Thesis for the Degree of Master of Science in Fundamental Physics

Free (2,0) Theory on a Circle Fibration

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CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg, Sweden 2011

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Typeset using LaTeX
Printed by Chalmers Reproservice
Göteborg, Sweden 2011

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Abstract
Using the geometry of a circle fibration the free (2,0) theory, a six dimensional superconformal field theory, is reduced to a five dimensional abelian Yang-Mills theory. The six dimensional theory is conjectured to exist as an interacting theory but as of this time no consistent formulation of the theory is known. In this thesis a simpler non-interacting, classical version of the theory is considered on a six dimensional space-time which is a circle fibration over a curved five dimensional manifold. It is found that the reduced theory in five dimensions has a supersymmetric Lagrangian description. This thesis starts by giving an introduction to the mathematical formalism used in field theory on curved space-time and continues to describe the free (2,0) theory in six dimensions. The reduction on the circle fibration is then presented together with the resulting five dimensional theory.

Acknowledgments

First I would like to thank my supervisor, Måns Henningson, for providing such an interesting topic to study and for all the wonderful insights into mathematical physics. A special thanks also to Fredrik Ohlsson, with whom I worked on the topics of this thesis and from whom I have learned so much. Thank you for giving me the opportunity to work together during this year, your dedication and knowledge in the field has been a constant source of inspiration!
Thank you Christian for a great year together as room mates, I wish you all the best for your PhD and what ever lies thereafter! I would also like to thank Pär Pettersson for all the great coffee discussions, I have missed them ever since you graduated!
Despite lunchtime being an illusion I would like to thank the lunch crew (you know who you are) for being the steady pillar of joy during working days. A special thanks also to Ronja Thies for the many tips for improvement in her thorough opposition of this thesis. Finally I would like to thank my family and friends without whom I would not be possible.

Göteborg, June 2011
Hampus Linander

"This is not for purists, this is for real!"

–Sidney Coleman (Quantum field theory lectures, Harvard, 1975)

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Chapter 1

Introduction

The main topic of this thesis is superconformal field theory. Field theory is a mathematical framework that many modern theories of fundamental physics is built upon. In particular the four known fundamental forces are all modeled by field theories: Einsteins theory of general relativity describing gravity and the quantum field theories describing the electromagnetic, weak and strong force.

Superconformal field theories are a special class of field theories that are very symmetric, they are both supersymmetric and conformally invariant.

Symmetry

The word symmetry in everyday life is mostly associated with geometry, for example the human body is approximately left/right symmetric. In physics a symmetry can be geometric but it can also be more abstract. The essence of a symmetry is that there is some kind of transformation that leaves an object unchanged. In the case of the human body this transformation is reflection through a vertical plane. This idea can be generalised to a more abstract setting, for example equations. A simple example is the equation

$$x^2 = 1.$$

This equation is symmetric under the transformation of changing x into $-x$, i.e. if we replace x by $-x$ the equation looks exactly the same due to the fact that $(-1)^2 = 1$. If x corresponds to a coordinate then this would be a reflection, but x doesn't necessarily need to correspond to something geometric. These kinds of abstract symmetries turn out to be a powerful tool in creating mathematical models of reality, and there is a simple reason for this.

Symmetry is often associated to beauty and one could argue that this is also the case in physics. When analyzing experimental data of how nature works at a small scale one finds an interesting structure, a simplicity in that most of what we observe can be explained in terms of a handful of elementary particles and their interactions. What's more, these interactions are highly systematic. One way of modeling this simplicity mathematically is by introducing symmetries. There is an analogy with geometry where symmetry is also associated with simplicity. Take for example a sphere, the most symmetric of all geometric objects. It has, besides reflection symmetries, complete rotational symmetry. Because of its symmetries a sphere is completely specified by a single point on its surface, i.e. with this information the whole sphere can be constructed. By demanding the theories describing elementary particles to have certain symmetries the simple structure observed in nature can be implemented.

Superconformal field theories have two symmetries that are not directly motivated by observations but rather for their mathematical consequences:

- Supersymmetry

  Not only does symmetry provide a natural way to implement constraints in a theory, it can also save them from some disasters. In fact sometimes it is absolutely necessary to impose a symmetry to get a consistent theory. One example of this is antiparticles. When combining special relativity and quantum mechanics one is led to a formulation where it becomes absolutely necessary to have a charge symmetry [1], i.e. there need to exist antiparticles. Supersymmetry is another possible symmetry of a theory that relates fermions[1] to bosons and it brings about many nice properties for quantum field theories [2]. It is yet to be seen if nature chooses to use this symmetry, but that doesn't stop us from investigating its consequences.

- Conformal symmetry

  Apart from Poincaré[2] symmetry and supersymmetry this is the last remaining possible space-time symmetry. Conformal transformations change the scale of things from point to point. In addition to simplifying a theory from a mathematical point of view conformal symmetry also shows up in macroscopic systems, often related to phase transitions in condensed matter physics [3].

There is an interesting reason for why these two particular types of symmetries are studied. It turns out that under some natural requirements for a field theory these symmetries are the only possibilities in addition to Poincaré symmetry and internal symmetries, i.e. symmetries not related to space and time [4].

(2,0) theory

As in the geometric case, symmetry constrains a theory and since superconformal theories are as symmetric as a theory can be, one would expect that maybe there doesn't exist so many examples of them. The analogy in geometry would be to ask the question: What objects are completely rotationally symmetric? The answer in geometry is a single object, the sphere. The situation for superconformal theories is similar. In fact, in 1977 all superconformal algebras[3] were completely classified by Nahm [5] and it turns out superconformal symmetry can only exist up to 6 space-time dimensions. Even though a symmetry exists

[1] Fermions are particles with half-integer spin whereas bosons have integer spin. Broadly speaking, fermions are associated with matter and bosons with the forces. For example the electron is a fermion and the photon is a boson.

[2] Poincaré transformations include translations, rotations and boosts. These are always symmetries of any relativistic field theory by the postulates of special relativity.

[3] Mathematically the symmetry transformations of an object form what is called a group. An algebra in this context is another mathematical construction related to the group. In particular all the infinitesimal symmetry transformations form an algebra and it is the classification of these that was carried out by Nahm in the case of the superconformal group.

abstractly there is no reason for a consistent field theory to have the algebra as its symmetries and the only examples of superconformal field theories today are defined in 4 or less space-time dimensions. In fact it was not believed there would exist any consistent theories in 5 and 6 dimensions [6]. This changed in 1995 when Witten [7] gave strong evidence for their existence as limits of string theory. In particular there seemed to be reason to believe that one particular superconformal algebra, the so called (2,0) algebra, should exist as an interacting theory in six dimensions. It is this theory that is the topic of this thesis.

The classification by itself is enough reason to study this theory but today there are many other reasons [8, 9]. One of these comes from a conjecture that was made two years later, in 1997, by Maldacena [10]. The conjecture states that there is a duality between string theory on a particular space and a conformal field theory on its boundary. The duality means that the two theories are essentially the same in a certain limit and instead of calculating a quantity in one of the theories, where maybe for various reasons it's difficult, we can instead calculate it in the other one if it is easier there.

A particular case that has been studied is the duality between M-theory [7] on the space-time $\mathrm{AdS}_7 \times S^4$ and a conformal field theory on the boundary of the AdS_7 [10]. Since AdS_7 is 7 dimensional its boundary is a 6 dimensional space and thus the dual theory in this case is a conformal field theory in 6 dimensions. The fact that these dualities relate theories with weak interactions to theories with strong interactions makes them very useful since with the current mathematical framework it is hard to do calculations in strongly interacting theories. Another exiting feature of this duality is that it relates a theory containing gravity to a theory that does not. This might be the correct setting to understand one of the biggest questions in physics today, how to formulate a quantum theory of gravity.

If this $(2, 0)$ theory is such an interesting theory, then certainly someone has written it down explicitly and investigated its properties? As it turns out there are problems with just formulating the theory at all. From the classification and from the properties inferred by the dualities there are many hints at how the theory should look like, for example what kind of fields it should contain and how they should behave (e.g [11], [12]), but when applying the standard machinery problems immediately appear. In particular there are problems with constructing a Lagrangian formulation, even for the free theory, as will be seen in chapter 4. Because of this there is at present no explicit formulation of the theory even though many of its properties have been investigated.

1.1 Overview

In this thesis a highly simplified version of this theory will be considered, a theory without any interactions. In addition to only looking at the free theory, only the classical field theory is considered. In other words the fields in the theory are not quantum fields, i.e. operator valued fields, but instead real and complex valued fields. This is normally the starting point for a quantum field theory but in the end a full quantum field theory might behave rather differently compared to its classical version. For example the quantum theory need not always inherit all the symmetries [13]. The classical version can often be used to say something qualitatively about the quantum version but in the end the full theory needs to be investigated to give the final answer.

The main goal of this thesis is to investigate what happens to the free theory when the 6 dimensional space-time locally is the product of a 5 dimensional space-time and a periodic dimension. The periodicity of the sixth dimension will enable a low energy limit to be taken so that only a theory living on the 5 dimensional space-time will remain. The hope is then that this reduced theory can tell us something about the theory in 6 dimensions.

This thesis, supervised by Måns Henningson, is based on the work presented in the included paper done together with Fredrik Ohlsson.

1.2 Outline

In the first part of the thesis some of the necessary mathematical formalism will be presented. Chapter 2 contains a summary of the standard construction of spinor representations and their properties in the dimensions under consideration. Chapter 3 introduces the framework that enables the construction of field theories on curved space-time. In chapter 4 the 6-dimensional superconformal theory is described and some of its properties are verified explicitly. Chapter 5 describes how the specific splitting of space-time under consideration is constructed. Finally in chapter 6 and 7 the results and the description of the calculations of the 5-dimensional reduced theory are presented, the special case of a product metric is also considered.

1.3 Conventions

Indices denoted by capital roman starting from M and lower-case Greek will be coordinate indices, also referred to as curved indices, in the range

$$\begin{aligned} M, N, \ldots &\in \{0, 1, \ldots, 5\} \\ \mu, \nu, \ldots &\in \{0, 1, \ldots, 4\}. \end{aligned}$$

The sixth coordinate index, corresponding in the fibration to the periodic coordinate, will be denoted by φ so that $x^5 = x^\varphi$.

Flat indices corresponding to non-coordinate bases will be denoted by capital and lower-case roman letters in the range

$$\begin{aligned} A, B, \ldots &\in \{0, 1, \ldots, 5\} \\ a, b, \ldots &\in \{0, 1, \ldots, 4\}. \end{aligned}$$

Indices corresponding to the fundamental representation of the internal symmetry group Sp(4) are denoted by lower-case roman letters starting from i in the range

$$i, j, \ldots \in \{0, 1, 2, 3\}.$$

The signature convention for the Minkowski metric is mostly plus, i.e.

$$\eta_{MN} = \mathrm{diag}(-1, 1, \dots, 1). \tag{1.1}$$

The 6-dimensional metric is denoted G_{MN} and the 5-dimensional metric in the fibration by $g_{\mu\nu}$. The determinant of these metrics will be defined by G and g respectively. Since the determinants are negative by the choice of signature, the invariant volume elements are

$$\begin{aligned} \mathrm{vol}_6 &= d^6x\sqrt{-G} \\ \mathrm{vol}_5 &= d^5x\sqrt{-g}. \end{aligned}$$

When the integration limits are suppressed it is understood that the integration should be carried out over all of space.

$$\int d^6x\sqrt{-G} = \int_M d^6x\sqrt{-G}$$

The Levi-Civita symbol is defined by

$$\varepsilon^{012345} = 1,$$

and that its totally antisymmetric. The corresponding tensor is defined by

$$\epsilon^{MNOPQR} = \frac{1}{\sqrt{-G}}\varepsilon^{MNOPQR}.$$

The commutator and anticommutator are defined respectively by

$$\begin{aligned} [X, Y] &= XY - YX \\ \{X, Y\} &= XY + YX \end{aligned}$$

Antisymmetrisation of indices is denoted by brackets and defined by

$$A_{[M_1 M_2 \dots M_p]} = \frac{1}{p!}\left(A_{M_1 M_2 \dots M_p} \pm \dots\right).$$

If some of the indices should be left out of the antisymmetrisation this is denoted with bars as in $A_{[M_1|M_2|M_3]} = \frac{1}{2}\left(A_{M_1 M_2 M_3} - A_{M_3 M_2 M_1}\right)$.

Complex conjugation swaps order of two Grassman variables so that for spinors

$$\begin{aligned} (\lambda_\alpha \alpha_\beta)^\star &= \alpha^\star_\beta \lambda^\star_\alpha \\ &= -\lambda^\star_\alpha \alpha_\beta. \end{aligned}$$

Chapter 2

Spinors

> "A spinor is thus a sort of 'directed' or 'polarised' isotropic vector; a rotation about an axis through an angle 2π changes the polarisation of this isotropic vector."
>
> -Élie Cartan: The Theory of Spinors.

When an object, this thesis for example, is rotated around an axis exactly one time it returns to it's original configuration. That is, it is impossible to distinguish between the object before and after we rotate it. This is what everyday experience tells us, but it turns out that there are things that behave differently under rotations, namely spinors. These are mathematical objects that when rotated one full revolution *are* distinguishable from the original, but when rotated two full revolutions return to the original configuration.

The reason for studying these kinds of objects is that some of the elementary particles behave in exactly this way, the electron and the quarks for example.

This chapter will review the construction of spin representations using a Clifford algebra and set the notation and conventions that will be used in later parts of this thesis. The goal will be to give a description of symplectic Majorana spinors that will be used later in the construction of the (2,0) theory.

2.0.1 Spin group

Rotations in space are transformations that leave the length of vectors unchanged. If you turn this around and ask what transformations leave the length of vectors unchanged you get rotations but also reflections. The group of all transformations preserving the length of vectors in n dimensions is called the orthogonal group, denoted $\mathrm{O}(n)$. The group of consisting of only rotations is called the special orthogonal group, denoted $\mathrm{SO}(n)$.

Transformations that preserve the Minkowski inner product of two vectors is called the Lorentz group, denoted by $\mathrm{O}(1, d-1)$. The Minkowski inner product is given by

$$
\begin{aligned}
< V, W > &= \begin{pmatrix} V^1, V^2, \dots, V^d \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{d\times d \text{ matrix}} \begin{pmatrix} W^1 \\ W^2 \\ \vdots \\ W^d \end{pmatrix} \\
&= V^M W^N \eta_{MN}.
\end{aligned}
$$

The subgroup of the Lorentz group which consists of elements that can be continuously deformed to the identity is called the proper orthosynchronous Lorentz group, denoted by $\mathrm{SO}^+(1, d-1)$. This consists of transformations that in addition to preserving the inner product also preserve orientation and the direction of time. Since these are the transformations that is of main interest and since that name is tiresome to write every time it is usually just referred to as the Lorentz group.

It turns out that to every special orthogonal group there is an associated group called the spin group. For the Lorentz group $\mathrm{SO}(1, d-1)$ the associated group is denoted $\mathrm{Spin}(1, d-1)$. They are almost the same but the spin group contains two transformations for every transformation in the orthogonal group. Because of this the spin group is called a double cover.

2.0.2 Spin representations

If there is a vector space on which the elements of $\mathrm{Spin}(1, d-1)$ acts as linear transformations, we say that there is a representation of the group on the vector space. For example $\mathbb{R}^3$ is a representation of the group $\mathrm{SO}(3)$, the group of rotations, when it acts on vectors by the rotation matrices. Spinors are simply elements of a vector space on which the group $\mathrm{Spin}(1, d-1)$ is represented.

In practise one seldom works with the full Lorentz and spin group but instead only consider infinitesimal transformations. These are transformations that are infinitesimally close to the identity transformation. The reason for doing this is that these are much easier to work with and they still contain almost all the information of the full group.

2.0.3 Infinitesimal transformations

Here a small motivation for working with the algebra of a group will be given. This part is not essential for the rest of the thesis, it only serves as a reminder of the motivations behind the constructions of spinors (and many other things).

Given a transformation T that is infinitesimally close to the identity we can write it as

$$T = 1 + \varepsilon A + \mathcal{O}(\varepsilon^2).$$

For example if T is a rotation then ε would be an infinitesimal angle. If ε is small, the higher order terms are neglectible and can be dropped. Given two such infinitesimal transformations A and B, the composition acts linearly:

$$(1 + \varepsilon_a A)(1 + \varepsilon_b B) = 1 + \varepsilon_a A + \varepsilon_b B + \mathcal{O}(\varepsilon^2).$$

A finite transformation of length l can be constructed by breaking l down into infinitesimal pieces,

$$T_n = \underbrace{\left(1 + \frac{l}{n}A\right)\left(1 + \frac{l}{n}A\right)\dots\left(1 + \frac{l}{n}A\right)}_{n\,\text{times}}.$$

By letting $n \to \infty$ each piece becomes infinitesimal and the above expression becomes more and more accurate. By expanding T_n we find

$$\begin{aligned}\lim_{n\to\infty} T_n &= \lim_{n\to\infty}\left(1 + lA + \binom{n}{2}\frac{l^2}{n^2}A^2 + \dots\right)\\ &= 1 + lA + \frac{1}{2!}l^2A^2 + \dots\\ &= e^{lA}.\end{aligned}$$

Thus the exponential map gives a finite transformation from an infinitesimal one. How does the infinitesimal transformations contain the information from the full group? The defining property of a group is that two transformations become a new transformation in the group. To see how this is described infinitesimally, let A and B be two infinitesimal transformations. Exponentiated they will give rise to a combined finite transformations, the question is now what infinitesimal transformation C does this new finite transformation correspond to. In equations,

$$e^A e^B = e^C.$$

The problem is that since A, B, C are operators they might not commute so $C \neq A + B$ in general. To find out what C is, we can simply take the logarithm (defined by its Taylor series) of both sides.

$$\begin{aligned}C &= \log\left(e^A e^B\right)\\ &= \log\left(\left[1 + A + \frac{1}{2}A^2 + \dots\right]\left[1 + B + \frac{1}{2}B^2 + \dots\right]\right)\\ &= 1 + A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 - \frac{1}{2}\left(A + B + \dots\right)^2 + \dots\\ &= 1 + A + B + \frac{1}{2}\left[A, B\right] + \dots\end{aligned}$$

So C is almost $A + B$ but with corrections in terms of the commutator. The wonderful thing is now that all of the higher order corrections can also be expressed in terms of the commutator. In conclusion, if we know the infinitesimal transformations and their commutators we can construct the full group. This last statement is not true for any group, but for compact simply connected groups it is. The gain from this construction is that instead of working with the group which in general is non-linear we can work with linear maps and their commutators instead. The infinitesimal transformations together with the commutator is called an algebra.

2.0.4 Clifford algebra

Suppose we have matrices $\Gamma^A, A \in \{0, 1, \dots d-1\}$ acting as linear transformations on a complex vector space with the property

$$\left\{\Gamma^A, \Gamma^B\right\} = 2\eta^{AB} I.$$

The algebra generated by these matrices is called a Clifford algebra and the matrices themselves are called gamma matrices. If such matrices can be found a representation of the Lorentz algebra can be constructed with the following combination.

$$\Sigma^{AB} = \frac{1}{4}\left[\Gamma^A, \Gamma^B\right]$$

These satisfy the Lorentz algebra,

$$\left[\Sigma^{AB}, \Sigma^{CD}\right] = -\eta^{AC}\Sigma^{BD} + \eta^{AD}\Sigma^{BC} - \eta^{BD}\Sigma^{AC} + \eta^{BC}\Sigma^{AD},$$

and thus constitute a representation of the Lorentz algebra.

It can be shown [14] that the gamma matrices can always be chosen such that

$$\begin{aligned}\left(\Gamma^0\right)^\dagger &= -\Gamma^0\\ \left(\Gamma^A\right)^\dagger &= \Gamma^A \quad A > 0.\end{aligned}$$

Antisymmetrized products of gamma matrices will be denoted by

$$\Gamma^{M_1 M_2 \dots M_n} = \Gamma^{[M_1}\Gamma^{M_2}\dots\Gamma^{M_n]}.$$

The generators of the Lorentz algebra can then be written as

$$\begin{aligned}\Sigma^{AB} &= \frac{1}{2}\frac{1}{2}\left(\Gamma^{AB} - \Gamma^{BA}\right)\\ &= \frac{1}{2}\Gamma^{AB}.\end{aligned}$$

In d dimensions the vector space on which these generators act is $2^{\left\lfloor \frac{d}{2}\right\rfloor}$, where the brackets indicate the integer part of a number. For $d = 6$ the vector space is 8 dimensional. Spinors are elements of this vector space.

2.0.5 Charge conjugation

In all dimensions there exists [14] a charge conjugation operator C with the property

$$\left(C\Gamma^{(r)}\right)^T = -t_r C\Gamma^{(r)}, \tag{2.1}$$

where $\Gamma^{(r)}$ is an antisymmetrized product of r gamma matrices and $t_r = \pm 1$ depending on the dimension and signature of the metric.

In 5 dimensions in particular the numerical factors are given by

$$\begin{aligned}t_0 &= t_1 = 1\\ t_2 &= t_3 = -1.\end{aligned}$$

In 6 dimensions there are two choices where one is equal to the 5 dimensional t_r. For convenience in the spinor reduction we choose t_r to be the same in both 5 and 6 dimensions.

Thus for example

$$C^T = -C \tag{2.2}$$
$$\left(C\Gamma^A\right)^T = -C\Gamma^A \tag{2.3}$$
$$\left(C\Gamma^{AB}\right)^T = C\Gamma^{AB} \tag{2.4}$$

These can then be used to relate a spinor to its transpose. Relation (2.3) implies

$$\begin{aligned} \left(\Gamma^A\right)^T C^T &= -C\Gamma^A \\ \text{[using (2.2)]} \Longleftrightarrow & \\ \left(\Gamma^A\right)^T &= C\Gamma^A C^{-1} \end{aligned} \tag{2.5}$$

This enables the definition of a conjugate spinor in the following useful way:

$$\bar{\lambda} \equiv \lambda^T C. \tag{2.6}$$

Note that this is not the standard definition, where the bar would indicate the Dirac conjugate spinor $\bar{\lambda} = \lambda^\dagger \Gamma^0$, but they are related as will be seen later.

With this definition we can form a scalar from two spinors that is Lorentz invariant,

$$\begin{aligned} \bar{\lambda}\varepsilon &= \lambda^T C\varepsilon \\ &= \lambda^\alpha C_{\alpha\beta}\varepsilon^\beta. \end{aligned} \tag{2.7}$$

Where α and β are indices in spinor space, i.e. the vector space where λ and ε are elements. This is the only time that the indices associated with the spinor space will be written out. All the properties of spinors can be conveniently expressed in matrix notation and henceforth all the spinor indices will be suppressed.

To check that (2.7) is invariant we perform an infinitesimal Lorentz transformation $\Lambda = 1 + \frac{1}{2}\omega_{MN}\Sigma^{MN} = \frac{1}{8}\omega_{MN}\Gamma^{MN}$. A spinor bilinear will then transforms as

$$\begin{aligned} \delta\left(\bar{\lambda}\varepsilon\right) &= \overline{\delta\lambda}\varepsilon + \bar{\lambda}\delta\varepsilon \\ &= \frac{1}{8}\left(\overline{\omega_{MN}\Gamma^{MN}\lambda}\right)\varepsilon + \frac{1}{8}\bar{\lambda}\omega_{MN}\Gamma^{MN}\varepsilon \\ &= \frac{1}{8}\omega_{MN}\lambda^T\left(\Gamma^{MN}\right)^T C\varepsilon + \frac{1}{8}\bar{\lambda}\varepsilon_{MN}\Gamma^{MN}\varepsilon \\ &= \frac{1}{8}\omega_{MN}\lambda^T CC^{-1}\left(\Gamma^{MN}\right)^T C\varepsilon + \frac{1}{8}\bar{\lambda}\varepsilon_{MN}\Gamma^{MN}\varepsilon \end{aligned}$$

Note that (2.2) and (2.4) implies that $\quad C^{-1}\left(\Gamma^{MN}\right)^T C = -\Gamma^{MN}$

$$\begin{aligned} &= \frac{1}{8}\bar{\lambda}\varepsilon_{MN}\Gamma^{MN}\varepsilon - \frac{1}{8}\bar{\lambda}\varepsilon_{MN}\Gamma^{MN}\varepsilon \\ &= 0 \end{aligned}$$

Note also that with these definitions we can flip a spinor bilinear, start for example with a bilinear

$$\bar{\lambda}\Gamma_M\varepsilon.$$

This is a complex number and therefore equal to it's transpose.

$$\begin{aligned}\left(\bar{\lambda}\Gamma_M\varepsilon\right)^T &= \left(\lambda^T C\Gamma_M\varepsilon\right)^T \\ &= -\varepsilon^T\left(\Gamma_M\right)^T C^T\lambda \\ &= \varepsilon^T C C^{-1}\Gamma_M C\lambda \\ &= \varepsilon^T C\Gamma_M\lambda \\ &= \bar{\varepsilon}\Gamma_M\lambda\end{aligned}$$

Where the first sign change comes from the fact that we have defined complex conjugation to swap the order of the Grassman components of the spinors.

2.0.6 Majorana spinors

In four dimensions there is the possibility to impose a reality condition on spinors. The spinors satisfying such a condition are called Majorana spinors. It will turn out that it is not possible to impose this kind of condition on spinors in 5 and 6 dimensions, but there is a generalisation. To motivate the introduction of symplectic Majorana spinors this section will first investigate why the normal approach fails in these dimensions.

A general reality condition for a spinor λ takes the form

$$\lambda^\star = B\lambda. \tag{2.8}$$

The reason for the matrix B is that the simplest condition $\lambda^\star = \lambda$ is not Lorentz invariant and thus a real spinor in one frame is necessarily not real in another. To find out what B needs to satisfy we perform a Lorentz transformation of (2.8).

$$\begin{aligned}\omega_{MN}\left(\Gamma^{MN}\right)^\star\lambda^\star &= \omega_{MN}B\Gamma^{MN}\lambda \\ &\Longleftrightarrow \\ \omega_{MN}\left(\Gamma^{MN}\right)^\star\lambda^\star &= \omega_{MN}B\Gamma^{MN}B^{-1}B\lambda\end{aligned}$$

Thus B must satisfy

$$B\Gamma^{MN}B^{-1} = \left(\Gamma^{MN}\right)^\star$$

With C such a matrix can be constructed as follow.

$$B = C\Gamma^0.$$

Actually,

$$\begin{aligned}B\Gamma^M B^{-1} &= C\Gamma^0\Gamma^M\left(-\Gamma^0\right)C^{-1} \\ &= -C\left(\Gamma^M\right)^\dagger C^{-1} \\ &= -C\left(\left(\Gamma^M\right)^\star\right)^T C^{-1} \\ &= -\left(\Gamma^M\right)^\star \qquad (2.9) \\ &\Longrightarrow \\ B\Gamma^M\Gamma^N B^{-1} &= B\Gamma^M B^{-1}B\Gamma^N B^{-1} \\ &= \left(\Gamma^M\Gamma^N\right)^\star.\end{aligned}$$

Now "charge conjugation" can be defined as

$$\lambda^C = B^{-1}\lambda^\star. \tag{2.10}$$

Note that with these definitions,

$$\left(\lambda^C\right)^C = -\lambda.$$

The reality condition then becomes

$$\begin{aligned} \lambda^C &= \lambda \\ B^{-1}\lambda^\star &= \lambda \\ \lambda^\star &= B\lambda \end{aligned} \tag{2.11}$$

It turns out that it is not only a Lorentz transformation that can spoil this relation. By just taking the complex conjugate of (2.11) and inserting back into itself we get

$$\lambda^\star = BB^\star\lambda^\star$$

So for a spinor of this type to be non-zero, B needs to satisfy

$$BB^\star = 1 \tag{2.12}$$

But in 5 or 6 dimensions this is not the case with B defined as above,

$$\begin{aligned} BB^\star &= C\gamma^0C^\star\left(\gamma^0\right)^\star \\ [\text{insert identity}] &= C\gamma^0C^{-1}CC^\star\left(\gamma^0\right)^\star \\ [\text{use}(2.3)] &= \left(\gamma^0\right)^T CC^\star\left(\gamma^0\right)^\star \\ [\text{use}(2.2)] &= -\left(\gamma^0\right)^T CC^\dagger\left(\gamma^0\right)^\star \\ [C\,\text{unitary}] &= -\left(\gamma^0\right)^T\left(\gamma^0\right)^\star \\ &= -\left(\left(\gamma^0\right)^\dagger\gamma^0\right)^\star \\ &= -\left(-\gamma^0\gamma^0\right)^\star \\ &= -1. \end{aligned}$$

So we can not consistently impose this kind of reality condition on spinors in 5 or 6 dimensions. There is another way to impose a reality condition between an even number of spinors instead of relating a spinor to itself. The resulting spinors are called symplectic Majorana spinors and these are the irreducible spinors in 5 dimensions. There are also symplectic Majorana spinors in 6 dimensions but here we can also have chirality and the two are compatible so that the irreducible spinors are symplectic Majorana-Weyl.

Before turning to symplectic Majorana spinors we make some useful definitions for reality calculations. The Majorana condition will be stated in terms of charge conjugation it would be nice to be able to express complex conjugation in terms of charge conjugation when checking the reality of bilinears. To arrive at such a relation, regard a general bilinear term.

$$\begin{aligned} \left(\bar\lambda M\varepsilon\right)^\star &= -\left(\bar\lambda\right)^\star M^\star\varepsilon^\star \\ &= -\left(\bar\lambda\right)^\star M^\star BB^{-1}\varepsilon^\star \\ &= -\left(\bar\lambda\right)^\star M^\star B\varepsilon^C \end{aligned}$$

Then note that,

$$
\begin{aligned}
(\bar{\lambda})^\star &= \left(\lambda^T C\right)^\star \\
&= \lambda^{\star T} C^\star \\
&= \left(B^{-1}\lambda^\star\right)^T B^T C^\star \\
&= \left(\lambda^C\right)^T B^T C^\star B B^{-1} \\
&= \left(\lambda^C\right)^T \left(\Gamma^0\right)^T C^T C^\star \left(C\Gamma^0\right) B^{-1} \\
&= \left(\lambda^C\right)^T \left(\Gamma^0\right)^T C\Gamma^0 B^{-1} \\
&= \left(\lambda^C\right)^T C C^{-1} \left(\Gamma^0\right)^T C\Gamma^0 B^{-1} \\
&= -\overline{\lambda^C} B^{-1}.
\end{aligned}
$$

Combining this gives

$$\left(\bar{\lambda} M \varepsilon\right)^\star = \overline{\lambda^C} B^{-1} M^\star B \varepsilon^C.$$

If we then define $M^C = B^{-1} M^\star B$, complex conjugation of a bilinear can be replaced with charge conjugation and we have

$$
\begin{aligned}
\left(\bar{\lambda} M \varepsilon\right)^\star &= \left(\bar{\lambda} M \varepsilon\right)^C \\
&= \overline{\lambda^C} M^C \varepsilon^C
\end{aligned}
$$

In particular we can calculate the action of charge conjugation on a gamma matrix:

$$
\begin{aligned}
\left(\Gamma^M\right)^C &= B\left(\Gamma^M\right)^\star B^{-1} \\
\text{[using (2.9)]} &= -\Gamma^M
\end{aligned}
$$

2.1 Symplectic Majorana spinors

The crucial thing that prevented the construction of Majorana spinors in 5 and 6 dimensions was the consistency check (2.12). As will be investigated in this section there is a way around this. We will introduce an antisymmetric, non-degenerate form M_{ij} and then change the reality condition to

$$\left(\lambda^i\right)^\star = M_{ij} B \lambda^j. \tag{2.13}$$

In order not to loose any information by imposing this relation the form should also be invertible. A form with these conditions only exists in even dimensions, thus for this construction to work there needs to be an even number of spinors. A non-degenerate antisymmetric form is called a symplectic form [15].

In the (2,0) theory under consideration, the above structure comes from the internal R-symmetry group Sp(4).

2.1.1 Symplectic group and representations

We will take the spinor fields to live in the fundamental representation of Sp(4). In other words they will be vectors in a 4 dimensional complex vector space on which the symplectic transformations act. The vector space will be denoted by

4, referring to its dimension. The symplectic transformations are defined by that they preserve an antisymmetric form which for example can be taken to be

$$M_{ab} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

This bilinear form acts on two vectors $x, y \in \mathbf{4}$ in the fundamental representation in the following way,

$$\begin{aligned}(x,y)_M &= x^i M_{ij} y^j \\ &= x^T M y \quad \in \mathbb{C}.\end{aligned} \tag{2.15}$$

The symplectic form is very similar to an inner product except that by definition it is antisymmetric. To be preserved under a transformation A means that

$$(x,y)_M = (Ax, Ay)_M,$$

or in matrix notation using definition (2.15),

$$\begin{aligned}x^T M y &= (Ax)^T M (Ay) \\ &= x^T A^T M A y.\end{aligned} \tag{2.16}$$

Which says that the inner product of the vectors are unchanged by a symplectic transformation. A symplectic transformation A satisfies (2.16) for all vector x and y i.e

$$A^T M A = M.$$

As with a metric the symplectic form can be used to define upper and lower indices. Any vector space V has a dual vector space $V^\star$ where elements are linear maps from V to $\mathbb{C}$. When the vector space has a non-degenerate bilinear form defined there is a natural isomorphism between V and $V^\star$. In this case the isomorphism takes the form

$$\begin{aligned}V &\to V^\star \\ x &\mapsto M_{ij} x^j.\end{aligned}$$

The resulting object $M_{ij}x^j$ is indeed a linear map from V to $\mathbb{C}$ since we can contract it with a vector to form a complex number. The standard definition is then to say that vectors in V have upper indices and vectors in $V^\star$ have lower.

The matrix M_{ij} has an inverse denoted by T^{ij} that satisfy

$$M_{ij} T^{jk} = \delta_i{}^k.$$

This in turn can be used to map a vector in $V^\star$ to V, or in the language of indices, to raise the index of vectors. In practice this means that we can have a notation that looks exactly like that for space-time vectors, Einstein summation convention for contractions etc.

From the theory of representations we can choose our representation to be unitary ([15] chap. 13.4), that is all the transformations are unitary matrices.

Indeed we must if the theory is to preserve probabilities. For the algebra of infinitesimal transformations this means that the representation matrices will be antihermitian. A matrix is antihermitian if $R = -R^\dagger = -R^{\star T}$.

A complex representation naturally has a conjugate representation, constructed by simply taking the complex conjugate of the matrices. These also form a representation and in the case of unitary representations this is related to the dual vector space. To see this lets start by taking the complex conjugate of a representation matrix R.

$$\begin{aligned} R^\star &= \left(-R^\dagger\right)^\star \\ &= -R^T \end{aligned}$$

Thus the complex conjugate of a representation matrix is associated with it's transpose, but since the transpose naturally acts on the dual this means that something that transforms under $R^\star$ transforms in the same way as something in the dual.

In particular we can look at a vector v^i, it transforms under symplectic transformations as

$$w^i = R^i{}_j v^j.$$

Taking the complex conjugate of this equation gives

$$\begin{aligned} \left(w^i\right)^\star &= \left(R^i{}_j\right)^\star \left(v^j\right)^\star \\ &= -\left(R^T\right)^i{}_j \left(v^j\right)^\star \end{aligned}$$

So $(v^i)^\star$ transforms as a vector in the dual representation. In this way we arrive at the conclusion that it is natural to associate a lower index to the complex conjugate of a vector, $\left(v^i\right)^\star = \tilde{v}_i$. Note that this object need not be the same as lowering the index using the symplectic form, hence the tilde on the right hand side.

2.1.2 The symplectic Majorana condition

With the above definitions for the symplectic group the consistency of (2.13) can be checked. First by complex conjugation of (2.13) we find

$$\lambda^i = \left(M_{ij}\right)^\star B^\star \left(\lambda^j\right)^\star$$

inserting this back into (2.13):

$$\begin{aligned} \left(\lambda^i\right)^\star &= M_{ij} B \left(M_{jk}\right)^\star B^\star \left(\lambda^k\right)^\star \\ &= B B^\star M_{ij} \left(M_{jk}\right)^\star \left(\lambda^k\right)^\star \\ &= -M_{ij} \left(M_{jk}\right)^\star \left(\lambda^k\right)^\star , \end{aligned}$$

where M_{ij} is just a complex number and thus commutes with everything.

For this to be consistent we thus require M_{ij} to satisfy

$$M_{ij} \left(M_{jk}\right)^\star = -\delta_i{}^k. \tag{2.17}$$

That M_{ij} satisfies this can be checked by noting that (2.14) satisfies it and that any other $\tilde{M}_{ij}$ can be brought to this form by a unitary change of basis. For the components of $(M_{ij})^\star$, (2.17) implies that

$$(M_{ij})^\star = -T^{ij}.$$

In conclusion we now have a consistent reality condition for chiral spinors in 6 dimensions,

$$\left(\psi^i\right)^\star = M_{ij} B \psi^j.$$

Multiplying with B^{-1} gives

$$B^{-1}\left(\psi^i\right)^\star = M_{ij}\psi^j.$$

Comparing this to the definition of charge conjugation (2.10), the condition is simply

$$\begin{aligned}\left(\psi^i\right)^C &= M_{ij}\psi^j \\ &= \psi_i\end{aligned} \tag{2.18}$$

Which also implies

$$\begin{aligned}\left(\psi_i\right)^C &= \left(M_{ij}\psi^j\right)^C \\ &= (M_{ij})^\star \left(\psi^j\right)^C \\ &= -T^{ij}\psi_j \\ &= -\psi^i\end{aligned} \tag{2.19}$$

Expression (2.18) and (2.19) are the conditions that will be used in calculations. A spinor that satisfies this condition is called a symplectic Majorana spinor.

Note that in all calculations involving the internal indices it is important to note the order of contracted indices since the symplectic form and it's inverse are antisymmetric.

Spinor bilinears To create spinor bilinears that are invariant under both Lorentz and symplectic transformations we simply write down the natural contraction of two symplectic Majorana spinors using the symplectic form,

$$M_{ji}\overline{\psi^i}\varepsilon^j = \overline{\psi_j}\varepsilon^j.$$

Note the choice of order in the contraction which will be adopted as convention when the internal indices are suppressed, so that

$$\bar{\psi}\varepsilon = \overline{\psi_i}\varepsilon^i.$$

Finally we can put all of this framework together to easily verify if a given bilinear in terms of symplectic Majorana spinors is real. For example,

$$\begin{aligned}\left(\bar{\psi}\varepsilon\right)^\star &= \left(\overline{\psi_i}\varepsilon^i\right)^\star \\ &= \left(\overline{\psi_i}\varepsilon^i\right)^C \\ &= \overline{\left(\psi_i\right)^C}\left(\varepsilon^i\right)^C \\ &= -\overline{\psi^i}\varepsilon_i \\ &= -M_{ij}\overline{\psi^i}\varepsilon^j \\ &= \overline{\psi_i}\varepsilon^i\end{aligned}$$

It is now possible to check if a given term in the action functional is real. In the end a well defined Lagrangian should be real scalar and it is now possible to verify that easily. This is useful as a consistency check on the reduction as well, since all the resulting terms should be real or possibly sum together to be real.

Dirac bar The connection between definition (2.6) of conjugation and the standard Dirac bar can be seen by regarding the specific combination

$$\begin{aligned}
\overline{\lambda^C} &= \overline{B^{-1}\lambda^\star} \\
&= \lambda^\dagger \left(B^{-1}\right)^T C \\
&= -\lambda^\dagger \left(\Gamma^0 C^\dagger\right)^T C \\
&= -\lambda^\dagger C^\star \left(\Gamma^0\right)^T C \\
&= \lambda^\dagger C^\dagger \left(\Gamma^0\right)^T C \\
&= \lambda^\dagger C^\dagger \left(\Gamma^0\right)^T C \\
&= \lambda^\dagger \Gamma^0 .
\end{aligned}$$

Which is the standard Dirac bar. The two definitions agree for normal Majorana spinors, since then $\lambda^C = \lambda$, but in the case of symplectic Majorana spinors the definition employed here is simply more convenient.

Chapter 3

Field theory on curved space-time

"We'll do a parallel transport on the Captain."

-Worf: Star Trek TNG, 1988

The first part of this chapter will briefly review the relevant parts of differential geometry relevant to this thesis and introduce the concept of vielbeins. The vielbeins will then be used to carry over the construction of spinors to the setting of curved space-time. This chapter is based on standard material that can be found for example in [16], [17] and [18]. The discussion of spinors in curved space-time is based on [19].

3.1 Mathematics of curved space-time

A smooth manifold is something that looks locally like flat space $\mathbb{R}^d$. A smooth Lorentzian manifold is also equipped with a non-degenerate metric of indefinite signature, in this thesis taken to be $(-1, 1, \ldots, 1)$. This means that in a small neighbourhood around a point the manifold is equivalent to a neighbourhood of $\mathbb{R}^d$ together with an indefinite metric. This enables the manifold to be described in terms of the coordinates on $\mathbb{R}^d$.

The coordinates on a smooth Lorentzian manifold M of dimension d will be denoted by $x^0, x^1, \ldots, x^{d-1}$. Indices for tensors in the set of capital roman letters $L, M, \ldots$ will take values $\{0, 1, \ldots d-1\}$ in anticipation of the reduction in chapter 6 where two sets of indices will be needed.

At each point p on the manifold there is a tangent space, denoted T_pM, which is the vector space spanned by vectors tangent to the manifold at this point. Here one can think of the example when the manifold is a 2 dimensional surface in space. In this case the tangent space at a point is the tangent plane, i.e the best linear approximation to the surface at that point.

The local coordinates x^M induce a basis of the tangent space at a point denoted by ∂_M. A vector at this point can be expressed in terms of the basis,

$$V = V^M \partial_M.$$

This notation stems from the fact that when the manifold is not described by an embedding into $\mathbb{R}^n$ one can still define the tangent space in terms of directional derivatives.

Usually one does not write out the geometric information about the vector but instead choose to work only with the component V^M.

The metric tensor will be denoted G_{MN} and the scalar product that corresponds to it will be denoted

$$< V, W > = V^M W^N G_{MN}.$$

The inverse metric will be denoted by G^{MN}.

There is a covariant derivative on M denoted D_N defined by its action on tangent vectors,

$$D_N V^P = \partial_N V^P + \Gamma^P_{NL} V^L.$$

The connection coefficients Γ^P_{MN} are the Christoffel symbols, uniquely determined by the condition that the covariant derivative should be metric compatible,

$$D_N < V, W > = < D_N V, W > + < V, D_N W >, \tag{3.1}$$

and that $\Gamma^P_{NL} = \Gamma^P_{LN}$, i.e. it should be torsion free.

Solving (3.1) in local coordinates for the connection coefficients gives

$$\Gamma^P_{MN} = \frac{1}{2} G^{PL} \left(\partial_M G_{NL} + \partial_N G_{ML} - \partial_L G_{MN} \right).$$

Condition (3.1) is also equivalent in component form to the condition

$$D_M G_{NP} = 0.$$

There is a canonical tensor defined on any smooth manifold with a covariant derivative that characterises it's curvature, the Riemann tensor. It is defined by the amount that two covariant derivatives fail to commute,

$$\left[D_M, D_N \right] V^P = R_{MN}{}^P{}_L V^L. \tag{3.2}$$

This is the infinitesimal form of parallel transporting the vector V around a small loop and comparing it to the original vector. The new vector will differ by a linear transformation specified by the Riemann tensor.

By writing out the left hand side of (3.2), an expression for the Riemann tensor in terms of the Christoffel symbols is found,

$$R^P{}_{SMN} = \partial_M \Gamma^P_{NS} - \partial_N \Gamma^P_{MS} + \Gamma^Q_{SN} \Gamma^P_{QM} - \Gamma^Q_{SM} \Gamma^P_{QN}$$

From the Riemann tensor one additional tensor and one scalar can be constructed by contraction.

$$\begin{aligned} R_{MN} &= R^P{}_{MPN} \\ R &= G^{MN} R_{MN} \end{aligned}$$

These are the Ricci tensor and the Ricci scalar respectively.

3.2 Vielbeins

The coordinate basis for the tangent space defined above is just one of many possible choices. The freedom to choose a basis can be used to create a new tensor called the vielbein that will turn out to be very useful. The starting point is to decribe the metric information of a manifold locally by choosing a basis of orthonormal tangent vectors at each point.

$$\begin{aligned} E_A\,(x) &\in T_x M \\ E_A(x) &= E_A^M(x)\partial_M \\ < E_A, E_B > &= \eta_{AB} \end{aligned} \tag{3.3}$$

The E_A are called vielbeins, German for many legs, and are as stated simply vectors in the tangent space.

In components the orthonormality means

$$\begin{aligned} < E_A, E_B > &= < E_A^M \partial_M, E_B^N \partial_N > \\ &= E_A^M E_B^N G_{MN} \\ &= \eta_{AB}. \end{aligned} \tag{3.4}$$

Now, since it was assumed that the vielbeins formed a basis, the coordinate basis can also be expressed in the vielbeins

$$\partial_M = (\partial_M)^C\, E_C.$$

Substituting this into (3.3) shows that

$$E_A(x) = E_A^M(x)\,(\partial_M)^C\, E_C(x).$$

For this to be consistent, the matrix $(\partial_M)^C$ is seen to be the inverse of $E_A^M(x)$. In other words

$$E_A^M(x)\,(\partial_M)^C = \delta_A^C.$$

The same line of reasoning leads to

$$E_A^M(x)\,(\partial_N)^A = \delta_N^M.$$

To adhere to the standard notation found in literature, and also following the notation for Lorentz transformations, we now make the slightly confusing choice to name

$$(\partial_M)^C = E_M^C.$$

One could have argued that the matrix E_A^M by definition is invertible and just name it's inverse E_M^A but the above construction emphasises the geometric origin. It should be noted that the vielbeins also go by the name of frame fields.

3.3 Spinors on curved space-time

The starting point for introducing spinors on curved manifolds is to note that the tangent space at a point is a flat Minkowski vector space by the definition of a Lorentzian manifold. So it might seem like there is no problem in doing the standard construction of spinors on the tangent space using a Clifford algebra as in chapter 2. But there are some obstacles.

The next step is to note that two coordinate systems on M induce two different bases for the tangent space at a point. These two bases are related by a general linear transformation. Since the spin group is only related to the group of orthogonal transformations we will need some way to restrict the general linear transformation above to something that is only an orthogonal transformation so that we know how it acts in spin space. This is where the vielbeins come in. They define an orthonormal base of the tangent space, and any two different sets of vielbeins will thus be related by an orthogonal transformation.

We start by associating with every point on the manifold a complex vector space of dimension $2^{\frac{d}{2}}$. On this vector space we define d operators, the gamma matrices. Finally we couple these gamma matrices to the vielbeins to create gamma matrices with a coordinate index.

$$\Gamma^M \equiv E_A^M \Gamma^A \tag{3.5}$$

The point of this is that by the construction of the vielbeins, they will transform under general coordinate transformations since they are related to the coordinate basis. The gamma matrices on the other hand is only related to a non-coordinate basis on the tangent space, namely the vielbeins themselves, and so they will only transform under local Lorentz transformations. These local Lorentz transformations are implemented exactly as a change of orthonormal basis in the tangent space.

The gamma matrices with indices $A, B, \ldots$ will be referred to as the flat gamma matrices, and the newly defined ones as the curved gamma matrices.

By the commutation relations of the flat gamma matrices and the properties of the vielbeins,

$$\begin{aligned}
\left\{\Gamma^M, \Gamma^N\right\} &= E_A^M E_B^N \left\{\Gamma^A, \Gamma^B\right\} \\
&= 2E_A^M E_B^N \eta^{AB} \\
&= 2G^{MN}.
\end{aligned}$$

Which would be the natural generalisation of $\left\{\Gamma^A, \Gamma^B\right\} = 2\eta^{AB}$ to the curved case.

3.3.1 Spin connection

To be able to sensibly differentiate spinor fields there is a need for something to relate a spinor at one point to a spinor at another. This new tool will be called the spin connection and it is completely analogous to the Christoffel connection for vectors. The definition is

$$\begin{aligned}
D_M\psi &= \partial_M\psi + \frac{1}{2}\Omega_M^{AB}\Sigma_{AB}\psi \\
&= \partial_M\psi + \frac{1}{4}\Omega_M^{AB}\Gamma_{AB}\psi.
\end{aligned} \tag{3.6}$$

Here Σ_{AB} are generators for the Lorentz group in the spin representation and Ω_M^{AB} is the spin connection. In words this definition says that the change in a spinor field at a point is the combination of a change due to the component functions and a change corresponding to a linear transformation of the spinor. This is completely analogous the Christoffel connection and can be made more clear by a slight change in notation for the vector case,

$$D_M V^P = \partial_M V^P + (\Gamma_M)^P{}_L V^L.$$

Another way to think about the spin connection in the language of gauge theory is that the spin connection is the gauge field associated with local Lorentz transformations. Thus we can think of the spin connection as the required object to be introduced if we want to create a derivative that transforms covariantly under local Lorentz transformations,

$$D_M \left(\Lambda(x)\psi\right) = \Lambda(x) D_M \psi.$$

Exactly as the vector potential is introduced to create a suitable covariant derivative when the global $U(1)$ symmetry of QED is promoted to a local symmetry [1].

The spin connection also defines a covariant derivative for objects with flat indices associated with the vielbeins. The only difference to the definition (3.6) is that the generators for the Lorentz group are taken in the vector representation instead of the spin representation.

$$D_M X^A = \partial_M X^A + \frac{1}{2}\Omega_M^{CD} \left(T_{CD}\right)^A{}_L X^L$$

For the particular vector representation $(T_{CD})^{AB} = \delta_C^A \delta_D^B - \delta_C^B \delta_D^A$ this simply becomes

$$D_M X^A = \partial_M X^A + \Omega_{MD}^C X^D$$

Since we can write $X^M = E_A^M X^A$, a consistency condition of the spin connection is that $D_N E_A^M = 0$ so that the covariant derivative gives the same result no matter what representation we choose to present the vector X in.

$$D_N X^M = E_A^M D_N X^A$$

This condition when written out,

$$\begin{aligned} D_M E_A^N &= \partial_M E_A^N + \Gamma_{ML}^N E_A^L + \Omega_{NA}^B E_B^N \\ &= 0, \end{aligned}$$

determines the spin connection uniquely in terms of the vielbeins.

$$\Omega_M^{AB} = 2E^{N[A}\partial_{[M} E_{N]}^{B]} - E^{P[A} E^{|Q|B]} \partial_P E_{QC} E_M^C$$

Since the spin connection is defined to agree with the Christoffel connection they both have the same curvature tensor. This results in the following relation for spinors,

$$\left[D_M, D_N\right]\psi = \frac{1}{4} R_{MNAB}\Gamma^{AB}\psi,$$

which can be verified by direct computation.

3.3.2 Obstructions from topology

The above construction sweeps a few requirements under the rug. It turns out that to be able to introduce spinors on the whole manifold in a consistent way the manifold needs to satisfy certain properties. There is a precise classification of when a manifold admits a spin structure as it is called. The classification can be given in terms of the cohomology groups of the space. Specifically the manifold admits a spin structure if a certain element in the second cohomology group with $\mathbb{Z}_2$ coefficients is trivial.

$$H^2(M, \mathbb{Z}_2) \ni w_2 = 1$$

The failure in trying to introduce a spin structure on certain manifold is similar in nature to the problem of defining a nowhere vanishing vector field on a manifold. The sphere for example does not admit any nowhere vanishing vector fields. This fact is usually expressed colloquially as the statement "you cannot comb a sphere" which is a consequence of the aptly named hairy ball theorem.

The main point is that spinors can not be introduced on all smooth manifolds.

For a full treatment of these concepts see ref. [20].

3.4 Differential forms

> "These are the things which occur under integral signs."
>
> -Harley Flanders: Differential Forms with Applications to the Physical Sciences.

A differential form is a totally antisymmetric tensor field. As indicated they are the generalisation of the differentials encountered in calculus, guided especially by their use in integration. They are conveniently represented by introducing the basis 1-forms dx^M and the wedge product $\wedge$. The wedge product is defined by

$$dx^M \wedge dx^N = -dx^N \wedge dx^M .$$

A general differential form of degree p is then a linear combination of the basis forms

$$\alpha = \frac{1}{p!} a_{M_1 \dots M_p} dx^{M_1} \wedge \dots \wedge dx^{M_p} .$$

By the antisymmetry of the wedge product the coefficient functions $\alpha_{M_1 \dots M_p}$ can be taken to be antisymmetric.

Note that by antisymmetry, $dx^M \wedge dx^M = -dx^M \wedge dx^M = 0$. In particular this means that in d dimensions, differential forms of degree higher than d will be identically zero since there will need to be some repetition in the wedge product of basis forms.

The space of p-forms on M is denoted $\Omega^p(M)$ and forms a $\binom{d}{p}$-dimensional vector space. The algebra generated by the forms together with the wedge product is the collection of all these spaces $\oplus_p \Omega^p(M)$ called the exterior algebra.

There is a derivative operator for differential forms whose action on a p-form gives a $(p+1)$-form. It is defined by its action on functions

$$df(x) = \partial_M f(x) dx^M,$$

and then extended to a general form by

$$d\left(\frac{1}{p!} a_{M_1 \ldots M_p} \mathrm{dx}^{M_1} \wedge \ldots \wedge \mathrm{dx}^{M_p}\right) = \frac{1}{p!} \partial_N a_{M_1 \ldots M_p} dx^N \wedge dx^{M_1} \wedge \ldots \wedge dx^{M_p}.$$

From these definitions it follows that $d^2 = 0$ on any form. It also follows that for $A \in \Omega^q(M)$ and $B \in \Omega^l(M)$,

$$d(A \wedge B) = dA \wedge B + (-1)^q A \wedge dB. \tag{3.7}$$

The operator d is called the exterior derivative.

Finally we can make contact with integration by defining

$$\int_M f(x) dx^{M_1} \wedge \ldots \wedge dx^{M_d} \equiv \int_M f(x) dx^{M_1} \ldots dx^{M_d},$$

where the right hand side is the Riemann integral. Strictly speaking the above definition only makes sense in a coordinate patch of the manifold so that integration over all of M needs to be broken down into smaller pieces. The main point of differential forms is exactly that the above definition does not depend on the choice of coordinates so that integration makes sense.

For a more complete treatment of differential forms see for example [21], [18] and [16].

3.4.1 Levi-Civita tensor

There is a tensor that can be defined on any smooth manifold called the Levi-Civita tensor. Start by introducing the totally antisymmetric symbol $\varepsilon^{M_1 M_2 \ldots M_d}$, defined by

$$\varepsilon^{01 \ldots d-1} = 1,$$

and that it is totally antisymmetric in all indices. This is not a tensor but a tensor density, transforming with an additional factor of the determinant of the coordinate transformation. By combining it with the determinant of the metric, which is also a tensor density, we can create a tensor $\epsilon^{M_1 M_2 \ldots M_d}$ called the Levi-Civita tensor.

$$\epsilon^{M_1 M_2 \ldots M_d} = \frac{1}{\sqrt{-G}} \varepsilon^{M_1 M_2 \ldots M_d}$$

3.4.2 Hodge dual

There is a natural operator acting on differential forms on any manifold with a metric, this operator is called the Hodge dual and it will be used later when introducing the action and equations of motion for a form field.

If there is a metric present on a smooth manifold then there is a natural map from p-forms to $(d-p)$-forms. One way to see this is to note that we have a canonical totally antisymmetric tensor with d indices, the Levi-Civita tensor. We can use this to construct a $(d-p)$-form α from a p-form β according to (3.8).

$$\alpha_{M_1\ldots M_{d-p}} = C\epsilon_{M_1\ldots M_{d-p}}{}^{N_{d-p+1}\ldots N_d}\beta_{N_{d-p+1}\ldots N_d} \qquad C\in\mathbb{C} \tag{3.8}$$

Because the Levi-Civita tensor is totally antisymmetric, α will also be antisymmetric and thus we have constructed a new differential form. This new form will be called the Hodge dual of β, denoted $\star\beta$. Note that the Hodge dual of a 0-form is a d-form, and so it will be proportional to the volume form.

$$(\star 1)_{M_1\ldots M_d} = C\epsilon_{M_1\ldots M_d}$$

Writing this out in terms of the basis forms gives

$$\begin{aligned}
\star 1 &= \frac{1}{d!}C\epsilon_{M_1\ldots M_d}dx^{M_1}\wedge\ldots\wedge dx^{M_d}\\
&= \frac{1}{d!}CG_{M_1N_1}\ldots G_{M_dN_d}\epsilon^{N_1\ldots N_d}dx^{M_1}\wedge\ldots\wedge dx^{M_d}\\
&= \frac{1}{d!}CG_{M_1N_1}\ldots G_{M_dN_d}\frac{1}{\sqrt{-G}}\varepsilon^{N_1\ldots N_d}dx^{M_1}\wedge\ldots\wedge dx^{M_d}\\
&= CG_{1N_1}\ldots G_{dN_d}\frac{1}{\sqrt{-G}}\varepsilon^{N_1\ldots N_d}dx^{1}\wedge\ldots\wedge dx^{d}\\
&= C\frac{G}{\sqrt{-G}}dx^1\wedge\ldots\wedge dx^d\\
&= -C\sqrt{-G}dx^1\wedge\ldots\wedge dx^d.
\end{aligned}$$

So if we choose $C=-1$ then this is exactly equal to the volume form of the manifold. Actually we choose to $C=\frac{-1}{(d-p)!}$, depending on the degree of the form. Definition (3.9) summarises the conventions used.

$$\begin{aligned}
\star\alpha &= \star\left(\frac{1}{p!}\alpha_{Q_1\ldots Q_p}dx^{Q_1}\wedge\ldots\wedge dx^{Q_p}\right)\\
&= -\frac{1}{p!}\frac{1}{(d-p)!}\epsilon_{M_1\ldots M_{d-p}}{}^{N_1\ldots N_p}\alpha_{N_1\ldots N_p}dx^{M_1}\wedge\ldots\wedge dx^{M_{d-p}}
\end{aligned} \tag{3.9}$$

Note that when writing out the full form with the basis elements it is actually more natural to think of the $\star$ operator acting on the basis elements instead of on the components functions. In this language the $\star$ operator can be defined equivalently as

$$\star dx^{M_1}\wedge\ldots\wedge dx^{M_p} = \frac{-1}{(d-p)!}\epsilon^{M_1\ldots M_p}{}_{N_1\ldots N_{d-p}}dx^{N_1}\wedge\ldots\wedge dx^{N_{d-p}}$$

If this is taken as the definition of the star operator on differential forms it will induce the same transformation on the component functions as defined in (3.8). This definition is somewhat more geometrical since it describes how the operator acts on the geometric objects, the basis forms, instead of the induced transformation on the component functions.

3.5 Conformal structure

A conformal transformation preserves angles but might distort the size of objects. To preserve angles means that if two curves cross with a certain angle, they will continue to do so after the conformal transformation. Globally the shape and size of things will change but locally angles are preserved. An example of a conformal transformation is the representation of the earth on a Mercator map. If a ship sails with a specific compass course, i.e. angle to the meridian, it will also do so on the Mercator map [22]. But the map does not preserve area, Africa looks about as big as Greenland while in reality it is ≈ 14 times larger.

The group of conformal transformations in flat space consists of the Lorentz transformations together with translations and 5 additional transformations: the scaling transformation and 4 special conformal transformations.

On curved space-time all the information about lengths and angles is contained in the metric. A conformal transformation can be characterised by that it preserves the metric up to a scaling

$$\begin{aligned} < V, W > &\rightarrow e^{-2\sigma(x)} < V, W > \\ &= e^{-2\sigma(x)} G_{MN} V^M W^N. \end{aligned} \tag{3.10}$$

Note that $\sigma(x)$ is a function of the coordinates. This is simply a rescaling of the metric at each point by a strictly positive smooth function.

From (3.10) it is clear that the transformation preserves angles since

$$\begin{aligned} \cos(\alpha) &= \frac{< V, W >}{\sqrt{V^2 W^2}} \\ &\rightarrow \frac{e^{2\sigma} < V, W >}{\sqrt{e^{2\sigma} e^{2\sigma} V^2 W^2}} \\ &= \frac{< V, W >}{\sqrt{V^2 W^2}}. \end{aligned}$$

A conformal structure is a collection of metrics that are related by local rescalings of this kind. Conformal invariance in this sense means that a theory only depends on the conformal structure. In other words it should not matter which metric in the conformal structure one chooses to use.

When checking that something is conformally invariant we can implement the transformation in (3.10) by letting the components of the metric transform as

$$G_{MN} \rightarrow e^{-2\sigma(x)} G_{MN}. \tag{3.11}$$

Also note that (3.11) implies that the inverse metric transforms as

$$G^{MN} \rightarrow e^{2\sigma} G^{MN}.$$

Apart from simply calculating the inverse this can be seen from the fact that $G^{MN} G_{MN} = d$ is an invariant.

The determinant of the metric is a sum of terms where there are d factors of G_{MN} and so it transforms as

$$G = e^{-2d\sigma} G.$$

Since the metric information is also contained in the vielbeins they will also transform under a conformal transformation. From the definition of the vielbeins we have the relation

$$E_{MA}E_N^A = G_{MN}.$$

The natural choice for the transformation of the vielbeins is then

$$\begin{aligned} E_M^A &\to e^{-\sigma}E_M^A \\ E_A^M &\to e^{\sigma}E_A^M \end{aligned} \tag{3.12}$$

Chapter 4

(2,0) theory

With the framework of the previous chapters the construction of the (2,0) theory in 6 dimensions can now be introduced. In this chapter the content of theory is given and some of its properties explored. In particular it will be seen why a Lagrangian formulation is not possible.

4.1 Field content

The theory contains symplectic Majorana-Weyl spinors, a 2-form field and scalar fields, called the tensor multiplet for the reason that its highest spin field is a totally antisymmetric tensor field. In six dimensions the fields will be denoted according to table 4.1.

Field	Type
ψ^i	spinor (anti-chiral, symplectic majorana)
B_{MN}	two form (real)
H_{MNP}	field strength of B_{MN} (self-dual)
ϕ^{ij}	scalar (real)

(4.1)

Both the scalars and the two form are real while the spinors are complex Grassman, i.e. the components are taken to anticommute. All the fields are massless as is to be expected from a conformally invariant theory.

The spinors ψ^i are anti-chiral, i.e. they satisfy $\Gamma\psi^i = -\psi^i$. The two form B_{MN} is self-dual, i.e. it has a self-dual field strength $H = \star H$.

In addition to the indicated transformation properties under the Lorentz group the fields also transform under an internal symmetry group Spin(5) $\cong$ Sp(4). This group will be referred to as the R-symmetry group, adhering to standard notation. The fields transform under the representations of the internal symmetry group given in (4.2), where the complex dimension of the representation is indicated.

$$\begin{aligned} \psi &\in \mathbf{4} \\ B_{MN} &\in \mathbf{1} \\ \phi &\in \mathbf{5} = \mathbf{4}\hat{\otimes}\mathbf{4} \end{aligned} \tag{4.2}$$

Where $\hat{\otimes}$ indicates the antisymmetric and traceless part of $\mathbf{4} \otimes \mathbf{4}$.

Thus all the fields carry, in addition to their Lorentz indices, an R-symmetry index. This index will be denoted by roman literals in the range $\{i, j, \ldots\}$. The reason for the internal symmetry group is that it is necessary in order to be able to accommodate a supersymmetric theory in six dimensions as was shown in chapter 2. One subtlety in the choice of notation is that instead of letting the scalar field have one index taking five values, we instead choose to view it as an antisymmetric tensor with two indices in the **4** representation. An antisymmetric two tensor in the **4** representation has $\frac{4 \cdot 3}{2} = 6$ independent components so in order to reduce that to 5 we impose the condition

$$M_{ij}\phi^{ij} = 0.$$

The reason for choosing this peculiar representation is simply that now it is easy to write down terms coupling a scalar field to a spinor field since now we can simply contract them. Since the scalar fields take values in a complex representation we also need to impose a reality condition for them. In this case there is no problem to simply impose $\left(\phi^{ij}\right)^{\star} = \phi_{ij}$.

A consequence of supersymmetry is that when the fields in the theory are on-shell, the bosonic and fermionic degrees of freedom need to match [2]. Since supersymmetry exchanges fermions and bosons this seems like a natural requirement. As a first step we can verify that the tensor multiplet of (2,0) theory satisfies this.

On-shell degrees of freedom A spinor in 6 dimensions has $2^{\frac{6}{2}} = 8$ complex components. Chirality halves this down to 4 complex components and the symplectic Majorana condition again halves this to 2 complex components. Finally the on-shell field satisfies the Dirac equation which halves this number one final time down to 1 complex on-shell degrees of freedom. Since the spinor fields also live in the **4** representation of the R-symmetry group the final number is

$$\begin{aligned} d.o.f_{\text{on - shell}}(\psi) &= 4 \quad \text{complex} \\ &= 8 \quad \text{real} \end{aligned}$$

The scalar fields simply represent 5 degrees of freedom by virtue of the R-symmetry group and the 2-form field has three on-shell degrees of freedom. It can be shown that a p-form in d dimensions has $\binom{d-2}{p}$ on-shell degrees of freedom ([23] appendix B.4). For a 2-form in 6 dimensions this is $\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$ d.o.f. Self duality then halves this down 3.

Thus the total number of fermionic and bosonic degrees of freedom on-shell are the same and the number is 8.

Symplectic structure One way to motivate the symplectic structure is to think of dimensionally reduced 11-dimensional supergravity. If we think of the space-time in this case as $M = M_6 \times M_5$ and reducing down to M_6 leaves 5 of the 11 dimensions as an internal symmetry. Spinors under the 11-dimensional Lorentz group will then get an internal Spin(5) symmetry and the scalars will get an internal SO(5) symmetry. As a group, Spin(5) is isomorphic to Sp(4). As was shown in chapter 2, a transformation A of Sp(4) preserves an antisymmetric bilinear form M,

$$A^T M A = M. \tag{4.3}$$

We will not check the isomorphism Spin(5) $\cong$ Sp(4) but we can as a first calculation see that the dimensions work out. The dimension of Spin(5), that is how many independent generators it contains, is the same as for SO(5) since it is just it's double cover. The algebra of SO(5) consists of antisymmetric 5×5 matrices which have $\frac{5 \cdot 4}{2} = 10$ independent components, and thus it takes 10 generators to span the algebra.

The symplectic group on the other hand is the group of 4×4 matrices that satisfies condition (4.3). A general 4 by 4 matrix contains 4^2 entries, these entries are then constrained by the symplectic condition. By regarding each entry in the matrix condition as an equation we get 4^2 conditions, but these are not all independent. Taking the transpose of the condition leaves it invariant so there are a number of relations between the equations equal to the number of independent entries in a symmetric 4×4 matrix. The remaining number of independent equations is then $4^2 - \frac{4 \cdot 5}{2} = 6$. Now we have 6 equations for the $4^2 = 16$ entries of the matrix of the symplectic transformation, leaving 10 independent components. This is the same number as we got for Spin(5) showing that they have the same dimension.

4.2 Equations of motion

The equations of motion for fields on a curved space-time are very similar to the corresponding equations for the flat space case. With the formalism for curved space-time introduced in chapter 3 the equations look exactly the same except for an extra term in the equation of motion for the scalar fields. During the introduction of the equations of motion the R-symmetry indices will sometimes be suppressed for clarity.

4.2.1 Scalars

The scalar equation of motion is the massless Klein-Gordon equation but with an extra term that is required to make the theory conformally invariant.

$$S_\phi = -\int d^6x\sqrt{-G}\left(D_M\phi D^M\phi + \frac{1}{5}R\phi^2\right)$$

The first term is the usual scalar kinetic term, which up to a numerical factor is the only possible Lorentz invariant term giving rise to a second order differential equation for ϕ. The second term looks like a mass term but contains the Ricci scalar, this term will turn out to be necessary for conformal invariance as will be investigated shortly.

This action gives rise to the following equation of motion

$$D_M D^M\phi - \frac{1}{5}R\phi = 0.$$

4.2.2 Spinors

The spinor action is the generalisation of the flat space version using the spin connection to create a covariant derivative and using the vielbeins to associate a coordinate index to the tangent space index carried by the gamma matrices.

$$
\begin{aligned}
S_\psi &= \int d^6x\sqrt{-G}\overline{\psi_i}E^M_A\Gamma^A D_M\psi^i \\
&\equiv \int d^6x\sqrt{-G}\overline{\psi_i}\not{D}\,\psi^i
\end{aligned}
$$

Fields at stationary point of this action satisfy the curved space-time Dirac equation,

$$\not{D}\,\psi^i = 0.$$

4.2.3 2-form

To motivate the equation of motion for the 2-form it is usefull to recall the Lagrangian formulation of electro-magnetism.

In the language of differential forms the action that gives rise to the Maxwell equations in flat 4-space is

$$
\begin{aligned}
S_{\text{maxwell}} &= \int F\wedge\star F \\
&= \int d^4x\frac{1}{4}F^{\mu\nu}F_{\mu\nu}
\end{aligned}
$$

where $F = dA$ is the field strength corresponding to the potential A and $\star$ is the Hodge dual. In the case of Maxwell theory the potential is a one-form and therefore the field strength is a two form.

This action gives rise to the equation of motion for A.

$$
\begin{aligned}
d\star F &= 0 \\
&\Longleftrightarrow \\
\partial_\mu F^{\mu\nu} &= 0
\end{aligned}
\tag{4.4}
$$

This correspond to one of the two pairs of Maxwell's equations. The other pair is given by $dF = 0$, which is fulfilled by construction since $dF = d^2A = 0$ since $d^2 = 0$.

This action generalises directly to a curved manifold since differential forms keep transforming tensorially under general coordinate transformations. The problems of the two form arises when we declare it to be self dual. If we now specialise to 6 dimensions and name the two form B and it's field strength H the above action generalises to

$$S_H = \int H\wedge\star H$$

But if H is self dual, that is $H = \star H$, then this action is identically zero!

$$
\begin{aligned}
H\wedge\star H &= H\wedge H \\
&= H_{M_1M_2M_3}H_{N_1N_2N_3}dx^{M_1}\wedge dx^{M_2}\wedge dx^{M_3}\wedge dx^{N_1}\wedge dx^{N_2}\wedge dx^{N_3} \\
&= (-1)^9 H_{M_1M_2M_3}H_{N_1N_2N_3}dx^{N_1}\wedge dx^{N_2}\wedge dx^{N_3}\wedge dx^{M_1}\wedge dx^{M_2}\wedge dx^{M_3} \\
&= -H\wedge H
\end{aligned}
$$

The first step is just writing out the forms in terms of their components. In the second step the first three basis elements are moved to the far right, picking up a minus sign for every basis element it passes (by antisymmetry). So $H \wedge H = -H \wedge H = 0$.

Although the action does not generalise in the case of six dimensions the equations of motion (4.4) does. We can take the field strength H to satisfy

$$d \star H = 0.$$

This equation has no problems in 6 dimensions, it's only that there is no action that gives rise to it. From a classical point of view this would not pose any problems since it's only the equations of motion that determine the behavior of the theory. On the other hand if we try to construct a quantum theory, the action is essential since the quantum theory depends not only on the stationary points of the action. The approach here will be to start with what can be generalised directly, the equations of motion, and see what this implies for a compactified version of the theory in 5 dimensions.

Note that since the field strength is self-dual, the two equations

$$\begin{aligned} d \star H &= 0 \\ dH &= 0 \end{aligned}$$

are equivalent. So the complete equations of motion for the potential B is

$$\begin{aligned} dH &= 0 \\ H &= \star H. \end{aligned}$$

4.3 Supersymmetry

4.3.1 Penrose condition

In flat space-time we use constant supersymmetry parameters to implement global supersymmetry, i.e. the same supersymmetry transformation applies to all points. The condition of being constant is simply $\partial_M \varepsilon = 0$, but this is not good condition in curved space-time. Instead the natural generalisation is for the parameter to be covariantly constant,

$$D_M \varepsilon = 0.$$

But this condition is not conformally invariant so there is no hope of using it in a conformal theory. In a conformal theory, after a further generalisation, the condition is

$$P_M \varepsilon \equiv D_M \varepsilon - \frac{1}{d} \Gamma_M \not{D} \varepsilon = 0. \tag{4.5}$$

Where P_M is called the Penrose operator.

It turns out that there are not many first order differential operators acting on spinors that are conformally invariant. There are two that can be constructed naturally [24] and these are the Dirac operator and the Penrose operator.

The condition (4.5) is conformally invariant and this is what we will take our supersymmetry parameters to satisfy. In this thesis it will be referred to as the Penrose condition. For a mathematical treatment of this operator see [24].

4.3.2 Amount of supersymmetry

The number of supersymmetry transformations will depend on how many solution there are to the Penrose condition. Since we only want to regard fields of spin less than or equal to 1 the maximal amount of supersymmetry possible is $N = 4$ extended supersymmetry.

The name (2,0) refers to the fact that the supercharges have the same chirality and that there are two chiral supersymmetry parameters. This notation stems from how one usually labels representation of the supersymmetry algebra [25]. A spinor in 6 dimensions has $2^{\frac{6}{2}} = 8$ complex components, thus a chiral spinor has $\frac{8}{2} = 4$ components. Two chiral spinors together has 16 real components, resulting in 16 supercharges, corresponding to $N = 4$.

When the space-time manifold is not specified there can be everything from maximal to no supersymmetry at all.

4.3.3 Supersymmetry variations

Solution to the equations of motion have a supersymmetry given by

$$\begin{aligned} \delta\psi^i &= i\frac{1}{12}\Gamma^{MNP}H_{MNP}\varepsilon^i + iD_M\phi^{ij}\Gamma^M\varepsilon_j + \frac{2}{3}\phi^{ij}\not{D}\,\varepsilon_j \\ \delta H_{MNP} &= 3i\partial_{[M}\left(\overline{\psi_i}\Gamma_{NP]}\varepsilon^i\right) \\ \delta\phi^{ij} &= \overline{\psi^{[i}}\varepsilon^{j]} - \frac{1}{4}T^{ij}\overline{\psi_k}\varepsilon^k. \end{aligned}$$

Where ε^i are parameters that satisfy $P_M\varepsilon^i = 0$.

Under a transformation $\psi^i \to \psi^i + \delta\psi^i$, the new field will still be a solution to the Dirac equation, provided H_{MNP} and ϕ satisfy their equations of motion. Likewise for the other fields. Since the theory is supersymmetric only when the equations of motion are satisfied we say that the theory is supersymmetric on-shell.

These transformations are also conformally invariant and in fact that fixes the relative coefficients between the second and third term in the variation for the spinor field.

4.4 Conformal invariance

We will now show that the free classical version of (2,0) theory is conformally invariant in the sense of section 3.5, i.e. the theory only depends on the conformal structure.

4.4.1 Scalars

The scalar Lagrangian is given by

$$S = -\int d^6x\sqrt{-G}\left(G^{MN}\partial_M\phi\partial_N\phi - cR\phi^2\right). \tag{4.6}$$

It will now be seen why the term proportional to the Ricci scalar is needed for the conformal invariance. This can be done in any dimension, but as will be seen the constant c is dimension dependent.

The scalar field is taken to transform as

$$\phi \to e^{s\sigma}\phi,$$

with s yet to be determined. By first doing a global transformation, i.e. $\partial_M \sigma = 0$, we find

$$\sqrt{-G}G^{\mathrm{MN}}\partial_M\phi\partial_N\phi \to e^{-d\sigma}e^{2\sigma}e^{2s\sigma}\sqrt{-G}G^{\mathrm{MN}}\partial_M\phi\partial_N\phi.$$

For this to be conformally invariant the exponential factors must cancel so that

$$\begin{aligned}
-d\sigma + 2\sigma + 2s\sigma &= 0 \\
&\Longrightarrow \\
s &= \frac{d-2}{2} \\
[d=6] &= 2.
\end{aligned}$$

This is a necessary condition, it remains to be seen if this implies invariance under local rescalings.

Ricci scalar The Riemann tensor is given in terms of the Christoffel symbols as

$$\begin{aligned}
R^P{}_{SMN} &= \partial_M\Gamma^P_{NS} - \partial_N\Gamma^P_{MS} + \Gamma^Q_{SN}\Gamma^P_{QM} - \Gamma^Q_{SM}\Gamma^P_{QN} \\
\Gamma^S_{LM} &= \frac{1}{2}G^{NS}\left(\partial_L G_{MN} + \partial_M G_{LN} - \partial_N G_{ML}\right).
\end{aligned}$$

The Ricci tensor and Ricci scalar are defined as the following contractions.

$$\begin{aligned}
R_{MN} &= R^Q{}_{MQN} \\
R &= G^{MN}R_{MN}
\end{aligned}$$

Under a conformal transformation the Christoffel symbols transforms as

$$\begin{aligned}
\Gamma^P_{MN} &\to \frac{1}{2}e^{2\sigma}G^{QP}\left(\partial_M(e^{-2\sigma}G_{NQ}) + \partial_N(e^{-2\sigma}G_{MQ}) - \partial_Q(e^{-2\sigma}G_{MN})\right) \\
&= \Gamma^P_{MN} + \left(-\partial_M\sigma G_{NQ}G^{QP} - \partial_N\sigma G_{MQ}G^{QP} + \partial_Q\sigma G_{MN}G^{QP}\right) \\
&= \Gamma^A_{LM} + C^P_{MN} \\
C^P_{MN} &= -\partial_M\sigma\delta^P_N - \partial_N\sigma\delta^P_M + \partial_Q\sigma G^{QP}G_{MN}
\end{aligned}$$

Plugging this back into the definition of the Riemann tensor and contracting, one finds that the Ricci scalar transforms as

$$\begin{aligned}
R &\to R + \delta R \\
\delta R &= D_M\partial_N\sigma\left(2\left[d-1\right]G^{MN}\right) + \partial_M\sigma\partial_N\sigma\left(-\left(d-2\right)(d-1)G^{MN}\right)
\end{aligned}$$

Kinetic term Looking again at the scalar action:

$$S = -\int d^6x\sqrt{-G}\left(G^{MN}\partial_M\phi\partial_N\phi - cR\phi^2\right) \tag{4.7}$$

The first term transforms as

$$
\begin{aligned}
&\sqrt{-G}G^{MN}\partial_M\phi\partial_N\phi \longrightarrow\\
\longrightarrow\ & e^{2\sigma}e^{-d\sigma}\sqrt{-G}G^{MN}\partial_M\left(e^{s\sigma}\phi\right)\partial_N\left(e^{s\sigma}\phi\right)\\
=\ & \sqrt{-G}G^{MN}\partial_M\phi\partial_N\phi+\\
& +e^{2\sigma}e^{-d\sigma}\sqrt{-G}G^{MN}\left(e^{s\sigma}s\partial_M\sigma\phi+e^{s\sigma}\partial_M\phi\right)\left(e^{s\sigma}s\partial_N\sigma\phi+e^{s\sigma}\partial_N\phi\right)\\
=\ & \sqrt{-G}\left(G^{MN}\partial_M\phi\partial_N\phi+G^{MN}\left(s^2\partial_M\sigma\partial_N\sigma\phi^2+2s\partial_M\sigma\left(\partial_N\phi\right)\phi+\partial_M\phi\partial_N\phi\right)\right)
\end{aligned}
$$

Finally the total variation of the Lagrangian density is given by

$$
\begin{aligned}
\delta S=-\int d^6x\sqrt{-G}\{&G^{MN}\left(s^2\partial_M\sigma\partial_N\sigma\phi^2+2s\partial_M\sigma\left(\partial_N\phi\right)\phi\right)+\\
&-c\left(D_M\partial_N\sigma(2\left[d-1\right]G^{MN})+\partial_M\sigma\partial_N\sigma\left(-(d-2)(d-1)G^{MN}\right)\right)\phi^2\}
\end{aligned}
$$

Collecting terms this becomes

$$
\begin{aligned}
\delta S=-\int d^6x\sqrt{-G}G^{MN}\{&\left(s^2+c(d-2)(d-1)\right)\phi^2\partial_M\sigma\partial_N\sigma+\\
&+2s\phi\partial_M\sigma\left(\partial_N\phi\right)-2c(d-1)D_M\partial_N\sigma\phi^2\}
\end{aligned}
$$

From the first term the condition for conformal invariance requires

$$
\begin{aligned}
s^2+c(d-2)(d-1) &= \frac{(d-2)^2}{4}+c(d-2)(d-1)=0\\
&\Longrightarrow\\
c &= -\frac{1}{4}\frac{d-2}{d-1}\\
&= -\frac{1}{5}
\end{aligned}
$$

With this definition the remaining part becomes

$$
\begin{aligned}
\delta S &= -\int d^6x\sqrt{-G}G^{MN}\left\{(d-2)\phi\partial_M\sigma\left(\partial_N\phi\right)+\frac{1}{2}(d-2)D_M\partial_N\sigma\phi^2\right\}\\
&= -\int d^6x\sqrt{-G}(d-2)G^{MN}D_M\left(\frac{1}{2}\partial_N\sigma\phi^2\right)
\end{aligned}
$$

The action is thus invariant up to a total derivative and under the assumption that the fields vanish at infinity the action is invariant.

4.4.2 Spinors

The spinor term in the action is

$$
S_\psi=\int d^6x\sqrt{G}\bar{\psi}\not{D}\,\psi.
$$

The only metric dependence apart from the determinant is contained in the spin connection in the covariant derivative. Let's single out that factor and expand it.

$$
\begin{aligned}
\not{D}\,\psi &= \Gamma^M D_M\psi\\
&= E_A^M\Gamma^A D_M\psi\\
&= E_A^M\Gamma^A\left(\partial_M\psi+\frac{1}{4}\Omega_M^{\mathrm{BC}}\Gamma_{BC}\psi\right)
\end{aligned}
\tag{4.8}
$$

The spin connection Ω is given in terms of the vielbeins as

$$\Omega_M^{AB} = 2E^{N[A}\partial_{[M}E_{N]}^{B]} - E^{P[A}E^{QB]}\partial_P E_{QC}E_M^C$$

Recall from equation (3.12) that the vielbeins transform as

$$\begin{aligned} E_A^M &\to e^{\sigma}E_A^M \\ E_M^A &\to e^{-\sigma}E_M^A. \end{aligned}$$

Using this the spin connection transforms as

$$\begin{aligned} \Omega_M^{AB} &\to 2e^{\sigma}E^{N[A}\partial_{[M}\left(e^{-\sigma}E_{N]}^{B]}\right) - e^{2\sigma}E^{P[A}E^{QB]}\partial_P\left(e^{-\sigma}E_{QC}\right)e^{-\sigma}E_M^C \\ &= \Omega_M^{AB} + 2E^{N[A}E_M^{B]}\partial_N\sigma. \end{aligned}$$

Thus the Dirac term transforms as

$$\begin{aligned} \not{D}\,\psi \;&=\; E_A^M\Gamma^A\left(\partial_M\psi + \frac{1}{4}\Omega_M^{\mathrm{BC}}\Gamma_{BC}\right)\psi \\ &\longrightarrow e^{\sigma}E_A^M\Gamma^A\left(\partial_M\left(e^{s\sigma}\psi\right) + \frac{1}{4}\left(\Omega_M^{BC} + 2E^{N[B}E_M^{C]}\partial_N\sigma\right)\Gamma_{BC}\right)e^{s\sigma}\psi \\ &=\; e^{\sigma(1+s)}\not{D}\,\psi + e^{\sigma(1+s)}\left(E_A^M\Gamma^A s\partial_M\sigma + \frac{1}{2}E^{NB}\partial_N\sigma\Gamma^A\Gamma_{BA}\right)\psi \end{aligned} \tag{4.9}$$

The product of gamma matrices can be simplified,

$$\begin{aligned} \Gamma^A\Gamma_{BA} &= \frac{1}{2}\left(\Gamma^A\Gamma_B\Gamma_A - \Gamma^A\Gamma_A\Gamma_B\right) \\ &= \frac{1}{2}\left(\left[2\delta_B^A - \Gamma_B\Gamma^A\right]\Gamma_A - \Gamma^A\Gamma_A\Gamma_B\right) \\ &= (1-d)\,\Gamma_B. \end{aligned}$$

Inserting back into (4.9) gives

$$\begin{aligned} \not{D}\,\psi \to e^{\sigma(1+s)}\not{D}\,\psi &+ e^{\sigma(1+s)}\left(E_A^M\Gamma^A s\partial_M\sigma + \frac{1}{2}E^{NB}\partial_N\sigma\,(1-d)\,\Gamma_B\right)\psi \\ = e^{\sigma(1+s)}\not{D}\,\psi &+ e^{\sigma(1+s)}\left(s + \frac{1-d}{2}\right)E_A^M\Gamma^A\psi\partial_M\sigma. \end{aligned}$$

For this to be conformally covariant the second term needs to vanish. The Dirac equation is thus conformally covariant if the spinor transforms as

$$\begin{aligned} \psi &\to e^{s\sigma}\psi \\ s &= \frac{d-1}{2}. \end{aligned}$$

Which in 6 dimensions is

$$s = \frac{6-1}{2} = \frac{5}{2}.$$

4.4.3 Form field

The first of the equations of motion

$$dH = 0$$

contains no metric information and so is automatically conformally invariant if H itself does not transform under a conformal transformation. The self-duality condition on the other hand needs to be checked.

$$\begin{aligned}
H &= \star H \\
&\Longleftrightarrow \\
H_{MNP} &= -\frac{1}{3!}\frac{1}{3!}\frac{1}{\sqrt{-G}}G_{MM_1}G_{NM_2}G_{PM_3}\varepsilon^{\;M_1M_2M_3QRS}H_{QRS} \\
&= -\frac{1}{3!}\frac{1}{3!}\frac{1}{\sqrt{-G}}G_{MM_1}G_{NM_2}G_{PM_3}E^{M_1}_{A_1}E^{M_2}_{A_2}E^{M_3}_{A_3}E^{Q}_{B_1}E^{R}_{B_2}E^{S}_{B_3}\varepsilon^{\;A_1A_2A_3B_1B_2B_3}H_{QRS} \\
&\to -\frac{1}{3!}\frac{1}{3!}\frac{1}{e^{-d\sigma}\sqrt{-G}}e^{-2\sigma-2\sigma-2\sigma}G_{MM_1}G_{NM_2}G_{PM_3}E^{M_1}_{A_1}E^{M_2}_{A_2}E^{M_3}_{A_3}E^{Q}_{B_1}E^{R}_{B_2}E^{S}_{B_3}\cdot \\
&\quad\cdot\varepsilon^{\;A_1A_2A_3B_1B_2B_3}H_{QRS} \\
&= -e^{d\sigma-6\sigma}\frac{1}{3!}\frac{1}{3!}\frac{1}{\sqrt{-G}}\varepsilon_{MNP}{}^{QRS}H_{M_1M_2M_3} \\
[d=6] &= 0
\end{aligned}$$

It might look as though this was only possible in exactly 6 dimensions but the above reasoning works out equally well in d dimensions for $\frac{d}{2}-$forms.

4.5 Summary of (2,0) theory in 6 dimensions

The 6 dimensional action for the scalar and fermion fields is

$$S = \int d^6x\sqrt{-G}\left[i\overline{\psi_i}\Gamma^M D_M\psi^i - D_M\phi^{ij}D^M\phi_{ij} - \frac{1}{5}R\phi^{ij}\phi_{ij}\right]$$

The equations of motion for the two form is given in terms of its three form field strength,

$$\begin{aligned} dH &= 0 \\ H &= \star H. \end{aligned}$$

This theory is invariant on-shell under the following supersymmetry transformations,

$$\begin{aligned}
\delta\psi^i &= i\frac{1}{12}\Gamma^{MNP}H_{MNP}\varepsilon^i + iD_M\phi^{ij}\Gamma^M\varepsilon_j + \frac{2}{3}\phi^{ij}\not{D}\,\varepsilon_j \\
\delta H_{MNP} &= 3i\partial_{[M}\left(\bar{\psi}_i\Gamma_{NP]}\varepsilon^i\right) \\
\delta\phi^{ij} &= \overline{\psi^{[i}}\varepsilon^{j]} - \frac{1}{4}T^{ij}\overline{\psi^C_k}\varepsilon^k,
\end{aligned}$$

where the supersymmetry parameters satisfy $P_M\varepsilon^i = 0$ with

$$P_M\varepsilon^i = D_M\varepsilon^i - \frac{1}{d}\Gamma_M\not{D}\,\varepsilon^i.$$

The theory is also scale invariant under the transformation,

$$
\begin{aligned}
G_{MN} &\to e^{-2\sigma(x)} G_{MN} \\
\psi &\to e^{\frac{5}{2}\sigma} \psi \\
\phi &\to e^{2\sigma} \phi \\
H &\to H \\
\varepsilon &\to e^{-\frac{1}{2}\sigma} \varepsilon .
\end{aligned}
$$

Chapter 5

Circle fibration

A concrete example of a fibration is a brush. The brush consists of a base cylinder and many fibers attached to it. The mathematical version of this, attaching many copies of a space to a base space is what is called a fibration.

A circle fibration over a space X is a new space M which consists of the space X with a circle attached at each point. More precisely the circle fibration over X looks locally like

$$M = X \times S^1.$$

The space X is called the base manifold and the circle S^1 is called the fibre.

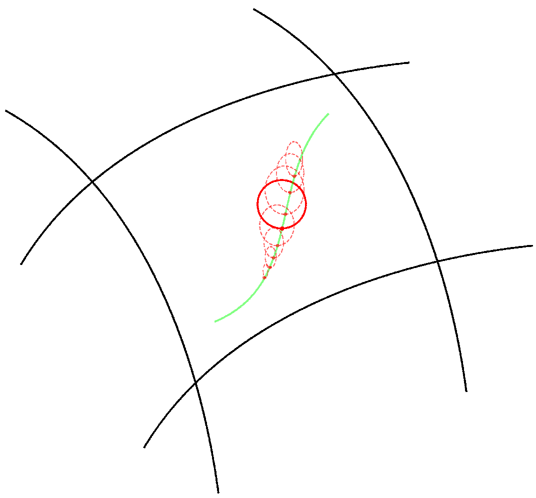

Figure 5.1: The green line represents a path in space-time and over each point on the path is an attached circle.

In figure (5.1) a piece of a surface is drawn together with a path on it. The space is a circle fibration so over every point on the path we find a circle. The circles can vary in radius and orientation from point to point.

In our case the base manifold is a 5-dimensional manifold denoted by M_5 and the fibration looks locally like

$$M_6 = M_5 \times S^1.$$

The coordinate on the circle is taken to be $x^5 = \varphi$. And the range is chosen so that $0 \leqslant \varphi < 2\pi$.

5.1 Fibre metric

To specify how the circles are connected to the 5-dimensional base manifold we need to specify the metric on the product $M_6 = M_5 \times S^1$.

When choosing the fibre coordinate φ on the six dimensional manifold the metric can be written in local coordinates as

$$\mathrm{ds}^2 = G_{\mu\nu}(x^\mu)\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu + G_{\mu\varphi}(x^\mu)\,\mathrm{dx}^\mu\,d\varphi + G_{\varphi\varphi}(x^\mu)d\varphi d\varphi.$$

Note that the metric components are taken to be function on the base manifold M_5.

Thus we have split the metric into terms containing the φ dependence and the 5 dimensional part. This would be the natural way to do the splitting but it turns out that there is a smarter way to parametrize the metric. Regard instead the following

$$\mathrm{ds}^2 = g_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu + \left(r d\varphi + \tilde{\theta}_\mu\,\mathrm{dx}^\mu\right)^2,$$

which by a redefinition of $\tilde{\theta}$ can be written as

$$\begin{aligned} &= g_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu + r^2\left(d\varphi + \theta_\mu\,\mathrm{dx}^\mu\right)^2 \qquad (5.1)\\ &= g_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu + r^2 d\varphi^2 + 2r^2\theta_\mu\,\mathrm{dx}^\mu\,d\varphi + r^2\theta_\mu\theta_\nu\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu\,. \end{aligned}$$

Note that in the last expression it is clear that this choice is just a different way of parametrizing the degrees of freedom in G. The good thing about this is that r and θ have natural interpretations. The term $r^2 d\varphi^2$ is exactly the metric on a circle with radius r, and θ now behaves as a gauge potential under coordinate transformations on the circle. To see this lets do a coordinate transformation in the circle direction φ.

$$\varphi \to \varphi + f(x) \Rightarrow d\varphi \to d\varphi + \partial_\mu f(x) dx^\mu$$

Substituting this back into the metric in the form (5.2) gives

$$\mathrm{ds}^2 \to g_{\mu\nu}\,\mathrm{dx}^\mu\,\mathrm{dx}^\nu + r^2\left(d\varphi + (\theta_\mu + \partial_\mu f)\,\mathrm{dx}^\mu\right)^2.$$

So we see that $\theta \to \theta + df$, which looks exactly like a gauge transformation for θ.

To summarise the choice of metric the components are given by

$$G_{\mu\nu} = g_{\mu\nu} + r^2\theta_\mu\theta_\nu \qquad G_{\varphi\mu} = r^2\theta_\mu \qquad G_{\varphi\varphi} = r^2\ . \qquad (5.2)$$

The inverse metric can then be calculated to be

$$G^{\mu\nu} = g^{\mu\nu} \qquad G^{\varphi\mu} = -\theta^\mu \qquad G^{\varphi\varphi} = \tfrac{1}{r^2} + \theta^\mu\theta_\mu\ . \qquad (5.3)$$

This also implies that the determinant of G_{MN} splits into

$$G = rg. \qquad (5.4)$$

If this was Kaluza reduction of gravity [26] then θ would become the electromagnetic potential. Here it is a convenient way to parametrize the metric so that it is easy to see when a term is invariant under coordinate transformations on the circle. And of course we want all the resulting terms from the reduction to be invariant since in the end we want to find something that only depends on the five dimensional manifold. This can now be verified by checking that all the resulting terms in the 5-dimensional theory should be invariant under $\theta \to \theta + df$.

5.2 Vielbeins

The vielbeins are defined by that they form an orthonormal basis in the tangent space as defined in (3.4). By a slight rearrangement of this condition we get

$$E_M^A E_{AN} = G_{\mathrm{MN}}. \tag{5.5}$$

Since the metric has already been chosen on the fibration, some appropriate vielbeins needs to be found to satisfy relation (5.5). By looking at the individual components of the metric in the form (5.2), the components of the vielbeins can be found.

$G_{\varphi\varphi}$:

$$E_\varphi^A E_{A\varphi} = G_{\varphi\varphi} = r^2$$

To satisfy this we choose $E_\varphi^5 = r$ and $E_\varphi^a = 0$.

$G_{\mu\varphi}$:

$$E_\mu^A E_{A\varphi} = G_{\mu\varphi} = r^2\theta_\mu$$

By the previous choice of E_φ^A this becomes

$$\begin{aligned} E_\mu^5 E_{5\varphi} &= r^2\theta_\mu. \\ &\Longrightarrow \\ E_\mu^5 &= r\theta_\mu \end{aligned}$$

Finally the $G_{\mu\nu}$ component:

$$\begin{aligned} E_\mu^A E_{A\nu} &= E_\mu^a E_{a\nu} + E_\mu^5 E_{5\nu} \\ &= E_\mu^a E_{a\nu} + r^2\theta_\mu\theta_\nu \\ &= G_{\mu\nu} = g_{\mu\nu} + r^2\theta_\mu\theta_\nu. \\ &\Longrightarrow \\ E_\mu^a E_{a\nu} &= g_{\mu\nu} \end{aligned}$$

Thus the 5-dimensional part of the vielbeins needs to satisfy the 5-dimensional version of condition (5.5).

To make the distinction clear, name the vielbeins with only 5 dimensional components with a lowercase e_μ^a. With this choice the vielbeins now have the following appearance,

$$\begin{array}{ll} E_\mu^a = e_\mu^a & E_\varphi^a = 0 \\ E_\mu^5 = r\theta_\mu & E_\varphi^5 = r \end{array}, \tag{5.6}$$

where $e^a_\mu e_{a\nu} = g_{\mu\nu}$.

The inverse vielbeins are then given by

$$\begin{array}{ll} E^\nu_b = e^\nu_b & E^\nu_5 = 0 \\ E^\varphi_b = -\theta_\mu e^\mu_b & E^\varphi_5 = \frac{1}{r}. \end{array}$$

The inverse vielbeins can be found either by calculating the inverse of the first form or by just raising or lowering with the fibre metric.

5.3 Spin connection

The spin connection is given in terms of contractions of the vielbeins and so it will split up into a 5-dimensional part and φ-part. Using expressions (5.6) the spin connection, given by

$$\omega^{\mathrm{AB}}_M = 2E^{N[A}\partial_{[M}E^{B]}_{N]} - E^{P[A}E^{|Q|B]}\partial_P E_{QC}E^C_M, \tag{5.7}$$

reduces on the fibration to

$$\begin{array}{rcl} \omega^{ab}_\varphi & = & -\frac{1}{2}r^2 e^{\rho a}e^{\sigma b}\mathfrak{F}_{\rho\sigma} \\ \omega^{5b}_\varphi & = & e^{\rho b}\partial_\rho r \\ \omega^{5b}_\mu & = & -\frac{1}{2}re^{\rho b}\mathfrak{F}_{\mu\rho} + \theta_\mu e^{\rho b}\partial_\rho r \\ \omega^{ab}_\mu & = & \tilde{\omega}^{ab}_\mu - \frac{1}{2}r^2\theta_\mu e^{\rho a}e^{\sigma b}\mathfrak{F}_{\rho\sigma}. \end{array} \tag{5.8}$$

Where $\mathfrak{F}_{\mu\nu} = \partial_\mu\theta_\nu - \partial_\nu\theta_\mu$ and $\tilde{\omega}^{ab}_\mu$ is the spin connection in 5-dimensions, given by expression (5.7) but with the 5-dimensional vielbeins e^μ_a instead of E^M_a.

Chapter 6

Reduction

The dynamics of the 6 dimensional theory is described by the equation of motion for the fields. When the theory is reduced to five dimensions these equations of motion will split into parts that depend only on the 5 dimensional manifold and parts involving the circle coordinate φ. As an example lets look at a contraction of two tensors A^M and B^N.

$$\begin{aligned} A^M B_N &= A^\mu B_\mu + A^\varphi B_\varphi \\ M &\in \{0, 1, \dots 5\} \\ \mu &\in \{0, 1, \dots 4\} \end{aligned}$$

Note that the index corresponding to the circle coordinate is denoted by φ.

The goal of the reduction will be to find a Lagrangian description of the resulting 5 dimensional theory. Thus, since we already have an action for the scalars and the spinors we might as well do the reduction of the action directly. The action for the scalars and the spinors is

$$S = \int d^6x \sqrt{-G} \left[i\overline{\psi_i} \Gamma^M D_M \psi^i - D_M \phi^{ij} D^M \phi_{ij} - \frac{1}{5} R \phi^{ij} \phi_{ij} \right].$$

It is not obvious that this will reduce to something nice in terms of the five remaining coordinates but as we will see the result is quite simple although the calculations are lengthy.

To get an overview of what needs to be done, lets look at the action and try to see how it depends on φ. First there is the determinant of the metric. From (5.4) we know it splits nicely into the 5-dimensional metric. Next there is the Dirac terms for the spinors. The contraction splits trivially but the covariant derivative contains the spin connection which in turn is a contraction of the metric and it's derivatives. The spinors live in spin representations of the 6 dimensional Lorentz group and these will split into two parts as well. The scalar kinetic term is easy since the covariant derivative acts on scalars does not depend on the connection. Finally the last term contains the Ricci scalar which is a contraction of the Riemann tensor which in turn is a contraction of the metric and it's derivatives.

For form field the equations of motion needs to be reduces and the main part will consist of the implications of the self-duality condition.

6.1 Field reduction

From Fourier analysis we know that a periodic function can be represented by a sum of plane waves with discrete periods. One way to think of a smooth periodic function is as a smooth function on the circle, in particular a smooth function is continuous so that when we get back to the same point on the circle the function necessarily gets back to the same value again. Now let's say that we have a wave function on the circle describing a particle, then only momenta that corresponds to wavelengths that fits in nicely on the circumference of the circle will be able to propagate. This is exactly the above statement about periodic function in physical terms.

When one of the coordinates is taken to be a circle the fields will be restricted to a discrete set of momenta in this direction. If we then take the limit of low energy only the zero modes of the fields will survive since there will not be enough energy to excite the first discrete momenta on the circle. This means that in the low energy limit the fields will not depend on the circle coordinate since the zero modes of the fields are constant in the circle direction. Lets see this in more mathematical terms.

Let $\phi(x^M)$ be a field on the manifold, where $M = \{0, 1, \ldots, 5\}$. If now one of the coordinates, $x^5 = \varphi$ is periodic we can single it out and write the field as $\phi(x^\mu, \varphi)$, where $\mu = \{0, 1, \ldots, 4\}$. If φ is the angle coordinate of a circle the field will have a 2π periodicity and so we can expand it in a Fourier series.

$$\begin{aligned}\phi(x^\mu, \varphi) &= \sum_{p\in\mathbb{Z}} \phi_p(x^\mu) \cdot e^{p\varphi i} \\ &= \phi_0(x^\mu) + \sum_{p>0} \phi_p(x^\mu) \cdot e^{p\varphi i}\end{aligned}$$

The coefficients of this series is of course still functions of x^μ since it is only an expansion in the circle coordinate. In the low energy limit we can then throw away the $p > 0$ part and thus get

$$\begin{aligned}\phi_{\text{low - energy}}(x^\mu, \varphi) &= \phi_0(x^\mu) \\ &\equiv \phi(x^\mu).\end{aligned}$$

The argument is similar for the form field and the scalar. For a more complete treatment of these arguments see [27].

In addition to throwing away the non-zero modes we will also rescale the scalar and the fermion field by the radius $r(x)$ to ensure that five dimensional fields will have their canonical scaling dimensions. The rescaling takes the form

$$\begin{aligned}\psi &\to \frac{\psi}{r\sqrt{2\pi}} \\ \phi &\to \frac{\phi}{r\sqrt{2\pi}} \\ H &\to \frac{H}{\sqrt{2\pi}}.\end{aligned}$$

The extra factor of $\frac{1}{\sqrt{2\pi}}$ is included for convenience to cancel the 2π coming from integrating over φ as will be seen later. In the reduction we will look at the low energy behavior of the theory and thus only retain the zero modes of the fields. This means that in the low energy limit we get a theory that looks 5 dimensional. It is the appearance of this 5 dimensional theory that is the goal of this chapter.

6.2 Scalar reduction

Ricci scalar In the reduction of the terms involving the scalar field we need to work out the behaviour or the Ricci scalar curvature under the reduction.

The Ricci scalar is a fully contracted Riemann tensor. To find out what parts depend on the lower dimensions these contractions needs to be split up into the five dimensional contractions and the φ direction. As before the sixth index is denoted $\varphi = 5$.

$$\begin{aligned} R &= G^{\mathrm{MN}} R_{\mathrm{MN}} \\ &= G^{\mu\nu} R_{\mu\nu} + 2G^{\varphi\nu} R_{\varphi\nu} + G^{\varphi\varphi} R_{\varphi\varphi} \\ &= G^{\mu\nu} R^{P}{}_{\mu P\nu} + 2G^{\varphi\nu} R^{P}{}_{\varphi P\nu} + G^{\varphi\varphi} R^{P}{}_{\varphi P\varphi} \\ &= \ldots \end{aligned}$$

The Riemann tensor is itself a contraction of the Christoffel symbols which in turn is a contraction of the metric and it's derivative. Using the expressions (5.2) for the metric on the fibration results in that the Ricci scalar consists of two parts, one which is just the five dimensional Ricci scalar and another part which depends on the parameters of the fibration, $r(x)$ and $\theta_\mu(x)$.

$$R = \tilde{R} - \frac{1}{4} r^2 \mathfrak{F}_{\mu\nu} \mathfrak{F}_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - \frac{2}{r} g^{\mu\nu} D_\mu D_\nu r \tag{6.1}$$

Here $\tilde{R}$ is the 5 dimensional Ricci scalar curvature, all the covariant derivatives are taken with respect to the 5 dimensional geometry and where

$$\mathfrak{F}_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu .$$

6.2.1 Action

Using (6.1) we can see how the full action looks on the fibration,

$$\begin{aligned} S_\phi \;&=\; - \int d^6x \sqrt{-G} \left[G^{MN} \partial_M \phi \partial_N \phi + \frac{1}{5} R \phi^2 \right] \\ \longrightarrow& \\ & - \int_0^{2\pi} d\varphi \int d^5x \sqrt{-g}\, r \left[g^{\mu\nu} \partial_\mu \left(\frac{\phi}{r\sqrt{2\pi}} \right) \partial_\nu \left(\frac{\phi}{r\sqrt{2\pi}} \right) + \right. \\ & \left. + \frac{1}{5} \left(\tilde{R} - \frac{1}{4} r^2 \mathfrak{F}_{\mu\nu} \mathfrak{F}^{\mu\nu} - \frac{2}{r} D^\mu D_\mu r \right) \left(\frac{\phi}{r\sqrt{2\pi}} \right)^2 \right] \\ =& \; - \int d^5x \sqrt{-g} \left[\frac{1}{r} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2 \frac{1}{r^2} g^{\mu\nu} \phi \partial_\mu r \partial_\nu \phi + \frac{1}{r^3} g^{\mu\nu} \phi^2 \partial_\mu r \partial_\nu r \right. \\ & \left. + \frac{1}{5} \frac{1}{r} \left(\tilde{R} - \frac{1}{4} r^2 \mathfrak{F}_{\mu\nu} \mathfrak{F}^{\mu\nu} - \frac{2}{r} D^\mu D_\mu r \right) \phi^2 \right] \end{aligned}$$

To simplify this we can use partial integration to bring the terms into the same form. Partial integration on the second term gives

$$-2 \int \frac{1}{r^2} g^{\mu\nu} \phi \partial_\mu \phi \partial_\nu r = \int (-4) \frac{\partial_\mu r}{r^3} g^{\mu\nu} \phi^2 \partial_\nu r + \frac{2}{r^2} g^{\mu\nu} \left(\partial_\mu \phi \right) \phi \partial_\nu r + \frac{2}{r^2} g^{\mu\nu} \phi^2 \partial_\mu \partial_\nu r .$$

Rearranging terms results in the following identity,

$$-2\int \frac{1}{r^2} g^{\mu\nu}\phi\partial_\mu\phi\partial_\nu r = \int \frac{2}{r^3} g^{\mu\nu}\phi^2\partial_\mu r\partial_\nu r - \frac{1}{r^2} g^{\mu\nu}\phi^2\partial_\mu\partial_\nu r.$$

Substituting this back into the action gives

$$\begin{aligned} S_\phi = -\int d^5x\sqrt{-g}\Bigg[&\frac{1}{r} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{r} D_\mu\left(\frac{1}{r} D^\mu r\right)\phi^2 \\ &+\frac{1}{5}\frac{1}{r}\left(\tilde{R} - \frac{1}{4} r^2\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} - \frac{2}{r} D^\mu D_\mu r\right)\phi^2\Bigg]. \end{aligned}$$

6.3 Three form reduction

Start by splitting up the three form so as to single out the φ dependence of the basis.

$$H = E + d\varphi \wedge F$$

This is simply a way of organising the components of H into a three form E and a two form F. From the general arguments of section (6.1) E and F will not contain any dependence on φ. It now remains to see what the self-duality condition and the equations of motion imply for E and F.

6.3.1 Self-duality condition

The Hodge dual of the three form H in terms of E and F is

$$\star H = \star E + \star(d\varphi \wedge F). \tag{6.2}$$

Since H is self-dual, this will impose a relation between E and F:

$$\star E + \star(d\varphi \wedge F) = E + d\varphi \wedge F.$$

It turns out that using this relation we can completely solve for E in terms of F. The final relation is

$$E = \frac{1}{r} \star_g F + \theta \wedge F.$$

Where $\star_g$ is the Hodge dual with respect to the 5-dimensional manifold. This makes sense since in 5 dimensions a 2-form and a 3-form has $\binom{5}{2} = \binom{5}{3} = 10$ components, which also coincides with a self-dual 3-form in 6 dimensions. This means that we can take all the degrees of freedom in F and forget about E as long as we use the fact that $dE = 0$ as will be done in the next section.

6.3.2 Equation of motion for F

From the fact that

$$dH = 0$$

It follows that

$$dF = dE = 0$$

From the self duality of H it followed that there is a linear relation between E and F. Thus the condition that E is closed becomes an equation of motion for F.

$$\begin{aligned} dE &= d\left(\frac{1}{r}\star_g F + \theta \wedge F\right) \\ &= -\frac{dr}{r^2} \wedge \star_g F + \frac{1}{r} d \star_g F + d\left(\theta \wedge F\right) \\ &= 0 \end{aligned}$$

This equation follows from the action

$$S_F = \int_M \left\{ -\frac{1}{r} F \wedge \star F - \theta \wedge F \wedge F \right\}$$

As usual F is regarded as the field strength of some connection A so that

$$F = dA.$$

This is true at least locally since we now that $dF = 0$.

A variation of the connection then simply induces a variation in F as

$$\delta F = d\delta A$$

The variation of the action S_F then becomes

$$\delta S_F = \int_M \left\{ -\frac{1}{r} \left[\delta F \wedge \star_g F + F \wedge \star_g \delta F\right] - 2\theta \wedge F \wedge \delta F \right\}$$

The θ-term follows from that F is a two form, so in the term where the variation hits the middle F it can be moved to the right picking up two sign changes.

From the definition of the Hodge dual one can calculate that for p-forms A and B,

$$A \wedge \star B = < A, B > \mathrm{vol}_d \,.$$

Where vol_d is the volume element, e.g. $\mathrm{vol}_d = \sqrt{-g} d^5 x$ in 5 dimensions. Since the inner product is symmetric, it then follows that

$$A \wedge \star B = \star A \wedge B.$$

Using this fact the variation of the action can be written

$$\begin{aligned} \delta S_F &= 2 \int_M \left\{ -\frac{1}{r} \star_g F \wedge \delta F - \theta \wedge F \wedge \delta F \right\} \\ &= 2 \int_M \left\{ -\frac{1}{r} \star_g F \wedge d\delta A - \theta \wedge F \wedge d\delta A \right\} \\ &= 2 \left[\int_M d \left\{ \frac{1}{r} \left[\star_g F \wedge \delta A\right] + \theta \wedge F \wedge \delta A \right\} - \int_M \left\{ d\left(\frac{1}{r}\star_g F\right) \wedge \delta A + d\left(\theta \wedge F\right) \wedge \delta A \right\} \right] \\ &= 2 \left[\underbrace{\int_{\partial M} \left\{ \frac{1}{r} \left[\star_g F \wedge \delta A\right] + \theta \wedge F \wedge \delta A \right\}}_{0} - \int_M \left\{ d\left(\frac{1}{r}\star_g F\right) + d\left(\theta \wedge F\right) \right\} \wedge \delta A, \right. \end{aligned}$$

using relation (3.7).

Demanding this variation to be zero for every δA gives the desired equation of motion for F.

6.4 Spinor reduction

Since the spinors are elements in a representation of the Spin group, their reduction on the circle fibration is not as transparent as for the scalar and the 2-form. The vector space of spinors is 8 dimensional in six dimensions and 4 dimensional in five, so the reduction is not as simple as throwing away one component. The correct way is instead to view the 6 dimensional representation as a tensor product of two spinor representations in 5 and 2 space-time dimensions. Such a tensor product representation will be a vector space of dimension $4 \cdot 2 = 8$ which is the correct dimensionality for spinors on a 6 dimensional space-time.

6.4.1 Clifford algebra reduction

Start with a 4 dimensional Clifford algebra represented by the matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$.

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab} \tag{6.3}$$

To construct a five dimensional Clifford algebra append a fifth matrix consisting of the four multiplied together.

$$\gamma^4 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

The i ensures that

$$\left(\gamma^4\right)^2 = 1,$$

as can be checked by using (6.3).

The full six dimensional algebra can now be created as a tensor product construction which makes the splitting under the fibration $6 = 5 + 1$ explicit.

Take as the six dimensional gamma matrices

$$\begin{aligned} \Gamma^a &= \gamma^a \otimes \rho_1 \\ \Gamma^5 &= 1 \otimes \rho_2, \end{aligned}$$

where ρ_1 and ρ_2 satisfy $\{\rho_i, \rho_j\} = 2\delta_{ij}$.

These have the correct anticommutation relations to form a six dimensional algebra. The two dimensional matrices ensure that Γ^5 has the right relation to the other 5 matrices. For example:

$$\begin{aligned} \left\{\Gamma^a, \Gamma^b\right\} &= \gamma^a\gamma^b \otimes (\rho_1)^2 + \gamma^b\gamma^a \otimes (\rho_1)^2 \\ &= \gamma^a\gamma^b \otimes 1 + \gamma^b\gamma^a \otimes 1 \\ &= \left\{\gamma^a, \gamma^b\right\} \otimes 1 \\ &= 2\eta^{ab} (1 \otimes 1) \\ &= 2\eta^{ab} I \end{aligned}$$

The chirality operator in 6 dimensions now take the form:

$$\begin{aligned} \Gamma &= \Gamma^0\Gamma^1 \ldots \Gamma^5 \\ &= \gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 \otimes (\rho_1)^5 \rho_2 \\ &= -i\left(\gamma^4\right)^2 \otimes \rho_1\rho_2 \\ &= 1 \otimes -i\rho_1\rho_2 \end{aligned}$$

Thus it is natural to define the two dimensional chirality operator as

$$\rho = -i\rho_1\rho_2. \tag{6.4}$$

For the reduction calculations a particular choice for this two dimensional algebra needs to be done. Here the first two Pauli matrices will serve that purpose.

$$\begin{aligned}\rho_1 &= \sigma_1 \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \rho_2 &= \sigma_2 \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\end{aligned}$$

These anticommute and square to one:

$$\begin{aligned}\rho_1\rho_2 - \rho_2\rho_1 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= 0 \\ \rho_1^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \rho_2^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

For the particular choice of chirality operator in (6.4):

$$\begin{aligned}\rho &= -i\rho_1\rho_2 \\ &= -i\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

The eigenvectors of this operator is

$$\begin{aligned}\eta_+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \eta_- &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned}$$

with eigenvalue ± 1 as indicated. Thus chiral 6 dimensional spinors lie in the η_+ direction, and antichiral spinors in the η_- direction.

The following spinors will come up often in the calculations:

$$\begin{aligned}\rho_1\eta_+ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \eta_- \\ \rho_2\eta_+ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ i \end{pmatrix} \\ &= i\eta_-\end{aligned}$$

So $\rho_1\eta_+$ and $\rho_2\eta_+$ transforms as antichiral spinors.

6.4.2 Charge conjugation

Both the charge conjugation matrix and the B matrix need to be split as well into the framework of the above reduction. One possible choice is to start with

$$B = B_5 \otimes 1.$$

Letting $C = C_5 \otimes C_2$, where C_5 is the unique charge conjugation matrix in 5 dimensions, we see that the six dimensional relation $B = C\Gamma^0$ implies

$$B_5 \otimes 1 = C_5\gamma^0 \otimes C_2\rho_1.$$

For this to be consistent we choose $C_2 = \rho_1$ so that

$$B_5 = C_5\gamma^0.$$

It can now be checked that B and C in terms of their splitting satisfies all the properties of the original B and C. For example,

$$\begin{aligned} C\Gamma^\mu C^{-1} &= C_5\gamma^\mu C_5^{-1} \otimes \rho_1\rho_1\rho_1 \\ &= (\gamma^\mu)^T \otimes \rho_1 \\ &= (\gamma^\mu)^T \otimes \rho_1^T \\ &= (\Gamma^\mu)^T, \end{aligned}$$

which is the same result as equation (2.5).

6.4.3 Spinors

By the choice basis for the 2-dimensional Clifford algebra we can choose the spinors ψ and the supersymmetry parameters ε to split according to

$$\begin{aligned} \psi^i &= \lambda^i \otimes \eta_- \\ \varepsilon^i &= \varepsilon^i \otimes \eta_+. \end{aligned}$$

With these definitions they have the same chirality properties as in the full 6-dimensional theory. For example,

$$\begin{aligned} \Gamma\psi^i &= (1 \otimes \rho)\left(\lambda^i \otimes \eta_-\right) \\ &= \lambda^i \otimes (-\eta_-) \\ &= -\lambda^i \otimes \eta_- \\ &= -\psi^i. \end{aligned}$$

6.4.4 Spinor action

The action for the six dimensional spinors is

$$\begin{aligned} S_\psi &= \int d^6x\sqrt{-G}i\overline{\psi_i}\Gamma^M D_M\psi^i \\ &= \int d^6x\sqrt{-G}i\overline{\psi_i}\Gamma^A E_A^M D_M\psi^i. \end{aligned}$$

Using the explicit splitting of the spinor representation from the reduction of the Clifford algebra and rescaling with the radius $r(x)$ the spinor ψ splits as

$$\psi^i = \frac{1}{r\sqrt{2\pi}}\lambda^i \otimes \eta_-.$$

Using the expressions for the vielbeins (5.6) and the spin connection (5.8) the action above can be expanded to find that

$$S_\psi = \frac{1}{2\pi}\int d^6x\sqrt{-G}\left[\frac{i}{r^2}\overline{\lambda_i}\gamma^\mu D_\mu\lambda^i - \frac{1}{2}\frac{i}{r^3}\partial_\mu r\overline{\lambda_i}\gamma^\mu\lambda^i + \frac{1}{8}\frac{1}{r}e^{\rho b}e^{\mu c}\mathfrak{F}_{\mu\rho}\overline{\lambda_i}\gamma_{\text{cb}}\lambda^i\right]\otimes\eta_-^\dagger\eta_-.$$

The second term is identically zero since,

$$\begin{aligned}
\overline{\lambda_i}\gamma^\mu\lambda^i &= \left(\overline{\lambda_i}\gamma^\mu\lambda^i\right)^T \\
&= -\left(\lambda^i\right)^T(\gamma^\mu)^T C^T\lambda_i \\
&= \left(\lambda^i\right)^T CC^{-1}(\gamma^\mu)^T C\lambda_i \\
&= \left(\lambda^i\right)^T C\gamma^\mu\lambda_i \\
&= -\overline{\lambda_i}\gamma^\mu\lambda^i.
\end{aligned}$$

The determinant of the metric splits according to (5.4) and since nothing depends on the φ direction we can perform the φ integral.

$$S_\psi = \int d^5x\sqrt{-g}\left[\frac{i}{r}\overline{\lambda_i}\gamma^\mu D_\mu\lambda^i + \frac{1}{8}e^{\rho b}e^{\mu c}\mathfrak{F}_{\mu\rho}\overline{\lambda_i}\gamma_{\text{cb}}\lambda^i\right]\otimes\eta_-^\dagger\eta_-$$

Finally note that we can choose the basis of the two dimensional algebra $\eta_\pm$ to be normalised so that $\eta_-^\dagger\eta_- = 1$. Thus the right tensor factor reduces to the identity and we are left with the 5 dimensional action

$$S_\lambda = \int d^5x\sqrt{-g}\left(\frac{i}{r}\overline{\lambda_i}\gamma^\mu D_\mu\lambda^i + \frac{1}{8}\mathfrak{F}_{\mu\rho}\overline{\lambda_i}\gamma^{\mu\rho}\lambda^i\right).$$

This action contains the usual Dirac term, although with a varying coupling $\frac{1}{r}$, but also contains an additional term depending on θ_μ.

6.5 Supersymmetry reduction

6.5.1 Scalars

The supersymmetry transformation for the scalar field reduces to

$$\begin{aligned}
\delta\phi^{ij} &= \overline{\psi^{[i}}\varepsilon^{j]} - \frac{1}{4}T^{ij}\overline{\psi_k}\,\varepsilon^k \\
&= \left(\psi^{[i}\right)^T C\varepsilon^{j]} - \frac{1}{4}T^{ij}\left(\psi_k\right)^T C\,\varepsilon^k \\
&= \left(\left(\lambda^{[i}\right)^T\otimes\eta_-^T\right)(C_5\otimes\rho_1)\left(\tilde{\varepsilon}^{j]}\otimes\eta_+\right) - \frac{1}{4}T^{ij}\left((\lambda_k)^T\otimes\eta_-^T\right)(C_5\otimes\rho_1)\left(\tilde{\varepsilon}^k\otimes\eta_+\right) \\
&= \overline{\lambda^{[i}}\tilde{\varepsilon}^{j]}\otimes\eta_-^T\rho_1\eta_+ - \frac{1}{4}T^{ij}\overline{\lambda_k}\,\varepsilon^k\otimes\eta_-^T\rho_1\eta_+ \\
&= \left(\overline{\lambda^{[i}}\tilde{\varepsilon}^{j]} - \frac{1}{4}T^{ij}\overline{\lambda_k}\,\varepsilon^k\right)\otimes 1.
\end{aligned}$$

6.5.2 Two form

Starting with the supersymmetry transformation for the field strength H,

$$\delta H_{MNP} = 3\partial_{[M}\left(E_N^A E_{P]}^B \bar{\psi}\Gamma_{\mathrm{AB}}\varepsilon\right).$$

From section 6.3 the remaining degrees of freedom after reducing the three form are contained in just the two form F. Therefore we want to find what the above transformation implies for F.

From the splitting,

$$H = E + d\varphi \wedge F,$$

we see that the transformation for F will be given by the φ components of δH.

$$\delta F_{\mu\nu} = \delta H_{\mu\nu\varphi}$$

Carrying out this computation, using the reduced expressions for the vielbeins and spinor bilinear it is found that

$$\delta F_{\mu\nu} = 2iD_{[\mu}\left(\bar{\lambda}\gamma_{\nu]}\tilde{\varepsilon}\right).$$

6.5.3 Penrose condition

The Penrose condition for the supersymmetry parameters ε in 6 dimensions is given by

$$P_M\varepsilon = 0,$$

where

$$P_M\varepsilon = D_M\varepsilon - \frac{1}{6}\Gamma_M \not{D}\,\varepsilon.$$

The reduction of P_M on the circle fibration consists of two parts, $P_\mu\varepsilon$ and $P_\varphi\varepsilon$. Combining the information from the two we get a condition in five dimensions

$$\begin{aligned} P_\mu\varepsilon = D_\mu\tilde{\varepsilon} - \frac{1}{6}\gamma_\mu\widetilde{\not{D}}\tilde{\varepsilon} + \frac{i}{4}r\mathfrak{F}_{\mu\rho}\gamma^\rho\tilde{\varepsilon} - \frac{i}{8}\frac{1}{d}r\mathfrak{F}_{\rho\sigma}\gamma_\mu\gamma^{\rho\sigma}\tilde{\varepsilon} + \\ -\frac{1}{2}\frac{1}{d}\frac{1}{r}\partial^\rho r\gamma_{\mu\rho}\tilde{\varepsilon} - \frac{1}{2}\frac{1}{d}\frac{1}{r}\partial_\mu r\tilde{\varepsilon}. \end{aligned}$$

This can be further reduced by contracting with γ^μ to give

$$D_\mu\varepsilon = \frac{1}{2}\frac{1}{r}\partial^\rho r\gamma_{\mu\rho}\varepsilon + \frac{1}{2}\frac{1}{r}\partial_\mu r\varepsilon - \frac{i}{8}r\mathfrak{F}_{\rho\sigma}\gamma_\mu\gamma^{\rho\sigma}\tilde{\varepsilon} - \frac{i}{4}r\mathfrak{F}_{\mu\rho}\gamma^\rho\varepsilon. \tag{6.5}$$

6.5.4 Spinors

Starting from super symmetry transformation in 6 dimensions for the spinor fields,

$$\delta\psi^i = i\frac{1}{12}\Gamma^{MNP}H_{MNP}\varepsilon^i + iD_M\phi^{ij}\Gamma^M\varepsilon_j + \frac{2}{3}\phi^{ij}\not{D}\,\varepsilon_j,$$

going through the same procedure of reduction as before it is found after some computation that

$$\delta\lambda = -\frac{1}{2}F_{\nu\rho}\gamma^{\nu\rho}\tilde{\varepsilon} - \frac{1}{12}r\mathfrak{F}_{\nu\rho}\phi\gamma^{\nu\rho}\tilde{\varepsilon} - \frac{2i}{3}r\partial_\mu r\phi\gamma^\mu\tilde{\varepsilon} + i\partial_\mu\phi\gamma^\mu\tilde{\varepsilon} + \frac{2}{3}\phi\widetilde{\not{D}}\tilde{\varepsilon}.$$

Using relation (6.5) this can be further reduced to

$$\delta\lambda = -\frac{1}{2}F_{\nu\rho}\gamma^{\nu\rho}\tilde{\varepsilon}^i + \frac{1}{2}r\mathfrak{F}_{\nu\rho}\phi^{ij}\gamma^{\nu\rho}\tilde{\varepsilon}_j + i\frac{1}{r}\partial_\mu r\phi^{ij}\gamma^\mu\tilde{\varepsilon}_j + i\partial_\mu\phi^{ij}\gamma^\mu\tilde{\varepsilon}_j.$$

Chapter 7

Results

7.1 Summary of reduction

Combining the result from the scalar, form and spinor reductions the action is

$$S = \int d^5x\sqrt{-g}\left[\frac{i}{r}\overline{\lambda_i}\gamma^\mu D_\mu\lambda^i + \frac{1}{8}\mathfrak{F}^{\mu\nu}\overline{\lambda_i}\gamma_{\mu\nu}\lambda^i - \frac{1}{2r}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\theta_\mu F_{\nu\rho}F_{\sigma\tau}\varepsilon^{\mu\nu\rho\sigma\tau} + \right.$$
$$\left. -\frac{1}{r}g^{\mu\nu}\partial_\mu\phi^{ij}\partial_\nu\phi_{ij} - \frac{1}{r}D_\mu\left(\frac{1}{r}D^\mu r\right)\phi^2 - \frac{1}{5}\frac{1}{r}\left(\tilde{R} - \frac{1}{4}r^2\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} - \frac{2}{r}D^\mu D_\mu r\right)\phi^2\right].$$

The six dimensional Penrose condition on the supersymmetry parameter reduces to the following condition in five dimensions for the supersymmetry parameters ε^i.

$$D_\mu\varepsilon^i = \frac{1}{2}\frac{1}{r}\partial^\rho r\gamma_{\mu\rho}\varepsilon^i + \frac{1}{2}\frac{1}{r}\partial_\mu r\varepsilon^i - \frac{i}{8}r\mathfrak{F}_{\rho\sigma}\gamma_\mu\gamma^{\rho\sigma}\varepsilon^i - \frac{i}{4}r\mathfrak{F}_{\mu\rho}\gamma^\rho\varepsilon^i.$$

The 6 dimensional supersymmetry variations reduce to

$$\begin{aligned}
\delta\lambda^i &= -\frac{1}{2}F_{\nu\rho}\gamma^{\nu\rho}\varepsilon^i + \frac{1}{2}r\mathfrak{F}_{\nu\rho}\phi^{ij}\gamma^{\nu\rho}\varepsilon_j + i\frac{1}{r}\partial_\mu r\phi^{ij}\gamma^\mu\varepsilon_j + i\partial_\mu\phi^{ij}\gamma^\mu\varepsilon_j \\
\delta F_{\mu\nu} &= 2i\tilde{D}_{[\mu}\left(\overline{\lambda_i}\gamma_{\nu]}\varepsilon^i\right) \\
\delta\phi^{ij} &= \overline{\lambda^{[i}}\varepsilon^{j]} - \frac{1}{4}T^{ij}\overline{\lambda_k}\varepsilon^k.
\end{aligned}$$

The main result is now that the action as given above is invariant under these supersymmetry transformations if ε satisfies the reduced Penrose condition. The calculations, although lengthy, are straight forward variation and substitution with the above definitions.

Note that the coupling "constant" is no longer constant, the function $\frac{1}{r(x)}$ depends on x. In the same manner the terms containing $\mathfrak{F}$ are also x dependent, but neither x nor θ_μ are dynamic fields.

Also note the appearance of the θ-term in the action for the 2-form F. This is to be compared with the θ-term discussed in [28] which in $d = 4$ super Yang-Mills gives rise to S-duality. Here the term arises naturally from the geometry of the 6 dimensional manifold. Upon further compactification this term would result in exactly the θ-term of the 4 dimensional theory but also additional terms.

In conclusion the classical 6-dimensional free (2,0) theory on a circle fibration over a 5-dimensional manifold reduces in the low energy limit to a 5-dimensional theory with a Lagrangian description where the Lagrangian is supersymmetric.

7.1.1 Product metric

Note also that if we take the fibre metric to be the product metric by assuming r constant and $\theta_\mu = 0$, the Lagrangian reduces to

$$S = \int d^5x\sqrt{-g}\left[\frac{i}{r}\overline{\lambda_i^C}\gamma^\mu D_\mu\lambda^i - \frac{1}{2r}F_{\mu\nu}F^{\mu\nu} - \frac{1}{r}g^{\mu\nu}\partial_\mu\phi^{ij}\partial_\nu\phi_{ij}\right]. \quad (7.1)$$

And the supersymmetry variations to

$$\begin{aligned}\delta\lambda^i &= -\frac{1}{2}F_{\nu\rho}\gamma^{\nu\rho}\tilde{\varepsilon}^i\\ \delta F_{\mu\nu} &= 2i\tilde{D}_{[\mu}\left(\overline{\lambda_i}\gamma_{\nu]}\tilde{\varepsilon}^i\right)\\ \delta\phi^{ij} &= \overline{\lambda^{[i}}\tilde{\varepsilon}^{j]} - \frac{1}{4}T^{ij}\overline{\lambda_k}\tilde{\varepsilon}^k.\end{aligned}$$

With ε satisfying $D_\mu\varepsilon^i = 0$.

Note that the reduced Penrose condition now reduces further to the condition that ε should be covariantly constant. This turns out to have quite strong implications for the geometry, in particular we can form the combination

$$[D_\mu, D_\nu]\,\varepsilon = \frac{1}{4}\tilde{R}_{\mu\nu ab}\gamma^{ab}\varepsilon.$$

Since the left hand side is zero, after some manipulations, this gives the condition $\tilde{R}_{\mu\nu} = 0$. This of course also implies $\tilde{R} = 0$ so there is no such term in (7.1).

The resulting theory is the free $N = 4$ supersymmetric Yang-Mills theory in 5 dimensions.

7.2 Outlook

One could say that the existence of a supersymmetric Lagrangian formulation on the circle fibration of the free theory is a positive sign of a consistent theory in 6 dimensions. It is certainly not suprising that the reduced equations of motions are supersymmetric, but that they extend to a supersymmetric action is at least a first step.

In the end one would like to make contact with the interacting theory and one of the motivations of the circle fibration is that this extension comes about naturally. In particular, since the reduced theory in 5 dimensions looks exactly like a free supersymmetric Yang-Mills theory the extension to an interacting theory in 5 dimensions is immidiate. The 2-form in 6 dimensions reduced to a 1-form in 5 dimensions which is precisely the correct setting for a gauge potential. If the fields are then placed in representations of a gauge group, and the 1-form is promoted to a connection, we have an interacting theory in 5 dimensions. The hope is that there will be natural extension of the reduced action to a supersymmetric interacting theory in 5 dimensions. This interacting theory in 5 dimensions would then hopefully give some new hints for the interacting theory in 6 dimensions.

Bibliography

[1] S.P. Weinberg. *The quantum theory of fields: Foundations*. Number v. 1. Cambridge Univ. Press, 2004.

[2] M.F. Sohnius. Introducing supersymmetry. *Physics reports*, 128(2-3):39–204, 1985.

[3] S.P. Weinberg. Critical Phenomena for Field Theorists. Lectures presented at Int. School of Subnuclear Physics, Ettore Majorana, Erice, Sicily, Jul 23 - Aug 8, 1976.

[4] R. Haag, J.T. Lopuszanski, and M. Sohnius. All possible generators of supersymmetries of the s-matrix. *Nuclear Physics B*, 88(2):257 – 274, 1975.

[5] W. Nahm. Supersymmetries and their representations. *Nuclear Physics B*, 135(1):149–166, 1978.

[6] E. Witten. Conformal Field Theory In Four And Six Dimensions. *ArXiv e-prints*, December 2007.

[7] E. Witten. Some comments on string dynamics. 1995.

[8] P. Arvidsson. Superconformal theories in six dimensions. 2006.

[9] E. Flink. *Strings and Particles in Six Dimensions*. PhD thesis, Chalmers University of Technology, 2006.

[10] J.M. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[11] M. Henningson. The quantum Hilbert space of a chiral two-form in d = 5+1 dimensions. *JHEP*, 03:021, 2002.

[12] M. Henningson, B.E.W. Nilsson, and Per Salomonson. Holomorphic factorization of correlation functions in (4k+2)-dimensional (2k)-form gauge theory. *JHEP*, 09:008, 1999.

[13] M. Henningson. Surface observables and the Weyl anomaly. *Arxiv preprint hep-th/9908183*, 1999.

[14] J. Scherk. Extended supersymmetry and extended supergravity theories. In *NATO ASIB Proc. 44: Recent Developments in Gravitation*, volume 1, page 479, 1979.

[15] J. Fuchs and C. Schweigert. *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003.

[16] M. Nakahara. *Geometry, topology, and physics*. Graduate student series in physics. Institute of Physics Publishing, 2003.

[17] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. Gravitation. 1973.

[18] J.C. Baez and J.P. Muniain. *Gauge fields, knots, and gravity*. K & E series on knots and everything. World Scientific, 1994.

[19] M.B. Green, J.H. Schwarz, and E. Witten. *Superstring Theory: Introduction*. Cambridge monographs on mathematical physics. Cambridge University Press, 1988.

[20] H.B. Lawson and M.L. Michelsohn. *Spin geometry*. Princeton mathematical series. Princeton University Press, 1989.

[21] H. Flanders. *Differential forms with applications to the physical sciences*. Dover books on mathematics. Dover Publications, 1989.

[22] V. Frederick Rickey and Philip M. Tuchinsky. An application of geography to mathematics: History of the integral of the secant. *Mathematics Magazine*, 53(3):pp. 162–166, 1980.

[23] J. Polchinski, P. Landshoff, and D.R. Nelson. *String Theory, Vol. 2: Superstring Theory and Beyond*. Cambridge Univ. Pr., 1998.

[24] H. Baum. *Twistor and Killing spinors in Lorentzian geometry*. SFB 288, 2000.

[25] N. Seiberg. Notes on theories with 16 supercharges. *Nuclear Physics B-Proceedings Supplements*, 67(1-3):158–171, 1998. p. 4 in article.

[26] T. Kaluza. Zum unitätsproblem der physik. *Sitz. Preuss. Akad. Wiss. Phys. Math., K*, 1:966, 1921.

[27] C. Pope. Lectures on Kaluza-Klein. `http://faculty.physics.tamu.edu/pope/`.

[28] E. Witten. Geometric Langlands From Six Dimensions. *ArXiv e-prints*, 2009.