The Information-Theoretic Cost of Learning Fading Channels

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Front Cover: A MIMO fading channel with no a priori channel state information.

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To my parents and Wanlu

“The art of doing mathematics consists in finding that special case which contains all the germs of generality.”

– David Hilbert
Abstract

Recent results in communication theory suggest that significant throughput gains in wireless fading networks can be achieved by exploiting network coordination (e.g., CoMP, network MIMO, and interference alignment), provided that each node in the network has perfect channel knowledge. In practice, however, the channels are not known a priori and must be estimated. Lack of a priori channel knowledge determines a penalty on the throughput compared to the case where perfect channel knowledge is available.

In this thesis, we take a fresh look at the problem of learning fading channels. We characterize the cost of learning fading channels in an information-theoretic way by determining the maximal achievable rate over point-to-point fading channels under the assumption that neither the transmitter nor the receiver have a priori channel knowledge.

Paper A and Paper B characterize the capacity of fading channels in the high signal-to-noise ratio (SNR) regime. Specifically, in paper A, we establish an upper bound on the capacity pre-log (i.e., the asymptotic ratio between capacity and the logarithm of SNR as SNR goes to infinity) of Rayleigh-fading correlated block-fading single-input multiple-output (SIMO) channels. The upper bound matches the lower bound reported in Riegler et al. (2011), and, hence, yields a complete characterization of the SIMO capacity pre-log, provided that the channel covariance matrix satisfies a mild technical condition. In Paper B, we characterize the capacity of Rayleigh-fading constant block-fading multiple-input multiple-output (MIMO) channels in the high SNR regime, and provide the input distribution that achieves capacity up to a term that vanishes as SNR grows large.

The insights gained from the high-SNR analysis allows us to obtain tight bounds on the maximal achievable rate $R^*(n, \epsilon)$ for a given blocklength $n$ and block error probability $\epsilon$ at finite SNR. Specifically, in Paper C, we establish upper and lower bounds on $R^*(n, \epsilon)$ for a single-antenna Rayleigh-fading constant block-fading channel. Our results show that for a given block-length and error probability, $R^*(n, \epsilon)$ is not monotonic in the channel coherence time, but there exists a rate maximizing coherence time that optimally trades between diversity and cost of estimating the channel. Finally, in Paper D, we consider the special case when the fading channel does not vary over the transmission of each codeword (quasi-static fading channels). We characterize the channel dispersion in the SIMO setting, and provide tight bounds on $R^*(n, \epsilon)$ together with an easy-to-compute approximation.

Keywords: Shannon capacity, outage capacity, block-fading channel, finite blocklength, quasi-static fading, multiple antennas
List of Included Publications

The thesis is based on the following appended papers:


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Wei Yang
Gothenburg, August 2013
Acronyms

3GPP: 3rd Generation Partnership Project
AWGN: Additive white Gaussian noise
BSTM: Beta-variate space time modulation
cdf: Cumulative distribution function
CoMP: Coordinated multi-point
CSI: Channel state information
DT: Dependence testing
MIMO: Multiple-input multiple-output
pdf: Probability density function
SIMO: Single input multiple output
SISO: Single input single output
SNR: Signal to noise ratio
SVD: Singular value decomposition
USTM: Unitary space time modulation
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Part I

Overview
Chapter 1

Introduction

The surge in the use of broadband services in the public and private sectors poses high demands on future wireless communication networks, from the core (backhaul) to the periphery (cellular base stations). Ericsson, a wireless network manufacturer, predicts that mobile data traffic will have to double every year to match future demands [1]. At the same time, wireless networks also need to provide ubiquitous connectivity for the end user at a minimum guaranteed data rate. It is widely agreed that these throughput gains as well as the expectation of ubiquitous user experience can only be achieved by a dense and heterogeneous deployment of the wireless network infrastructure. Under this scenario, the availability of effective interference management schemes becomes crucial.

Recent results in communication theory suggest that throughput gains of up to two orders of magnitude are obtainable by exploiting interference management techniques such as CoMP [2], network MIMO [3], and interference alignment [4]. However, over-the-air trials have only demonstrated disappointingly low throughput improvements (typically not exceeding 30%) [2] or no improvements at all [5].

One possible explanation for this disconnect between theory and practice lies in the assumption of perfect channel state information (CSI) under which state-of-the-art interference management techniques are developed. This assumption is not realistic because, in most wireless communication networks, CSI is not known a priori at each node and must be estimated, for example through the transmission of pilot symbols. The time and/or bandwidth spent sending pilot symbols is not negligible, particularly in systems with a large number of nodes or antenna elements. Also, determining the amount of resources to be used for channel estimation is crucial in the design of wireless communication systems operating over fading channels. It is therefore of great importance to understand the cost of learning fading channels.

In this thesis, we take a fresh look at the problem of learning fading channels. Determining the Shannon capacity [6, Ch. 7] of fading channels under the assumption that neither the transmitter nor the receiver have a priori channel knowledge is a fundamental way to assess the cost of learning fading channels. We emphasize that in our setup of no a priori channel knowledge, the receiver is allowed to try and estimate the channel. The employed communication strategy does not necessarily need to be noncoherent. The
typical approach of transmitting pilot symbols to estimate the channel is simply viewed
as a specific form of coding. Hence, capacity in the absence of a priori CSI is the ultimate
limit on the rate of reliable communication over fading channels, regardless of whether
channel estimation is performed or not.

Unfortunately, capacity in the absence of a priori CSI is notoriously difficult to char-
acterize analytically and no closed-form expressions are available to date, even for the
simplest channel models [7, 8]. Most of the results in this area pertain to the asymptotic
regime of low or high signal-to-noise ratio (SNR) [9–12].

In order to study channel capacity of fading channels with no a priori CSI, we need
to specify a channel model that describes the channel variations in time. In this the-
esis, we consider two models, namely, the constant block-fading model and the corre-
lated block-fading model. In the constant block-fading model, the fading process takes on inde-
pendent realizations across blocks of a certain length, and remains constant within each
block. This model is accurate for systems employing frequency hopping or time-division
multiple access, and is widely used in the literature due to its simplicity [8, 9, 13]. The
correlated block-fading model, which was first introduced in [14], generalizes the constant
block-fading model by allowing more general forms of temporal correlation between dif-
ferent fading gains within each block. Hence, channel variations in time can be captured
in a more accurate way than with the constant block-fading model. The correlated block-
fading model arises naturally in the analysis of cyclic-prefix orthogonal frequency-division
multiplexing (OFDM) systems operating over frequency-selective channels [12, 15].

Channel capacity is a relevant performance metric only when sufficiently long code-
words are used. In emerging applications such as machine-type and vehicle-to-vehicle
communications, the requirement of long codewords is often too stringent, and short
codewords are needed to meet the delay constraints. Indeed, the 4G wireless standard
LTE-Advanced employs codes with blocklengths as short as 100 symbols [16, Sec. 5.1.3].
For such short blocklengths, the fundamental limit of practical interest is the maximal
achievable rate \( R^*(n, \epsilon) \) for a given blocklength \( n \) and block error probability \( \epsilon \).

In the nonasymptotic regime of finite blocklength, there are no exact formulas for
the maximal achievable rate as a function of blocklength and error probability. In fact,
the exact computation of \( R^*(n, \epsilon) \) is an NP-hard problem [17], and is prohibitive already
for very short blocklengths. Nevertheless, computable upper and lower bounds on the
maximal achievable rate are available in the literature [18–23]. Of particular interest for
this thesis are the new bounds recently developed by Polyanskiy, Poor and Verdú [23]
such as the meta-converse bound, the dependence-testing bound, and the \( \kappa/\beta \) bound.
Using these bounds, they showed that for various channels with positive capacity \( C \), the
maximal achievable rate can be tightly approximated by

\[
R^*(n, \epsilon) \approx C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon). \tag{1.1}
\]

Here, \( Q^{-1}(\cdot) \) denotes the inverse of the complementary cumulative distribution function
(cdf) of a standard normal random variable, and \( V \) is the channel dispersion (see [23,
Def. 1] or (2.17)).
1.1 Objectives

The aim of this thesis is to study—under the assumption of no a priori CSI—the capacity and the maximal achievable rate at finite blocklength of point-to-point fading channels. To this end, we first study the capacity in the high SNR regime. For the correlated block-fading model, we determine the pre-log (also known as number of degrees of freedom or multiplexing gain), which is defined as the asymptotic ratio between capacity and the logarithm of the SNR as SNR goes to infinity, in the single-input multiple-output (SIMO) setting. For the constant block-fading model, we characterize the capacity of multiple-input multiple-output (MIMO) channel in the high SNR regime and provide the input distribution that achieves capacity up to a term that vanishes as SNR grows large. The insights gained from the high-SNR analysis allow us to obtain tight bounds on the maximal achievable rate $R^*(n, \epsilon)$ at finite SNR and at finite blocklength. Specifically, we establish upper and lower bounds on $R^*(n, \epsilon)$ for a single-input single-output (SISO) constant block-fading channel at finite SNR and at finite blocklength. Finally, for the special case that the fading channel does not vary over the transmission of a codeword (quasi-static fading channels), we characterize the dispersion of the channel in the SIMO setting and provide tight bounds as well as an easy-to-compute approximation for $R^*(n, \epsilon)$.

1.2 Thesis Outline

The thesis is organized as follows: in Chapter 2, we introduce the fading channel models studied in this thesis, define the fundamental information-theoretic limits, and review previous results. In Chapter 3, we present the information-theoretic tools we shall use in this thesis. Finally, in Chapter 4, we summarize our contributions.
Chapter 2

Channel Model, Fundamental Limits, and Known Results

This thesis deals with the problem of characterizing the maximal achievable rate over point-to-point fading channels. In Section 2.1, we introduce the two channel models that will be studied in this thesis, namely, the constant block-fading model and the correlated block-fading model. In Section 2.2, we define the notion of channel capacity, outage capacity, and maximal achievable rate at finite blocklength, and review some known results.

2.1 Channel Model

The capacity of fading channels under no \textit{a priori} CSI is known to be sensitive to the channel model used, especially in the high SNR regime [10, 24, 25]. In our quest for simplicity of exposition, we focus on block-fading channel models, which we will describe in the following two sections.

2.1.1 Constant Block-Fading Model

The \textit{constant} block-fading model is perhaps the simplest model to capture channel variations. The input-output relation of a constant block-fading MIMO channel with $T$ transmit antennas, $R$ receive antennas within a block of $N$ channel uses is [8, 9, 13]:

$$y_r = \sqrt{\frac{\rho}{T}} \sum_{t=1}^{T} h_{r,t} x_t + w_r, \quad r \in \{1, \ldots, R\}. \quad (2.1)$$

Here, $x_t \in \mathbb{C}^N$ is the signal vector originating from the $t$th transmit antenna over the block of $N$ channel uses; $y_r \in \mathbb{C}^N$ is the signal vector at the $r$th receive antenna; $h_{r,t}$ is the channel coefficient between receive antenna $r$ and transmit antenna $t$; $w_r \sim \mathcal{CN}(0, I_N)$ is the noise vector at the $r$th receive antenna; finally, $\rho \in \mathbb{R}_+$ denotes the SNR. In most of
the thesis (with the exception of Paper D) we shall assume $h_{r,t} \sim \mathcal{CN}(0, 1)$, i.e., Rayleigh fading. In our setup, the transmitted signal vectors $x_t$ do not depend on the channel coefficients $h_{r,t}$ and the noise vectors $w_r$. The channel coefficients $h_{r,t}$ are assumed to be mutually independent and independent across $r \in \{1, \ldots, R\}$ and $t \in \{1, \ldots, T\}$, and to change in an independent fashion from block to block (“block-memoryless” assumption). We assume that neither the transmitter nor the receiver have a priori knowledge of the realizations of $\{h_{r,t}\}$ and $\{w_r\}$, but both know their statistics.

In most of the thesis, we shall consider an ergodic scenario where the coding is performed over sufficiently many blocks. A different scenario considered in Section 2.2.2 and Paper D is the so-called quasi-static setting, originally proposed in [26], where it is assumed that $N$ is large and that coding is performed over one block only. In this scenario, the fading is a non-ergodic process.

### 2.1.2 Correlated Block-Fading Model

The correlated block-fading model extends the constant block-fading model to allow for the $N$ fading coefficients in each block to be jointly Gaussian with a covariance matrix of rank $Q \geq 1$. The input-output relation of a correlated block-fading MIMO channel with $T$ transmit antennas, $R$ receive antennas within a block of $N$ channel uses is [14]:

$$y_r = \sum_{t=1}^{T} \text{diag}(h_{r,t}) x_t + w_r, \quad r \in \{1, \ldots, R\}.$$  \hfill (2.2)

Here, the vector $h_{r,t} \sim \mathcal{CN}(0, PP^H)$ contains the fading coefficients between the receive antenna $r$ and transmit antenna $t$, and its realizations are assumed to be unknown by the transmitter and the receiver. The $N \times Q$ matrix $P$, which is deterministic and of full rank $1 \leq Q \leq N$, describes the correlation structure within a block and is assumed to be known perfectly to the transmitter and the receiver. We further assume that the rows of $P$ have unit norm, and hence, the entries of $h_{r,t}$ are identically distributed. The fading vectors $\{h_{r,t}\}$ between all antenna pairs are assumed to be independent and identically distributed (i.i.d.),\(^1\) and are independent across different blocks. The correlated block-fading model just described captures channel variation in time in a more accurate (yet simple) way than the constant block-fading model: large $Q$ corresponds to fast channel variations. Furthermore, (2.2) models accurately the IO relation in the frequency domain of a cyclic-prefix OFDM system that operates over a multipath channel with $Q$ uncorrelated taps. Note that (2.1) is a special case of (2.2) with $P = [1, \ldots, 1]^T$ and $Q = 1$.

Before ending this section, we remark that in both the constant block-fading model and the correlated block-fading model the fading is modeled as a non-stationary process. An alternative approach to capturing channel variations in time is to assume that the fading process is symbol-by-symbol stationary. The key differences between block-fading and stationary fading is discussed in [28]. The high-SNR capacity of the stationary channel with no a priori CSI is studied in [10, 24, 29, 30]. Somewhat surprisingly, this

\(^1\)This assumption can be further relaxed by allowing for the fading vectors between different antenna pairs to have different covariance matrices of the same rank [27].
model can lead in some cases to capacity estimates that are drastically different from the
ones obtained using the block-fading models.

2.2 Fundamental Limits and Known Results

2.2.1 Channel Capacity

To keep notation simple, we write

\[ X \triangleq [x_1, \ldots, x_T] \in \mathbb{C}^{N \times T} \text{ and } Y \triangleq [y_1, \ldots, y_R] \in \mathbb{C}^{N \times R}. \]

Because of the block-memoryless assumption, the ergodic capacity of the channel in (2.1) and (2.2) is given by

\[ C(\rho) = \frac{1}{N} \sup I(X;Y). \]  

(2.3)

Here, \( I(X;Y) \) denotes the mutual information [6, p. 251] and the supremum is taken over all input distributions on \( \mathbb{C}^{N \times T} \) satisfying the average power-constraint

\[ \sum_{t=1}^{T} \mathbb{E}[\|x_t\|^2] \leq NT. \]  

(2.4)

The capacity pre-log is defined as

\[ \chi = \lim_{\rho \to \infty} \frac{C(\rho)}{\log \rho}. \]  

(2.5)

The capacity of the constant block-fading MIMO channel (2.1) with no a priori CSI
has been studied extensively in the literature for the Rayleigh fading case [8, 9, 13, 31].
However, no closed-form capacity expression is available to date. Marzetta and
Hochwald [8] proved that the capacity-achieving input matrix \( X \) is of the form
\( X = \Phi D \), where \( \Phi \) is an \( N \times T \) isotropically distributed unitary matrix and \( D \) is a \( T \times T \) nonnegative diagonal matrix independent of \( \Phi \). In the same paper, they also conjectured
that allocating equal powers to different transmit antennas, i.e., choosing \( D = \sqrt{N} I_T \),
is optimal at high SNR. The corresponding probability distribution on \( X \) is sometimes
referred to as unitary space-time modulation (USTM) [31–33]. Zheng and Tse [9] studied
capacity of the channel (2.1) at high SNR and proved that the pre-log is given by

\[ \chi = M^* \left( 1 - \frac{M^*}{N} \right), \quad \text{with } M^* = \min\{T, R, \lfloor N/2 \rfloor\}. \]  

(2.6)

For the case \( N \geq T + R \), Zheng and Tse [9] proved that USTM is indeed optimal at high
SNR, and that

\[ C(\rho) = \chi \log \rho + c + o(1), \quad \rho \to \infty \]  

(2.7)

where [9, Eq. (24)]

\[ c = \frac{1}{N} \log \frac{\Gamma_T(T)}{\Gamma_T(N)} + T \left( 1 - \frac{T}{N} \right) \log \frac{N}{Te} + \left( 1 - \frac{T}{N} \right) \mathbb{E} \left[ \log \det(HH^H) \right] \]  

(2.8)
is a constant that does not depend on \( \rho \), \( \Gamma_m(a) = \pi^{m(m-1)/2} \prod_{k=1}^{m} \Gamma(a - k + 1) \) with \( \Gamma(\cdot) \)
being the Gamma function \([34, \text{Eq. (6.1.1)}]\), and \( o(1) \) denotes a function of \( \rho \) that vanishes as \( \rho \to \infty \). Differently from the pre-log result (2.6), the high-SNR expression (2.7) describes capacity accurately already at moderate SNR values \([35]\).

For the correlated block-fading channel model, it was shown in \([14]\) that the capacity pre-log for the SISO case is given by

\[
\chi = \left(1 - \frac{Q}{N}\right).
\]

(2.9)

A lower bound on the SIMO capacity pre-log is reported in \([36, 37]\), and generalized in \([27]\) to the MIMO case. Finding a capacity characterization that is tight up to \( o(1) \) \([\text{cf. (2.7)}]\) for the correlated block-fading channel is an open problem.

While all results reviewed in this section pertain to the Rayleigh fading case, it is shown in \([38]\) that among all fading distributions whose law has no mass point at zero, the Rayleigh distribution gives rise to the smallest pre-log. So Rayleigh fading is the worst fading in terms of capacity pre-log.

### 2.2.2 Outage Capacity

For the quasi-static fading model introduced at the end of Section 2.1.1, the channel capacity is zero whenever the closure of the support of the fading distribution includes zero, because reliable communication cannot be guaranteed for any positive data rate. In this scenario, a more meaningful performance metric may be the outage capacity (also referred to as \( \epsilon \)-capacity \([39]\)). Let \( H \triangleq [h_{r,t}]_{1 \leq r \leq R, 1 \leq t \leq T} \in \mathbb{C}^{R \times T} \). The outage capacity of MIMO quasi-static fading channels is given by \([40]\)

\[
C_\epsilon = \sup \left\{ \xi : P_{\text{out}}(\xi) \leq \epsilon \right\}
\]

(2.10)

where \( P_{\text{out}}(\xi) \) denotes the outage probability\(^2\) corresponding to a given rate \( \xi \)

\[
P_{\text{out}}(\xi) = \inf_{Q \triangleright 0, \text{tr}(Q) \leq T} \mathbb{P} \left[ \log \det \left( I_R + \sqrt{\frac{\rho}{T}} HQH^H \right) \leq \xi \right].
\]

(2.11)

Note that (2.10) holds for the case when CSI is not available at the transmitter and regardless of whether CSI is available at the receiver or not. The matrix \( Q \) that minimizes the right-hand-side (RHS) of (2.11) is in general not known. For the Rayleigh-fading case, Telatar \([40]\) conjectured that the optimal \( Q \) is of the form\(^3\)

\[
\frac{\rho}{T} \text{ diag} \left\{ 1, \ldots, 1, 0, \ldots, 0 \right\}, \quad 1 \leq T^* \leq T
\]

(2.12)

---

\(^2\)Strictly speaking, (2.10) holds for all \( \epsilon > 0 \) for which \( C_\epsilon \) is a continuous function of \( \epsilon \).

\(^3\)This conjecture has recently been proved in \([41]\) for the multiple-input single-output case.
and that for small $\epsilon$ values or for high SNR, all available transmit antennas should be used (see also [42, p. 367]).

When CSI is available at the transmitter, the well known water-filling power allocation achieves the infimum in (2.11).

### 2.2.3 Maximal Achievable Rate

We introduce the notion of the maximal achievable rate $R^*(n, \epsilon)$ for a given blocklength $n$ and error probability $\epsilon$. As in [23, 43], we define it for an abstract random transformation $P_{Y|X}$ with input and output sets $A$ and $B$. An $(M, \epsilon)_{\max}$ code for the random transformation $P_{Y|X}$ is a pair of random transformations $P_{X|W}: \{1, \cdots, M\} \rightarrow X$ and $P_{W|Y}: Y \mapsto \{1, \cdots, M\}$ such that

$$\max_{1 \leq j \leq M} P[\hat{W} \neq W | W = j] = \max_{1 \leq j \leq M} \left(1 - P_{W|Y}(j | j)\right) \leq \epsilon. \quad (2.13)$$

The random transformations $P_{X|W}$ and $P_{W|Y}$ are called the encoder and the decoder of the code, respectively. An $(M, \epsilon)_{\mathrm{avg}}$ code is defined in a similar way, except that (2.13) is replaced with the average probability of error criterion:

$$P[\hat{W} \neq W] = \frac{1}{M} \sum_{j=1}^{M} \left(1 - P_{W|Y}(j | j)\right) \leq \epsilon \quad (2.14)$$

where $W$ is equiprobable on $\{1, \cdots, M\}$.

A channel is a sequence of random transformations, $\{P_{Y_n|X_n}, n = 1, 2, \ldots\}$, parameterized by the blocklength $n$. An $(M, \epsilon)$ code for the $n$-th random transformation of the channel is called an $(n, M, \epsilon)$ code. The maximal achievable rate $R^*(n, \epsilon)$ is defined as

$$R^*(n, \epsilon) \triangleq \sup \left\{ \frac{\log M}{n} : \exists (n, M, \epsilon) \text{ code} \right\}. \quad (2.15)$$

As already mentioned, the capacity $C$ and $\epsilon$-capacity $C_\epsilon$ are the asymptotic limits of $R^*(n, \epsilon)$:

$$C = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} R^*(n, \epsilon), \quad C_\epsilon = \lim_{n \rightarrow \infty} R^*(n, \epsilon). \quad (2.16)$$

As noted in Chapter 1, the exact value of $R^*(n, \epsilon)$ is unknown. In fact, the search for $R^*(n, \epsilon)$ in (2.15) is doubly exponential in the blocklength $n$. Nevertheless, bounds and easy-to-compute approximations for $R^*(n, \epsilon)$ are available in the literature [18–23]. In particular, Polyanskiy, Poor and Verdù [23] recently derived several new bounds on the maximal achievable rate, such as the meta-converse bound, the dependence-testing bound, and the $\kappa \beta$ bound. By analyzing these bounds using the central limit theorem, they obtained a simple closed-form approximation, given in (1.1), for the maximal achievable rate. To compute the approximation (1.1), one only needs to know one more

---

4In addition, one should also specify which probability of error criterion, (2.13) or (2.14), is used.
parameter of the channel (in addition to channel capacity), i.e., the channel dispersion [23, Def. 1], which is defined as

$$V = \lim_{\epsilon \to 0} \lim_{n \to \infty} n \left( \frac{C - R^*(n, \epsilon)}{Q^{-1}(\epsilon)} \right)^2. \quad (2.17)$$

The rationale behind the approximation (1.1) and the definition (2.17) of channel dispersion is the following expansion [23]

$$R^*(n, \epsilon) = C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon) + \mathcal{O}\left(\frac{\log n}{n}\right) \quad (2.18)$$

which is valid for various channels with positive channel capacity. Here, the notation \( f(x) = \mathcal{O}(g(x)) \) means that \( \limsup_{x \to \infty} |f(x)/g(x)| < \infty \).
Chapter 3

Information Theoretic Tools

In this chapter, we introduce the information theoretic tools we shall need for the analysis of the capacity and of the maximal achievable rate over block-memoryless fading channels.

3.1 The Duality Technique

For the convenience of notation, consider a memoryless channel $P_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y}$. Here, $\mathcal{X}$ and $\mathcal{Y}$ denote the input and output alphabets, respectively. Let $P_Y$ be the probability distribution induced on $Y$ by the input distribution $P_X$ and by the channel $P_{Y|X}$. By the definition of mutual information [6, Sec. 8.5] [21, Eq. (1.8)], we have that

$$I(X; Y) \triangleq D(P_{Y|X} \| P_{Y|P_X})$$

(3.1)

$$= \int dP_X(x) \int dP_{Y|X=x}(y) \log \frac{dP_{Y|X=x}(y)}{dP_Y(y)} dx$$

(3.2)

where $D(\cdot \| \cdot)$ stands for the relative entropy between two probability distributions and

$$D(V\|W|P_X) \triangleq E_{P_X}[D(V(\cdot|X)\|W(\cdot|X))]$$

(3.3)

denotes the conditional relative entropy [21, Eq. (2.4)]. The capacity of the channel $P_{Y|X}$ is given by

$$C = \sup_{P_X} D(P_{Y|X} \| P_Y|P_X)$$

(3.4)

where the supreme is taken over all probability measures $P_X$ on $\mathcal{X}$ that satisfy the given cost constraint (e.g., an average power constraint).

Establishing lower bounds on channel capacity is relatively simple: every input distribution $P_X$ leads to a lower bound on $C$

$$C \geq D(P_{Y|X} \| P_Y|P_X).$$

(3.5)

To obtain a tight lower bound, we need to choose $P_X$ to be close to a capacity-achieving input distribution and such that $D(P_{Y|X} \| P_Y|P_X)$ is tractable.
Finding upper bounds on channel capacity is more difficult because we need to perform the maximization over all possible input distributions, and because the function we need to maximize depends on $P_X$ in an intricate way (see (3.2)). The main idea behind the duality technique is the following dual expression for capacity

$$C = \inf_{Q_Y} \sup_{P_X} D(P_{Y|X} \| Q_Y | P_X).$$  \hspace{1cm} (3.6)$$

We see from (3.6) that every probability distribution $Q_Y$ on the output $Y$ yields an upper bound on $C$

$$C \leq \sup_{P_X} D(P_{Y|X} \| Q_Y | P_X).$$  \hspace{1cm} (3.7)$$

Comparing (3.4) and (3.7), we see that in the duality bound (3.7) the distribution $P_Y$ is replaced by an auxiliary distribution $Q_Y$ that does not depend on the input distribution $P_X$. This makes the RHS of (3.7) much easier to deal with than the RHS of (3.4). As illustrated in many examples (see, e.g., [10, 12] and Paper B), for an appropriate choice of $Q_Y$, it is possible to establish tight upper bounds on $D(P_{Y|X} \| Q_Y | P_X)$ that hold for every input distribution $P_X$. In other words, the maximization in (3.7) can be circumvented through appropriate bounding steps.

Obviously, the crucial step in deriving capacity upper bounds using the duality technique is to choose an appropriate output distribution $Q_Y$. In principle, one should choose $Q_Y$ to be close to the capacity-achieving output distribution and so as to guarantee that the RHS of (3.7) is tractable. At high SNR, a typical choice of $Q_Y$ is the output distribution induced by the input distribution without the additive Gaussian noise (see, e.g., [12]). In Paper A and Paper B, we used a more refined choice of $Q_Y$, which is motivated by geometric arguments.

### 3.2 Finite Blocklength Bounds

In this section, we review some of the bounds on the maximal achievable rate $R^*(n, \epsilon)$ developed in [23]. These results apply to arbitrary channels and arbitrary blocklengths. As in [23], we state them for an abstract random transformation $P_{Y|X}$ with input and output sets $A$ and $B$.

For a joint distribution $P_{XY}$ on $A \times B$ we denote the information density by

$$i(x; y) = \log \frac{dP_{Y|X=x}}{dP_Y}(y).$$  \hspace{1cm} (3.8)$$

The following result, referred to as the dependence testing (DT) bound, provides an achievability (lower) bound on $R^*(n, \epsilon)$.

---

1When we apply these results, we shall take $A$ and $B$ to be $n$-fold Cartesian products of some alphabets $\mathcal{X}$ and $\mathcal{Y}$. Thus, the element of $A$ and $B$ and the realizations of the random variables $X$ and $Y$ can be thought of as vectors of dimension equal to the blocklength $n$. 
Theorem 3.1 (DT bound, [23, Th. 17]) For every distribution $P_X$ on $A$, there exists a code with $M$ codewords and average probability of error not exceeding

$$\epsilon \leq \mathbb{E} \left[ \exp \left\{-\left| i(X;Y) - \log \frac{M - 1}{2}\right|^{+}\right\} \right]$$

(3.9)

where $| \cdot |^{+} \triangleq \max\{0, \cdot\}$.

The DT bound uses a threshold decoder that sequentially tests all messages and returns the first message whose likelihood exceeds a pre-determined threshold. Applications of the DT bound to various channels (mainly channels without input constraints) can be found in [23] [44]. In Paper C, we use the DT bound to derive lower bounds on $R^*(n,\epsilon)$ for constant block-fading channels.

For channels with input constraints, an achievability bound, called $\kappa\beta$ bound, that is easier to analyze analytically than the DT bound is established in [23]. To state the $\kappa\beta$ bound, we need some definitions regarding the performance of hypothesis testing. Given two distributions $P$ and $Q$ on a common measurable space $W$, we define a randomized test between $P$ and $Q$ as a random transformation $P_{Z|W}:W \mapsto \{0,1\}$ where 0 indicates that the test chooses $Q$. As performance metric for the test between $P$ and $Q$, we take the Neyman-Pearson ROC curve [45]

$$\beta_\alpha(P,Q) = \min \int P_{Z|W}(1|w)Q(dw)$$

(3.10)

where the minimum is over all probability distributions $P_{Z|W}$ satisfying

$$\int P_{Z|W}(1|w)P(dw) \geq \alpha.$$  

(3.11)

The minimum in (3.10) is guaranteed to be achieved by the Neyman-Pearson Lemma [46]. Each per-codeword cost constraint can be defined by specifying a subset $F \subset A$ of permissible inputs. For an arbitrary $F \subset A$, we define the following measure of performance for the composite hypothesis test between $Q_Y$ and the collection $\{P_{Y|X=x}\}_{x \in F}:

$$\kappa_\tau(F,Q_Y) = \inf \int P_{Z|Y}(1|y)Q_Y(dy).$$

(3.12)

Here, the infimum is over all probability distributions $P_{Z|Y}$ satisfying

$$\int P_{Z|Y}(1|y)P_{Y|X=x}(dy) \geq \tau, \quad \forall x \in F.$$  

(3.13)

We are now ready to state the $\kappa\beta$ bound.

Theorem 3.2 ($\kappa\beta$ bound, [23, Th. 25]) For every $0 < \epsilon < 1$, there exists an $(M,\epsilon)_{\text{max}}$ code with codewords chosen from $F \subset A$, satisfying

$$M \geq \sup_{0<\tau<\epsilon} \sup_{Q_Y} \frac{\kappa_\tau(F,Q_Y)}{\sup_{x \in F} \beta_{1-\epsilon+\tau}(P_{Y|X=x},Q_Y)}.$$  

(3.14)
The $\kappa \beta$ is used in [23] to derive an achievability bound on $R^*(n, \epsilon)$ for the AWGN channel and in [47] for point-to-point fading channels with perfect a priori CSI. In Paper D, we shall use a modified version of this bound to derive an achievability bound on $R^*(n, \epsilon)$ for quasi-static SIMO fading channels.

Next, we introduce a general converse bound called meta-converse bound.\footnote{The maximal probability of error counterparts to Theorem 3.3 and Theorem 3.4 can be found in [23, Th. 30 and Th. 31].} The meta-converse bound allows one to establish a converse bound for the channel $P_{Y|X}$, provided that a converse bound for an auxiliary channel $Q_{Y|X}$ is available.

**Theorem 3.3 (Meta-converse, [23, Th. 26])** Let $Q_{Y|X} : A \mapsto B$ be an auxiliary channel. For a given code, let

$$\epsilon = \text{average error probability with } P_{Y|X}$$

$$\epsilon' = \text{average error probability with } Q_{Y|X}$$

$$P_X = \text{encoder output distribution with equiprobable codewords.}$$

Then

$$\beta_1 - \epsilon(P_X P_{Y|X}, P_X Q_{Y|X}) \leq 1 - \epsilon'.$$

(3.15)

Choosing $Q_{Y|X}$ in Theorem 3.3 to be independent of the input $X$, we get the following result.

**Theorem 3.4 (Minmax converse, [23, Th. 27])** Every $(M, \epsilon)_{\text{avg}}$ code with codewords belonging to $A$ satisfies

$$\log M \leq -\log \left\{ \inf_{P_X} \sup_{Q_Y} \beta_1 - \epsilon(P_X, P_X Q_{Y|X}) \right\}.$$  

(3.16)

where $P_X$ ranges over all probability measures on $A$, and $Q_Y$ ranges over all probability measures on $B$.

Note that $\beta_\alpha(\cdot, \cdot)$ satisfies the following saddle point property [48] (under regularity conditions on the input set $A$)

$$\inf_{P_X} \sup_{Q_Y} \beta_1 - \epsilon(P_X, P_X Q_Y) = \sup_{Q_Y} \inf_{P_X} \beta_1 - \epsilon(P_X P_{Y|X}, P_X Q_Y).$$

(3.17)

Hence, (3.16) can be written equivalently as

$$\log M \leq -\log \left\{ \sup_{Q_Y} \inf_{P_X} \beta_1 - \epsilon(P_X P_{Y|X}, P_X Q_Y) \right\}.$$  

(3.18)

The bound (3.18) can be further relaxed by fixing a convenient $Q_Y$ as follows

$$\log M \leq -\log \left\{ \inf_{P_X} \beta_1 - \epsilon(P_X P_{Y|X}, P_X Q_Y) \right\}.$$  

(3.19)
As in the duality technique, a crucial step in using (3.19) to derive upper bounds on 
\( R^*(n, \epsilon) \) is to choose an appropriate output distribution \( Q_Y \). A natural choice for \( Q_Y \) is the capacity achieving output distribution (see, e.g., [23, p. 2324] and [47]). Other choices of \( Q_Y \) are motivated by symmetries in the channel input-output relations and geometry arguments.

It is interesting to compare the duality bound (3.7) on the channel capacity with the converse bound (3.19) on the maximal achievable rate at finite blocklength. The duality bound (3.7) can be seen as the asymptotic counterpart of the bound (3.19). In fact, the Chernoff-Stein Lemma [6, Th. 11.8.3] implies that

\[
- \log \beta_{1-\epsilon}(P_{XY}^n, P_X^n Q_Y^n) = n D(P_{XY} || P_X Q_Y) + o(n) \tag{3.20}
\]

\[
= n D(P_{Y|X} || Q_Y | P_X) + o(n) \tag{3.21}
\]

where \( P_{XY}^n, P_X^n \) and \( Q_Y^n \) are \( n \) i.i.d. copies of \( P_{XY}, P_X \) and \( Q_Y \). In particular, we expect that the output distributions \( Q_Y \) that yield tight upper bounds on capacity via the duality bound perform also well at finite blocklength via the bound (3.19).
Chapter 4

Contributions

This thesis aims at studying—under the assumption of no a priori CSI—the capacity and the maximal achievable rate at finite blocklength of point-to-point fading channels. In Section 4.1, we list the papers that are appended to this thesis and summarize their contributions. Additional publications by the author, which are not included in this thesis, are listed in Section 4.2.

4.1 Included Publications

1. **Paper A: “Capacity pre-log of SIMO correlated block-fading channels”**
   We establish an upper bound on the noncoherent capacity pre-log of temporally correlated block-fading single-input multiple-output (SIMO) channels. The upper bound matches the lower bound recently reported in Riegler et al. (2011), and, hence, yields a complete characterization of the SIMO capacity pre-log, provided that the channel covariance matrix satisfies a mild technical condition. This result allows one to determine the optimal number of receive antennas to be used to maximize the capacity pre-log for a given block-length and a given rank of the channel covariance matrix.

2. **Paper B: “On the capacity of large-MIMO block-fading channels”**
   We characterize the capacity of Rayleigh block-fading multiple-input multiple-output (MIMO) channels in the noncoherent setting where transmitter and receiver have no a priori knowledge of the realizations of the fading channel. We prove that unitary space-time modulation (USTM) is not capacity-achieving in the high signal-to-noise ratio (SNR) regime when the total number of antennas exceeds the coherence time of the fading channel (expressed in multiples of the symbol duration), a situation that is relevant for MIMO systems with large antenna arrays (large-MIMO systems). This result settles a conjecture by Zheng & Tse (2002) in the affirmative. The capacity-achieving input signal, which we refer to as Beta-variate space-time modulation (BSTM), turns out to be the product of a unitary isotropically distributed random matrix, and a diagonal matrix whose nonzero entries are distributed as the square-root of the eigenvalues of a Beta-distributed random matrix of appropriate size. Numerical results
illustrate that using BSTM instead of USTM in large-MIMO systems yields a rate gain as large as 13% for SNR values of practical interest.

3. **Paper C: “Diversity versus channel knowledge at finite block-length”**
   We study the maximal achievable rate $R^*(n, \epsilon)$ for a given block-length $n$ and block error probability $\epsilon$ over Rayleigh block-fading channels in the noncoherent setting and in the finite block-length regime. Our results show that for a given block-length and error probability, $R^*(n, \epsilon)$ is not monotonic in the channel’s coherence time, but there exists a rate maximizing coherence time that optimally trades between diversity and cost of estimating the channel.

4. **Paper D: “Quasi-static SIMO fading channels at finite blocklength”**
   We investigate the maximal achievable rate for a given blocklength and error probability over quasi-static single-input multiple-output (SIMO) fading channels. Under mild conditions on the channel gains, it is shown that the channel dispersion is zero regardless of whether the fading realizations are available at the transmitter and/or the receiver. The result follows from computationally and analytically tractable converse and achievability bounds. Through numerical evaluation, we verify that, in some scenarios, zero dispersion indeed entails fast convergence to outage capacity as the blocklength increases. In the example of a particular $1 \times 2$ SIMO Rician channel, the blocklength required to achieve 90% of capacity is about an order of magnitude smaller compared to the blocklength required for an AWGN channel with the same capacity.

4.2 **Publications Not Included**

Publications by the author, which are not included in this thesis, are listed below.


References


