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Comments on the stochastic characteristics of fission chamber signals

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Abstract

This report presents a theoretical investigation of the stochastic properties of the signal series of ionisation chambers, in particular fission chambers. The signals of the detector are assumed to be generated by incoming particles corresponding to an inhomogeneous Poisson distribution. Each incoming particle generates a current pulse with constant shape and random amplitude, and the detector signal consists of the time series of such current signal pulses incurring also a pile-up effect in the case of high intensity of the primary events. Exact relationships are derived for the higher order moments of the detector signal, which constitute a generalisation of the so-called higher order Campbelling techniques. The probability distribution of the number of time points when the signal exceeds a certain level is also derived. Assuming that the incoming particles form a homogeneous Poisson process, explicit expressions are given for the higher order moments of the signal and the number of level crossings in a given time interval for a few selected pulse shapes.

1. Introduction

The recent interest in fast reactor systems and in many other fields drew an increased attention to the development of neutron detectors which are particularly suitable for deployment in such systems, especially to the technical improvement of the *fission chambers*. The fission chamber is an *ionization chamber* in which the electron-ion pairs are generated by fission fragments in the gaseous volume of the detector. At low neutron intensities, the fissions generate individual current signals, i.e. pulses, that are generally separated and can be counted with a given efficiency. At high neutron intensities, a continuous current is formed between the electrodes with its mean value being represented by a *direct current* value. The *current mode* is based on the measurement of the direct current. In addition, the detector current shows particular fluctuations, which can be separated as an *alternating current* component. The *Campbell technique* utilizes the information content in the alternating current part. However, it has to be noted here and now that the use of the Campbell technique through the variance calculated from these fluctuations is not without complications. This is because, in a measurement in a reactor core, the cumulative signals are not independent due to the branching process of the neutrons in the core, and hence the temporal distribution of the number of neutrons inducing fissions in the detector is not Poissonian. Some consequences of the non-Poissonian character of time series of neutrons crossing the fission chamber in a given time interval will be discussed in an other report.

The theoretical aspects of the physical processes that take place inside the fission chambers have been studied by several authors [1], [2], [3]. By using the basic charge transport equations, they are trying to calculate both the *mean* shape of an individual signal and the mean value of the saturation current, but they do not investigate the effect of the temporal randomness of the *output signals* of the fission chamber.

In contrast, we are not concerned with the processes taking place inside the fission chamber. We will consider the fission chamber from this point of view essentially as a *black box*. Hence, the objective of the present work is the study of the stochastic properties of the *output current (voltage) signal* of the fission chambers, which are special, gas-filled ionization chambers. The starting point is the working principle of the ionization chamber, which will be referred to as *detector*: the nuclear particles passing through the fission chamber generate positive and negative charges, which give rise to a signal of *electric current* due to the electric field between the electrodes of the chamber. In order to derive expressions for the statistics of the detector signal, consisting of the sum of randomly following individual pulses, we shall make some assumptions on the statistics of the incoming particles and on the shape and the amplitude statistics of the individual pulses generated by the incoming particles.

The *practical* way of classifying the different detector operation modes can be found in Chapter 4 of Knoll's book [4]. If the incoming particle flux is low, then the detectors are used in *pulse mode*, and if the incoming flux is high, in *current mode*. In the latter case it is practical to measure the *variance* which is specific to the *fluctuations* of the current, to quantify the incoming particle flux. As it will be seen shortly, the advantage of this approach is that the contributions from the gamma radiation, which in this context count as noise, and which produce less charge in the detector per incoming particle than the neutrons, and can be reduced to a negligible level. This measurement method is usually called the *Campbell-method*. It is important to note that the Campbell-method gives correct results only if the particles arrive to the detector according to a *Poisson process*.

Actually, in principle it is not necessary to distinguish between these two operational modes, because if the probability distribution function of the sum of the random signals in the detector can be determined for a given moment $t > 0$, then its expected value can be identified with the measurement in the current mode, and its variance can be identified with the measurement in the Campbell mode. Obviously, the number of signals higher than a given level in a given time interval can be associated with measurements in the pulse mode.

Numerous articles can be found in the literature on the theory of fission chambers, and on the derivation of the so-called higher order Campbelling techiques, which contain unnecessarily complicated and often incorrect calculations, and which will not be listed here. In this report a simple *detector model* will be used, which is amenable to concise analytical derivations to obtain exact results. These will, among others, reproduce the results published by Lux and Baranyai in 1982 [5], [6] and by Bärs in 1989 [7] in a simple and transparent way.

Before turning to the model and the derivations from first principles, it might be interesting to recall the empirical methodology used in Knoll's book [4] to obtain expressions for the current and the Campbell mode. Assume that ν detections took place during the measurement time T, each creating a charge Q, then the current η can be estimated as

$$\eta = \frac{\nu}{T}\, Q. \tag{1.1}$$

The moments of the current can be estimated from that of the number of detections. Assuming that ν is a random variable which follows a Poisson distribution (such that $\mathbf{E}\{\nu\} = s_0\, T$), the expectation of the current is given as

$$\mathbf{E}\{\eta\} = i = s_0\, Q, \tag{1.2}$$

where s_0 is the *intensity* of the detection events. Thus, the mean value of the current is proportional to the incoming particle flux ψ, since this latter is linearly proportional to the intensity s_0 of the detection events. That is, one has $s_0 = r\,\psi$, where r is a constant proportionality factor, taking into account the physical properties of the detector.

Since, due to the Poisson distribution one has $\mathbf{D}^2\{\nu\} = s_0\, T$, the variance of the current η is equal to

$$\mathbf{D}^2\{\eta\} = \frac{s_0}{T}\, Q^2. \tag{1.3}$$

The *relative standard deviation* of η is given as:

$$\frac{\mathbf{D}\{\eta\}}{\mathbf{E}\{\eta\}} = \frac{1}{\sqrt{s_0\, T}}. \tag{1.4}$$

This formula indicates that the relative standard deviation of the current is inversely proportional to the square root of the measurement time T. Hence a long measurements time (often called sampling time) reduces the relative standard deviation as long as the system is stationary. However, in practice a long measurement period has the disadvantage that the expectation of the current may change in time, which will go unnoticed, and hence will have undesirable technical consequences.

Eq. (1.3) gives an expression for the variance which is necessary for a measurement in *Campbell-mode*. It describes a proportionality between the variance of the detector current and the intensity s_0 of the incoming radiation and, what is more important, a proportionality between the variance and the *square of the charge*. In this simple calculation the fluctuation of the charge Q, generated in the individual detection events, was neglected, which is acceptable in the present case. The variance in (1.3) reflects only the temporal fluctuations of the number of the individual detection events. The fact that the variance is proportional to the square of the charge created by the detected events indicates the suppression of the intensity of the particles creating less charge per event in the case of several types of radiation being present simultaneously. For example, if the detector is a *fission chamber*, then in the presence of gamma background radiation the neutron intensity can be measured more precisely in Campbell-mode than in current mode, since in the former, the detection of the gamma events will be suppressed. However, it is important to emphasize that these simple calculations are valid only if the number of particles arriving to the detector follow a *Poisson distribution*.

It is worth to note that the *Campbell theorem* [8], which is often referred to in the literature, is given by the following expressions for the expected value and variance of the random process $\eta(t)$ which consists of random sum of deterministic signals $f(t)$ created according a homogenous Poisson process with intensity s_0:

$$\mathbf{E}\{\eta)\} = s_0 \int_{-\infty}^{+\infty} f(t)\, dt \qquad \text{and} \qquad \mathbf{D}^2\{\eta\} = s_0 \int_{-\infty}^{+\infty} f(t)^2\, dt. \tag{1.5}$$

Expressions (1.5) are valid if the integrals exist. In the simple case descried above, the function $f(t)$ is equal to a window of the same length as the measurement period T, i.e.

$$f(t) = \frac{Q}{T}\, \Delta(t)\, \Delta(T-t), \tag{1.6}$$

Using this in (1.5) leads to

$$\mathbf{E}\{\eta\} = s_0 \int_{-\infty}^{+\infty} f(t)\, dt = s_0\, \frac{Q}{T} \int_0^T dt = s_0\, Q,$$

and

$$\mathbf{D}^2\{\eta\} = s_0 \int_{-\infty}^{+\infty} f(t)^2\, dt = s_0\, \frac{Q^2}{T^2} \int_0^T dt = \frac{s_0}{T}\, Q^2.$$

which, obviously, are identical with (1.2) and (1.3).

The *current signal* (or *voltage signal*) generated by the particles arriving to the detector can be considered as the *response function of the detector*. In reality, this response function cannot be given by a deterministic function $f(t)$; rather, it can only be described by a function $\varphi(\boldsymbol{\xi}, t)$ which depends on the possible realisations of a random variable $\boldsymbol{\xi}$. The continuously arriving particles generate the detector current as the aggregate of such response function current signals, each related to a different realisation of $\boldsymbol{\xi}$. In the general case, the resulting random detector signal is complicated to calculate and it is hardly possible to handle it analytically. For the sake of simplicity, in this report we will study a class of random response functions, in which the dependence of $\varphi(\boldsymbol{\xi}, t)$ on the random variable $\boldsymbol{\xi}$ and time t is factorised into a form $\varphi(\boldsymbol{\xi}, t) = a(\boldsymbol{\xi})\, f(t)$ where a is the random amplitude of the pulse and $f(t)$ is the pulse shape. Although this assumption restricts somewhat the generality of the description, it will lead to a formalism which, for several basic signal shapes $f(t)$ is amenable to an analytical treatment, while still representing a realistic model of the detector signal.

The objective of this work is thus to determine the probability distribution function of the sum of random response signals of randomly appearing particles in a simple detector model. We assume that the number of incoming particles within a given time period follows an inhomogeneous Poisson distribution, and that the detector counts all arriving particles. We do not deal with the charge generating processes in the ionization chamber. For the simplicity, the interaction between the charges generated by consecutive particles will also be neglected. This means that the random response signals related to different particles are considered to be independent and identically distributed. The question of correlated detection events, induced by incoming neutrons generated in a branching process, will be treated in a forthcoming publication.

2. General theory

As indicated in the Introduction, the detector response signal (current or voltage), induced by the particle detected is assumed to depend on time t and one single random variable. This random variable will be indicated with $\boldsymbol{\xi}$, and let t denote the time from the beginning of the response signal. If x is a realization of the random variable $\boldsymbol{\xi}$, then the time dependence of the response signal is given by the real function $\varphi(x,t)$. For the most basic signal shapes to be considered in this work, i.e those that are constant or monotonically decreasing for $t > 0$ (square, exponential, triangle) $\boldsymbol{\xi}$ will be the (random) initial value of the response signal. For other, non-monotonically varying signal shape it can be identified with a given parameter of the signal pulse. We assume that $\boldsymbol{\xi} \in \Re$, where $\Re$ is the set of real numbers, and it has a finite expected value and variance. Let

$$W(x) = \int_{-\infty}^{x} w(x')\, dx' = \mathcal{P}\left\{\boldsymbol{\xi} \le x\right\} \tag{2.1}$$

denote the distribution function of the random variable $\boldsymbol{\xi}$. The probability that the value of the response at time t after the arrival of the particle is not greater than y is given by the degenerate distribution function

$$H(y,t) = \int_{-\infty}^{+\infty} \Delta[y - \varphi(x,t)]\, w(x)\, dx. \tag{2.2}$$

The probability density function $h(y,t)$ of $H(y,t)$ is given as

$$h(y,t) = \int_{-\infty}^{+\infty} \delta\left[y - \varphi(x,t)\right]\, w(x)\, dx. \tag{2.3}$$

For the sake of the generality of the treatment we assume that the sequence of particle arrivals constitutes an *inhomogeneous Poisson process*. In this case the probability that no particle arrives at the detector during the time interval $[t_0, t],\ t \ge t_0$ is given by

$$T(t_0,t) = \exp\left\{-\int_{t_0}^{t} s(t')\, dt'\right\}. \tag{2.4}$$

Here $s(t')$ is the intensity of the particle arrivals at time t'. Denoting the sum of the signals at $t \ge t_0$ as $\eta(t)$, we shall seek the probability of the event that $\eta(t)$ is less than or equal to y with the condition that its value was zero at t_0, i.e. the quantity

$$\mathcal{P}\left\{\eta(t) \le y | \eta(t_0) = 0\right\} = P(y,t|0,t_0) = \int_{-\infty}^{y} p(y',t|0,t_0)\, dy'. \tag{2.5}$$

Straightforward considerations yield the following backward-type integral Chapman-Kolmogorov-equation for the probability density function $p(y,t|0,t_0)$:

$$p(y,t|0,t_0) = T(t_0,t)\,\delta(y) + \int_{t_0}^{t} T(t_0,t')s(t') \int_{-\infty}^{y} h(y',t-t')\, p(y-y',t|0,t')\, dy'\, dt'. \tag{2.6}$$

The r.h.s. of Eq. (2.6) consists of the sum of the probabilities of the mutually exclusive events that there will not, or there will be a first detection sometime between t_0 and t, whereas the product of the probabilities under the last integral sign is due to the fact that the contribution from the first detection at time t' to the total detector signal resulting from all detection events at the terminal time t is independent from that of the subsequent detections.

Introduce now the characteristic functions

$$\pi(\omega, t|0, t_0) = \int_{-\infty}^{+\infty} e^{\imath \omega y}\, p(y, t|0, t_0)\; dy, \tag{2.7}$$

$$\rho(\omega) = \int_{-\infty}^{+\infty} e^{\imath \omega x}\, w(x)\; dx, \tag{2.8}$$

and

$$\chi(\omega, t) = \int_{-\infty}^{+\infty} e^{\imath \omega y}\, h(y, t)\; dy =$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\imath \omega y}\, \delta\left[y - \varphi(x, t)\right]\; dy\, w(x)\; dx = \int_{-\infty}^{+\infty} e^{\imath \omega \varphi(x,t)}\, w(x)\; dx. \tag{2.9}$$

Then, from Eq. (2.6) one obtains

$$\pi(\omega, t|0, t_0) = T(t_0, t) + \int_{t_0}^{t} T(t_0, t') s(t')\, \chi(\omega, t - t')\, \pi(\omega, t|0, t')\; dt'. \tag{2.10}$$

From this integral equation it is seen that

$$\lim_{t \downarrow t_0} \pi(\omega, t|0, t_0) = 1. \tag{2.11}$$

Further, by derivation w.r.t. t_0, one obtains the differential equation

$$\frac{\partial \pi(\omega, t|0, t_0)}{\partial t_0} = s(t_0)\, \pi(\omega, t|0, t_0)\left[1 - \chi(\omega, t - t_0)\right] \tag{2.12}$$

By accounting for the initial condition (2.11), the solution of (2.12) is obtained as

$$\pi(\omega, t|0, t_0) = \exp\left\{-\int_{t_0}^{t} s(t')\left[1 - \chi(\omega, t - t')\right]\; dt'\right\}, \tag{2.13}$$

or, equivalently,

$$\pi(\omega, t|0, t_0) = \exp\left\{-\int_{0}^{t - t_0} s(t - t')\left[1 - \chi(\omega, t')\right]\; dt'\right\} \tag{2.14}$$

where $\chi(\omega, t')$ is defined in (2.10) as

$$\chi(\omega, t') = \int_{-\infty}^{+\infty} \exp\left\{\imath \omega\, \varphi(x, t')\right\}\, w(x)\; dx. \tag{2.15}$$

Eq. (2.14) can be considered to be the characteristic function of the generalized inhomogeneous Poisson-process.

Assume that the time instants of the particle arrivals correspond to a homogeneous Poisson-process with constant intensity s_0, and further that the condition

$$s_0 \left| \int_{-\infty}^{+\infty} [1 - \chi(\omega, t)] \; dt \right| < +\infty$$

is fulfilled. In this case the characteristic function $\pi(\omega, +\infty|0, -\infty)$ exists and is given by

$$\pi(\omega, +\infty|0, -\infty) = \pi_{st}(\omega) = \exp\left\{ -s_0 \int_{-\infty}^{+\infty} [1 - \chi(\omega, t)] \; dt \right\}. \tag{2.16}$$

This also means that there exists an *asymptotically stationary signal level* $\eta^{(st)}$, with the probability density function

$$\pi_{st}(y) = \mathcal{L}^{-1}\left\{\pi_{st}(\omega)\right\}. \tag{2.17}$$

For the simplification of the subsequent calculations it is practical to introduce the logarithm $\gamma_{st}(\omega)$ of the characteristic function (2.16), i.e.

$$\gamma_{st}(\omega) = \ln \pi_{st}(\omega) = s_0 \int_{-\infty}^{+\infty} [\chi(\omega, t) - 1] \; dt. \tag{2.18}$$

2.1. Expected value, variance and cumulants

From the formula (2.18) one can easily calculate the expected value of the stationary signal $\eta^{(st)}$

$$\mathbf{E}\{\eta^{(st)}\} = i_1^{(st)} = \frac{1}{\imath}\left[\frac{d\gamma_{st}(\omega)}{d\omega}\right]_{\omega=0} = s_0 \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \varphi(x,t)\, w(x)\; dx\right] dt, \tag{2.19}$$

and its variance as

$$\mathbf{D}^2\left\{\eta^{(st)}\right\} = \sigma_{st}^2 = -\left[\frac{d^2\gamma_{st}(\omega)}{d\omega^2}\right]_{\omega=0} = s_0 \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \varphi(x,t)^2\, w(x)\; dx\right] dt. \tag{2.20}$$

If the value of the signal amplitude $\boldsymbol{\xi}$ is always unity, that is $w(x) = \delta(x-1)$, then (2.19) and (2.20) revert to the formulae of the Campbell's theorem mentioned in the foregoing in the form

$$\mathbf{E}\{\eta^{(st)}\} = s_0 \int_{-\infty}^{+\infty} f(t)\; dt \tag{2.21}$$

and

$$\mathbf{D}^2\left\{\eta^{(st)}\right\} = s_0 \int_{-\infty}^{+\infty} f(t)^2\; dt \tag{2.22}$$

with $f(t) = \varphi(1,t)$. After proper calibration, both of these forms are suitable to determine the particle intensity s_0.

As is known [9], through the formula

$$\kappa_n^{(st)} = \frac{1}{\imath^n}\left[\frac{d^n\gamma_{st}(\omega)}{d\omega^n}\right]_{\omega=0} = s_0 \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \varphi(x,t)^n\, w(x)\; dx\right] dt$$

from the logarithmic characteristic function $\gamma_{st}(\omega)$ one can determine the *cumulants* or *semiinvariants* $\kappa_n^{(st)}$ of $\eta^{(st)}$, which can be expressed by the moments of $\eta^{(st)}$). This way we immediately arrive at the results referred to in the literature as *higher order Campbell techniques* [5], [6]. It is readily seen that all cumulants are linearly proportional to the intensity s_0. A few cumulants are given below for illustration.

$$\begin{aligned}
\kappa_1^{(st)} &= i_1^{(st)}, \\
\kappa_2^{(st)} &= i_2^{(st)} - \left(i_1^{(st)}\right)^2, \\
\kappa_3^{(st)} &= i_3^{(st)} - 3\, i_2^{(st)}\, i_1^{(st)} + \left(i_1^{(st)}\right)^3, \\
\kappa_4^{(st)} &= i_4^{(st)} - 4\, i_3^{(st)}\, i_1^{(st)} - 3\, \left(i_2^{(st)}\right)^2 + 12\, i_2^{(st)}\, \left(i_1^{(st)}\right)^2 - 6\, \left(i_1^{(st)}\right)^4,
\end{aligned}$$

where

$$i_n^{(st)} = \mathbf{E}\{\left(\eta^{(st)}\right)^n\}, \qquad n = 1, 2, \ldots \quad .$$

2.2. Autocorrelation function

In many cases it is desirable to know the autocorrelation function of the detector signal. For the determination of the autocorrelation function one needs the probability

$$\mathcal{P}\left\{\eta(t_1) \leq y_1, \eta(t_2) \leq y_2 | \eta(0) = 0\right\} = P_2(y_1, y_2, t_1, t_2) \tag{2.23}$$

that the value of the detector signal $\eta(t)$ is not larger than y_1 at t_1 and it is not larger than y_2 at t_2, on the condition that its value was zero at $t = 0$, for non-negative time intervals

$$t_2 - t_1 = \theta.$$

Further, it is assumed that the distribution function $P_2(y_1, y_2, t_1, t_2)$ is absolute continuous, hence one can write

$$P_2(y_1, y_2, t_1, t_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} p_2(y_1', y_2', t_1, t_2)\, dy_1'\, dy_2'. \tag{2.24}$$

As before, denote the distribution function of the random variable $\boldsymbol{\xi}$ as

$$\mathcal{P}\left\{\boldsymbol{\xi} \leq x\right\} = W(x) = \int_{-\infty}^{x} w(x')\, dx' \tag{2.25}$$

and let $s_0\, \Delta t + o(\Delta t)$ be the probability that an impulse occurs in the time interval $(t, t + \Delta t)$. With the standard methods of the derivation of the backward equation one obtains

$$p_2(y_1, y_2, t_1, t_2) = e^{-s_0 t_2}\, \delta(y_1)\, \delta(y_2) +$$

$$s_0 \int_0^{t_2} e^{-s_0 t'} \left[\Delta(t_1 - t')\, U_2(y_1, y_2, t_1 - t', t_2 - t') + \Delta(t' - t_1)\, U_1(y_2, t_2 - t')\right] dt', \tag{2.26}$$

where

$$U_2(y_1, y_2, t_1 - t', t_2 - t') =$$

$$\int_{-\infty}^{y_1}\int_{-\infty}^{y_2} h_2(y_1',y_2',t_1-t',t_2-t')\,p_2(y_1-y_1',y_2-y_2',t_1-t',t_2-t')\;dy_1'\;dy_2' \tag{2.27}$$

and

$$U_1(y_2,t_2-t') = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} \delta(y_1')\,h_1(y_2',t_2-t')\,p_1(y_2-y_2',t_2-t')\;dy_1'\;dy_2'. \tag{2.28}$$

It is easily seen that the relationships

$$h_2(y_1',y_2',t_1-t',t_2-t') = \int_{-\infty}^{+\infty} \delta\left[y_1'-\varphi(x,t_1-t')\right]\,\delta\left[y_2'-\varphi(x,t_2-t')\right]\,w(x)\;dx \tag{2.29}$$

and

$$h_1(y_2',t_2-t') = \int_{-\infty}^{+\infty} \delta\left[y_2'-\varphi(x,t_2-t')\right]\,w(x)\;dx. \tag{2.30}$$

hold. For completeness, write also down the relationship

$$p_1(y,t) = e^{-s_0\,t}\,\delta(y) + s_0\int_0^t e^{-s_0\,(t-t')}\left[\int_{-\infty}^{y} h(y',t')\,p_1(y-y',t')\;dy'\right]\,dt', \tag{2.31}$$

where

$$h(y',t')\;dy' = \int_{-\infty}^{+\infty} \delta\left[y'-\varphi(x,t')\right]\,w(x)\;dx\;dy'.$$

Introducing the characteristic functions

$$\pi_2(\omega_1,\omega_2,t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{\imath\,\omega_1 y_1+\imath\,\omega_2 y_2\}p_2(y_1,y_2,t_1,t_2)\;dy_1\;dy_2, \tag{2.32}$$

$$\chi_2(\omega_1,\omega_2,t_1-t',t_2-t') =$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{\imath\,\omega_1 y_1+\imath\,\omega_2 y_2\}h_2(y_1,y_2,t_1-t',t_2-t')\;dy_1\;dy_2 =$$
$$\int_{-\infty}^{+\infty} \exp\left\{\imath\,\omega_1\,\varphi(x,t_1-t')+\imath\,\omega_2\,\varphi(x,t_2-t')\right\}\,w(x)\;dx, \tag{2.33}$$

as well as

$$\pi_1(\omega_2,t_2-t') = \int_{-\infty}^{+\infty} \exp\{\imath\,\omega_2 y_2\}p_1(y_2,t_2-t')\;dy_2 \tag{2.34}$$

and

$$\chi_1(\omega_2,t_2-t') =$$
$$\int_{-\infty}^{+\infty} \exp\{\imath,\omega_2 y_2\}\,h_1(y_2,t_2-t')\;dy_2 = \int_{-\infty}^{+\infty} \exp\left\{\imath\,\omega_2\,\varphi(x,t_2-t')\right\}\,w(x)\;dx, \tag{2.35}$$

from (2.26) one obtains the equation

$$\pi_2(\omega_1,\omega_2,t_1,t_2) = e^{-s_0 t_2} +$$

$$s_0 \int_0^{t_2} e^{-s_0 t'} \; [\Delta(t_1 - t')\,\chi_2(\omega_1, \omega_2, t_1 - t', t_2 - t')\,\pi_2(\omega_1, \omega_2, t_1 - t', t_2 - t') +$$
$$\Delta(t' - t_1)\,\chi_1(\omega_2, t_2 - t')\,\pi_1(\omega_2, t_2 - t')] \; dt'$$

Introducing the variables

$$t = t_2, \qquad t - \theta = t_1, \qquad \text{where} \qquad \theta > 0, \qquad \text{and} \qquad t' = t - v$$

one obtains the equation

$$\pi_2(\omega_1, \omega_2, t - \theta, t) = e^{-s_0 t} +$$
$$s_0 \int_0^{t} e^{-s_0 (t - v)} \; [\Delta(v - \theta)\,\chi_2(\omega_1, \omega_2, v - \theta, v)\,\pi_2(\omega_1, \omega_2, v - \theta, v) +$$
$$\Delta(\theta - v)\,\chi_1(\omega_2, v)\,\pi_1(\omega_2, v)] \; dv. \tag{2.36}$$

In view of the relationship

$$\pi_2(\omega_1, \omega_2, t - \theta, t) = \pi_1(\omega_2, t), \qquad \text{if} \qquad 0 < t \leq \theta,$$

it is seen that Eq. (2.36) corresponds to two integro-differential equations. Namely, if $t > \theta$, one has

$$\frac{\partial \pi_2(\omega_1, \omega_2, t - \theta, t)}{\partial t} = -s_0\,\pi_2(\omega_1, \omega_2, t - \theta, t) + s_0\,\chi_2(\omega_1, \omega_2, t - \theta, t)\,\pi_2(\omega_1, \omega_2, t - \theta, t), \tag{2.37}$$

whereas for $t \leq \theta$

$$\frac{\partial \pi_1(\omega_2, t)}{\partial t} = -s_0\,\pi_1(\omega_2, t) + s_0\,\chi_1(\omega_2, t)\,\pi_1(\omega_2, t). \tag{2.38}$$

The solution represented by (2.37) and (2.38) can be compactly written in the form of one single equation as

$$\pi_2(\omega_1, \omega_2, t - \theta, t) =$$
$$\exp\left\{-s_0 \int_0^{t} \{\Delta(t' - \theta)\,[1 - \chi_2(\omega_1, \omega_2, t' - \theta, t')] + \Delta(\theta - t')\,[1 - \chi_1(\omega_2, t')]\} \; dt'\right\}, \tag{2.39}$$

where

$$\chi_2(\omega_1, \omega_2, t' - \theta, t') = \int_{-\infty}^{+\infty} \exp\{\imath\,\omega_1\,\varphi(x, t' - \theta) + \imath\,\omega_2\,\varphi(x, t')\} \; w(x) \; dx \tag{2.40}$$

and

$$\chi_1(\omega_2, t') = \int_{-\infty}^{+\infty} \exp\{\imath\,\omega_2\,\varphi(x, t')\} \; w(x) \; dx. \tag{2.41}$$

For our purposes, the density function of the *asymptotically stationary distribution* is relevant:

$$\lim_{t \to \infty} p_2(y_1, y_2, t - \theta, t) = p_2^{(st)}(y_1, y_2, \theta). \tag{2.42}$$

If the characteristic function of (2.42) exists then it is given by

$$\lim_{t \to \infty} \pi_2(\omega_1, \omega_2, t - \theta, t) = \pi_2^{(st)}(\omega_1, \omega_2, \theta) =$$

$$\exp\left\{-s_0\int_\theta^\infty\left[1-\chi_2(\omega_1,\omega_2,t'-\theta,t')\right]\,dt'-s_0\int_0^\theta\left[1-\chi_1(\omega_2,t')\right]\,dt'\right\}. \tag{2.43}$$

The *covariance function* and the cumulants can be determined from the logarithm of the characteristic function

$$\Phi_2^{(st)}(\omega_1,\omega_2,\theta)=\ln\pi_2^{(st)}(\omega_1,\omega_2,\theta)=$$

$$s_0\int_\theta^\infty\left[\chi_2(\omega_1,\omega_2,t'-\theta,t')-1\right]\,dt'+s_0\int_0^\theta\left[\chi_1(\omega_2,t')-1\right]\,dt'. \tag{2.44}$$

Let $\eta_t^{(st)}$ denote the stationary expectation at an arbitrary time instant t (note that notation of time is the same for both the non-stationary and stationary cases).

The following expressions can be written down for the expected values:

$$\mathbf{E}\left\{\eta_{t-\theta}^{(st)}\right\}=\frac{1}{\imath}\left[\frac{\partial\Phi_2^{(st)}(\omega_1,\omega_2,\theta)}{\partial\omega_1}\right]_{\omega_1=\omega_2=0}=s_0\int_\theta^\infty\int_{-\infty}^{+\infty}\varphi(x,t'-\theta)\,w(x)\,dx\,dt'=$$

$$s_0\int_0^\infty\int_{-\infty}^{+\infty}\varphi(x,t)\,w(x)\,dx\,dt, \tag{2.45}$$

and

$$\mathbf{E}\left\{\eta_t^{(st)}\right\}=\frac{1}{\imath}\left[\frac{\partial\Phi_2^{(st)}(\omega_1,\omega_2,\theta)}{\partial\omega_2}\right]_{\omega_1=\omega_2=0}=$$

$$s_0\int_\theta^\infty\int_{-\infty}^{+\infty}\varphi(x,t')\,w(x)\,dx\,dt'+s_0\int_0^\theta\int_{-\infty}^{+\infty}\varphi(x,t')\,w(x)\,dx\,dt'=$$

$$s_0\int_0^\infty\int_{-\infty}^{+\infty}\varphi(x,t)\,w(x)\,dx\,dt. \tag{2.46}$$

The result is trivial since the expectation of a stationary process is constant, independent of time. The same holds for the variance, which is given by

$$\mathbf{D}^2\left\{\eta_{t-\theta}^{(st)}\right\}=-\left[\frac{\partial^2\Phi_2^{(st)}(\omega_1,\omega_2,\theta)}{\partial\omega_1^2}\right]_{\omega_1=\omega_2=0}=$$

$$s_0\int_\theta^\infty\int_{-\infty}^{+\infty}\left[\varphi(x,t'-\theta)\right]^2\,w(x)\,dx\,dt'=s_0\int_0^\infty\int_{-\infty}^{+\infty}\left[\varphi(x,t)\right]^2\,w(x)\,dx\,dt, \tag{2.47}$$

and

$$\mathbf{D}^2\left\{\eta_t^{(st)}\right\}=-\left[\frac{\partial^2\Phi_2^{(st)}(\omega_1,\omega_2,\theta)}{\partial\omega_2^2}\right]_{\omega_1=\omega_2=0}=$$

$$s_0\int_\theta^\infty\int_{-\infty}^{+\infty}\left[\varphi(x,t')\right]^2\,w(x)\,dx\,dt'+s_0\int_0^\theta\int_{-\infty}^{+\infty}\left[\varphi(x,t')\right]^2\,w(x)\,dx\,dt'=$$

$$s_0\int_0^\infty\int_{-\infty}^{+\infty}\left[\varphi(x,t)\right]^2\,w(x)\,dx\,dt. \tag{2.48}$$

For the covariance function $\mathbf{Cov}\left\{\eta_{t-\theta}^{(st)},\, \eta_t^{(st)}\right\}$, one can immediately write

$$\mathbf{Cov}\left\{\eta_{t-\theta}^{(st)},\, \eta_t^{(st)}\right\} = -\left[\frac{\partial^2 \Phi_2^{(st)}(\omega_1, \omega_2, \theta)}{\partial\omega_1\, \partial\omega_2}\right]_{\omega_1=\omega_2=0} =$$

$$s_0 \int_\theta^\infty \int_{-\infty}^{+\infty} \varphi(x, t'-\theta)\, \varphi(x, t')\, w(x)\; dx\; dt' =$$

$$s_0 \int_0^\infty \int_{-\infty}^{+\infty} \varphi(x, t)\, \varphi(x, t+\theta)\, w(x)\; dx\; dt, \tag{2.49}$$

From this it follows that the correlation function is given as

$$\mathbf{Corr}\left\{\eta_{t-\theta}^{(st)},\, \eta_t^{(st)}\right\} = \mathbf{Corr}\left\{\eta_t^{(st)},\, \eta_{t+\theta}^{(st)}\right\} = R_{st}(\theta) =$$

$$\frac{\displaystyle\int_0^\infty \int_{-\infty}^{+\infty} \varphi(x, t)\, \varphi(x, t+\theta)\, w(x)\; dx\; dt}{\displaystyle\int_0^\infty \int_{-\infty}^{+\infty} \left[\varphi(x, t)\right]^2\; w(x)\; dx\; dt}. \tag{2.50}$$

It is worth mentioning that if $\varphi(x, t) = x\, f(t)$ then one has

$$R_{st}(\theta) = \frac{\displaystyle\int_0^\infty f(t)\, f(t+\theta)\; dt}{\displaystyle\int_0^\infty \left[f(t)\right]^2\; dt}. \tag{2.51}$$

Hence the autocorrelation function of the stationary signal is independent of the distribution function of the random variable $\boldsymbol{\xi}$. The properties of the correlation are solely determined by the deterministic function $f(x)$. As an illustration, one can note that if $f(t) = \exp\{-\alpha t\}$, one has

$$R_{st}(\theta) = e^{-\alpha|\theta|}.$$

Remark It can be easily shown that Eq. (2.44) is not necessary for the derivation of the autocorrelation function $R_{st}(\theta)$. To this end, define the random function

$$\zeta_t^{(st)} = \eta_t^{(st)} + \eta_{t-\theta}^{(st)},$$

and the deterministic function corresponding to the realisation of the random variable $\boldsymbol{\xi} = x$

$$g(x, t) = \varphi(x, t) + \varphi(x, t-\theta)$$

Taking into account the relationship $\varphi(x, t) = 0$ if $t < 0$, then, first, according to (2.20) one can write

$$\mathbf{D}^2\left\{\zeta_t^{(st)}\right\} = s_0 \int_0^\infty \int_{-\infty}^{+\infty} \left[g(x, t)\right]^2\; w(x)\; dx\; dt =$$

$$2\,s_0\int_0^\infty\int_{-\infty}^{+\infty}[\varphi(x,t)]^2\;w(x)\;dx\;dt+2\,s_0\int_\theta^\infty\int_{-\infty}^{+\infty}\varphi(x,t)\,\varphi(x,t-\theta)\,w(x)\;dx\;dt;\quad(2.52)$$

further, it is seen that

$$\mathbf{D}^2\left\{\zeta_t^{(st)}\right\}=\mathbf{D}^2\left\{\eta_t^{(st)}+\eta_{t-\theta}^{(st)}\right\}=2\,s_0\;[1+R_{st}(\theta)]\int_0^\infty\int_{-\infty}^{+\infty}[\varphi(x,t)]^2\;w(x)\;dx\;dt.\quad(2.53)$$

Thus one obtains

$$R_{st}(\theta)=\frac{\displaystyle\int_\theta^\infty\int_{-\infty}^{+\infty}\varphi(x,t)\,\varphi(x,t-\theta)\,w(x)\;dx\;dt}{\displaystyle\int_0^\infty\int_{-\infty}^{+\infty}[\varphi(x,t)]^2\;w(x)\;dx\;dt}=$$

$$\frac{\displaystyle\int_0^\infty\int_{-\infty}^{+\infty}\varphi(x,t+\theta)\,\varphi(x,t)\,w(x)\;dx\;dt}{\displaystyle\int_0^\infty\int_{-\infty}^{+\infty}[\varphi(x,t)]^2\;w(x)\;dx\;dt},\quad(2.54)$$

which is the same as (2.50). For mixed moments higher than second order, use of (2.44) is necessary.

2.3. Non-negative detector signals

In the forthcoming we will only deal with processes of the form $\varphi(x,t)=x\,f(t)$ where $f(t)$ is a deterministic signal function. We will also assume that the realizations x of the random variable $\boldsymbol{\xi}$, as well as the signal function $f(t)$ take only non-negative real values. With the signal forms assumed in the concrete work, with one exception, this means that the arrival of a particle to the detector incurs a jump of the signal level with the value x. In this case, i.e. when the detector signals are non-negative, it is practical to consider the *Laplace-transform* of the density function $p(y,t|0,t_0)$, defined in (2.5) as

$$\widetilde{p}(s,t|0,t_0)=\int_0^\infty e^{-sy}\,p(y,t|0,t_0)\;dy\quad(2.55)$$

as the characteristic function. Introducing the transforms

$$\widetilde{h}(s,t)=\int_0^\infty e^{-sy}\,h(y,t)\;dy\qquad\text{and}\qquad\widetilde{w}(s)=\int_0^\infty e^{-sx}\;w(x)\;dx,\quad(2.56)$$

from (2.14) we obtain

$$\widetilde{p}(s,t|0,t_0)=\exp\left\{-\int_0^{t-t_0}s(t-t')\left[1-\widetilde{h}(s,t')\right]\;dt'\right\},\quad(2.57)$$

where $\widetilde{h}(s,t')$ is defined as

$$\widetilde{h}(s,t')=\int_0^\infty\exp\left\{-s\,f(t')\,x\right\}\;w(x)\;dx=\widetilde{w}[sf(t')].\quad(2.58)$$

To simplify the further considerations let us choose

$$s(t-t') = s_0 \qquad \text{and} \qquad t_0 = 0$$

and use the notation

$$\widetilde{p}(s,t|0,0) = \widetilde{p}(s,t). \tag{2.59}$$

From equation (2.57) one immediately obtains

$$\widetilde{p}(s,t) = \exp\left\{-s_0 \int_0^t \left[1-\widetilde{h}(s,t')\right]\, dt'\right\} = \exp\left\{-s_0 \int_0^t \{1-\widetilde{w}[sf(t')]\}\, dt'\right\}. \tag{2.60}$$

For the case when

$$\lim_{t\to\infty} \int_0^t \left[1-\widetilde{h}(s,t')\right]\, dt' < \infty,$$

the Laplace-transform

$$\lim_{t\to\infty} \widetilde{p}(s,t) = \widetilde{p}_{st}(s) =$$

$$\exp\left\{-s_0 \int_0^\infty \left[1-\widetilde{h}(s,t)\right]\, dt\right\} = \exp\left\{-s_0 \int_0^\infty \{1-\widetilde{w}[sf(t)]\}\, dt\right\} \tag{2.61}$$

exists, from which it follows that there exists a *asymptotically stationary signal level* $\eta^{(st)}$ with a density function

$$p_{st}(y) = \mathcal{L}^{-1}\left\{\widetilde{p}_{st}(s)\right\}. \tag{2.62}$$

Even in this case, for the determination of the cumulants it is practical to use the logarithm of the Laplace-transform $\widetilde{p}_{st}(s)$ of the density function $p_{st}(y)$:

$$\widetilde{g}_{st}(s) = \ln \widetilde{p}_{st}(s) = s_0 \int_0^\infty \left[\widetilde{h}(s,t) - 1\right]\, dt, \tag{2.63}$$

where

$$\widetilde{h}(s,t) = \widetilde{w}[sf(t)].$$

2.4. Signal discrimination

Often one needs to know the intensity $n_{st}(V)$ of the events which occur when the signal level jumps above a certain level V from a value $y \leq V$. In the stationary case, this is obviously given by

$$n_{st}(V) = s_0 \int_0^V p_{st}(y)\, \left[1 - W(V-y)\right]\, dy, \tag{2.64}$$

whose Laplace-transform is

$$\widetilde{n}_{st}(s) = s_0\, \frac{1-\widetilde{w}(s)}{s}\, \widetilde{p}_{st}(s). \tag{2.65}$$

One can illustrate the physical meaning of the intensity $n_{st}(V)$ by defining the process being in state $\mathfrak{L}$ when $\eta^{(st)} \leq V$ and its state $\mathfrak{U}$ when $\eta^{(st)} \geq V$. The level V is usually called the *threshold*. The function $n_{st}(V)$ gives the intensity of the jumps from state $\mathfrak{L}$ to state $\mathfrak{U}$. It can be expected that $n_{st}(V)$ is small both for low and high threshold values, since the density function $p_{st}(y)$ is close to zero for both cases. From this it follows that the intensity

$n_{st}(V)$ has a maximum at a threshold value V_{max}. It is worth noting that the noise mixed to the useful signal can only be reduced by using a threshold chosen by practical considerations. This threshold is obviously greater than the threshold V_{max}, corresponding to the maximum of the intensity $n_{st}(V)$.

In many cases the value of the jump $\boldsymbol{\xi}$ can be assumed to be constant, i.e.

$$\mathcal{P}\{\boldsymbol{\xi} \leq x\} = W(x) = \Delta\left(x - \frac{1}{\mu}\right), \tag{2.66}$$

hence one has

$$n_{st}(V) = s_0\,[P_{st}(V) - P_{st}(V - 1/\mu)], \tag{2.67}$$

since

$$\int_0^V p_{st}(y)\,\Delta\left(V - y - \frac{1}{\mu}\right)\,dy = \int_0^{V-1/\mu} p_{st}(y)\,dy = P_{st}(V - 1/\mu).$$

It will be seen that in the case of a constant jump, the determination of the density function $p_{st}(y)$ from the Laplace-transform $\widetilde{p}_{st}(s)$ is not an easy task. The problems encountered will be shown for the pulse shape $f(t) = e^{-\alpha t}$.

In order to study the characteristics of the detector signal functions in more detail, in the next subsection we perform detailed calculations for *rectangular, exponential and triangular pulse functions* $f(t)$.

3. Rectangular pulses

In this case the particles, arriving at the detector according to a Poisson process, generate a signal with a constant width T_0 and random height $\boldsymbol{\xi}$. Two different realisations of such a pulse are shown on Fig. 1. For simplicity, assume an exponential distribution of the amplitudes, i.e.

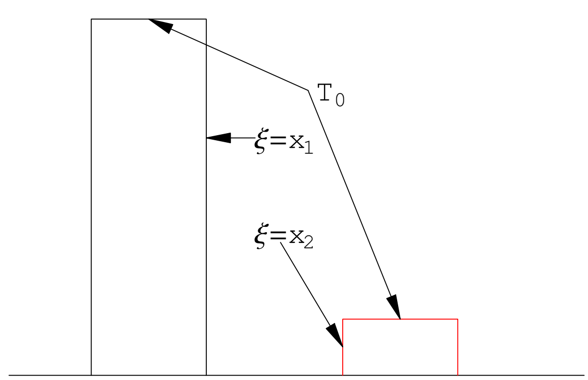

Figure 1: Illustration of the rectangular pulses with random heights.

$$\mathcal{P}\{\boldsymbol{\xi} \leq x\} = W(x) = 1 - e^{-\mu x}. \tag{3.1}$$

Since

$$f(t) = \Delta(T_0 - t), \tag{3.2}$$

from (2.60) one obtains

$$\widetilde{p}(s,t) = \exp\left\{-s_0 \int_0^t \frac{s\,\Delta(T_0-t')}{s\,\Delta(T_0-t')+\mu}\right\} dt'. \tag{3.3}$$

Thus one arrives at

$$\widetilde{p}(s,t) = \begin{cases} \exp\left\{s_0\, t\left(\frac{\mu}{s+\mu}-1\right)\right\}, & \text{if } t \le T_0, \\ \\ \exp\left\{s_0\, T_0\left(\frac{\mu}{s+\mu}-1\right)\right\}, & \text{if } t > T_0. \end{cases} \tag{3.4}$$

Using the relationship

$$\mathcal{L}^{-1}\left\{\exp\left\{\frac{a}{s+b}\right\}\right\} = \delta(y) + \sqrt{\frac{a}{y}}\, e^{-by}\, I_1\left(2\sqrt{a\,y}\right), \tag{3.5}$$

one obtains

$$p(y,t) = \begin{cases} e^{-s_0 t}\, e^{-\mu y}\left[\delta(y) + \sqrt{\frac{s_0 t\mu}{y}}\, I_1\left(2\sqrt{s_0 t\mu y}\right)\right], & \text{if } t \le T_0, \\ \\ e^{-s_0 T_0}\, e^{-\mu y}\left[\delta(y) + \sqrt{\frac{s_0 T_0\mu}{y}}\, I_1\left(2\sqrt{s_0 T_0\mu y}\right)\right], & \text{if } t > T_0. \end{cases} \tag{3.6}$$

It is notable that $p(y,t)$ converges rather fast to the asymptotically stationary density function $p_{st}(y)$ with increasing t. As the second part of (3.6) shows, the sum of the individual signals of particles arriving according to a homogeneous Poisson process with intensity s_0 has a stationary distribution already for $t > T_0$.

From the Laplace transform (3.4) we can get immediately the expected value of the sum of detector pulses at time t as

$$\mathbf{E}\{\eta(t)\} = -\left[\frac{\partial \ln \widetilde{p}(s,t)}{\partial s}\right]_{s=0} = \frac{s_0}{\mu}\left[t\,\Delta(T_0-t) + T_0\,\Delta(t-T_0)\right] \tag{3.7}$$

and its variance as

$$\mathbf{D}^2\{\eta(t)\} = \left[\frac{\partial^2 \ln \widetilde{p}(s,t)}{\partial s^2}\right]_{s=0} = 2\,\frac{s_0}{\mu^2}\left[t\,\Delta(T_0-t) + T_0\,\Delta(t-T_0)\right]. \tag{3.8}$$

It is also worth noting that the *Fano factor* for this case is equal to

$$\mathcal{F} = \frac{2}{\mu}, \tag{3.9}$$

where $1/\mu$ is the expected value of the pulse jump. It is seen that the Fano factor does not depend on time.

For the illustration of the signal discrimination in stationary case, we shall calculate the intensity $n_{st}(V)$ of particle arrivals which induce a jump of the signal from a level $y \le V$ to

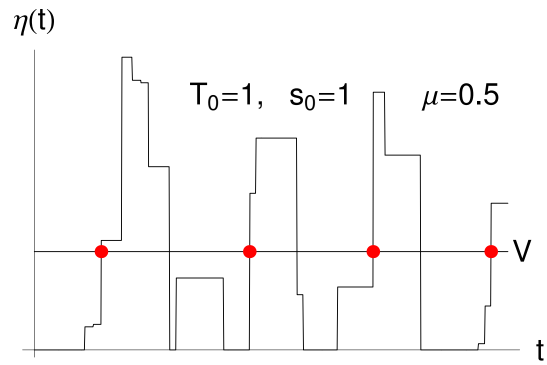

Figure 2: A possible realization of the sum of signals in an arbitrary time interval.

a signal level higher than V. Fig. 2. shows a possible realization of the sum of pulses within a stationary time interval. The red dots mark the particles which induce the jump of the signal level from a state $y \leq V$ to above the threshold V. By using Eq. (2.64) we obtain

$$n_{st}(V) = s_0 \int_{+0}^{V} e^{-\mu(V-y)}\, p_{st}(y)\, dy, \tag{3.10}$$

where

$$p_{st}(y) = e^{-s_0 T_0}\, e^{-\mu y} \left[\delta(y) + \sqrt{\frac{s_0 T_0 \mu}{y}}\, I_1\left(2\sqrt{s_0 T_0 \mu y}\right) \right].$$

From this it follows that

$$n_{st}(V) = s_0\, e^{-s_0 T_0}\, e^{-\mu V} \int_0^V \left[\delta(y) + \sqrt{\frac{s_0 T_0 \mu}{y}}\, I_1\left(2\sqrt{s_0 T_0 \mu y}\right) \right] dy =$$

$$s_0\, e^{-s_0 T_0}\, e^{-\mu V}\, I_0\left(2\sqrt{s_0 T_0 \mu V}\right), \tag{3.11}$$

where $I_0(x)$ is the modified Bessel function of order zero. The dependence of the intensity

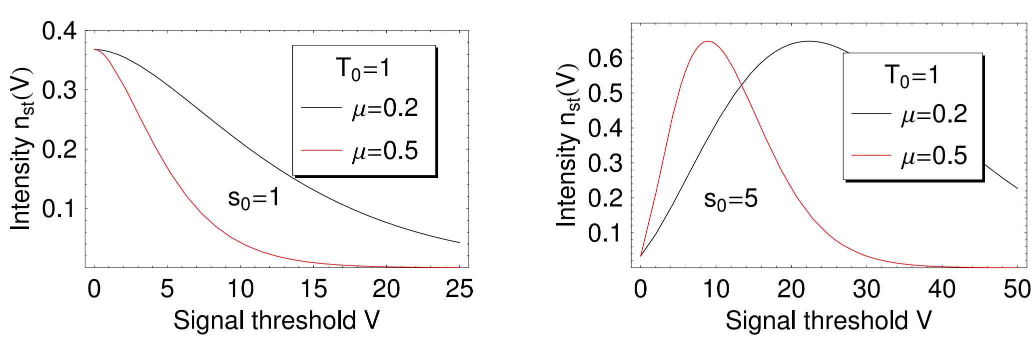

Figure 3: Dependence of the intensity $n_{st}(V)$ on the signal threshold V at two different input intensities s_0.

$n_{st}(V)$ on the threshold V is shown in Fig. 3 for two different values of μ and at two different input intensities s_0. It is seen how the values of μ and s_0 influence the dependence of the

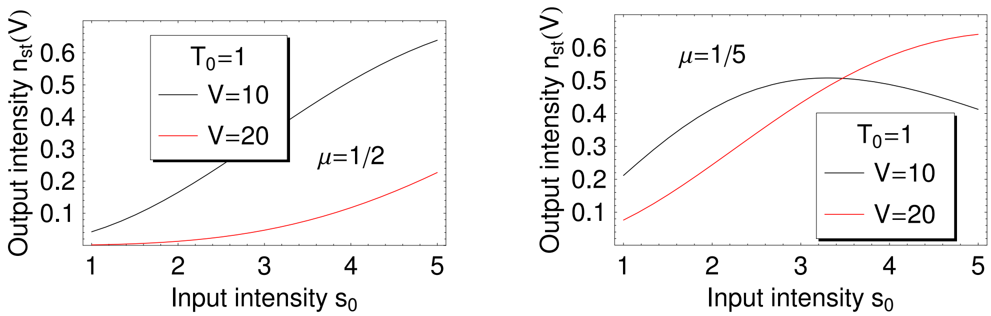

Figure 4: Dependence of the output intensity $n_{st}(V)$ on the input intensity s_0 for two different values of μ and at two different threshold values V.

output intensity $n_{st}(V)$ on the threshold value V. In order to show the influence of the input intensity s_0 on the output intensity $n_{st}(V)$, in Fig. 4 it is plotted the dependence of $n_{st}(V)$ on s_0 at two threshold values V. One concludes that the mean amplitude $1/\mu$ of the rectangular signal must be chosen very carefully.

4. Exponential pulses

We will treat now the case when the pulses have exponential decay shape, the initial values of which being the realizations of the random variable $\boldsymbol{\xi}$. For simplicity, assume again exponential distribution of the amplitudes

$$\mathcal{P}\left\{\boldsymbol{\xi} \leq x\right\} = W(x) = \int_0^x w(x')\, dx' = 1 - e^{-\mu x},$$

where $1/\mu$ is expectation of the starting amplitude of a single signal. Fig. 5. shows two possible pulses with $\alpha = 2$.

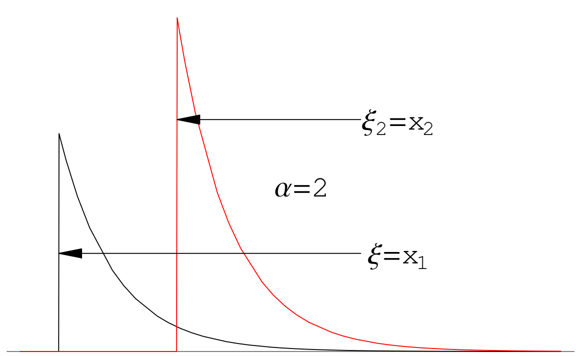

Figure 5: Exponential pulses generated by incoming particles.

Since one now has

$$f(t) = e^{-\alpha t}, \tag{4.1}$$

the probability density of the signal induced by one particle is given as

$$h(y,t) = \int_0^\infty \delta\left(y - xe^{-\alpha t}\right)\, w(x)\, dx \tag{4.2}$$

whose Laplace transform

$$\widetilde{h}(s,t) = \int_0^\infty e^{-sy}\, h(y,t)\, dy \tag{4.3}$$

can be written as

$$\widetilde{h}(s,t) = \int_0^\infty \exp\left\{-s\,x\,e^{-\alpha t}\right\}\, w(x)\, dx = \mu \int_0^\infty \exp\left\{-s\,x\,e^{-\alpha t}\right\}\, e^{-\mu x}\, dx. \tag{4.4}$$

From this one immediately obtains

$$\widetilde{h}(s,t) = \frac{\mu}{\mu + s\,e^{-\alpha t}}. \tag{4.5}$$

Using (2.60) yields the Laplace transform of the density function $p(y,t)$ as

$$\widetilde{p}(s,t) = \exp\left\{-s_0 \int_0^t \left[1 - \frac{\mu}{\mu + s\,e^{-\alpha v}}\right]\, dv\right\} = \exp\left\{-s_0 \int_0^t \frac{s\,e^{-\alpha v}}{\mu + s\,e^{-\alpha v}}\, dv\right\}. \tag{4.6}$$

Accounting for the identity

$$\frac{s\,e^{-\alpha v}}{\mu + s\,e^{-\alpha v}} = -\frac{1}{\alpha}\,\frac{d\ln\left(\mu + s\,e^{-\alpha v}\right)}{dv},$$

Eq. (4.6) can be written as

$$\widetilde{p}(s,t) = \exp\left\{\frac{s_0}{\alpha}\,\ln\,\frac{\mu + s\,e^{-\alpha 1 t}}{\mu + s}\right\} = \left(\frac{\mu + s\,e^{-\alpha t}}{\mu + s}\right)^{s_0/\alpha}. \tag{4.7}$$

It is immediately seen that

$$\widetilde{p}(0,t) = \int_0^\infty p(y,t)\, dy = 1.$$

From (4.7) it is also obvious that a *stationary density function* exists with the Laplace transform

$$\lim_{t\to\infty} \widetilde{p}(s,t) = \widetilde{p}_{st}(s) = \left(\frac{\mu}{\mu + s}\right)^{s_0/\alpha}. \tag{4.8}$$

Eq. (4.7) can be rewritten as

$$\widetilde{p}(s,t) = \left[1 - \left(1 - e^{-\alpha t}\right)\,\frac{s}{s+\mu}\right]^q, \tag{4.9}$$

where

$$q = \frac{s_0}{\alpha} > 0. \tag{4.10}$$

If q *is not an integer* and the following inequality holds:

$$\left(1 - e^{-\alpha t}\right)\left|\frac{s}{s+\mu}\right| < q,$$

then (4.9) is identical with the absolute convergent series

$$\widetilde{p}(s,t) = 1 + \sum_{k=0}^{\infty} (-1)^k \, \frac{q(q-1)\cdots(q-k+1)}{k!} \, \left(1 - e^{-\alpha t}\right)^k \left(\frac{s}{s+\mu}\right)^k \tag{4.11}$$

in which, according to Erdélyi [10]

$$(-1)^k \, q(q-1)\cdots(q-k+1) = \frac{\Gamma(-q+k)}{\Gamma(-q)}. \tag{4.12}$$

It can be shown that

$$\mathcal{L}^{-1}\left\{\frac{s^k}{(s+\mu)^k}\right\} = \delta(y) - \mu e^{-\mu y} \, L_{k-1}^{(1)}(\mu y), \tag{4.13}$$

where $\mathcal{L}^{-1}$ stands for the operator of the inverse Laplace transform and $L_{k-1}^{(1)}(\mu y)$ is the so called *generalized Laguerre polynomial*. It is shown in Erdélyi's "Tables of Integral Transforms Volume 1" [11] that

$$\int_0^{\infty} e^{-rx} \, L_{k-1}^{(1)}(x) \, dx = \widetilde{L}_{k-1}^{(1)}(r) = \sum_{j=0}^{k-1} \frac{(r-1)^{k-1-j}}{r^{k-j}}, \tag{4.14}$$

which can be written as

$$\widetilde{L}_{k-1}^{(1)}(r) = \frac{1}{r-1} \sum_{j=0}^{k-1} \left(\frac{r-1}{r}\right)^{k-j} = \frac{1}{r} \sum_{j=0}^{k-1} \left(\frac{r-1}{r}\right)^{j} = 1 - \left(1 - \frac{1}{r}\right)^k. \tag{4.15}$$

By the substitution

$$r = \frac{s+\mu}{\mu} \quad \text{and} \quad x = \mu y,$$

one arrives at

$$\int_0^{\infty} e^{-sy} \, \mu e^{-\mu y} \, L_{k-1}^{(1)}(\mu y) \, dy = 1 - \frac{s^k}{(s+\mu)^k}, \tag{4.16}$$

from which (4.13) immediately arises. Using this relationship, the inverse Laplace transform of (4.11) can be written as

$$p(y,t) = \delta(y) \left[1 + \sum_{k=1}^{\infty} \frac{\Gamma(-s_0/\alpha + k)}{\Gamma(-s_0/\alpha)\, \Gamma(k+1)} \, (1 - e^{-\alpha t})^k \right] -$$

$$\mu \, e^{-\mu y} \sum_{k=1}^{\infty} \frac{\Gamma(-s_0/\alpha) + k)}{\Gamma(-s_0/\alpha)\, \Gamma(k+1)} \, L_{k-1}^{(1)}(\mu y) \, (1 - e^{-\alpha t})^k. \tag{4.17}$$

Fig. 6. shows the dependence of the of the density function $p(y,t)$ on the parameter αt for $q = s_0/\alpha = 0.8$, for three different signal levels. It is interesting to note that the density function becomes constant relatively fast; in the present case the stationary behaviour is already reached for $\alpha t \approx 5$.

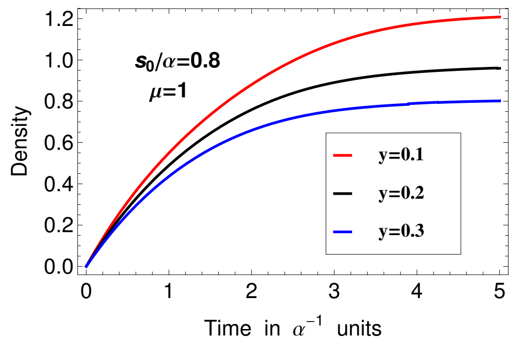

Figure 6: Dependence of the density function $p(y,t)$ on the time parameter αt at three signal levels and at $q = s_0/\alpha = 0.8$.

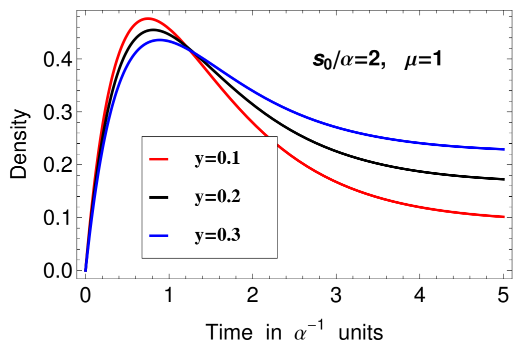

Figure 7: Dependence of the density function $p(y,t)$ on the time parameter αt at three signal levels for $q = s_0/\alpha = 2$.

If $q = s_0/\alpha$ is a *positive integer*, then expression (4.17) cannot be used. One has to return to Eq. (4.9) and use the rearrangement

$$\widetilde{p}(s,t) = 1 - \sum_{k=1}^{q} \binom{q}{k} (-1)^{k-1} \left(1 - e^{-\alpha t}\right)^k \frac{s^k}{(s+\mu)^k}. \tag{4.18}$$

From this one obtains

$$p(y,t) = \delta(y) \left[1 - \sum_{k=1}^{q} \binom{q}{k} (-1)^{k-1} (1 - e^{-\alpha t})^k \right] +$$

$$\mu \, e^{-\mu y} \sum_{k=1}^{q} \binom{q}{k} (-1)^{k-1} L_{k-1}^{(1)}(\mu y) \, (1 - e^{-\alpha t})^k, \tag{4.19}$$

which does not contain singular Gamma functions. Fig. 7. shows the dependence of the density function $p(y,t)$ on the parameter αt for $q = s_0/\alpha = 2$, for three different signal levels. One finds again that the density function is close to stationary already at $\alpha t \approx 5$.

We will now investigate the properties of the *stationary* signal sequence. The Laplace transform (4.8) of the density function $p_{st}(y)$ can easily be inverted. One obtains

$$p_{st}(y) = \frac{(\mu y)^{\frac{s_0}{\alpha} - 1}}{\Gamma(s_0/\alpha)} \, e^{-\mu y} \, \mu, \tag{4.20}$$

which shows that the stationary distribution of the sum of exponential pulses is given by

$$P_{st}(y) = \int_0^y p_{st}(y')\, dy' = \frac{\Gamma(s_0/\alpha) - \Gamma(\mu y, s_0/\alpha)}{\Gamma(s_0/\alpha)}, \tag{4.21}$$

in which

$$\Gamma(\mu y, s_0/\alpha) = \int_{\mu y}^{\infty} \frac{v^{\frac{s_0}{\alpha}-1}}{\Gamma(s_0/\alpha)}\, e^{-v}\, dv$$

is the incomplete Gamma function.

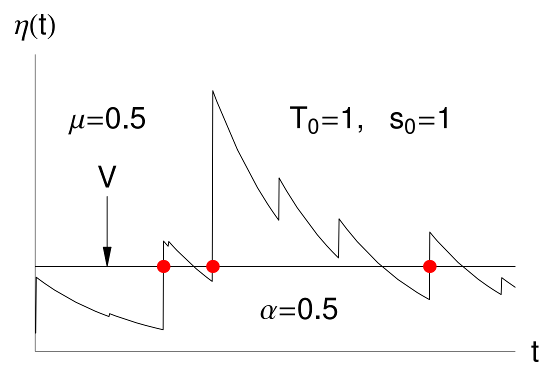

Figure 8: A possible realization of the sum of signals in stationary case in an arbitrary time interval.

Fig. 8 shows a possible realization of the exponential pulse train in the stationary state. The red dots mark the particles which induce a jump of the signal level to above the threshold V from a level under V. By using (2.65) we obtain the Laplace transform

$$\widetilde{n}_{st}(s) = s_0\, \frac{1 - \widetilde{w}(s)}{s}\, \widetilde{p}_{st}(s) = s_0\, \frac{1}{\mu + s} \left(\frac{\mu}{\mu + s}\right)^{s_0/\alpha}, \tag{4.22}$$

whose inverse is given as

$$n_{st}(V) = s_0\, \frac{(\mu V)^{\frac{s_0}{\alpha}}}{\Gamma(s_0/\alpha + 1)}\, e^{-\mu V}. \tag{4.23}$$

Fig. 9 shows the dependence of the intensity $n_{st}(V)$ on the threshold V for two different values of the parameter μ. It is seen that the intensity $n_{st}(V)$ has a distinct maximum at the threshold

$$V_{max} = \frac{s_0}{\alpha\mu}.$$

The knowledge of this maximum could be important for the design of the detector electronics.

The expectation and the variance, important for practical applications, will now be calculated for both the non-stationary and the stationary case. From the logarithm of the Laplace transform (4.7) one obtains

$$-\left[\frac{d \ln \widetilde{p}(s,t)}{ds}\right]_{s=0} = \mathbf{E}\{\eta(t)\} = \begin{cases} \dfrac{s_0}{\alpha\mu}\,(1 - e^{-\alpha t}), & \text{if } t \le \infty, \\[2ex] \dfrac{s_0}{\alpha\mu}, & \text{if } t = \infty, \end{cases} \tag{4.24}$$

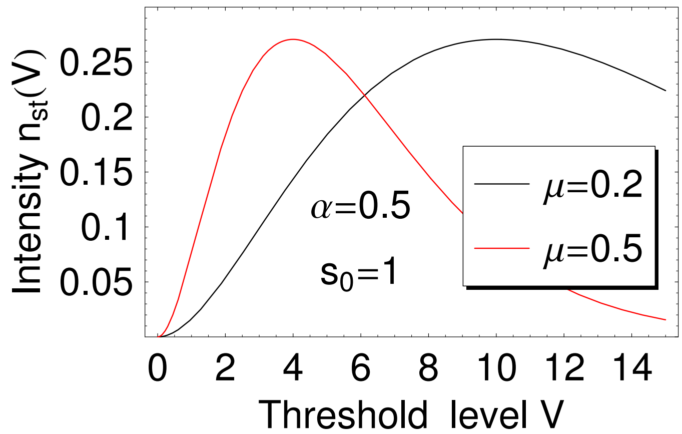

Figure 9: Dependence of the intensity $n_{st}(V)$ on the signal threshold V.

and

$$\left[\frac{d^2 \ln \widetilde{p}(s,t)}{ds^2}\right]_{s=0} = \mathbf{D}^2 \left\{\eta(t)\right\} = \begin{cases} \frac{s_0}{\alpha\mu^2} \left(1 - e^{-2\alpha t}\right), & \text{if } t \leq \infty, \\ \\ \frac{s_0}{\alpha\mu^2}, & \text{if } t = \infty. \end{cases} \tag{4.25}$$

For illustration, the Fano factor for this case is also given. It reads as

$$\mathcal{F} = \begin{cases} \frac{1}{\mu} \left(1 + e^{-\alpha t}\right), & \text{if } t \leq \infty, \\ \\ \frac{1}{\mu}, & \text{if } t = \infty. \end{cases} \tag{4.26}$$

For didactic purposes consider the special case when the random variable $\boldsymbol{\xi}$ is constant, i.e. its density function is given as $w(x) = \delta(x - x_0)$. From (4.2) one has

$$h(y,t) =$$

$$\int_0^\infty \delta\left[y - xe^{-\alpha t}\right] w(x)\, dx = \int_0^\infty \delta\left[y - xe^{-\alpha t}\right] \delta(x - x_0)\, dx = \delta\left[y - x_0\, e^{-\alpha t}\right], \tag{4.27}$$

whose Laplace transform is obtained as

$$\widetilde{h}(s,t) = \exp\left[-s\, x_0\, e^{-\alpha t}\right]. \tag{4.28}$$

Substituting into (2.61) yields

$$\widetilde{p}_{st}(s) = \exp\left\{-s_0 \int_0^\infty \left[1 - \widetilde{h}(s,t)\right]\, dt\right\} = \exp\left\{-s_0 \int_0^\infty \left[1 - \exp\left[-s\, x_0\, e^{-\alpha t}\right]\right]\, dt\right\},$$

from which one obtains

$$\widetilde{g}_{st}(s) = \ln \widetilde{p}_{st}(s) = s_0 \int_0^\infty \left[\exp\left(-s\, x_0\, e^{-\alpha t}\right) - 1\right]\, dt. \tag{4.29}$$

Introducing the variable

$$u = e^{-\alpha t}$$

one arrives at

$$\widetilde{g}_{st}(s) = \ln \widetilde{p}_{st}(s) = -\frac{s_0}{\alpha} \int_0^1 \frac{1 - e^{-s\, x_0\, u}}{u}\, du. \tag{4.30}$$

Using the known relationship

$$\int_0^1 \frac{1 - e^{-s\, x_0\, u}}{u}\, du = C + \ln x_0\, s + \int_{x_0}^{\infty} \frac{e^{-s\, u}}{u}\, du$$

will yield, accounting for (4.30), the result

$$\widetilde{p}_{st}(s) = \exp\left[-\frac{s_0}{\alpha}\left(C + \ln x_0\, s + \int_{x_0}^{\infty} \frac{e^{-s\, u}}{u}\, du\right)\right] = \frac{\exp\left(-\frac{s_0}{\alpha} \int_{x_0}^{\infty} \frac{e^{-s\, u}}{u}\, du\right)}{\left(x_0\, e^C\, s\right)^{s_0/\alpha}}. \tag{4.31}$$

From (4.29), one can immediately determine the expectation and the variance of $\eta^{(st)}$. One obtains

$$\mathbf{E}\{\eta^{(st)}\} = s_0\, \frac{x_0}{\alpha} \qquad \text{and} \qquad \mathbf{D}^2\left\{\eta^{(st)}\right\} = s_0\, \frac{x_0^2}{2\,\alpha}, \tag{4.32}$$

whereas the Fano factor equals $\mathcal{F} = x_0/2$.

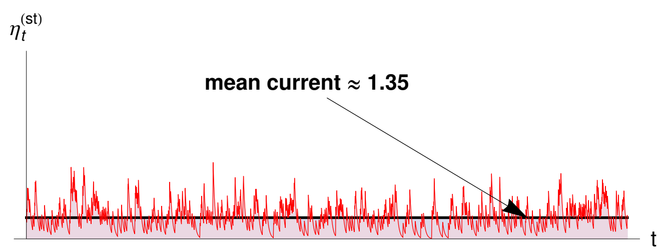

Figure 10: Simulation of the stationary current $\eta_t^{(st)}$ of the ionization chamber as a function of time, with exponentially decaying pulses of unit initial value.

Fig. 10. shows the time dependence of the stationary current $\eta_t^{(st)}$ of the ionisation chamber, when the current is formed by exponentially decaying pulses of unit initial value. For this case one has

$$\mathbf{E}\left\{\eta_t^{(st)}\right\} \approx 1.35 \qquad \text{and} \qquad \mathbf{D}^2\left\{\eta_t^{(st)}\right\} \approx 0.58.$$

Calculation of the inverse Laplace transform of (4.31) is rather laborious. Let us write $\widetilde{p}_{st}(s)$ in the following, absolute convergent, infinite series:

$$\widetilde{p}_{st}(s) = \frac{1}{\left(x_0\, e^C\right)^{s_0/\alpha}} \left\{\frac{1}{s^{s_0/\alpha}} + \sum_{n=1}^{\infty} (-1)^n\, \frac{1}{n!} \left(\frac{s_0}{\alpha}\right)^n \frac{1}{s^{s_0/\alpha}} \left(\int_{x_0}^{\infty} \frac{e^{-s\, u}}{u}\, du\right)^n\right\}. \tag{4.33}$$

For the sake of simplicity, let us introduce the notations:

$$\frac{s_0}{\alpha} = a, \qquad e^C = \gamma \qquad \text{and} \qquad x_0 = b. \tag{4.34}$$

The *first member* of the r.h.s. in (4.33) can be immediately inverted. One obtains that

$$f_1^{(st)}(x) = \frac{1}{(b\,\gamma)^a}\,\mathcal{L}^{-1}\left\{\frac{1}{s^a}\right\} = \frac{1}{\Gamma(a)}\,\frac{x^{a-1}}{(b\,\gamma)^a}, \tag{4.35}$$

and by using the *convolution theorem*, the inversion of the *second member* can be written in the form

$$-\frac{a}{(b\,\gamma)^a}\mathcal{L}^{-1}\left\{\frac{1}{s^a}\int_b^\infty \frac{e^{-su}}{u}\,du\right\} = -\frac{a}{(b\,\gamma)^a\,\Gamma(a)}\int_0^x (x-u)^{a-1}\,h_1(u)\,du, \tag{4.36}$$

where

$$h_1(u) = \mathcal{L}^{-1}\left\{\int_b^\infty \frac{e^{-su}}{u}\,du\right\} = \frac{\Delta(u-b)}{u}, \tag{4.37}$$

consequently

$$f_2^{(st)}(x) = -\frac{a}{(b\,\gamma)^a}\mathcal{L}^{-1}\left\{\frac{1}{s^a}\int_b^\infty \frac{e^{-su}}{u}\,du\right\} = -\frac{a}{(b\,\gamma)^a\,\Gamma(a)}\int_b^x (x-u)^{a-1}\,h_1(u)\,du. \tag{4.38}$$

The (4.37) is obvious, since

$$\int_0^\infty e^{-su}\,h_1(u)\,du = \int_0^\infty e^{-su}\,\frac{\Delta(u-b)}{u}\,du = \int_b^\infty \frac{e^{-su}}{u}\,du.$$

The *third member* is nothing else, than

$$\frac{a^2}{2!\,(b\,\gamma)^a}\,\mathcal{L}^{-1}\left\{\frac{1}{s^a}\left(\int_b^\infty \frac{e^{-su}}{u}\,du\right)^2\right\} = \frac{a^2}{2!\,(b\,\gamma)^a\,\Gamma(a)}\int_0^x (x-u)^{a-1}\,h_2(u)\,du, \tag{4.39}$$

where

$$h_2(u) = \int_0^u h_1(u-v)\,h_1(v)\,dv = \int_0^u \frac{\Delta(u-v-b)\,\Delta(v-b)}{(u-v)\,v}\,dv. \tag{4.40}$$

From the inequality

$$b \le v \le u-b,$$

it follows the equation

$$\int_0^u \frac{\Delta(u-v-b)\,\Delta(v-b)}{(u-v)\,v}\,dv = \int_b^{u-b} \frac{1}{(u-v)\,v}\,dv = h_2(u), \tag{4.41}$$

which is non-zero when $u > 2\,b$, hence

$$f_3^{(st)}(x) = \frac{a^2}{2!\,(b\,\gamma)^a}\,\mathcal{L}^{-1}\left\{\frac{1}{s^a}\left(\int_b^\infty \frac{e^{-su}}{u}\,du\right)^2\right\} =$$

$$\frac{a^2}{2!\,(b\,\gamma)^a\,\Gamma(a)}\int_{2\,b}^x (x-u)^{a-1}\,h_2(u)\,du. \tag{4.42}$$

Following this treatment, by *induction* it can be proved that the inverse Laplace transform of the $(n+1)th$ *member* in the series (4.33) is given by the formula

$$f_{n+1}^{(st)}(x) = \frac{a^n}{n!\,(b\,\gamma)^a}\,\mathcal{L}^{-1}\left\{\frac{1}{s^a}\,\left(\int_b^\infty \frac{e^{-s\,u}}{u}\,du\right)^n\right\} =$$

$$\frac{a^n}{n!\,(b\,\gamma)^a\,\Gamma(a)}\int_{n\,b}^{x}(x-u)^{a-1}\,h_n(u)\,du, \tag{4.43}$$

where

$$h_n(u) = \int_b^{u-(n-1)b}\frac{h_{n-1}(u-v)}{v}\,dv \tag{4.44}$$

which is corresponding to the rule expressed by (4.41). Obviously, $h_n(u)$ is non-zero when $u > n\,b$.

After replacing back in the formula (4.43) the original values of the quantities a, γ, and b given by (4.34), one obtains the inverse Laplace-transform of (4.33) in the following form:

$$p_{st}(x) =$$

$$\frac{1}{\left(x_0\,e^C\right)^{s_0/\alpha}\,\Gamma(s_0/\alpha)}\left\{x^{s_0/\alpha-1} + \sum_{n=1}^{[x/x_0]}\frac{(-1)^n\,(s_0/\alpha)^n}{n!}\int_{n\,x_0}^{x}(x-u)^{s_0/\alpha-1}\,h_n(u)\,du\right\}, \tag{4.45}$$

where $[x/x_0]$ is the largest integer less or equal to x/x_0. This follows from the inequality $x > n\,x_0$ which defines the maximal value of the summing index in (4.45). In order to determine the density function $p_{st}(x)$, the first task is solving the recursive equation

$$h_n(u) = \int_{x_0}^{u-(n-1)\,x_0}\frac{h_{n-1}(u-v)}{v}\,dv \tag{4.46}$$

with the starting function

$$h_1(u) = \frac{1}{u}\,\Delta(x-x_0).$$

It follows immediately, that

$$h_2(u) = \int_{x_0}^{u-x_0}\frac{dv}{(u-v)\,v} = 2\,\frac{\ln\left(\dfrac{u}{x_0}-1\right)}{u}\,\Delta(u-2\,x_0), \tag{4.47}$$

but the following step brings about already a too complex expression, namely

$$\begin{aligned} h_3(u) = \frac{1}{6\,u}\Bigg\{\pi^2 + 24\,\ln\,\frac{x_0}{u-2\,x_0}\,\ln\,\frac{x_0}{u-x_0} - \\ 12\,\mathrm{Poly}\ln\,\left[2, 1+\frac{x_0}{x_0-u}\right] + \\ 12\,\mathrm{Poly}\ln\,\left[2, \frac{x_0}{u-x_0}\right] + 12\,\mathrm{Poly}\ln\,\left[2, 2-\frac{u}{x_0}\right]\Bigg\}\,\Delta(u-3\,x_0). \end{aligned} \tag{4.48}$$

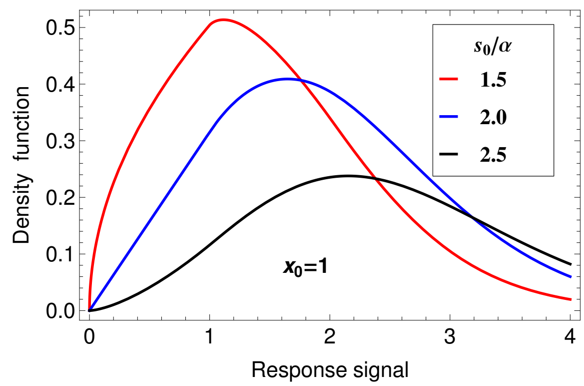

Figure 11: Probability density function of the stationary current $\eta_t^{(st)}$ of the ionization chamber with exponentially decaying pulses of constant $x_0 = 1$ amplitude with three different s_0/α parameter values.

Surprisingly, the next step results in such a monster formula for $h_4(x)$ which is unreasonable to write down.

By using the *Mathematica code* the dependence of the stationary density function $p_{st}(x)$ on x has been calculated in the case of constant $x_0 = 1$ amplitude for three different s_0/α parameter values. In Figure 11 one can see that the value of the ratio s_0/α sensitively influences the shape of $p_{st}(x)$. This sensitivity has to be taken into the count by the construction of the detector electronics.

5. Triangular pulses

As an exercise, in this section pulses with *right triangle* shape will be studied. The base of the triangles will be a constant time duration T_0 whereas their height is a random variable denoted by $\boldsymbol{\xi}$.

5.1. First version

Consider first the case when the random variable $\boldsymbol{\xi}$ is *uniformly distributed* in the interval $[0, \mathfrak{a}]$. Fig. 12 shows two possible triangle pulses when $\mathfrak{a} = 1$.

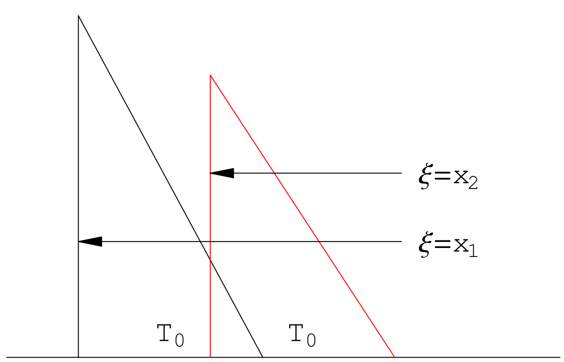

Figure 12: Pulses of rectangular triangles generated by incoming particles in the case of $\mathfrak{a} = 1$.

Since now one has

$$f(t) = \left(1 - \frac{t}{T_0}\right) \Delta(T_0 - t), \tag{5.1}$$

the density function of the signal is equal to

$$h(y,t) = \int_0^{\mathfrak{a}} \delta\left[y - x f(t)\right] \, \frac{dx}{\mathfrak{a}}, \tag{5.2}$$

whose Laplace transform

$$\widetilde{h}(s,t) = \int_0^{\infty} e^{-sy}\, h(y,t)\, dy \tag{5.3}$$

reads as

$$\widetilde{h}(s,t) = \int_0^{\mathfrak{a}} \exp\left\{-sx\left(1 - \frac{t}{T_0}\right)\, \Delta(T_0 - t)\right\} \, \frac{dx}{\mathfrak{a}} =$$

$$\frac{1 - \exp\left\{-s\mathfrak{a}\left(1 - \dfrac{t}{T_0}\right)\, \Delta(T_0 - t)\right\}}{s\mathfrak{a}\left(1 - \dfrac{t}{T_0}\right)\, \Delta(T_0 - t)} = \frac{1 - \exp\left\{-s\mathfrak{a}\left(1 - \dfrac{t}{T_0}\right)\right\}}{s\mathfrak{a}\left(1 - \dfrac{t}{T_0}\right)}\, \Delta(T_0 - t) + \Delta(t - T_0) =$$

$$1 - \left[1 - \frac{1 - \exp\left\{-s\mathfrak{a}\left(1 - \dfrac{t}{T_0}\right)\right\}}{s\mathfrak{a}\left(1 - \dfrac{t}{T_0}\right)}\right] \Delta(T_0 - t). \tag{5.4}$$

Substituting this into equation (2.60) we obtain

$$\widetilde{p}(s,t) = \exp\left\{-s_0 \int_0^t \left[1 - \frac{1 - \exp\left\{-s\mathfrak{a}\left(1 - \dfrac{v}{T_0}\right)\right\}}{s\mathfrak{a}\left(1 - \dfrac{v}{T_0}\right)}\right] \Delta(T_0 - v)\, dv\right\}, \tag{5.5}$$

from which the distribution of the random function $\eta(t)$ and its moments can be determined.

Derive now the stationary form $\widetilde{p}_{st}(s)$ of the Laplace transform (5.5) of the distribution function $p(y,t)$. Introducing the notation

$$1 - \frac{v}{T_0} = x$$

and taking into account the effect of the Haeviside function $\Delta(T_0 - v)$, we obtain

$$\widetilde{p}_{st}(s) = \exp\left\{-s_0\, T_0 \int_0^1 \left(1 - \frac{1 - e^{-s\mathfrak{a}x}}{s\mathfrak{a}x}\right)\, dx\right\}.$$

To calculate the integral in the exponent one can use the known relationship

$$\int_0^1 \frac{1 - e^{-u}}{u}\, du - \int_1^{\infty} \frac{e^{-u}}{u}\, du = C,$$

where $C = 0.5772\cdots$ is the Euler constant. With $u = s\mathfrak{a}x$, one has

$$dx = \frac{du}{s\mathfrak{a}}.$$

With this from the above we arrive at

$$C = \int_0^{1/s\mathfrak{a}} \frac{1-e^{-s\mathfrak{a}x}}{x}\,dx - \int_{1/s\mathfrak{a}}^{\infty} \frac{e^{-s\mathfrak{a}x}}{x}\,dx =$$

$$\int_0^1 \frac{1-e^{-s\mathfrak{a}x}}{x}\,dx + \int_1^{1/s\mathfrak{a}} \frac{1-e^{-s\mathfrak{a}x}}{x}\,dx - \int_1^{\infty} \frac{e^{-s\mathfrak{a}x}}{x}\,dx + \int_1^{1/s\mathfrak{a}} \frac{e^{-s\mathfrak{a}x}}{x}\,dx =$$

$$\int_0^1 \frac{1-e^{-s\mathfrak{a}x}}{x}\,dx - \int_1^{\infty} \frac{e^{-s\mathfrak{a}x}}{x}\,dx - \ln s\,\mathfrak{a},$$

from which we finally get

$$\widetilde{p}_{st}(s) = \exp\left\{-s_0\,T_0\left(1 - \frac{1}{s\mathfrak{a}}\int_0^1 \frac{1-e^{-s\mathfrak{a}x}}{x}\,dx\right)\right\} =$$

$$\exp\left\{-s_0\,T_0 + \frac{s_0\,T_0}{s\mathfrak{a}}\,\left[C + \ln s\mathfrak{a} + Z(s\mathfrak{a})\right]\right\}, \tag{5.6}$$

where

$$Z(s\mathfrak{a}) = \int_1^{\infty} \frac{e^{-s\mathfrak{a}x}}{x}\,dx = \int_{s\mathfrak{a}}^{\infty} \frac{e^{-x'}}{x'}\,dx' = -Ei(-s\mathfrak{a}).$$

From the logarithm of the Laplace transform (5.6),

$$\ln \widetilde{p}_{st}(s) = -s_0 T_0 + \frac{s_0\,T_0}{s\mathfrak{a}}\,\left[C + \ln s\mathfrak{a} + Z(s\mathfrak{a})\right] = \widetilde{g}_{st}(s) \tag{5.7}$$

one can calculate the cumulants of the stationary signal $\eta^{(st)}$. However, the determination of the density function $p_{st}(y)$ from the function $\widetilde{p}_{st}(s)$ is a challenging task, with which we will not concern in this work.

5.1.1. Expectation and variance

For the determination of the *expectation* and the *variance* of $\eta(t)$ it appears to be practical to use the following "ad hoc" method instead of the usual standard procedure. Define the function

$$R(a) = 1 - \frac{1-e^{-a}}{a} \tag{5.8}$$

which can be considered as the distribution function of a non-negative random variable **a**. For the calculation of the cumulants, the use of the logarithm of Laplace transform $\widetilde{p}(s,t)$ is expedient:

$$\widetilde{g}(s,t) = \ln \widetilde{p}(s,t) = -s_0 \int_0^t R\left[s\mathfrak{a}\left(1 - \frac{v}{T_0}\right)\right]\,\Delta(T_0 - v)\,dv. \tag{5.9}$$

From this the expectation is derived as

$$\mathbf{E}\left\{\eta(t)\right\} = -\left[\frac{\partial \widetilde{g}(s,t)}{\partial s}\right]_{s=0} = s_0 \int_0^t \left\{\frac{\partial R\left[s\mathfrak{a}\left(1 - \frac{v}{T_0}\right)\right]}{\partial s}\right\}_{s=0} \Delta(T_0 - v)\,dv, \tag{5.10}$$

whereas the variance is given by the formula

$$\mathbf{D}^2\left\{\eta(t)\right\} = \left[\frac{\partial^2 \widetilde{g}(s,t)}{\partial s^2}\right]_{s=0} = s_0 \int_0^t \left\{ \frac{\partial^2 R\left[s\mathfrak{a}\left(1-\dfrac{v}{T_0}\right)\right]}{\partial s^2} \right\}_{s=0} \Delta(T_0 - v)\, dv. \tag{5.11}$$

Since

$$\left[\frac{\partial R(sb)}{\partial s}\right]_{s=0} = \frac{b}{2} \quad \text{and} \quad \left[\frac{\partial^2 R(sb)}{\partial s^2}\right]_{s=0} = \frac{b^2}{3},$$

where

$$b = \mathfrak{a}\left(1 - \frac{v}{T_0}\right),$$

from (5.10) and (5.11) we obtain

$$\mathbf{E}\left\{\eta(t)\right\} = \begin{cases} \frac{1}{2}\, s_0 t\mathfrak{a}\left(1 - \frac{1}{2}\frac{t}{T_0}\right), & \text{if } t \le T_0, \\ \\ \frac{1}{4}\, s_0 T_0 \mathfrak{a}, & \text{if } t > T_0 \end{cases} \tag{5.12}$$

and

$$\mathbf{D}^2\left\{\eta(t)\right\} = \begin{cases} \frac{1}{9}\, s_0 T_0 \mathfrak{a}^2 \left[1 - \left(1 - \frac{t}{T_0}\right)^3\right], & \text{if } t \le T_0, \\ \\ \frac{1}{9}\, s_0 T_0 \mathfrak{a}^2, & \text{if } t > T_0. \end{cases} \tag{5.13}$$

It is seen that the stationary state is reached already at the time $t = T_0$. In this case the Fano factor is equal to

$$\mathcal{F} = \left(\frac{2}{3}\right)^2 \mathfrak{a}. \tag{5.14}$$

5.2. Second version

Consider now the case when the random variable $\boldsymbol{\xi}$ is exponentially distributed, i.e.

$$\mathcal{P}\left\{\xi \le y\right\} = 1 - e^{-\mu y}.$$

Then one has

$$\widetilde{h}(s,t) = \mu \int_0^\infty e^{-xsf(t)}\, e^{-\mu x}\, dx = \frac{\mu}{sf(t) + \mu}, \tag{5.15}$$

where

$$f(t) = \left(1 - \frac{t}{T_0}\right) \Delta(T_0 - t).$$

The Laplace transform

$$\widetilde{p}(s,t) = \int_0^\infty e^{-sy}\, p(y,t)\, dy$$

of the probability density $p(y,t)$ reads as

$$\widetilde{p}(s,t) = \exp\left\{ s_0 \int_0^t \left[\frac{\mu}{s f(t') + \mu} - 1 \right] \, dt' \right\} = \exp\left\{ -s_0 \int_0^t \frac{s f(t')}{s f(t') + \mu} \, dt' \right\},$$

from which, by using the form of $f(t)$ given above, one arrives at

$$g(s,t) = \ln \widetilde{p}(s,t) = -s_0 \int_0^t \frac{s\,(T_0 - t')\,\Delta(T_0 - t')}{s\,(T_0 - t')\,\Delta(T_0 - t') + \mu\, T_0} \, dt'. \tag{5.16}$$

For $t \leq T_0$ one has

$$\ln \widetilde{p}(s,t) = -s_0 \int_0^t \left[1 - \frac{\mu T_0}{s\,(T_0 - t') + \mu T_0} \right] \, dt' =$$

$$-s_0 \left\{ t + \frac{\mu T_0}{s} \left[\ln\left(s\,(T_0 - t) + \mu T_0 \right) - \ln(s T_0 + \mu T_0) \right] \right\} =$$

$$-s_0 t - \ln \left[\frac{s\,(T_0 - t) + \mu T_0}{s T_0 + \mu T_0} \right]^{\frac{s_0 T_0 \mu}{s}} = -s_0 t + \ln \left[\frac{(s + \mu)\, T_0}{s\,(T_0 - t) + \mu T_0} \right]^{\frac{s_0 T_0 \mu}{s}}, \tag{5.17}$$

from which one immediately obtains

$$\widetilde{p}(s,t) = e^{-s_0 t} \left[\frac{(s + \mu)\, T_0}{s\,(T_0 - t) + \mu T_0} \right]^{\frac{s_0 T_0 \mu}{s}}. \tag{5.18}$$

Since $\widetilde{p}(0,t) = 1$, one can show that the following relationship holds:

$$\lim_{s \to 0} \left[\frac{s T_0 + \mu T_0}{s\,(T_0 - t) + \mu T_0} \right]^{\frac{s_0 T_0 \mu}{s}} = e^{s_0 t}.$$

For this we have to perform the rearrangement

$$\left[\frac{s T_0 + \mu T_0}{s\,(T_0 - t) + \mu T_0} \right]^{\frac{s_0 T_0 \mu}{s}} = \left[1 + \frac{st}{(s + \mu) T_0 - st} \right]^{\frac{s_0 T_0 \mu}{s}} =$$

$$\left[1 + \frac{s_0 t}{s_0 (T_0 - t) + s_0 T_0 \mu / s} \right]^{\frac{s_0 T_0 \mu}{s}},$$

from which it is seen that

$$\lim_{s \to 0} \left[1 + \frac{s_0 t}{s_0 (T_0 - t) + s_0 T_0 \mu / s} \right]^{\frac{s_0 T_0 \mu}{s}} = e^{s_0 t}.$$

The calculation of the inversion of the Laplace transform $\widetilde{p}(s,t)$ in (5.18) appears to be a hard task. We disregard from the approximate calculations here.

For $t > T_0$ from (5.16) one obtains

$$g(s,t) = \ln \widetilde{p}(s,t) = -s_0 T_0 - \ln \left[\frac{\mu}{s+\mu} \right]^{\frac{s_0 T_0 \mu}{s}}, \tag{5.19}$$

which is also the *stationary solution* since it does not depend on the variable t. Hence, introducing the notations

$$g(s,t) = g_{st}(s), \qquad \text{and} \qquad \widetilde{p}(s,t) = \widetilde{p}_{st}(s)$$

for t larger than T_0 one obtains from (5.19) the result

$$\widetilde{p}_{st}(s) = \exp \left\{ -s_0\, T_0 + \ln \left(1 + \frac{s}{\mu}\right)^{\frac{s_0 T_0 \mu}{s}} \right\}. \tag{5.20}$$

This can be written in a form which is more suitable for the calculation of the cumulants as

$$\ln \widetilde{p}_{st}(s) = -s_0\, T_0 + \ln \left(1 + \frac{s}{\mu}\right)^{\frac{s_0 T_0 \mu}{s}}. \tag{5.21}$$

For better readability, use the notations

$$\frac{s_0 T_0 \mu}{s} = x, \qquad \text{and thus} \qquad \frac{s}{\mu} = \frac{s_0 T_0}{x},$$

with which it is easy to prove the relationship

$$\lim_{s \to 0} \left(1 + \frac{s}{\mu}\right)^{\frac{s_0 T_0 \mu}{s}} = e^{s_0 T_0}.$$

It is seen that

$$\left(1 + \frac{s}{\mu}\right)^{\frac{s_0 T_0 \mu}{s}} = \left(1 + \frac{s_0 T_0}{x}\right)^{x},$$

which, for $s \to 0$, that is for $x \to \infty$, obviously converges to $e^{s_0 T_0}$.

5.2.1. Expectation and variance

From (5.17) and (5.21) with Mathematica one can easily calculate the expectation and variance of $\eta(t)$ and $\eta^{(st)}$:

$$\mathbf{E}\{\eta(t)\} = \begin{cases} s_0 t \left(1 - \frac{1}{2}\frac{t}{T_0}\right) \frac{1}{\mu}, & \text{if } t \le T_0, \\ \frac{1}{2}\, s_0 T_0\, \frac{1}{\mu}, & \text{if } t > T_0, \end{cases} \tag{5.22}$$

and

$$\mathbf{D}^2\{\eta(t)\} = \begin{cases} 2\, s_0 T_0 \left(1 - \frac{t}{T_0} + \frac{1}{3}\frac{t^2}{T_0^2}\right) \frac{1}{\mu^2}, & \text{if } t \le T_0, \\ \frac{2}{3}\, s_0 T_0\, \frac{1}{\mu^2}, & \text{if } t > T_0. \end{cases} \tag{5.23}$$

6. Conclusions and open questions

In this report the theory of the Campbell method and that of the so-called higher order Campbelling techniques was given for several random pulse shapes. Also the question of the threshold crossing frequency was addressed. Explicit results were derived for a few selected deterministic pulse shapes with random amplitudes of two different distributions.

It was shown that a pre-requisite of applying the Campbell method is that the particle arrivals to the detector constitute a Poisson process, and their responses are independent. In the reality these conditions are usually not fulfilled.

It appears therefore interesting to raise the question whether it is possible to define a theoretical model and calculate the distribution function of the detector signal if the individual responses are not independent. A further open question is whether interaction between the charges generated by the ionising particles, in the present case by the fission products, can be modelled by response functions, characterized by probability distributions. Some of these questions will be addressed in further work.

The majority of publications in this field in the past few years show that the authors usually seek only the mean current of the chamber with different Monte Carlo codes, but they do not concern with simple models capable of supplying exact results. The explicit results presented in this report can serve to benchmark the accuracy of the higher order moments of the Monte Carlo simulation results.

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