
| $\varepsilon$ | $(\mathbf{2},\mathbf{2})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{3})^{\pm 1/2} \oplus (\mathbf{1},\mathbf{1})^{\pm 1/2}$ | $\ldots, (\bar{\varepsilon}), \varepsilon$ |

Table 2: Representations after twisting.

supersymmetry parameters differ in their chirality on $C$ (which is not shown in table 1 and 2).

*If* we were in Euclidean signature, all of these new fields would also have charges under the $U(1)$ which would then be the Lorentz group of $C$. In the second step of the twisting procedure previously described, these charges would combine with the charges under $U(1)_R$. The charge under the new diagonal subgroup $U(1)'$ would then be given by the sum of these two charges. Hence the supercharge that would become scalar under such a twist would be the one with $U(1)_R$-charge of $-1/2$ whose parameter shall be denoted by $\varepsilon$. The other supersymmetry scalar on $M_4$, with $U(1)_R$-charge of $+1/2$, is denoted by $\bar{\varepsilon}$. That $\varepsilon$ is the parameter of interest can be seen by studying table 3 where the representations after the four-twist in the Euclidean scenario is written down. The superscript here denotes the charge under the $U(1)_R$ whereas the subscripts denote the charges under the $U(1)$ Lorentz group of $C$.

One may choose some chiral, constant spinors $e^\pm$ to generate the two spinor representations for the fermions which are scalar on $M_4$, namely $(\mathbf{1},\mathbf{1})^{\pm 1/2}_{-1/2}$. (Again, the subscript denotes the charge under a hypothetical $U(1)$ Lorentz group of $C$, and is what distinguishes the two fermionic singlet representations on $M_4$ from the ones of the supersymmetries.) In some

| | $U(1) \times SU(2)_l \times SU(2)' \times U(1)_R$ |
|---|---|
| $\Phi$ | $(\mathbf{1},\mathbf{3})^0_0 \oplus (\mathbf{1},\mathbf{1})^{\pm 1}_0$ |
| $\psi$ | $(\mathbf{2},\mathbf{2})^{\pm 1/2}_{1/2} \oplus (\mathbf{1},\mathbf{3})^{\pm 1/2}_{-1/2} \oplus (\mathbf{1},\mathbf{1})^{\pm 1/2}_{-1/2}$ |
| $H$ | $(\mathbf{3},\mathbf{1})^0_0 \oplus (\mathbf{1},\mathbf{3})^0_0 \oplus (\mathbf{2},\mathbf{2})^0_0$ |
| $\varepsilon$ | $(\mathbf{2},\mathbf{2})^{\pm 1/2}_{-1/2} \oplus (\mathbf{1},\mathbf{3})^{\pm 1/2}_{1/2} \oplus (\mathbf{1},\mathbf{1})^{\pm 1/2}_{1/2}$ |

Table 3: Hypothetical Euclidean twist.

cases, it is convenient to think of these two new base-spinors as complex linear combinations of constant symplectic Majorana-Weyl spinors, $e_1$ and $e_2$, such that $e^{\pm} = e_1 \pm ie_2$.

The two spinors $e^{\pm}$ will as mentioned need to be chiral in the six-dimensional sense to generate the fermionic representations. $\Gamma_+ e^{\pm}$ are then anti-chiral, constant spinors, which generate the $(\mathbf{1},\mathbf{1})^{\pm 1/2}_{+1/2}$ where the supersymmetry-charges that are of interest to us live.

This allows for a parametrisation of the two supercharges which are scalar on $M_4$ in terms of some Grassmann parameters $u$ and $v$, together with a $\Gamma$-matrix along $C$ to account for the six-dimensional chirality. These relations are given by:

$$\varepsilon = v\Gamma_+ e^- \qquad , \qquad \bar{\varepsilon} = u\Gamma_+ e^+, \tag{1}$$

where as repeatedly mentioned, the supersymmetry parameter that would become scalar on $C$ as well after a hypothetical further twist is $\varepsilon$.

## 2.1 Details of reinterpreting the fields

The next order of business is to create a dictionary, translating the original field content of the six-dimensional free tensor multiplet (table 1) to the field content of the twisted theory (table 2).

### Bosonic scalar

Let the indices $i, j \in \{1,2,3\}$. One can then quite easily see that the self-dual two-form $E_{\mu\nu}$ of the twisted theory can be related to the first three components of the six-dimensional scalar field $\Phi$ as follows:

$$\begin{aligned} E_{4i} =& - \Phi_i \\ E_{ij} =& \epsilon_{ijk}\Phi^k. \end{aligned} \tag{2}$$

Furthermore, the two last components of the six-dimensional scalar $\Phi$ are after twisting combined into a complex scalar $\sigma$:

$$\sigma = \frac{1}{\sqrt{2}}(\Phi_4 - i\Phi_5). \tag{3}$$

**Bosonic three-form**

Reinterpreting the six-dimensional bosonic three-form in terms of the new, twisted fields is only slightly more complicated than the case of the scalars above. By using the fact that $H_{MNP}$ is self-dual, (with respect to the orientation and Riemannian structure on $M_6$), one may show that $H_{+\mu\nu}$ is a self-dual two-form in four dimensions, and $H_{-\mu\nu}$ likewise is an anti-self-dual two-form on $M_4$ (all with respect to the orientation and Riemannian structure on $M_4$). This gives us a natural interpretation of the components of $H$ in terms of the twisted two-form $F$ as:

$$\begin{aligned} H_{+\mu\nu} =& \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}H_{+\rho\sigma} = F^+_{\mu\nu} \\ H_{-\mu\nu} =& -\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}H_{-\rho\sigma} = F^-_{\mu\nu}, \end{aligned} \tag{4}$$

where $F^{\pm}_{\mu\nu}$ denotes the self-dual and anti-self-dual parts respectively.

Moreover, one may in a similar fashion interpret $H_{\mu\nu\rho}$ and $H_{+-\sigma}$ in terms of the twisted one-form $A_\sigma$ and its dual as:

$$H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}H^{\sigma}_{+-} = \epsilon_{\mu\nu\rho\sigma}A^{\sigma}. \tag{5}$$

**Fermionic fields**

$\Psi$ may be expanded in terms of the twisted fields $\eta$, $\psi$, ... as follows:

$$\Psi = (\eta + \Gamma_+\Gamma_\mu\psi^\mu + \frac{1}{4}\Gamma_\mu\Gamma_\nu\chi^{\mu\nu})e^+ + (\tilde{\eta} + \Gamma_+\Gamma_\mu\tilde{\psi}^\mu + \frac{1}{4}\Gamma_\mu\Gamma_\nu\tilde{\chi}^{\mu\nu})e^-. \tag{6}$$

The terms in the above decomposition are precisely the twisted field content of the spinor field as given in table 2. By using how $e^{\pm}$ are related to symplectic Majorana-Weyl spinors, one can show that $\Psi$ indeed is a symplectic

Majorana-Weyl spinor as well under the condition that the fields with- and without twiddles are related by complex conjugation. This also is consistent with the $U(1)_R$-charges of these different fields.

However, in the case we wish to consider, namely the theory invariant under only the one supercharge that would become scalar in a Euclidean scenario, we must loosen these requirements on $\Psi$, since there is no such notion as a spinor being Majorana-Weyl on a six-dimensional Euclidean manifold. This means that the fields with- and without the twiddles will need to be considered as independent of one another non the less.

## 2.2 Some useful relations

To perform further calculations, we must first find ways to handle the $\Gamma$-matrices which arise both in (6) when reinterpreting the fermionic spinor field in terms of the new, twisted ones, as well as in the expression for how the relevant supersymmetry parameter is written down in terms of our base spinors (1). In this section, some useful formulas for handling these are presented.

The first, and maybe most important relation comes from the knowledge that our constant base spinors are singlets under all of the $SU(2)$'s after twisting, which gives us the relations

$$\frac{1}{2}(\Gamma_{4i} - \frac{1}{2}\epsilon_{ijk}\Gamma^{jk})e^{\pm} = 0 \tag{7}$$
$$\frac{1}{2}(\Gamma_{4i} + \frac{1}{2}\epsilon_{ijk}\Gamma^{jk})e^{\pm} + \frac{1}{2}\epsilon_{ijk}\Gamma_R^{jk}e^{\pm} = 0.$$

Here $\Gamma$ denotes the $\Gamma$-matrices of the Lorentz group, whereas $\Gamma_R$ denotes the gamma matrices of the $R$-symmetry group. Again, the indices $\{i,j,k\}$ take values in $\{1,2,3\}$. The top one of the above equations enforces that the $e^{\pm}$ are singlets under $SU(2)_l$, and the lower one reflects the same behaviour under $SU(2)'$.

Furthermore, the charge under the $U(1)_R$ is known for the two spinors, and it is thus known how the generator of this group acts on them:

$$i\Gamma_R^4\Gamma_R^5 e^{\pm} = \pm e^{\pm}. \tag{8}$$

A short calculation also shows that the action of one of these, say $\Gamma_R^4$, corresponds to flipping the $U(1)_R$-charge and thus:

$$\Gamma_R^4 e^{\pm} = e^{\mp}. \tag{9}$$

Now we move on to relations involving the Γ-matrices of the Lorentz group. The spinors are chiral in a six-dimensional manner, thus

$$\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 e^{\pm} = e^{\pm}. \tag{10}$$

This may be reduced to chirality along $C$ and $M_4$ individually by studying how these representations decompose under the twisting procedure. If we let the six-dimensional indices be divided such that $\{0,1\} \in C$ and $\{2,3,4,5\} \in M_4$ for the moment, this may be expressed as:

$$\Gamma_2\Gamma_3\Gamma_4\Gamma_5 e^{\pm} = -e^{\pm} \qquad\qquad \Gamma_0\Gamma_1 e^{\pm} = -e^{\pm}. \tag{11}$$

From the above relations, all information necessary to perform our desired calculations may be deduced.

It is convenient to define

$$\Gamma_{\pm} = \frac{1}{\sqrt{2}}(\Gamma_1 \pm \Gamma_0) \quad , \quad \Gamma^{\pm} = \Gamma_{\mp}, \tag{12}$$

since we as previously mentioned wish to use light-cone coordinates on the two-manifold, and to consider the action of these on the spinors instead. This may be derived in a straight-forward manner using (12) together with (11), leading to the expressions:

$$\Gamma_+ e^{\pm} = \frac{1}{\sqrt{2}}(\Gamma_1 + \Gamma_0)e^{\pm} = \sqrt{2}\Gamma_1 e^{\pm} \quad , \quad \Gamma_- e^{\pm} = \frac{1}{\sqrt{2}}(\Gamma_1 - \Gamma_0)e^{\pm} = 0. \tag{13}$$

The most favourable way to express these relations is not however in the form in which they are given now, but rather in terms of the relations for some spinor bilinears which they lead to. Below, the most commonly used ones of these are listed:

$$\begin{aligned} &\bar{e}^{\mp}\Gamma^- e^{\pm} = 1 && \bar{e}^{\mp}\Gamma_\mu\Gamma_\nu\Gamma_+ e^{\pm} = \delta_{\mu\nu} \\ &\bar{e}^{\mp}\Gamma^+ e^{\pm} = 0 && \bar{e}^{\mp}\Gamma_\mu\Gamma_\nu\Gamma_\rho\Gamma_\sigma\Gamma_+ e^{\pm} = \delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma} \\ &\bar{e}^{\pm}\Gamma^{\pm} e^{\pm} = 0 && \bar{e}^{\mp}\Gamma_+\Gamma^-\Gamma_+ e^{\pm} = 2. \end{aligned} \tag{14}$$

## 2.3 Compactifying on $C$

In the construction of the class $\mathcal{S}$ theories [4], $C$ is a Riemann surface of genus $g$ with punctures. The $\mathcal{N} = 2$ Yang-Mills theory arise in the IR limit of (2,0) theory compactified on this surface. When considering the theory on a flat $C$, this simply means that we take all derivatives in these directions to vanish. This seems to be necessary if we want the theory on $M_4$ to be topological since terms of this type spoil all the interesting properties of the theory: $Q$ invariance and exactness of $T^{\mu\nu}$ and the nilpotency of $Q$.

However, it may be interesting to point out that there are two supercharges that are Lorentz scalars on $M_4$, described by parameters $\varepsilon$ and $\bar{\varepsilon}$. We have herein chosen to only consider the observables which live in $Q$-cohomology, but one may likewise consider a linear combination of $Q$ and $\bar{Q}$, and choose to consider states which lie in the cohomology of this new operator (as done in the Donaldson-Witten twist of $\mathcal{N} = 2$ Yang-Mills [14]). In this work, we are however interested in only the specific linear combination which would become scalar if the signature of $C$ was Euclidean and we thus could twist along that direction too. Hence the choice to consider only the theory invariant under $\varepsilon$ and observables in $Q$-cohomology.

# 3 The theory after twisting

After having worked out the field content in the previous section we now turn to the formulation of the theory after the twist. Here we will use the known equations of motion and supersymmetry variations for the abelian tensor multiplet to derive the corresponding expressions for the twisted fields. With the explicit correspondences given in section 2.1 this is almost immediate.

## 3.1 Equations of motion

In the six dimensional formalism, the scalar fields fulfil the Klein-Gordon equation, and the self-dual bosonic three-form satisfies $dH = 0$. Furthermore, the fermionic field satisfies the Dirac equation.

$$D^M D_M \Phi = 0 \tag{15}$$
$$dH = 0 \tag{16}$$
$$\Gamma^M D_M \Psi = 0 \tag{17}$$

Translated into the language of the twisted theory, the bosonic two-form and the complex scalar also satisfies the Klein-Gordon equation. Since any derivatives in the first two directions will vanish identically due to the compactification, what remains is the Klein-Gordon equation along $M_4$, that is:

$$\partial_\rho \partial^\rho E_{\mu\nu} = 0 \tag{18}$$

$$\partial_\rho \partial^\rho \sigma = 0. \tag{19}$$

Moreover, we may split the six-dimensional equation of motion for the bosonic three-form according to the number of indices along $M_4$. The six-dimensional equation of motion are then easily reinterpreted in terms of the twisted fields as:

$$\begin{aligned} 2\partial_{[\mu} A_{\nu]} &= 0 \\ \partial_{[\mu} F^{\pm}_{\nu\rho]} &= 0 \\ \partial_\mu A^\mu &= 0. \end{aligned} \tag{20}$$

Likewise, the equations of motion for the twisted fermionic fields may, after some calculations, be written as:

$$\begin{aligned} \partial_\mu \tilde{\psi}^\mu &= 0 \\ \partial_\mu \tilde{\eta} - \partial_\nu \tilde{\chi}_\mu{}^\nu &= 0 \\ (\partial_\mu \tilde{\psi}_\nu)^+ &= 0, \end{aligned} \tag{21}$$

and equivalently for the fields without twiddles. The notation $(\partial_\mu \tilde{\psi}_\nu)^+$ refers to the self-dual part of $\partial_{[\mu} \tilde{\psi}_{\nu]}$. Furthermore, since all components of the six-dimensional fermions satisfy the Klein-Gordon equation, one can show that the same applies to all components of our twisted fermionic fields (and, as for the scalars, particularly along $M_4$).

## 3.2 Supersymmetry

After the twisting procedure we are left with two supercharges which are Lorentz scalars on $M_4$, and as explained in section 2, the one with positive $U(1)_R$ charge is the one which we focus on herein. We now derive the component expressions for this supercharge acting on the twisted fields starting from the six-dimensional expressions. In a flat space-time, these supersymmetry variations for the free tensor multiplet are given by:

$$\delta H_{MNP} = 3\partial_{[M}\left(\bar{\Psi}_\alpha \Gamma_{NP]}\epsilon^\alpha\right) \tag{22}$$

$$\delta \Phi_K = 2(\Gamma_K^R)_{\alpha\beta}\bar{\Psi}^\alpha \epsilon^\beta \tag{23}$$

$$\delta \Psi^\alpha = \frac{i}{12}H_{MNP}\Gamma^{MNP}\epsilon^\alpha + iM_{\beta\gamma}\partial_M(\Gamma_K^R)^{\alpha\beta}\Phi^K\Gamma^M\epsilon^\gamma. \tag{24}$$

Where $K$ denotes an index in the vector representation of the $R$-symmetry group $Spin(5)$. Using the twisted field content of definitions (2)-(6) together with the supersymmetry parameter $\varepsilon$ of (1) , these variations induce the following variations of the twisted fields:

$$\begin{aligned}
\delta\sigma &= \sqrt{2}\tilde{\eta}v & \delta\eta &= 0 \\
\delta\bar{\sigma} &= 0 & \delta\psi_\nu &= -vi\sqrt{2}\partial_\nu\bar{\sigma} \\
\delta E_{\mu\nu} &= i\chi_{\mu\nu}v & \delta\chi_{\mu\nu} &= 0 \\
\delta F^+_{\mu\nu} &= 0 & \delta\tilde{\eta} &= 0 \\
\delta F^-_{\mu\nu} &= -4\partial_{[\mu}\psi_{\nu]}v & \delta\tilde{\psi}_\nu &= ivA_\nu - v\partial_\mu E_{\nu\mu} \\
\delta A_\mu &= \partial_\mu\eta v & \delta\tilde{\chi}_{\mu\nu} &= 2ivF^+_{\mu\nu}
\end{aligned} \tag{25}$$

These can be verified to square to zero, which is equivalent to the supercharge $Q$ considered here indeed being nilpotent. Furthermore, these variations can be shown to induce an isomorphism on the space of solutions to the equations of motions presented in equations (18), (19), (20) and (21).

# 4 Stress tensor

A first step towards computing the stress tensor for the theory in a general background it is to first perform the calculations in the special case when $M_4$ has vanishing curvature. This is the subject of this section, and is something that will greatly facilitate the investigation of the general case (performed in section 5).

### 4.1 Actions

Since the main objective of this paper is to obtain an explicit expression for the stress tensor of the twisted theory, it would be highly convenient if we could formulate an action for it. The derivation of the desired stress tensor would in principle then be straight forward, and could be carried out by a standard metric variation of this action. However, as previously mentioned on repeated occasions, there are some well-known problems with giving a satisfactory formulation of (2,0) theory in general, and using a Lagrangian formalism in particular, and we cannot hope to do this here either. However, there is a well-defined action for both the fermionic fields as well as the scalar fields of the abelian (2,0) theory, and by writing these down we may find an Ansatz for the contributions to the stress tensor which arise from these fields.

#### Scalars

The action for the scalar field in six dimensions is given by the standard expression

$$\mathcal{L}_{\text{scalars}} = -\partial_M \Phi^K \partial^M \Phi_K. \tag{26}$$

By exploiting the fact that all derivatives in the $\pm$-directions vanish, together with the relations:

$$\begin{aligned} \Phi_i \Phi^i &= \frac{1}{4} E_{\mu\nu} E^{\mu\nu} \\ \Phi_4 \Phi^4 + \Phi_5 \Phi^5 &= 2\sigma\bar{\sigma}, \end{aligned} \tag{27}$$

the action for the scalar fields in the twisted theory may be written as:

$$\mathcal{L}_{\text{scalars}} = -\frac{1}{4} \partial_\rho E_{\mu\nu} \partial^\rho E^{\mu\nu} - 2\partial_\rho \sigma \partial^\rho \bar{\sigma}. \tag{28}$$

#### Fermions

In six dimensions, the fermionic part of the action may be written on the well-known form

$$\mathcal{L} = \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi. \tag{29}$$

Recall that these six-dimensional fields may be reinterpreted in terms of the twisted ones according to equation (6), which states:

$$\Psi = (\eta + \Gamma_+\Gamma_\mu\psi^\mu + \frac{1}{4}\Gamma_{\mu\nu}\chi^{\mu\nu})e^+ + (\tilde{\eta} + \Gamma_+\Gamma_\mu\tilde{\psi}^\mu + \frac{1}{4}\Gamma_{\mu\nu}\tilde{\chi}^{\mu\nu})e^-, \qquad (30)$$

where $e^+$ and $e^-$ as previously are constant spinors which span the two chiral spinor representations that are Lorentz scalars on $M_4$. From this, an expression for $\bar{\Psi}$ may be obtained as:

$$\bar{\Psi} = \bar{e}^+(\eta - \Gamma_+\Gamma_\mu\psi^\mu - \frac{1}{4}\Gamma_{\mu\nu}\chi^{\mu\nu}) + \bar{e}^-(\tilde{\eta} - \Gamma_+\Gamma_\mu\tilde{\psi}^\mu - \frac{1}{4}\Gamma_{\mu\nu}\tilde{\chi}^{\mu\nu}). \qquad (31)$$

By using the properties (14) derived for the Γ-matrices, integration by parts and the fact that all derivatives along $C$ vanish, the six-dimensional fermionic action may be written in terms of the twisted fields as:

$$\mathcal{L}_{\text{Fermions}} = -i\left( \eta\partial_\mu\tilde{\psi}^\mu + \psi^\mu\partial_\mu\tilde{\eta} - \psi_\mu\partial_\nu\tilde{\chi}^{\mu\nu} + \chi^{\mu\nu}\partial_\mu\tilde{\psi}^\nu\right). \qquad (32)$$

## 4.2 Ansatz and modifications

The stress tensor in the flat case is obtained by computing the individual contributions originating from the six-dimensional bosonic three-form, the bosonic scalar and the fermions separately, whereupon the relative coefficients are fixed by requiring supersymmetry invariance. However, which to us was somewhat unintuitive, some modifications to the terms containing the bosonic self-dual two-forms are required in order to obtain an expression which is both conserved and $Q$-closed. This final expression of $T^{\mu\nu}$ may then be shown to also be $Q$-exact as desired.

Another important feature is that since the theory has no other definition than in terms of the equations of motion, the stress tensor will only be considered on-shell.

For the fields where an action exists, an Ansatz of the stress tensor may be computed in a standard way, namely by using

$$T^{\mu\nu} = \frac{1}{2}g^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}. \qquad (33)$$

For the part arising from the bosonic three-form however, we are forced to take a slightly different approach. We may regard the action for a non-chiral 3-form in six dimensions, taking the familiar expression

$$\mathcal{L} = H_{MNP}H^{MNP}, \qquad (34)$$

which may be used to compute a stress tensor by the recipe stated in equation (33). After this is done, the condition that $H$ is self-dual in six dimensions is imposed, and thus the first term in equation (33) will vanish. The remaining terms on $M_4$ will, in the language of the twisted fields, be given by

$$T_H^{\mu\nu} = -4A^{(\mu}A^{\nu)} - 2F^{+\mu}{}_\rho F^{-\rho\nu} - 2F^{-\mu}{}_\rho F^{+\rho\nu} + 2g^{\mu\nu}A_\rho A^\rho. \tag{35}$$

For the scalars and the fermions, one arrives at the following expressions respectively

$$T_\Phi^{\mu\nu} = -g^{\mu\nu}\partial_\rho\sigma\partial^\rho\bar{\sigma} + 2\partial^{(\mu}\sigma\partial^{\nu)}\bar{\sigma} + \frac{1}{4}\partial^{(\mu}E_{\rho\sigma}\partial^{\nu)}E^{\rho\sigma} - \frac{1}{8}g^{\mu\nu}\partial_\lambda E_{\rho\sigma}\partial^\lambda E^{\rho\sigma}, \tag{36}$$

$$\begin{aligned} T_\Psi^{\mu\nu} = &\frac{i}{2}g^{\mu\nu}\left(\partial_\rho\eta\tilde{\psi}^\rho + \partial_\rho\tilde{\eta}\psi^\rho\right) - i\left(\partial^{(\mu}\eta\tilde{\psi}^{\nu)} + \partial^{(\mu}\tilde{\eta}\psi^{\nu)}\right) \\ &- \frac{i}{4}g^{\mu\nu}\left(\tilde{\chi}^{\rho\sigma}\partial_{[\rho}\psi_{\sigma]} + \chi^{\rho\sigma}\partial_{[\rho}\tilde{\psi}_{\sigma]}\right) \\ &+ \frac{i}{2}\left(\chi^{\sigma(\mu}\partial_\sigma\tilde{\psi}^{\nu)} + \tilde{\chi}^{\sigma(\mu}\partial_\sigma\psi^{\nu)} - \chi^{\sigma(\mu}\partial^{\nu)}\tilde{\psi}_\sigma - \tilde{\chi}^{\sigma(\mu}\partial^{\nu)}\psi_\sigma\right). \end{aligned} \tag{37}$$

It should be noted here that since we have self-dual fields, the variation of the metric is not as straight-forward as it would appear to be in equation (33). This is because the condition of self-duality contains an implicit metric dependence, and thus a variation of the metric must be accompanied by a variation of all self-dual fields present. A term consisting of such a self-dual field, $\chi_{\mu\nu}$, with indices contracted with some other rank-2 tensor, $X^{\mu\nu}$, will under a metric variation take the form:

$$X^{\mu\nu}\delta_g\chi_{\mu\nu} = -\frac{1}{4}\delta g_{\mu\nu}g^{\mu\nu}X_{\kappa\lambda}\chi^{\kappa\lambda} + \delta g_{\mu\nu}X^{[\mu\sigma]}\chi^\nu{}_\sigma. \tag{38}$$

The three pieces in (35), (36) and (37) are each conserved individually, which may be shown by straight-forward, but yet tedious calculations that are omitted here. In order to stand a chance of fulfilling supersymmetry invariance under the transformations listed in equation (25), the relative coefficients amongst the different contributions are fixed. The stress tensor one then finds is given by:

$$T^{\mu\nu} = \frac{1}{2}\Big( -g^{\mu\nu}\partial_\rho\sigma\partial^\rho\bar{\sigma} + 2\partial^{(\mu}\sigma\partial^{\nu)}\bar{\sigma} + \frac{1}{4}\partial^{(\mu}E_{\rho\sigma}\partial^{\nu)}E^{\rho\sigma} - \frac{1}{8}g^{\mu\nu}\partial_\lambda E_{\rho\sigma}\partial^\lambda E^{\rho\sigma}\Big) \tag{39}$$

$$+ \frac{1}{8}\left(-4A^{(\mu}A^{\nu)} - 2F^{+\mu}{}_\rho F^{-\rho\nu} - 2F^{-\mu}{}_\rho F^{+\rho\nu} + 2g^{\mu\nu}A_\rho A^\rho\right)$$
$$+ \frac{i}{2}g^{\mu\nu}\left(\partial_\rho\eta\tilde{\psi}^\rho + \partial_\rho\tilde{\eta}\psi^\rho\right) - i\left(\partial^{(\mu}\eta\tilde{\psi}^{\nu)} + \partial^{(\mu}\tilde{\eta}\psi^{\nu)}\right)$$
$$- \frac{i}{4}g^{\mu\nu}\left(\tilde{\chi}^{\rho\sigma}\partial_{[\rho}\psi_{\sigma]} + \chi^{\rho\sigma}\partial_{[\rho}\tilde{\psi}_{\sigma]}\right)$$
$$+ \frac{i}{2}\left(\chi^{\sigma(\mu}\partial_\sigma\tilde{\psi}^{\nu)} + \tilde{\chi}^{\sigma(\mu}\partial_\sigma\psi^{\nu)} - \chi^{\sigma(\mu}\partial^{\nu)}\tilde{\psi}_\sigma - \tilde{\chi}^{\sigma(\mu}\partial^{\nu)}\psi_\sigma\right).$$

However, some problematic terms still exist which prevents the above expression from being $Q$-closed. By a long and quite intricate calculation, one may show that this obstruction is solved if the part of the stress tensor containing the self-dual two-form which arose from the six-dimensional scalars, namely terms containing $E_{\mu\nu}$, is altered to:

$$T^{\mu\nu}_{EE\text{-terms}} = \frac{1}{4}g^{\mu\nu}\partial^\kappa E_{\rho\kappa}\partial_\sigma E^{\rho\sigma} - \frac{1}{2}\partial^\rho\Big(\partial^\kappa E^{(\mu}{}_\kappa E^{\nu)}{}_\rho\Big) + \frac{1}{2}\partial^{(\mu}\partial_\kappa E^{\rho\kappa}E^{\nu)}{}_\rho. \tag{40}$$

Also this part is conserved on its own, and so this alteration preserves the conservation of $T^{\mu\nu}$. This may be shown by a slightly more complicated calculation than for any of the other terms, which requires the repeated use of the self-duality of $E_{\mu\nu}$.

That this problem of supersymmetry invariance is solved by altering the terms containing the fields originating from the scalars, for which we had an action from which to derive a stress tensor, may seem quite unintuitive. However, we must bear in mind that even though we have an action for some fields in the theory, there is no action for the *entire* theory. Hence we do not have a supersymmetric quantity from which we may derive a supersymmetric stress tensor, and though using the actions presented in equations (32) and (28) provides us with a good Ansatz for a stress tensor for the entire theory, we should not expect this approach to give us a supersymmetric result.

The complete stress tensor for this theory when placed on a flat background may then finally be written down explicitly as

$$
\begin{aligned}
T^{\mu\nu} = &\frac{1}{2}\Big( - g^{\mu\nu}\partial_\rho\sigma\partial^\rho\bar{\sigma} + 2\partial^{(\mu}\sigma\partial^{\nu)}\bar{\sigma}\Big) \\
&+ \frac{1}{8}\left(-4A^{(\mu}A^{\nu)} - 2F^{+\mu}{}_\rho F^{-\rho\nu} - 2F^{-\mu}{}_\rho F^{+\rho\nu} + 2g^{\mu\nu}A_\rho A^\rho\right) \\
&+ \frac{i}{2}g^{\mu\nu}\left(\partial_\rho\eta\tilde{\psi}^\rho + \partial_\rho\tilde{\eta}\psi^\rho\right) - i\left(\partial^{(\mu}\eta\tilde{\psi}^{\nu)} + \partial^{(\mu}\tilde{\eta}\psi^{\nu)}\right) \\
&- \frac{i}{4}g^{\mu\nu}\left(\tilde{\chi}^{\rho\sigma}\partial_{[\rho}\psi_{\sigma]} + \chi^{\rho\sigma}\partial_{[\rho}\tilde{\psi}_{\sigma]}\right) \\
&+ \frac{i}{2}\left(\chi^{\sigma(\mu}\partial_\sigma\tilde{\psi}^{\nu)} + \tilde{\chi}^{\sigma(\mu}\partial_\sigma\psi^{\nu)} - \chi^{\sigma(\mu}\partial^{\nu)}\tilde{\psi}_\sigma - \tilde{\chi}^{\sigma(\mu}\partial^{\nu)}\psi_\sigma\right) \\
&+ \frac{1}{4}g^{\mu\nu}\partial^\kappa E_{\rho\kappa}\partial_\sigma E^{\rho\sigma} - \frac{1}{2}\partial^\rho\Big(\partial^\kappa E^{(\mu}{}_\kappa E^{\nu)}{}_\rho\Big) + \frac{1}{2}\partial^{(\mu}\partial_\kappa E^{\rho\kappa}E^{\nu)}{}_\rho,
\end{aligned}
\tag{41}
$$

where the last line above is the manually altered terms that are needed to make the stress tensor invariant under the supersymmetry transformations in equation (25).

## 4.3 $Q$-exactness

The stress tensor presented above in (41) is after an examination found to be $Q$-exact and may be written as

$$
T^{\mu\nu} = \Big\{Q, \lambda^{\mu\nu}\Big\}, \tag{42}
$$

where

$$
\begin{aligned}
\lambda^{\mu\nu} = &\frac{1}{2}\Big(\sqrt{2}i\psi^{(\mu}\partial^{\nu)}\sigma + \tilde{\psi}^{(\mu}\partial^\rho E^{\nu)}{}_\rho + \partial_\rho\tilde{\psi}^{(\mu}E^{\nu)}{}_\rho - \partial^{(\mu}\tilde{\psi}^\rho E^{\nu)}{}_\rho \\
&+ i\tilde{\psi}^{(\mu}A^{\nu)} - \frac{i}{2}\tilde{\chi}^{(\mu}{}_\rho F^{-\nu)\rho} - \frac{i}{\sqrt{2}}g^{\mu\nu}\psi_\rho\partial^\rho\sigma - \frac{1}{2}g^{\mu\nu}\tilde{\psi}_\rho\partial_\sigma E_{\rho\sigma} - \frac{i}{2}g^{\mu\nu}\tilde{\psi}_\rho A^\rho\Big).
\end{aligned}
\tag{43}
$$

To find $\lambda^{\mu\nu}$, an Ansatz was used in which all possible allowed, terms were included. These are however not as many as one may think, since there are constraints due to dimensionality and $U(1)$-charge. These constraints forces us to restrict ourselves to terms of dimensionality 11/2 and $U(1)$ charge of $-1/2$, (which all of the above terms clearly satisfy). In table 4, the dimensionality and $U(1)$-charge of the different fields, as well as the supersymmetry parameter and stress tensor, are listed.

| | dimensionality | $U(1)_R$-charge |
|---|---|---|
| $\eta, \psi_\mu, \chi_{\mu\nu}$ | $5/2$ | $+1/2$ |
| $\tilde{\eta}, \tilde{\psi}_\mu, \tilde{\chi}_{\mu\nu}$ | $5/2$ | $-1/2$ |
| $\bar{\sigma}$ | $2$ | $+1$ |
| $\sigma$ | $2$ | $-1$ |
| $E_{\mu\nu}$ | $2$ | $0$ |
| $A_\mu, F_{\mu\nu}$ | $3$ | $0$ |
| $T_{\mu\nu}$ | $6$ | $0$ |
| $\epsilon$ | $-1/2$ | $-1/2$ |

Table 4: Mass dimension and $U(1)_R$ charges of the fields, parameters and curvature tensors.

# 5 The case when $M_4$ is curved

In the previous section, an expression for the stress tensor when $M_4$ has vanishing curvature is obtained and shown to indeed be $Q$-exact. This was done by explicitly finding a $\lambda^{\mu\nu}$ such that $T^{\mu\nu} = \{Q, \lambda^{\mu\nu}\}$. Now we are faced with the question: How does this change in the case when $M_4$ is curved?

A simple starting point here would instead be to ask the question "How may $\lambda^{\mu\nu}$ change when $M_4$ becomes curved?". The restrictions imposed upon $\lambda^{\mu\nu}$ by dimensionality may be used here as well. Since $\lambda^{\mu\nu}$ is of fractional dimension, an odd number of fermionic fields must be included. Also, since we wish to add terms related to curvature, the Riemann-, Ricci-tensor or curvature scalar must be included in these, each of which is of dimension 2. The remaining part of these terms must be of dimension 1, which means that our only option is to incorporate a derivative. Terms like these are however not bilinears in the fields, and thus make no sense at all.

By the reasoning above, there are no terms which may possibly be added to $\lambda^{\mu\nu}$ in the case when $M_4$ is curved. Thus, the stress tensor even in this case will still be given by the expression $\{Q, \lambda^{\mu\nu}\}$.

It should be noted that there are two more places that could be modified in the curved case: the scalar equations of motion and the fermion supersymmetry variations.

The scalar equations of motion could be modified to replace the right hand side of the Klein-Gordon equation in both (18) and (19) with a multiple of the curvature scalar multiplying the fields. However, such a modification in

(19) would ruin the conservation properties of the part of the stress tensor containing the bosonic scalars and is thus not allowed. The same modification in (18),

$$D_\rho D^\rho E_{\mu\nu} = aRE_{\mu\nu}, \tag{44}$$

may be carried out, where $a$ is some constant. However, this will not be enough to rectify the problems arising when $M_4$ is curved, something which is further discussed in section 5.1.

The fermionic supersymmetry variations for the six-dimensional free tensor multiplet in a curved background may contain an extra term of the form

$$\delta\Psi = \cdots + \Phi\Gamma^M D_M \epsilon. \tag{45}$$

This term will not contribute to the twisted supersymmetry transformations since the whole point of the twisting is to manufacture a covariantly constant supercharge.

Thus, in the curved case, the stress tensor cannot be subject to any modifications and will still be given by $\left\{Q, \lambda^{\mu\nu}\right\}$, where all partial derivatives in $\lambda^{\mu\nu}$ are now replaced by covariant ones. This gives us $T^{\mu\nu}$ as in equation (41) but again, with partial derivatives replaced by covariant ones. The generalisation to a curved $M_4$ is thus:

$$\begin{aligned} T^{\mu\nu} = &\frac{1}{2}\Big( - g^{\mu\nu} D_\rho \sigma D^\rho \bar{\sigma} + 2D^{(\mu}\sigma D^{\nu)}\bar{\sigma}\Big) \\ &+ \frac{1}{8}\left(-4A^{(\mu}A^{\nu)} - 2F^{+\mu}{}_\rho F^{-\rho\nu} - 2F^{-\mu}{}_\rho F^{+\rho\nu} + 2g^{\mu\nu}A_\rho A^\rho\right) \\ &+ \frac{i}{2} g^{\mu\nu}\left(D_\rho \eta \tilde{\psi}^\rho + D_\rho \tilde{\eta}\psi^\rho\right) - i\left(D^{(\mu}\eta\tilde{\psi}^{\nu)} + D^{(\mu}\tilde{\eta}\psi^{\nu)}\right) \\ &- \frac{i}{4} g^{\mu\nu}\left(\tilde{\chi}^{\rho\sigma} D_{[\rho}\psi_{\sigma]} + \chi^{\rho\sigma} D_{[\rho}\tilde{\psi}_{\sigma]}\right) \\ &+ \frac{i}{2}\left(\chi^{\sigma(\mu} D_\sigma \tilde{\psi}^{\nu)} + \tilde{\chi}^{\sigma(\mu} D_\sigma \psi^{\nu)} - \chi^{\sigma(\mu} D^{\nu)}\tilde{\psi}_\sigma - \tilde{\chi}^{\sigma(\mu} D^{\nu)}\psi_\sigma\right) \\ &+ \frac{1}{4} g^{\mu\nu} D^\kappa E_{\rho\kappa} D_\sigma E^{\rho\sigma} - \frac{1}{2} D^\rho\Big(D^\kappa E^{(\mu}{}_\kappa E^{\nu)}{}_\rho\Big) + \frac{1}{2} D^{(\mu} D_\kappa E^{\rho\kappa} E^{\nu)}{}_\rho. \end{aligned} \tag{46}$$

That this stress tensor is still $Q$-exact is obvious, but it is not completely clear that it still fulfils the criteria of being covariantly conserved. Rather surprisingly, it would seem that it does *not*. Again, the complications lie in the part containing the self-dual bosonic two-forms. By considering the covariant derivative of these terms, the complications arising here for a curved $M_4$ will be apparent.

## 5.1 Covariant conservation of $T^{\mu\nu}$ in the curved case

Consider the covariant divergence of the terms containing the bosonic self-dual two-forms:

$$D^\mu T^{\text{curved } EE\text{-terms}}_{\mu\nu} = +\frac{1}{2} g_{\mu\nu} D^\mu D^\kappa E_{\rho\kappa} D_\sigma E^{\rho\sigma} - \frac{1}{2} D^{[\mu} D^{\rho]} \Big( D_\kappa E_{(\mu}{}^\kappa E_{\nu)\rho} \Big) \\ - \frac{1}{2} D^{(\mu} D^{\rho)} \Big( D_\kappa E_{(\mu}{}^\kappa E_{\nu)\rho} \Big) + \frac{1}{2} D^\mu \Big( D_{(\mu} D_\kappa E^{\rho\kappa} E_{\nu)\rho} \Big). \tag{47}$$

This can, as previously mentioned, be shown to vanish when $M_4$ is flat, but in the curved case, there are additional terms arising from commutating the derivatives which may yet cause problems. A few of the above terms will give rise to terms containing derivatives on the curvature tensors, which must cancel on their own for any chance to maintain conservation of $T^{\mu\nu}$. Such terms will arise from terms containing three derivatives acting on the same field, that is from the two last terms in the expression above.

Let us start by considering terms of this kind. By using two forms of the Bianchi identity, together with a basis expansion of the self-dual two-forms according to $E_{\mu\nu} = E_i T^i_{\mu\nu}$, (where $i \in \{1,2,3\}$ and the $T^i$'s form a basis on the space of self-dual two-forms) in the cases where the two bosonic fields are contracted, one may in a straight-forward manner show that all terms containing the derivatives on the curvature tensors may be written as:

$$-\frac{1}{4} D_\tau R_{\rho\kappa} E^{\tau\kappa} E_\nu{}^\rho + \frac{1}{8} D_\nu R_{\mu\kappa\rho\tau} E^{\tau\kappa} E^{\mu\rho} + \frac{1-2a}{4} D_\nu R E_i E^i. \tag{48}$$

To obtain this expression, the most general form of the equations of motions for $E_{\mu\nu}$ on a curved background were used, as given in (44).

This is in general non-zero, which may be easily shown by introducing a concrete example in which this quantity does not vanish. An example of such a configuration is $M_4 = \mathbb{R} \times M_3$, where index value 1 denotes the coordinate along $\mathbb{R}$, and $M_3$ is of non-vanishing curvature. Consider (48) in the case where $\nu = 1$. In such a case, the two last terms vanish, where as the first one in general does not. We have thus shown that the unique, $Q$-exact stress tensor of the theory is not conserved when the theory is placed upon a general four-manifold $M_4$.

# 6 Conclusion and outlook

Herein, we have shown that there is no possible covariantly conserved, $Q$-exact stress tensor when this twisted form of the theory is placed in a general background. The twisting in question is taken to be the one described, for example, in [12] where the free tensor multiplet of (2,0) theory is placed on $M_6 = C \times M_4$ with Minkowski signature. Compactification along $C$ and twisting along $M_4$ as described in section 2 is then done. Furthermore, the theory is only considered under the supercharge that would become scalar on $C$ *if* it were of Euclidean signature and further twistings could be performed. This must however remain an "*if*", because of the problems surrounding the formulation of (2,0) theory, especially in Euclidean signature. Because of these problems, all of our investigations were kept on the level of equations of motion.

This result is to us a surprising one, but it may be more logical than it appears at first glance. It is a well-known fact that (2,0)-theory compactified on a two-manifold $C$ results in $\mathcal{N} = 2$ Super Yang-Mills theory [4–6, 18], and another well-known result that this theory admits a unique topological twisting in four dimensions [14]. Herein, a slightly different linear combination of the supersymmetry charges is considered than the one used in the Donaldson-Witten twist. This is because the supersymmetry charge of interest herein is the one that would become scalar on the two-manifold $C$ as well, if that were of Euclidean signature and the twisting thereon could be performed. It is thus logical in some sense that the twisting we consider fails to give rise to a theory which is topological on the four-manifold. It should however be pointed out that this twist, from the viewpoint of the four-manifold is not, as previously has been claimed on some occasions, the Donaldson-Witten twist, but something which differs slightly from this. We believe this to be the cause of the unexpected behaviour.

One could then ask if this situation finds its remedy in the hypothetical twisting along $C$. This will however not be the case since this twisting would only result in different $U(1)$-charges of the fields, and all arguments done here for possible curvature corrections etc are not dependent on this, but rather on dimensionality which remains unchanged.

Another possible resolution of these difficulties may be found in a hypothetical formulation of the free (2,0) tensor multiplet in a Euclidean signature, which is problematic for obvious reasons. If one requires that this hypothetical theory should indeed give rise to a topological field theory under the twisting described herein, this investigation of the difficulties presented for its Minkowski analog may shed some light on desired properties of the Euclidean theory.


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