Effects of the Second Harmonic on the Geodesic Acoustic Modes in Electron Scale Turbulence

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Outline

• Goals and motivation
• Electron Temperature Gradient Modes
• Geodesic Acoustic Modes (GAM)
• Non-linearly Driven GAM
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Goals and motivation

• Goal 1: Derive the corrections to the dispersion relation for the electron branch of the GAM originating from coupling to the $m=2$ mode.
• Motivation: In simulations growth rates of the GAMs are significantly influenced by the $m=2$ interactions.
Motivation

• In the ASDEX-U tokamak, a periodic modulation of flow and turbulence level, with the characteristics of a limit cycle oscillation at the geodesic acoustic mode (GAM) frequency, has been observed preceding the L-H transition in low density plasmas. [1]

• A complex interaction between turbulence driven ExB zonal flow oscillations, i.e., geodesic acoustic modes (GAMs), the turbulence, and mean equilibrium flows is observed during the low to high (L-H) plasma confinement mode transition. [1]

• It was observed that GAMs are only somewhat less effective than the residual zonal flow in providing the non-linear saturation. [2]

1) G. Conway et al PRL 2011
2) Waltz et al PoP 2008
Generation of the GAM

- The basic mechanism is found in the poloidal component of the equation for parallel motion.
- In toroidal geometry, a coupling between the $n=0=m$ electric potential perturbation and the $m=1$, $n=0$ density perturbation is established by toroidicity resulting in the GAM.
Electron Temperature Gradient Modes

We use the electron continuity and energy equations adapted from the Braginskii’s model

\[
\frac{dn_e}{dt} + \nabla \cdot (n_e \vec{v}_E + n_e \vec{v}_{*e}) + \nabla \cdot (n_e \vec{v}_{pe} + n_e \vec{v}_{\pi e}) + \nabla \cdot (n_e \vec{v}_{le}) = 0
\]

\[
\frac{3}{2} n_e \frac{dT_e}{dt} + n_e T_e \nabla \cdot \vec{v}_e + \nabla \cdot \vec{q}_e = 0
\]

where \( \vec{q}_e = -\frac{5}{2} \frac{p_e}{2 m_e \Omega_e} \vec{e}_\parallel \times \nabla T_e \) and \( \frac{d}{dt} = \frac{\partial}{\partial t} + \rho_e c_e \vec{e} \times \nabla \phi \cdot \nabla \)

\[
\nabla^2 \tilde{A}_\parallel = -\frac{4\pi}{c} \tilde{J}_\parallel
\]

Ion and impurity non-adiabatic response with

\[
\tilde{n}_i = \delta n / n_{i0}, \quad \tilde{n}_1 = \delta n_{10} / n_1, \quad \tilde{\phi} = e\phi / T_e
\]

\[
\tilde{n}_j = -\left( \frac{z \tau_j}{1 - \omega^2 / (k_{\perp c_j}^2)} \right) \tilde{\phi}
\]

\[
z_{\text{eff}} \approx (n_i + z^2 n_1) / n_e.
\]
Normalized Electron Equations

\[- \frac{\partial \tilde{n}_e}{\partial t} - \nabla_\perp^2 \frac{\partial}{\partial t} \tilde{\phi} - (1 + (1 + \eta_e)\nabla_\perp^2) \frac{\partial \tilde{\phi}}{\partial y} - \nabla_\parallel \nabla_\perp \tilde{A}_\parallel + \]

\[\varepsilon_n \left( \cos \theta \frac{1}{r} \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial x} \right) (\tilde{\phi} - \tilde{n}_e - \tilde{T}_e) = 0 \]

\[\left( \frac{\beta_e}{2} - \nabla_\perp^2 \right) \frac{\partial}{\partial t} + (1 + \eta_e)(\beta_e/2) \frac{\partial}{\partial y} \tilde{A}_\parallel + \nabla_\parallel (\tilde{\phi} - \tilde{n}_e - \tilde{T}_e) = 0 \]

\[\frac{\partial \tilde{T}_e}{\partial t} + \frac{5}{3} \varepsilon_n \left( \cos \theta \frac{1}{r} \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial x} \right) \frac{1}{r} \frac{\partial}{\partial y} \tilde{T}_e + (\eta_e - \frac{2}{3}) \frac{1}{r} \frac{\partial}{\partial y} \tilde{\phi} - \frac{2}{3} \frac{\partial \tilde{n}_e}{\partial t} = 0 \]

\[ (\tilde{\phi}, \tilde{n}_e, \tilde{T}_e) = \left( \frac{L_n}{\rho_e} (e \delta \phi) / T_{e0}, \delta n_e / n_0, \delta T_e / T_{e0} \right) \]

\[ \tilde{A}_\parallel = (2e / \beta_e c \rho_e) eA_\parallel / T_{e0} \]

\[ \beta_e = 8\pi nT_e / B_0^2 \]

Anderson et al PoP 2012 and IAEA 2012
The $m=0$, $m=1$ and $m=2$ Equations

\begin{align*}
- \nabla^2_{\perp} \frac{\partial}{\partial t} \tilde{\phi}^{(0)} - \varepsilon_n \sin \theta \frac{\partial}{\partial x} (\tilde{n}_e^{(1)} + \tilde{T}_e^{(0)}) &= 0 \quad \text{m=0} \\
- \frac{\partial \tilde{n}_e^{(1)}}{\partial t} - \nabla \nabla_{\parallel} \tilde{A}_{\parallel}^{(1)} - \varepsilon_n \sin \theta \frac{\partial}{\partial x} (-\tilde{\phi}^{(0)} + \tilde{n}_e^{(2)} + \tilde{T}_e^{(2)}) &= 0 \\
\left( (\beta_e/2 - \nabla_{\parallel}^2) \frac{\partial}{\partial t} \right) \tilde{A}_{\parallel}^{(1)} - \nabla \tilde{n}_e^{(1)} + \tilde{T}_e^{(1)} &= 0 \quad \text{m=1} \\
\frac{\partial \tilde{T}_e^{(1)}}{\partial t} - \frac{2}{3} \frac{\partial \tilde{n}_e^{(1)}}{\partial t} &= 0 \\
- \frac{\partial \tilde{n}_e^{(2)}}{\partial t} - \nabla \nabla_{\parallel} \tilde{A}_{\parallel}^{(2)} - \varepsilon_n \sin \theta \frac{\partial}{\partial x} (\tilde{n}_e^{(1)} + \tilde{T}_e^{(1)}) &= 0 \\
\left( (\beta_e/2 - \nabla_{\parallel}^2) \frac{\partial}{\partial t} \right) \tilde{A}_{\parallel}^{(2)} - \nabla \tilde{n}_e^{(2)} + \tilde{T}_e^{(2)} &= 0 \quad \text{m=2} \\
\frac{\partial \tilde{T}_e^{(2)}}{\partial t} + \frac{5}{3} \varepsilon_n \sin \theta \frac{\partial}{\partial r} T_e^{(1)} - \frac{2}{3} \frac{\partial \tilde{n}_e^{(2)}}{\partial t} &= 0
\end{align*}
Solving for the el-GAM dispersion relation starting with the m=2 equations

There is a simple relation between the m=2 density perturbation and the m=1 density and temperature perturbation

\[ \tilde{n}_e^{(2)} = -\frac{\varepsilon_n q_r}{\Omega_q} \sin \theta \left( \frac{1}{C_1} \tilde{n}_e^{(1)} + \tilde{T}_e^{(1)} \right) \]

\[ C_1 = \frac{5}{3} \frac{q_{\parallel}^2 q_{\perp}^2}{\Omega_q^2 (\frac{\beta_e}{2} + q_{\perp}^2)} \]

We can use this to compute a relation between the m=1 density and temperature perturbations, where \( C_1 = 1 \) and \( C_0 = \frac{2}{3} \) if the m=2 contributions are neglected,

\[ \tilde{T}_e^{(1)} = \frac{2}{3} \frac{1 - \frac{5}{3} \frac{\varepsilon_n^2 q_r^2}{\Omega_q^2 C_1}}{1 - \frac{5}{3} \frac{\varepsilon_n^2 q_r^2}{\Omega_q^2}} \sin^2 \theta \tilde{n}_e^{(1)} \]

\[ C_0 = \frac{2}{3} \frac{1 - \frac{5}{3} \frac{\varepsilon_n^2 q_r^2}{\Omega_q^2 C_1}}{1 - \frac{5}{3} \frac{\varepsilon_n^2 q_r^2}{\Omega_q^2}} \sin^2 \theta \]
GAM Dispersion Relation

To find the dispersion relation, we must determine the relation between the m=1 components and the potential

\[ \tilde{n}_e^{(1)} = \frac{\mathcal{E}_n q_r \sin \theta}{\Omega q} \tilde{\phi}^{(0)} \]

\[ 1 - \frac{q_{\parallel} q_{\perp}}{\Omega_q^2} \frac{q_{\parallel} q_{\perp}}{1 + C_0} + \frac{5}{3} \frac{\mathcal{E}_n^2 q_r^2}{\Omega_q^2 C_1} \frac{\sin^2 \theta}{\beta_e} + q_{\perp}^2 \]

The dispersion relation is now:

\[ 1 - \frac{q_{\parallel}^2 q_{\perp}^2}{\Omega_q^2} \frac{1 + C_0}{\beta_e} + \frac{5}{3} \frac{\mathcal{E}_n^2 q_r^2}{\Omega_q^2 C_1} \langle \sin^2 \theta \rangle = \frac{q_r^2 \mathcal{E}_n^2}{q_{\perp}^2 \Omega_q^2} (1 + C_0) \langle \sin^2 \theta \rangle \]

Note that this dispersion relation reduces to the previous relation (see page 11) for \( C_1 = 1 \) and \( C_0 = 2/3 \).
Solutions to the linear dispersion relation including m=2 effects

The linear dispersion relation w/o m=2 contributions is

\[ \Omega_q^2 = \frac{5}{3} \frac{c_e^2}{R^2} \left( 2 + \frac{1}{q^2} \frac{1}{1 + \beta_e/(2q_r^2)} \right) \]

Figure 1. (color online) The linear el-GAM real frequency (with \( m = 2 \) harmonics included in black line and without represented by the red line) normalized to \( (c_e/R) \) as a function of the safety factor \( \tilde{q} \) is shown for the parameter \( \eta_e = 4.0 \) whereas the remaining parameters are \( \epsilon_n = 0.909, \beta = 0.01, q_x \rho_e = 0.3 \) in the strong ballooning limit \( g(\theta) = 1 \).
Summary

• A first derivation of the effects of higher harmonics on the electron branch of the Geodesic Acoustic Mode (el-GAM) is presented.

• An analytical dispersion relation for the el-GAM growth rate was derived.

• Allowing for interactions with the higher harmonics ($m = 2$) components moderates the decrease in the frequency. This effect is due to the third term on the left hand side arising from the $m = 2$ higher harmonics. Note that, the $C0$ term describes the effect of including temperature perturbations in the system and would vanish if these could be neglected.