Block-Fading Channels at Finite Blocklength

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***Abstract*—This tutorial paper deals with the problem of characterizing the maximal achievable rate $R^*(n,\epsilon)$ at a given blocklength n and error probability ϵ over block-fading channels. We review recent results that establish tight bounds on $R^*(n,\epsilon)$ and characterize its asymptotic behavior. Comparison between the theoretical results and the data rates achievable with the coding scheme used in LTE-Advanced are reported.**

I. INTRODUCTION

Channel capacity dictates the largest data rate at which reliable communication, i.e., communication with vanishing error probability, is possible [1, Ch. 7]. In a scenario where the random fading channel stays constant for the duration of each transmit codeword, capacity is zero for many fading distributions of practical interest, such as Rayleigh, Rician, and Nakagami, because reliable communication cannot be guaranteed for any positive data rate. In this scenario, a more appropriate performance metric may be the *outage* capacity (also known as ϵ-capacity [2]), which is the maximal achievable data rate for a given positive error probability. Although both channel capacity and outage capacity are concerned with the asymptotic regime of codeword length going to infinity, they have been traditionally used as benchmarks for the performance of coding schemes.

In emerging applications such as machine-type and vehicle-to-vehicle communications, the requirement of long codewords is often too stringent, and short codewords are needed to fulfill latency constraints. Indeed, the 4G wireless standard LTE-Advanced employs codes with blocklength as short as 100 symbols [3, Sec. 5.1.3]. For such short blocklengths, channel capacity and outage capacity are poor benchmarks; a more meaningful benchmark is the maximal achievable rate $R^*(n,\epsilon)$ for a given blocklength n and block error probability ϵ.

The aim of this paper is:

i) to review recent progress in the derivation of bounds on $R^*(n,\epsilon)$ for block-fading channels [4], [5];
ii) to compare the performance of the coding schemes used in LTE-Advanced with these bounds.

II. THE AWGN CHANNEL

Building upon classic asymptotic results, Polyanskiy, Poor, and Verdú showed recently that for the (non-fading) AWGN channel with capacity $C = \log(1+\rho)$, where ρ denotes the SNR, the maximal achievable rate can be tightly approximated by [6]

$$R^*(n,\epsilon) = C - \sqrt{\frac{V}{n}}Q^{-1}(\epsilon) + \mathcal{O}\left(\frac{\log n}{n}\right). \tag{1}$$

Here, $Q^{-1}(\cdot)$ denotes inverse of the Gaussian Q-function, $V = \rho(2+\rho)/(1+\rho)^2$ is the *channel dispersion* [6, Def. 1], and $f(x) = \mathcal{O}(g(x))$ means that $\limsup_{x\to\infty} |f(x)/g(x)| < \infty$. The approximation (1) is established by characterizing the asymptotic behavior of analytically tractable achievability and converse bounds (see [6, Sec. III]). It implies that the penalty (with respect to C) required to sustain the desired error probability ϵ at a finite blocklength n is—up to first order—proportional to $1/\sqrt{n}$.

III. THE BLOCK-FADING MODEL

The block-fading channel model assumes that the channel coefficients remain constant for a block of T consecutive symbols and change to an independent realization in the next block [7]. The parameter T can be thought of as the channel's *coherence time*, or more generally, the number of time-frequency slots over which the channel stays constant. A codeword of length $n = LT$ spans L independent channel realizations.

When channel state information (CSI) is available at the receiver (but not at the transmitter), the maximal achievable rate $R^*_{\text{csi}}(n,\epsilon)$ is asymptotically given by [4], [8]

$$R^*_{\text{csi}}(n,\epsilon) = C_{\text{csi}} - \sqrt{\frac{V_{\text{csi}}}{n}}Q^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right). \tag{2}$$

Here, $C_{\text{csi}} = \mathbb{E}_H\left[\log(1+\rho|H|^2)\right]$ is the channel capacity (where $|H|^2$ is the random channel gain),

$$V_{\text{csi}} = T\,\mathbb{V}\text{ar}[\log(1+\rho|H|^2)] + 1 - \mathbb{E}^2\left[\frac{1}{1+\rho|H|^2}\right] \tag{3}$$

is the channel dispersion, and $f(x) = o(g(x))$ means that $\lim_{x\to\infty}|f(x)/g(x)| = 0$. We see from (2) and (3) that a small T and, hence, fast channel dynamics and large diversity order, is advantageous when CSI is available at the receiver, because it yields a small dispersion V_{csi} and therefore a small penalty term in (2).

When CSI is not *a priori* available at the transmitter or the receiver—a setup that accounts for the cost of learning the fading channel [9], [10]—the analysis is more involved. Indeed, no closed-form expression is available to date even for capacity, although the structure of the capacity-achieving

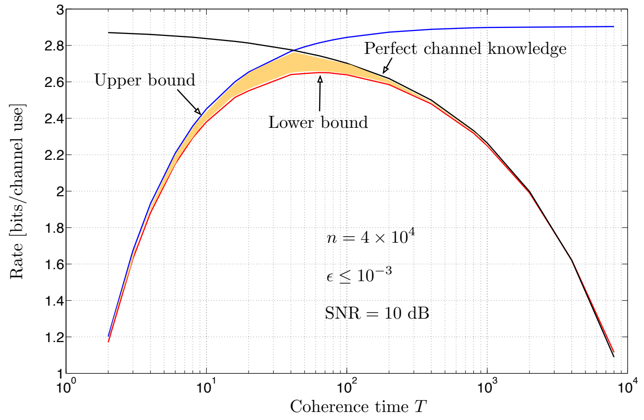

Fig. 1. Achievability and converse bounds for the block-fading Rayleigh-fading channel with no a priori CSI available at the transmitter and at the receiver; $n = 4 \times 10^4$, SNR = 10 dB, and $\epsilon = 10^{-3}$.

distribution is known for some fading distributions [7], [11]. Computationally tractable upper and lower bounds on $R^*(n, \epsilon)$ were recently developed in [4] for the Rayleigh-fading case. Fig. 1, which is taken from [4], shows upper and lower bounds on $R^*(n, \epsilon)$ together with the approximation (2) for $R^*_{\text{csi}}(n, \epsilon)$ as a function of the channel's coherence time T for a blocklength $n = 4 \times 10^4$. The SNR is set to 10 dB and the frame error rate ϵ is 10^{-3}. We see that for a given blocklength and error probability, the maximal achievable rate $R^*(n, \epsilon)$ is not monotonic in the channel's coherence time, and there exists a rate-maximizing coherence time that optimally trades between diversity and cost of estimating the channel.

IV. The Quasi-Static Fading Model

For the case of short data packets, it is reasonable to assume that the fading channel does not vary over the transmission of a codeword. This leads to the so-called quasi-static fading model, originally proposed in [12], which corresponds to a block-fading model with $L = 1$. In the quasi-static case, $R^*(n, \epsilon)$ can be characterized more accurately than for the general block-fading model, as we shall next review. Let $R^*_{\text{rt}}(n, \epsilon)$ be the maximal achievable rate for the case of perfect CSI at the transmitter and the receiver (CSIRT), and let $R^*_{\text{no}}(n, \epsilon)$ be the maximal achievable rate for the case of no CSI (neither at the transmitter nor at the receiver). Clearly,

$$R^*_{\text{no}}(n, \epsilon) \leq R^*_{\text{rt}}(n, \epsilon). \tag{4}$$

We next present two computationally and analytically tractable bounds on $R^*_{\text{rt}}(n, \epsilon)$ and $R^*_{\text{no}}(n, \epsilon)$ that were recently established in [5]. We start with an achievability bound on $R^*_{\text{no}}(n, \epsilon)$.

Theorem 1 ([5, Cor. 3]): Let $\boldsymbol{x}_0 \triangleq \left[\sqrt{\rho}\,\sqrt{\rho}\,\cdots\,\sqrt{\rho}\right]^{\mathsf{T}}$ and denote by $\boldsymbol{Y}$ the output vector induced by the input $\boldsymbol{x}_0$ through the channel. Then, for every $0 < \epsilon < 1$ and every $0 < \tau < \epsilon$, the maximal achievable rate $R^*_{\text{no}}(n, \epsilon)$ is lower-bounded by

$$R^*_{\text{no}}(n, \epsilon) \geq \log \tau - (n-1) \log \gamma_n \tag{5}$$

where $\gamma_n \in [0, 1]$ is chosen so that

$$\mathbb{P}\left[\frac{\left|\boldsymbol{x}_0^{\mathsf{H}} \boldsymbol{Y}\right|^2}{\|\boldsymbol{x}_0\|^2 \|\boldsymbol{Y}\|^2} \geq 1 - \gamma_n\right] = 1 - \epsilon + \tau. \tag{6}$$

Outline of the proof: In the absence of noise, the received vector $\boldsymbol{Y}$ is a scaled version of the transmitted codeword. The scaling factor is the channel gain, which is unknown to the decoder because we assumed no CSI. We exploit this geometry by using a decoder that measures the angle between each codeword and the received vector, and selects the codewords whose angle is below a certain threshold. If only one codeword is selected, then this codeword is declared to be the one that was transmitted. Otherwise, an error is declared. The bound (5) follows from the $\kappa\beta$-bound [6, Th. 25] applied to a channel whose output is the linear subspace spanned by $\boldsymbol{Y}$. $\square$

We next give a converse bound on $R^*_{\text{rt}}(n, \epsilon)$.

Theorem 2 ([5, Th. 1]): Let

$$L_n \triangleq n \log(1+\rho|H|^2) + \sum_{i=1}^{n} \left(1 - \left|\sqrt{\rho}|H|Z_i - \sqrt{1+\rho|H|^2}\right|^2\right) \tag{7}$$

and

$$S_n \triangleq n \log(1+\rho|H|^2) + \sum_{i=1}^{n} \left(1 - \frac{\left|\sqrt{\rho}|H|Z_i - 1\right|^2}{1+\rho|H|^2}\right) \tag{8}$$

where $\{Z_i\}_{i=1}^n$ are independent and identically distributed according to a zero-mean unit-variance circularly symmetric Gaussian distribution. For every n and every $0 < \epsilon < 1$, the maximal achievable rate $R^*_{\text{rt}}(n, \epsilon)$ for a quasi-static SISO fading channel with channel gain $|H|^2$ is upper-bounded by

$$R^*_{\text{rt}}(n-1, \epsilon) \leq \frac{1}{n-1} \log \frac{1}{\mathbb{P}[L_n \geq n\gamma_n]} \tag{9}$$

where γ_n is the solution of

$$\mathbb{P}[S_n \leq n\gamma_n] = \epsilon. \tag{10}$$

Outline of the proof: The converse bound (9) is based on the meta-converse theorem [6, Th. 30]. As *auxiliary channel*, we take one whose output $\boldsymbol{Y}$ is independent of the transmit codeword, with the entries of $\boldsymbol{Y}$ being conditionally independent and identically distributed complex Gaussian random variables, with zero mean and variance $(1 + \rho|H|^2)$ given the channel gain H. $\square$

The bounds in Theorems 1 and 2 match up to a $\mathcal{O}(\log(n)/n)$ term for a wide class of fading distributions [5]. Furthermore, it can be shown that [5]

$$\{R^*_{\text{rt}}(n, \epsilon), R^*_{\text{no}}(n, \epsilon)\} = C_\epsilon + \mathcal{O}\left(\frac{\log n}{n}\right). \tag{11}$$

Here, C_ϵ is the outage capacity given by [2, Th. 6]

$$C_\epsilon = \sup\{R : \mathbb{P}[\log(1+\rho|H|^2) \leq R] \leq \epsilon\}. \tag{12}$$

The result in (11) entails that, for the quasi-static fading case, the $1/\sqrt{n}$ penalty term is absent (compare (11) with (1) and (3)). In other words, the dispersion of quasi-static fading channels is zero.

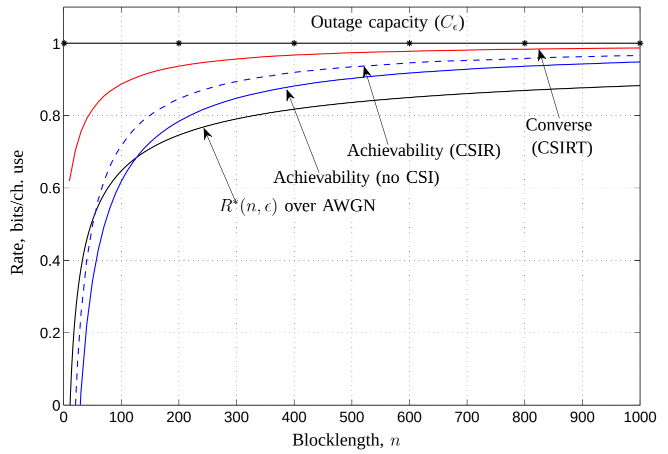

Fig. 2. Achievability and converse bounds for the quasi-static SIMO Rician-fading channel with K-factor equal to 20 dB, two receive antennas, $\text{SNR} = -1.55$ dB, and $\epsilon = 10^{-3}$.

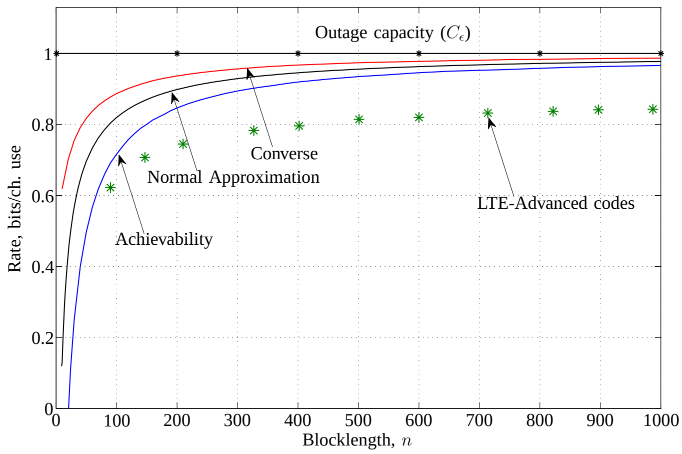

Fig. 3. Comparison between achievability and converse bounds and rate achievable with the coding schemes in LTE-Advanced. We consider a quasi-static SIMO Rician-fading channel with K-factor equal to 20 dB, two receive antennas, $\text{SNR} = -1.55$ dB, $\epsilon = 10^{-3}$, and receive CSI. The star-shaped markers indicate the rates achievable by the turbo codes in LTE-Advanced (QPSK modulation and ten iterations of a max-log-MAP decoder [15]).

Normal Approximation: A quasi-static channel is conditionally ergodic given the channel gain H. Hence, the asymptotic rate characterization (1) can be used as a basis for developing a simple-to-evaluate approximation $R^*_{\text{approx}}(n,\epsilon)$ for both $R^*_{\text{rt}}(n,\epsilon)$ and $R^*_{\text{no}}(n,\epsilon)$. Specifically, we shall take $R^*_{\text{approx}}(n,\epsilon)$ as the solution of

$$\epsilon = \mathbb{E}\left[Q\left(\frac{C(H) - R^*_{\text{approx}}(n,\epsilon)}{\sqrt{V(H)/n}}\right)\right] \tag{13}$$

where

$$C(H) = \log(1 + \rho|H|^2) \tag{14}$$

and

$$V(H) = \frac{\rho|H|^2(2+\rho|H|^2)}{(1+\rho|H|^2)^2}. \tag{15}$$

As we shall show by means of numerical simulations, this approximation is accurate.

Extension to Multiple-Antenna Systems: The bounds in Theorems 1 and 2 and the zero-dispersion result extend to channels with multiple antennas at the receiver (see [5, Th. 1 and Cor. 3]) and general multiple-input multiple-output systems under various assumption on the CSI availability [13].

Numerical Results: Fig. 2, which is taken from [5], shows the achievability bound and the converse bound for a quasi-static SIMO Rician-fading channel with two receive antennas and Ricean K-factor equal to 20 dB. The SNR is set to -1.55 dB and the frame error rate ϵ is 10^{-3}. For reference, we also plot a lower bound on $R^*_{\text{rt}}(n,\epsilon)$ obtained by using the $\kappa\beta$ bound [6, Th. 25] and assuming CSIR, together with the approximation (1) for $R^*(n,\epsilon)$ corresponding to an AWGN channel with the same capacity. It can be seen from this figure that the blocklength required to achieve 90% of capacity in the quasi-static fading channel is about an order of magnitude smaller compared to the blocklength required for an AWGN channel, which is about 1420.

Comparison with LTE-Advanced codes: Our achievability and converse bounds, as well as the normal approximation, can be used to benchmark the coding schemes adopted in current standards. In Fig. 3, we compare the performance of the coding schemes used in LTE-Advanced [3, Sec. 5.1.3.2], against our bounds for the same scenario as in Fig. 2. More specifically, we show in Fig. 3 the performance of a family of turbo codes combined with QPSK modulation. The decoder employs a max-log-MAP decoding algorithm [15] with ten iterations. We further assume that the decoder has perfect CSI. For the AWGN case, it was observed in [6, Fig. 12] that about half of the gap between the rate achieved by the best available channel codes[1] and capacity is due to the $(1/\sqrt{n})$-penalty in (1), and the other half is due to the suboptimality of the codes. From Fig. 3, we notice that for quasi-static fading channels, while the finite-blocklength penalty is significantly reduced (because of the zero-dispersion effect), the penalty due to the code suboptimality remains. In fact, we see that the gap between the rate achieved by LTE-Advanced turbo codes and our bound is approximately constant up to a blocklength of 1000.

In LTE-Advanced systems, hybrid automatic repeat request (HARQ) is employed to compensate for the event of deep fading. In this scenario, a typical frame error rate is 10^{-1} [16, p. 218]. To account for this, in Fig. 4, we set $\epsilon = 10^{-1}$, $\rho = 2.74$ dB, and consider Rayleigh fading (the other parameters are set as in Fig. 3). Again, we observe that there is a constant gap between the rate achieved by the LTE-Advanced turbo codes and our bounds.

[1]The codes used in [6, Fig. 12] are a certain family of multiedge low-density parity-check (LDPC) codes, whose performance is superior to that of the turbo codes used in LTE-Advanced.

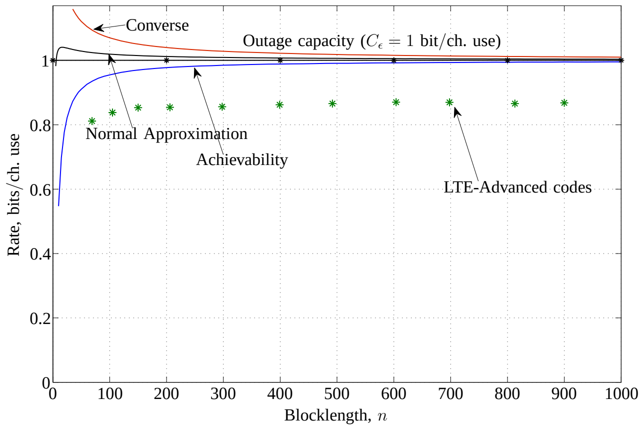

Fig. 4. Comparison between the achievability and converse bounds and the rates achievable by the coding schemes in LTE-Advanced. We consider a quasi-static SIMO Rayleigh-fading channel with two receive antennas, SNR $= 2.74$ dB, $\epsilon = 0.1$, and receive CSI . The star-shaped markers indicate the rates achievable by the turbo codes in LTE-Advanced (QPSK modulation and ten iterations of a max-log-MAP decoder [15]).

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