Harmonic regressor: Robust solution to least-squares problem

Alexander Stotsky

Abstract

A new robust and computationally efficient solution to least-squares problem in the presence of round-off errors is proposed. The properties of a harmonic regressor are utilized for design of new combined algorithms of direct calculation of the parameter vector. In addition, an explicit transient bound for estimation error is derived for classical recursive least-squares algorithm using the Lyapunov function method. Different initialization techniques of the gain matrix are proposed as an extension of the recursive least-squares algorithm. All the results are illustrated by simulations.

Introduction

Recursive least-squares (RLS) algorithms in which an estimate of the inverse of the information matrix is updated recursively are widely used in many applications such as adaptive control, signal processing, system identification and others.[1] One of the main drawbacks of RLS algorithms is a round-off error accumulation problem, which in turn has a direct impact on the estimation performance and even on system stability.[2–4] This problem might be solved via a frequent initialization of the gain matrix to the inverse of the information matrix. However, a computational burden associated with the inversion of the information matrix for large-scale systems is the main obstacle for implementation of this technique.

The periodic nature of a wide class of machines results in oscillations in their signals. The model of oscillating signals can be presented in the vector form with a harmonic regressor, which consists of trigonometric functions (sines and cosines) at different frequencies.[5] The information matrix of harmonic regressor, which is defined as the sum of rank 1 matrices calculated via multiplication of the regressor column vector by its transpose with a corresponding weighting sequence is a strictly diagonally dominant (SDD) matrix.[6] Strict diagonal dominance of the information matrix is used in this article for improvement of robustness of RLS algorithms in the presence of round-off errors. The least-squares problem is solved via direct calculation of the

parameter vector with an SDD information matrix. This solution is preferable for large-scale systems (for signals with a large number of frequencies), where high accuracy of calculation of the coefficients is required. One of the algorithms for direct calculation of the coefficients of the trigonometric polynomial is reported in Stotsky.[6] However, the convergence of algorithm described in that study[6] might be slow. A new class of algorithms (high-order algorithms) for direct calculation of the parameters with improved convergence rate is proposed in this article.

Moreover, a new combination of the matrix inversion technique with a recursive algorithm of direct parameter calculation is also proposed in this article. This combination represents a two-stage algorithm, where an SDD information matrix is rapidly inverted with a certain (relatively low) accuracy at the first stage. Notice that the convergence rate of the matrix inversion algorithm is much higher than the convergence rate of the recursive algorithm of direct calculation of the parameters. Therefore, the matrix inversion algorithm is used in the initial steps. Further

Department of Signals and Systems, Chalmers University of Technology, Gothenburg, Sweden

Corresponding author:
Alexander Stotsky, Department of Signals and Systems, Chalmers University of Technology, Gothenburg SE-412 96, Sweden.
Email: alexander.stotsky@chalmers.se

calculations of the inverse of an SDD information matrix give a minor contribution to the improvement of accuracy. A recursive algorithm for direct calculation of the parameter vector with improved convergence rate (high-order algorithm) is used in the second stage, and finally delivers the parameter vector with high accuracy. Simulations of new algorithms showed significant improvement of the estimation performance compared to well-known RLS algorithms in the presence of round-off errors.

Classical RLS algorithms are also studied in this article. Explicit transient bound for estimation error is derived using an assumption about positive definiteness of the gain update matrix. It is shown that the gain matrix, which is recursively updated, might be initialized differently to extend the capabilities of the recursive algorithm. It is also shown that RLS algorithm can be seen as a modification of the Kaczmarz projection algorithm, where the inverse of gain matrix is a positive definite and SDD matrix.

Main contributions of this article can be summarized as follows:

- A new class of high-order algorithms with improved convergence rate for direct calculation of the parameters in the least-squares problem;
- A new combined matrix inversion algorithm and recursive algorithm of direct parameter calculation;
- Explicit transient bound for estimation error in RLS algorithm;
- A new estimation algorithm in a form of RLS algorithm with an SDD gain matrix.

The article is organized as follows. The problem statement is presented in section "Problem statement." New properties of RLS algorithms are described in section "RLS algorithms." New recursive algorithms of direct calculation of the parameter vector are presented in section "Recursive algorithms of direct calculation of the parameter vector." Combined algorithms are presented in section "Combined recursive algorithms." The article ends with brief comparisons and conclusions in section "Other methods and conclusion."

Problem statement

Suppose that a measured oscillating signal y_k can be presented in the following form

$$y_k = \varphi_k^T \theta_* + \xi_k \tag{1}$$

where φ_k is the harmonic regressor and θ_* is the vector of constant unknown parameters, which are defined as follows

$$\varphi_k^T = [1 \; cos(q_1 k) \; sin(q_1 k) \; cos(q_2 k) \; sin(q_2 k) \ldots$$
$$cos(q_r k) \; sin(q_r k)] \tag{2}$$
$$\theta_*^T = [\theta_{0*} \; \theta_{1*} \; \theta_{2*} \; \theta_{3*} \; \theta_{4*} \; \ldots \; \theta_{(2r-1)*} \; \theta_{(2r)*}] \tag{3}$$

where $k = 1, 2, \ldots$ is the step number; $q_i \; i = 1, 2, \ldots, r$ are the frequencies; and ξ_k is a zero mean white Gaussian noise.

The model of the signal (1) is presented in the following form

$$\hat{y}_k = \varphi_k^T \theta_k \tag{4}$$

with adjustable parameters

$$\theta_k^T = [\theta_{0k} \; \theta_{1k} \; \theta_{2k} \; \theta_{3k} \; \theta_{4k} \; \ldots \; \theta_{(2r-1)k} \; \theta_{(2r)k}] \tag{5}$$

Minimization of the performance index

$$E_k = \sum_{j=1}^{k} \beta_j (y_j - \hat{y}_j)^2 \tag{6}$$

where β_j is a weighting sequence, results in the following least-squares solution

$$\theta_k = \left[\sum_{j=1}^{k} \beta_j \varphi_j \varphi_j^T\right]^{-1} \sum_{j=1}^{k} \beta_j \varphi_j y_j \tag{7}$$

Assigning weighting factor to one in step k and to λ_0 in the previous steps, the equation (7) can be written as follows

$$\theta_k = A_k^{-1} b_k \tag{8}$$
$$A_k = \lambda_0 \left[\sum_{j=1}^{k-1} \varphi_j \varphi_j^T\right] + \varphi_k \varphi_k^T \tag{9}$$
$$b_k = \lambda_0 \left[\sum_{j=1}^{k-1} \varphi_j y_j\right] + \varphi_k y_k \tag{10}$$

where the matrix A_k is information (Hessian) matrix, and $0 < \lambda_0 < 1$ is a forgetting factor, $k = 2, 3 \ldots$.

In order to calculate the parameter vector θ_k, this information matrix should be inverted in each step k. This is a computationally expensive procedure, if the size of the parameter vector is large. This matrix is an SDD matrix for a sufficiently large k and forgetting factor, which is close to one,[7] that facilitates matrix inversion and allows direct calculation of the parameter vector.

RLS algorithms

Algorithm description and transient bound on estimation error

RLS solution of equation (8) with $\Gamma_k = A_k^{-1}$ can be written as follows[1]

$$\theta_k = \theta_{k-1} + \frac{\Gamma_{k-1}\varphi_k}{\lambda_0 + \varphi_k^T \Gamma_{k-1}\varphi_k} \left(y_k - \theta_{k-1}^T \varphi_k\right) \tag{11}$$

$$\Gamma_k = \frac{1}{\lambda_0} \left[\Gamma_{k-1} - \frac{\Gamma_{k-1}\varphi_k \varphi_k^T \Gamma_{k-1}}{\left(\lambda_0 + \varphi_k^T \Gamma_{k-1}\varphi_k\right)}\right] \tag{12}$$

The properties of the systems (1), (4), (11) and (12) are summarized in the following Lemma.

Lemma. Consider the systems (1) and (4) with $\xi_k = 0$ and recursive estimation algorithms (11) and (12). Assume that the following inequalities for the inverse of the gain matrix are valid

$$0 < \alpha_1 I \leqslant \Gamma_k^{-1} \leqslant \alpha_2 I \tag{13}$$

where α_1 and α_2 are positive constants and I is the identity matrix. Then the following bound for estimation error $\tilde{\theta}_k = \theta_k - \theta_*$ is valid

$$\| \tilde{\theta}_k \| \leqslant \sqrt{\frac{\lambda_0^k V_0}{\alpha_1}} \tag{14}$$

where $V_0 = \tilde{\theta}_0^T \Gamma_0^{-1} \tilde{\theta}_0$ and $\tilde{\theta}_0 = \theta_0 - \theta_*$

Proof. The following relations are valid for algorithms (11) and (12)

$$\Gamma_k^{-1} = \lambda_0 \Gamma_{k-1}^{-1} + \varphi_k \varphi_k^T \tag{15}$$

$$\tilde{\theta}_k - \tilde{\theta}_{k-1} = -\frac{\Gamma_{k-1} \varphi_k \varphi_k^T}{\lambda_0 + \varphi_k^T \Gamma_{k-1} \varphi_k} \tilde{\theta}_{k-1} \tag{16}$$

$$\varphi_k \varphi_k^T \tilde{\theta}_k = \frac{\lambda_0 \varphi_k \varphi_k^T}{\lambda_0 + \varphi_k^T \Gamma_{k-1} \varphi_k} \tilde{\theta}_{k-1} \tag{17}$$

where equation (15) follows from relation (9), equation (16) follows from relation (11) and finally equation (17) follows from relation (16) when multiplying relation (16) by $\varphi_k \varphi_k^T$.

The performance of RLS algorithm is evaluated using Lyapunov function candidate $V_k = \tilde{\theta}_k^T \Gamma_k^{-1} \tilde{\theta}_k$, which can be written as $V_k = \tilde{\theta}_k^T (\lambda_0 \Gamma_{k-1}^{-1} + \varphi_k \varphi_k^T) \tilde{\theta}_k$. Taking into account equations (16) and (17), the first difference is evaluated as follows

$$\begin{aligned}
V_k - V_{k-1} &= \lambda_0 \left[\tilde{\theta}_k^T \Gamma_{k-1}^{-1} \tilde{\theta}_k - \tilde{\theta}_{k-1}^T \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1} \right] \\
&\quad + \tilde{\theta}_k^T \varphi_k \varphi_k^T \tilde{\theta}_k + (\lambda_0 - 1) \tilde{\theta}_{k-1}^T \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1} \\
&= \lambda_0 (\tilde{\theta}_k - \tilde{\theta}_{k-1})^T \Gamma_{k-1}^{-1} (\tilde{\theta}_k + \tilde{\theta}_{k-1}) \\
&\quad + \lambda_0 \tilde{\theta}_k^T \frac{\varphi_k \varphi_k^T}{\lambda_0 + \varphi_k^T \Gamma_{k-1} \varphi_k} \tilde{\theta}_{k-1} \\
&\quad + (\lambda_0 - 1) \tilde{\theta}_{k-1}^T \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1} \\
&= -\lambda_0 \tilde{\theta}_{k-1}^T \frac{\varphi_k \varphi_k^T}{\lambda_0 + \varphi_k^T \Gamma_{k-1} \varphi_k} \tilde{\theta}_{k-1} \\
&\quad + (\lambda_0 - 1) \tilde{\theta}_{k-1}^T \Gamma_{k-1}^{-1} \tilde{\theta}_{k-1}
\end{aligned}$$

and $V_k \leqslant \lambda_0 V_{k-1}$. Hence the bound (14) is true.

Initialization of the gain matrix Γ_k: two algorithms in one form

For convergence of the recursive algorithms (11) and (12), inequalities (13) should be satisfied. The gain matrix might be initialized in different ways in order to satisfy inequalities (13). Moreover, the diagonal dominance property of matrix A_k due to harmonic regressor facilitates the choice of initial gain matrix and extends the capabilities of the recursive algorithms (11) and (12).

Two initialization techniques can be used in algorithms (11) and (12), where the matrix Γ_k plays different roles. Initialization of the gain matrix Γ_k to the inverse of the information matrix A_k specifies the algorithm (12) as a recursive estimate of the inverse of the information matrix. This case corresponds to classical RLS algorithm. Initialization of the matrix Γ_k to a diagonal matrix with positive diagonal elements allows to treat this matrix as a gain matrix, and the properties of the algorithms (11) and (12) become similar to the properties of the Kaczmarz algorithm.[7,8] Classical RLS algorithms (11) and (12) are robust with respect to deviations of the gain matrix Γ_k from the inverse of the information matrix A_k, provided that inequalities (13) are valid.

Comparison of these two algorithms is presented in Figure 1, which shows time evolution of three estimated parameters defined in equation (5). Figure 1 shows that classical RLS algorithm provides better estimation performance compared to recursive estimation algorithm with an SDD gain matrix. The latter, however, is more robust against error accumulation problem, and its properties are similar to the properties of the Kaczmarz algorithm.[7,8]

Recursive algorithms of direct calculation of the parameter vector

Second-order algorithm

Least-squares problem is a problem of calculation of the parameter vector θ_k with high accuracy. The parameter vector θ_k in equation (8) can be recursively estimated as follows

$$\begin{aligned}
\vartheta_i &= \vartheta_{i-1} - 2D_k^{-1} A_k \vartheta_{i-1} + 2D_k^{-1} b_k + \\
&\quad \left(D_k^{-1} A_k\right)^2 \vartheta_{i-1} - D_k^{-1} A_k D_k^{-1} b_k
\end{aligned} \tag{18}$$

where ϑ_i is an estimate of θ_k; D_k is a diagonal matrix that contains diagonal elements of A_k, which is defined in equation (9); and b_k is defined in equation (10). Equations (8) and (18) can be combined as follows

$$\tilde{\vartheta}_i = \left(I - D_k^{-1} A_k\right)^2 \tilde{\vartheta}_{i-1} \tag{19}$$

where $\tilde{\vartheta}_i = \vartheta_i - \theta_k$ is estimation error.

The following inequality is valid $\|I - D_k^{-1} A_k\| \leqslant \kappa < 1$ due to the diagonal dominance of the matrix A_k,[6,9] where the norm $\|\cdot\|$ is defined as the maximum row sum matrix norm. This inequality guarantees the stability of the error model (19).

Notice that two errors are defined as estimation errors in this article. The first one is the error between estimated parameter vector θ_k and true parameter vector θ_*. This error is associated with estimation performance of RLS algorithm, see section "RLS algorithms." The second estimation error $\vartheta_i - \theta_k$ is
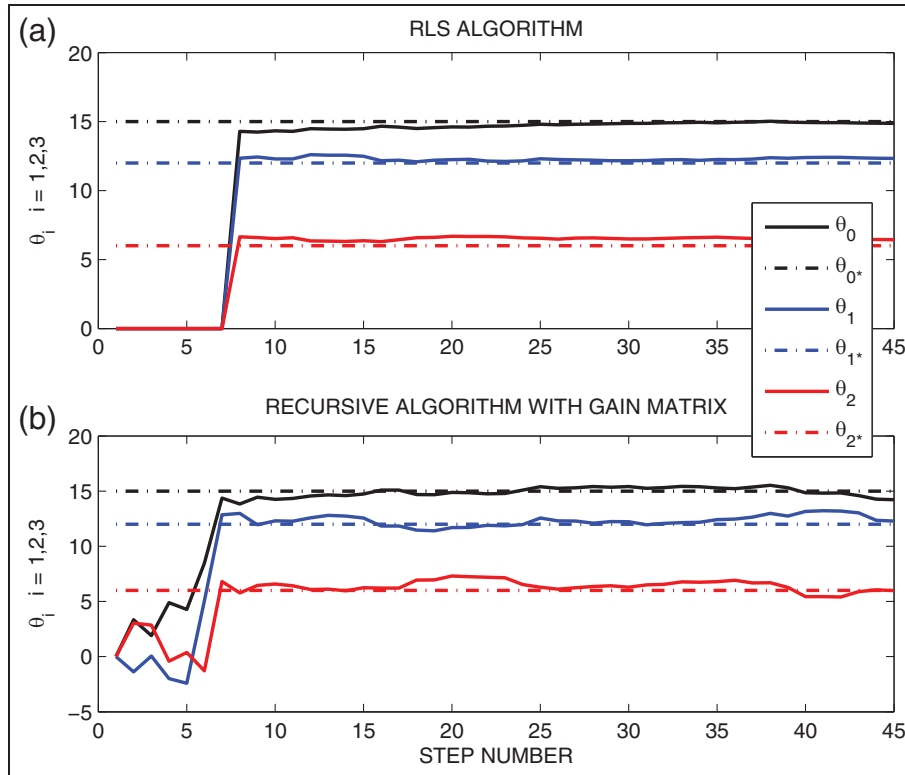
Figure 1. Comparison of (a) RLS algorithm and (b) recursive algorithm with an SDD gain matrix applied for estimation of the parameters of the noise-contaminated signal that contains three frequencies. Estimated parameters, defined in equation (5) are plotted with solid black, blue and red lines, whereas true parameters are plotted with dash-dot lines of the same colors.

associated with numerical calculation of the parameter vector θ_k defined in equation (8).

The following inequality is valid for estimation error $\tilde{\vartheta}_i$

$$\| \tilde{\vartheta}_i \| \leqslant \kappa^{2i} \| \tilde{\vartheta}_0 \| \tag{20}$$

and ϑ_i rapidly converges to θ_k.

Notice that algorithm (18) is named in this article as a second-order algorithm due to the square of the error matrix $(I - D_k^{-1} A_k)^2$ in the error model (19). This algorithm has improved converge rate compared to the first-order algorithm, described in Stotsky.[6] Notice that the second-order algorithm requires more computational efforts than a first-order algorithm.

The algorithm (18) is stopped if

$$\| A_k \vartheta_i - b_k \| < \epsilon \tag{21}$$

where ϵ is a sufficiently small positive number that controls the accuracy of estimation.

Notice that the diagonal matrix D_k^{-1} in algorithm (18) plays a role of a priori estimate of the inverse of A_k. Obviously, the convergence rate of algorithm (18) can be improved via improvement of the estimate of the inverse of A_k. Such algorithms are described in section "Combined recursive algorithms."

High-order algorithms

Algorithms of higher order can be worked out from the following error model

$$\tilde{\vartheta}_i = (I - GA_k)^n \tilde{\vartheta}_{i-1} \tag{22}$$

where $n = 1, 2, 3, \ldots$ and G is an estimate of the matrix A_k^{-1}, such that $\| I - GA_k \| \leqslant \kappa < 1$, for example, $G = D_k^{-1}$. Algorithms of the order n can be written as follows

$$\vartheta_i = \vartheta_{i-1} + \left\{ \sum_{d=0}^{n-1} \frac{n!}{d!(n-d)!}(-1)^{n-d}(GA_k)^{n-d} \right\} \vartheta_{i-1}$$
$$+ \sum_{d=0}^{n-1} \frac{n!}{d!(n-d)!}(-1)^{n-d-1}(GA_k)^{n-d-1}Gb_k \tag{23}$$

and are summarized in Table 1.

Notice that n iterations of the first-order algorithm are performed in each step in the nth-order algorithm. High-order algorithms might be useful in the estimation problems where a pre-specified accuracy should be achieved in a certain number of steps. The order of the algorithm can be chosen in this case using the error model (22).

Combined recursive algorithms

This algorithm starts with a recursive estimation of the inverse of matrix A_k with the following initialization

$$F_0 = I - G_0 A_k \tag{24}$$
$$G_0 = D_k^{-1} \tag{25}$$

Table 1. A family of high-order algorithms.

Order	High-order algorithms
1	$\vartheta_i = \vartheta_{i-1} - G(A_k\vartheta_{i-1} - b_k)$
2	$\vartheta_i = \vartheta_{i-1} - 2GA_k\vartheta_{i-1} + 2Gb_k + (GA_k)^2\vartheta_{i-1} - GA_kGb_k$
3	$\vartheta_i = \vartheta_{i-1} - (GA_k)^3\vartheta_{i-1} + 3(GA_k)^2\vartheta_{i-1} - 3GA_k\vartheta_{i-1} + (GA_k)^2Gb_k - 3GA_kGb_k + 3Gb_k$
	...
n	$\vartheta_i = \vartheta_{i-1} + \left\{ \sum_{d=0}^{n-1} \frac{n!}{d!(n-d)!}(-1)^{n-d}(GA_k)^{n-d} \right\}\vartheta_{i-1} + \sum_{d=0}^{n-1} \frac{n!}{d!(n-d)!}(-1)^{n-d-1}(GA_k)^{n-d-1}Gb_k$

$$\| F_0 \| \leqslant \kappa < 1 \tag{26}$$

where D_k is a diagonal matrix that contains diagonal elements of A_k. Matrix G_i is an estimate of the inverse of A_k, which is updated if the norm of the inversion error $I - G_{i-1}A_k$ is larger than a pre-specified number $\delta > 0$ as follows

$$\text{If } \| I - G_{i-1}A_k \| \geqslant \delta \tag{27}$$

$$G_i = G_{i-1} + F_{i-1}G_{i-1} \tag{28}$$

$$F_i = I - G_iA_k \tag{29}$$

else

$$G_i = G_{i-1}$$

The matrix G_i is used as an input to the update law for the parameter vector ϑ_i

$$\vartheta_i = \vartheta_{i-1} - 2G_iA_k\vartheta_{i-1} + 2G_ib_k +$$

$$(G_iA_k)^2\vartheta_{i-1} - G_iA_kG_ib_k \tag{30}$$

The error model for this algorithm (without condition (27)) can be written as

$$\tilde{\vartheta}_i = F_0^{2^{i+1}}\tilde{\vartheta}_{i-1} \tag{31}$$

which guarantees very fast convergence of ϑ_i to θ_k.

If algorithms (27)–(29) are executed for several steps $i = 1, \ldots, w$ for a sufficiently large δ, then the model for estimation error for $i = w + 1, w + 2, \ldots$ can be written as follows

$$\tilde{\vartheta}_i = F_0^{2^{w+1}}\tilde{\vartheta}_{i-1} \tag{32}$$

where $\| F_0^{2^{w+1}} \| \leqslant \kappa_1 < \kappa < 1$ and a positive constant κ_1 define the convergence rate of the model (32).

The algorithm is stopped, if $\| A_k\vartheta_i - b_k \| < \varepsilon$, where ε is a sufficiently small positive number. Notice that a recursive matrix inversion is a computationally expensive procedure, and therefore, the inverse of the information matrix might be calculated with low accuracy. Estimation of the inverse matrix is stopped if $\| I - G_iA_k \| < \delta$, where the norm $\| \cdot \|$ is the maximum row sum matrix norm, and the accuracy of the parameter calculation is much higher than an accuracy of the matrix inversion $\delta \ggg \varepsilon$.
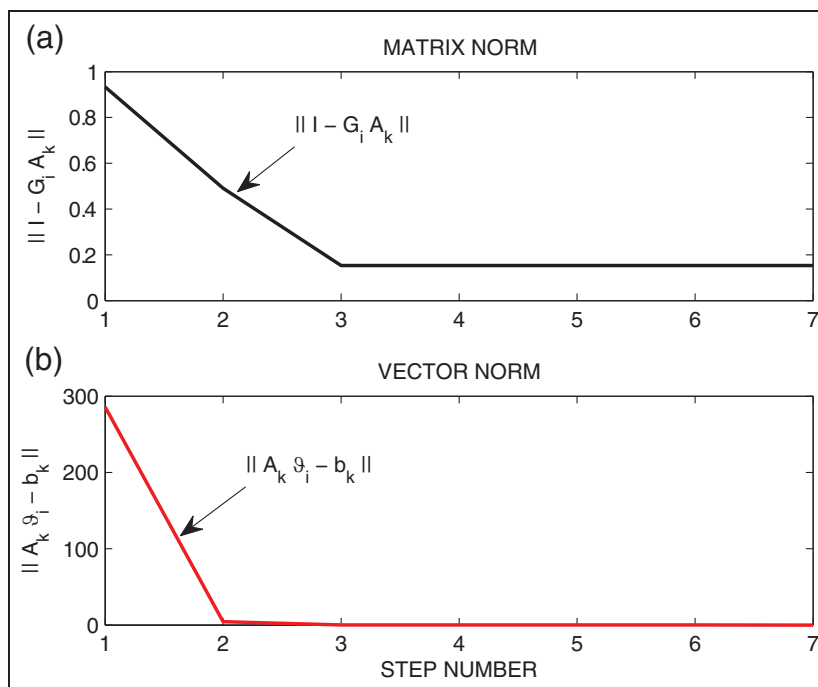
Figure 2. Time evolution of the (a) matrix norm $\| I - G_iA_k \|$ and (b) vector norm $\| A_k\vartheta_i - b_k \|$ for algorithms (24)–(30) with $\delta = 0.155$ and $\epsilon = 710^{-7}$.
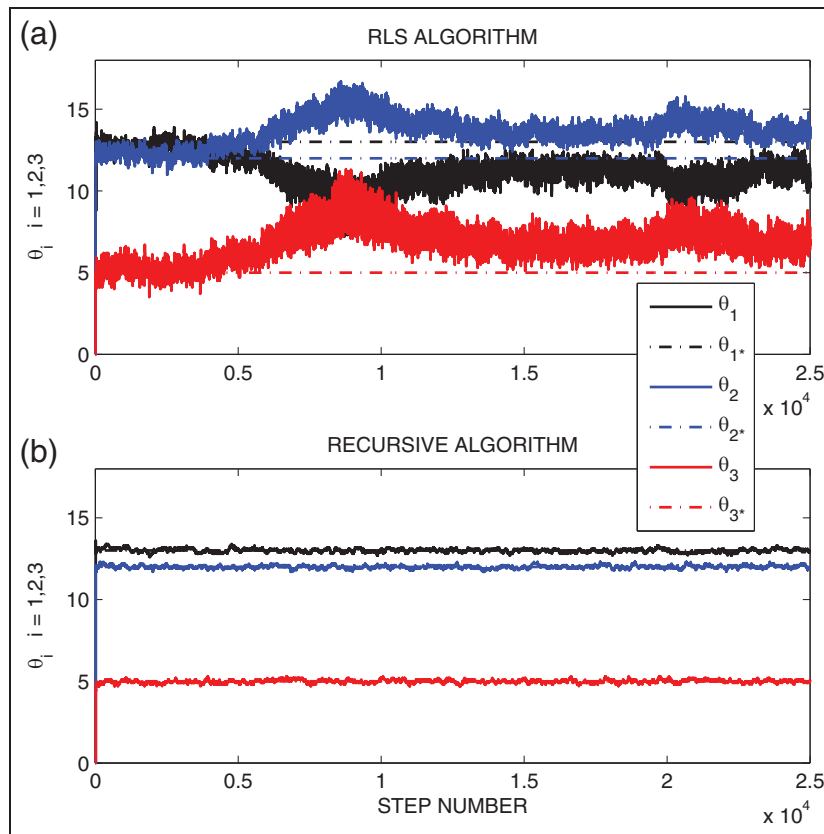
Figure 3. Comparison of (a) RLS algorithms (11) and (12) and (b) combined recursive algorithms (24)–(30) for direct calculation of the parameters of the noise-contaminated signal that contains three frequencies. Estimated parameters are plotted with solid black, blue and red lines, whereas true parameters are plotted with dash-dot lines of the same colors. All the variables in both algorithms are rounded to the first (leading) digit.

The performance of algorithms (24)–(30) is illustrated in Figure 2, where a time evolution of the norms $\| I - G_i A_k \|$ and $\| A_k \vartheta_i - b_k \|$, which represent the matrix inversion and parameter estimation errors, is presented. The matrix norm $\| I - G_i A_k \|$ is sufficiently reduced in three steps, and therefore, the matrix G_i gets sufficiently close to the inverse of information matrix A_k. Simultaneously, the vector norm $\| A_k \vartheta_i - b_k \|$ is essentially reduced in three steps. The accuracy which is required for a high-performance parameter estimation is achieved using algorithm (30) in the subsequent steps, see the second subplot of Figure 2.

Finally, a comparison of RLS algorithms (11) and (12) and combined recursive algorithms (24)–(30) for direct calculation of the parameters of the noise-contaminated signal that contains three frequencies is presented in Figure 3. The components of parameter vector (11) and the elements of gain matrix (12) of classical RLS algorithm are rounded to the first digit. All variables in combined recursive algorithms (24)–(30) are also rounded to the first digit, whereas the information matrix A_k, defined in equation (9) is calculated with higher accuracy. Rounding to the first digit is seldom met in practice, but it emphasizes the error accumulation problem in classical RLS algorithm. Figure 3 shows that the algorithms (24)–(30) provide better

estimation performance. This is maximum possible performance improvement when using the approach proposed in this article.

Other methods and conclusion

Two groups of methods might be used to solve linear algebraic equation $A_k \theta_k = b_k$: exact methods and recursive methods. Gaussian elimination method is the best known exact method for calculation of the parameter vector.[10,11] Unfortunately, this method does not give an exact solution of the algebraic equations due to round-off errors, which are always present in a finite precision implementation environment, and becomes computationally expensive for signals with a large number of frequencies.

Recursive methods for solution of linear algebraic equations, described in this article, that are based on strict diagonal dominance of the information matrix are preferable for large-scale systems as fast, robust and computationally efficient algorithms with controllable accuracy, which are suitable for real-time implementation.

This article shows that the properties of the harmonic regressor might be successfully used for

robustification of RLS algorithms. Moreover, explicit transient bound for estimation error was derived in this article using Lyapunov function method for classical RLS algorithms. It was also shown that RLS algorithms can be used for the parameter estimation without initialization of the gain matrix to the inverse of the information matrix. This in turn allows improvement of the computational efficiency and shows robustness of RLS algorithms.

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