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Eisenstein Series and Instantons in String Theory

Master of Science Thesis in Fundamental Physics

HENRIK P. A. GUSTAFSSON

Department of Fundamental Physics
Mathematical Physics
CHALMERS UNIVERSITY OF TECHNOLOGY
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Henrik P. A. Gustafsson

Department of Fundamental Physics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 - 772 1000

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Abstract

In this thesis we review U-duality constraints on corrections to the type IIB supergravity effective action on torus compactifications. The U-duality invariant corrections are described by automorphic forms whose construction and Fourier expansion we explain. The latter enables us to extract physical information about instanton states in the theory. In particular, we discuss the first non-vanishing higher-derivative correction to the Einstein-Hilbert action in ten dimensions, for which there is a unique automorphic form incorporating all the perturbative and non-perturbative corrections in the string coupling constant.

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Chapter 1

Introduction

The low energy limits of different string theories are described by certain supergravity theories which are important tools for understanding various aspects of string and M-theory [1, 2].

Quantum corrections to these effective theories are strongly constrained by symmetries such as supersymmetry and U-dualities which relate different string theories. In [3] for example, the large amount of supersymmetry in the ten dimensional type IIB supergravity is used to show that the first corrections in α' vanish.

Additionally, in [4, 5, 6], the first non-vanishing α' contributions are determined to all orders in the string coupling constant by requiring that the quantum corrected effective action is invariant under the discrete U-duality group. The contributions are summed up into functions called automorphic forms that have a very rich structure.

It is possible to extract a lot of information from these functions that is hard to calculate directly in string theory, such as non-perturbative, instanton corrections to scattering amplitudes which usually requires methods described in [7].

The automorphic forms also contain arithmetic information about instanton degeneracies in D dimensions, which, by compactification, is related to the counting of microstates of black holes in dimension $D+1$ and gives a detailed description of black hole entropy as discussed in [8].

Besides having great importance in string theory, automorphic forms are also mathematically very fascinating; connecting geometry, number theory and representation theory (see for example [9] for a good overview).

The purpose of this thesis is to understand how automorphic forms enter in string theory, how to construct them and to obtain physical information from them. This will be achieved by reviewing the type IIB supergravity effective action in ten dimension together with duality constraints on its quantum corrections, toroidal compactifications of supergravities, and instanton corrections in field theories and string theory.

Although we discuss automorphic forms on higher rank duality groups in general, we will make extensive use of (and sometimes limit ourselves to) the ten dimensional case with U-duality group $\mathrm{SL}(2,\mathbb{Z})$ as an example.

This thesis is organized as follows. In chapter 2, we review the tree-level effective action for type IIB supergravity in ten dimensions and, under the assumptions that the quantum corrections should be $\mathrm{SL}(2,\mathbb{Z})$ invariant, we find the coefficient in front of the $\mathcal{R}^4$ term (explained below) to all orders in the string coupling.

We briefly introduce the properties of the group $\mathrm{SL}(2,\mathbb{R})$ in chapter 3 that will be used throughout the text and in chapter 4 we present some relevant topics in the theory of Lie algebras. Chapter 5 is devoted to effective actions and the description of the scalar part of supergravities in different dimensions by non-linear sigma models on coset spaces.

In chapter 6, we discuss instantons in field theories and in string theory, describing how they contribute with non-perturbative corrections to the effective action. Chapter 7 then argues why the corrections to the effective action should be invariant under certain U-duality groups (motivating the assumption in chapter 2).

Then, chapter 8 and 9 introduce the theory of automorphic forms; how they are constructed and Fourier expanded to obtain physical information. The former chapter treats more general duality groups, while the latter concretizes with the example of $\mathrm{SL}(2,\mathbb{Z})$ referring back to chapter 2. Lastly, chapter 10 concludes the thesis with an outlook and discussion about recent progress and future prospects.

Chapter 2

Motivation from string theory

The type IIB supergravity (SUGRA) theory in 10 dimensions is the low energy effective theory of IIB string theory of the same dimension — a concept that will be explained in section 5.1. This theory will be our main example when discussing Eisenstein series in string theory because of the simplicity of its duality group as explained later.

We will focus on the bosonic part of the theory and especially the scalar fields: the dilaton ϕ (related to the string coupling by $g_s = e^{\phi}$)[1] and the axion $C_0 = \chi$. The other bosonic fields are the Neveu-Schwarz (NS) 2-form B_2, the Ramond-Ramond (RR) forms C_2 and C_4, and the metric $G_{\mu\nu}$, $\mu, \nu = 0, \ldots, 9$ meaning the graviton. The corresponding field strengths are defined as

$$H_3 = \mathrm{d}B_2 \qquad F_n = \mathrm{d}C_{n-1}\,. \tag{2.1}$$

The bosonic, effective action can be expressed as a sum of a NS, RR and Chern-Simons (CS) part

$$S_{\mathrm{IIB}} = S_{\mathrm{NS}} + S_{\mathrm{RR}} + S_{\mathrm{CS}}\,. \tag{2.2}$$

In the string frame (denoted by superscript s) these are [10, 11]

$$\begin{aligned}
S^{(s)}_{\mathrm{NS}} &= \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G^{(s)}}e^{-2\phi}\Big(R^{(s)} + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\Big)\\
S^{(s)}_{\mathrm{RR}} &= -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G^{(s)}}\Big(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\Big)\\
S^{(s)}_{\mathrm{CS}} &= -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\, C_4 \wedge H_3 \wedge F_3\,,
\end{aligned} \tag{2.3}$$

where $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. We use the notation that for a k-form A_k

$$|A_k|^2 := \frac{1}{k!}G^{\mu_1\nu_1}\cdots G^{\mu_k\nu_k}A_{\mu_1\cdots\mu_k}A_{\nu_1\cdots\nu_k}\,, \tag{2.4}$$

where $G^{\mu\nu}$ is either in the string frame or Einstein frame depending on context.

[1] More precisely, the string coupling should be the expectation value of this quantity at infinity.

Constructing the complex scalar field

$$\tau = \tau_1 + i\tau_2 = \chi + ie^{-\phi} = \chi + ig_s^{-1} \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\} \tag{2.5}$$

on the upper half plane called the axion-dilaton field and combing the 2-forms into

$$\mathcal{C}_2 = \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \qquad \mathcal{F}_3 = \mathrm{d}\mathcal{C}_2 \tag{2.6}$$

the action S_{IIB} is invariant under the global symmetry [11]

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \mathcal{C}_2 \to \Lambda\mathcal{C}_2 \qquad C_4 \to C_4 \qquad G_{\mu\nu} \to G_{\mu\nu} \tag{2.7}$$

where $G_{\mu\nu} = e^{-\phi/2}G^{(s)}_{\mu\nu}$ is the metric in the Einstein frame and

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) = \{\Lambda \in \mathbb{R}^{2\times 2} \mid \det\Lambda = 1\}\,. \tag{2.8}$$

This can be seen from a clever rewriting of the action that we will discuss more in section 5.2.2. In short, going to the Einstein frame (with no superscripts) and letting

$$M = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} |\tau|^2 & \operatorname{Re}\tau \\ \operatorname{Re}\tau & 1 \end{pmatrix} \tag{2.9}$$

which transforms as $M \to \Lambda M \Lambda^T$ the action can be written as [10, 11]

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-G}\Big(R + \frac{1}{4}\operatorname{Tr}(\partial_\mu M\partial^\mu M^{-1}) - \frac{1}{12}(F_3^T)_{\mu\nu\rho}M^{-1}(F_3)^{\mu\nu\rho} - \frac{1}{4}|\tilde{F}_5|^2\Big) \\ - \frac{\epsilon_{ij}}{8\kappa_{10}^2}\int C_4 \wedge \mathcal{F}_3^i \wedge \mathcal{F}_3^j \end{aligned} \tag{2.10}$$

where R is the Ricci scalar in the Einstein frame.

The S_{IIB} action presented above is only the tree-level effective action, meaning that it is only correct to the first order in the coupling constants of the theory.

Perturbations in string theory are expanded in two coupling constants: the worldsheet coupling α' which is related to the string length or the inverse string tension, and the string coupling g_s which is related to how often strings split and merge creating different handles on the worldsheet.

Let us now consider quantum corrections to the tree-level (classical) effective action (2.2) above in the string frame where the expansion is easier to see. We denote the corrected effective action by $S_{(n,k)}$ which is determined to the n^{th} order in α' and k^{th} order in g_s. [2]

For simplicity, we will only consider the first part of the action which only includes the metric, that is, the gravitational part

$$S^{(s)}_{(0,0)} = (\alpha')^{-4}\int d^{10}x\sqrt{G^{(s)}}e^{-2\phi}\Big(R^{(s)} + \dots\Big) \tag{2.11}$$

[2]not including the $(\alpha')^{-4}$ factor or, in the string frame, the $\exp(-2\phi) = g_s^{-2}$ factor.

where the ellipsis denote terms that are not purely gravitational.

It can be shown that, at this order in α', this action will not obtain any corrections from higher orders in g_s [6].

The next step is to consider corrections in α'. It turns out the first and second order corrections in α' vanish at all orders in g_s because of supersymmetry [6]. Thus, the first non-zero correction is of order $(\alpha')^3$ and has, at tree level in g_s, been calculated in [12] as a certain fourth order combination of $R^{(s)}_{\mu\nu\rho\sigma}$ whose exact expression can be found in [12, 4, 13], but will here be combined into the schematic $\mathcal{R}^4_{(s)}$. That is,

$$S^{(s)}_{(3,0)} = (\alpha')^{-4} \int \mathrm{d}^{10}x \sqrt{G^{(s)}} e^{-2\phi} \Big(R^{(s)} + (\alpha')^3 c_0 \mathcal{R}^4_{(s)} + \dots \Big) \tag{2.12}$$

where c_0 is a known constant.

The same combination of Riemann tensors also appear at first order in g_s [4] with

$$S^{(s)}_{(3,1)} = (\alpha')^{-4} \int d^{10}x \sqrt{G^{(s)}} e^{-2\phi} \Big(R^{(s)} + (\alpha')^3 (c_0 + c_1 g_s) \mathcal{R}^4_{(s)} + \dots \Big) \tag{2.13}$$

where c_1 also is a known constant.

Let us stop our expansion in g_s here for now[3], but we should still include non-perturbative (instanton) corrections whose expansions in g_s are identically zero. Already in [14], before it was understood how instanton corrections arise in string theory, it was argued that non-perturbative corrections are needed to repair divergent perturbation series.

As will be seen in chapter 6, the instanton corrections are expected to be accompanied by an imaginary exponent similar to that of the θ-angle in Yang-Mills theory.

$$S^{(s)}_{(3,1)} = (\alpha')^{-4} \int d^{10}x \sqrt{G^{(s)}} e^{-2\phi} \Big[R^{(s)} + (\alpha')^3 \Big(c_0 + c_1 g_s + \underbrace{\sum_{N \neq 0} a_N e^{-2\pi |N| (g_s)^{-1} + 2\pi i N \chi}}_{\text{instanton corrections}} \Big) \mathcal{R}^4_{(s)} + \dots \Big] \tag{2.14}$$

We now return to the Einstein frame where the tree level $\mathrm{SL}(2,\mathbb{R})$-symmetry could be made manifest. After calculating R and $\mathcal{R}^4$ in this frame in terms of ϕ, $R^{(s)}$ and $\mathcal{R}^4_{(s)}$, and explicitly writing out the constants c_0 and c_1 the effective action is found to be [6]

$$S_{(3,1)} = (\alpha')^{-4} \int d^{10}x \sqrt{G} \Big[R + (\alpha')^3 \Big(2\zeta(3) \tau_2^{3/2} + 4\zeta(2) \tau_2^{-1/2} + \sum_{N \neq 0} a_N e^{2\pi (i N \tau_1 - |N| \tau_2)} \Big) \mathcal{R}^4 + \dots \Big] \tag{2.15}$$

where $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \, . \tag{2.16}$$

[3] In fact, we will later argue that these are the only perturbative terms in g_s at this order in α'.

We see that the action above with its different powers of τ_2 cannot have a full $\mathrm{SL}(2,\mathbb{R})$ symmetry. By considering the inclusions of classical charges in the quantum theory (as will be explained in more detail in section 7.1) one can show that all symmetry is not lost but that we still have an $\mathrm{SL}(2,\mathbb{Z})$ symmetry. We see that such a discrete symmetry is hinted at in (2.15) where the translation symmetry of τ_1 breaks down to the discrete translation symmetry $\tau_1 \to \tau_1 + 1$ which is one of the two generators of $\mathrm{SL}(2,\mathbb{Z})$. This means that the parenthesized factor in front of $\mathcal{R}^4$ in (2.15) should combine to an $\mathrm{SL}(2,\mathbb{Z})$ invariant function $f(\tau)$ where $\tau = \chi + ie^{-\phi} \in \mathbb{H}$.

In section 3.2, we will show that $\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. This means that $f(\tau)$ can be considered a function on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ invariant under $\mathrm{SL}(2,\mathbb{Z})$ which is called an automorphic form on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. These objects will be the main topic of chapter 8.

We will see that the mathematical formalism of automorphic forms in this context is extremely powerful. Using the condition that $f(\tau)$ should have a perturbative expansion equal to that of the $\mathcal{R}^4$ factor, that is, asymptotically

$$f(\tau) \sim 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} \qquad \text{as } \tau_2 = g_s^{-1} \to \infty \tag{2.17}$$

it turns out [4] that we can uniquely decide $f(\tau)$ using the non-holomorphic Eisenstein series together with supersymmetry arguments

$$\mathcal{E}(\tau; s) = \sum_{\substack{(m,n)\in Z^2 \\ (m,n)\neq(0,0)}} \frac{\tau_2^s}{|m + n\tau|^{2s}} \tag{2.18}$$

with $f(\tau) = \mathcal{E}(\tau, 3/2)$ as will be discussed in section 9.3

After this brief overview of how automorphic forms enter in string theory, we will now dive into the different topics referred to above in more detail.

Chapter 3

The special linear group $\mathrm{SL}(2,\mathbb{R})$

As shown in chapter 2, type IIB supergravity in 10 dimensions is invariant under an $\mathrm{SL}(2,\mathbb{R})$ symmetry involving, among others, the classical scalar fields χ and ϕ corresponding the axion and the dilaton.

We will now make a more detailed review of the group $\mathrm{SL}(2,\mathbb{R})$ which will be useful for later exploring automorphic forms with this group as a recurring example.

Let us first generalize the definition used above. The special linear group of order n over the field F (e.g. $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{C}$), denoted by $\mathrm{SL}(n,F)$, is defined as the set of $n\times n$ matrices in F with determinant 1. For $\gamma\in\mathrm{SL}(2,\mathbb{R})$ we have that

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\qquad ad-bc=1\qquad a,b,c,d\in\mathbb{R}\,. \tag{3.1}$$

This specific notation for the matrix elements will be used throughout the text.

It is useful to also define the projective special linear group as $\mathrm{PSL}(2,\mathbb{R})=\mathrm{SL}(2,\mathbb{R})/\{\pm\mathbb{1}\}$ which removes an overall sign of the matrix γ.

Note that an element of the group $\mathrm{SL}(2,\mathbb{Z})$, called the modular group, has a well-defined inverse with integer values because the determinant is one. The modular group is generated by the elements T and S where [15]

$$T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\qquad S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{3.2}$$

3.1 Preliminaries on group theory

Before we can discuss some of the properties of the special linear group we should review some terminology in group theory.

Group theory is the mathematical framework used to describe symmetries and for continuous symmetries we turn to the theory of Lie groups. Besides being a group, Lie groups are also differentiable manifolds where the group operation and inversion are differentiable [16]. By considering the tangent space of such a manifold at the point of the identity element of the group one obtains the corresponding Lie algebra which is discussed in the next chapter.

Seen as a symmetry, a group acts on some space — transforming one point to another. A group action for a group G acting on a topological space X is a mapping $\phi : G \times X \to X$, usually denoted as simply $g(x)$ or gx for $g \in G$ and $x \in X$, that for all $x \in X$ satisfies

$$\begin{aligned} ex &= x \\ g(hx) &= (gh)x \quad \forall g, h \in G\,, \end{aligned} \tag{3.3}$$

where e is the identity element.

One says that a group action is faithful if there does not exist any $g \in G$ where $g \neq e$ such that $gx = x$ for all $x \in X$. Also, if there for all x and y in X exists $g \in G$ such that $gx = y$ the action is said to be transitive.

Now that we can describe the symmetries by group elements, it is also often useful to see the space as constructed from certain Lie groups. As we will soon see, the points of a symmetric space can with great benefit be described as the group of symmetries of the space if we identify the symmetries that stabilize a certain point. Think for example of the points $\boldsymbol{r}$ on the sphere S^2 which can be seen as the group of rotations SO(3) (the symmetries of the sphere) if we do not allow rotations around the $\boldsymbol{r}$-axis, that is, SO(2) (the symmetries that leave the north pole invariant). This is conveniently described using coset spaces with $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$ where the notation will be described below.

Let H be a subgroup of a group G. For $g \in G$ we define an equivalence class $[g] = \{gh \mid h \in H\}$ which will be denoted by gH and called a left coset (in contrast to Hg classes defined similarly and that are called right cosets). We can then define the coset space $G/H := \{gH \mid g \in G\}$ as the set of all such left cosets. The space $H\backslash G$ is similarly constructed from the right cosets.

When H is a normal subgroup, meaning that $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$, the left and right cosets coincides. And the resulting quotient G/H is also a group. The normality condition is needed to make $[g][g'] = [gg']$ for all $g, g' \in G$.

We will often consider coset spaces where H is the maximal compact subgroup of G, denoted by K, meaning the compact subgroup that contains all other compact subgroups in G.

As discussed above, a coset in G/K is defined by the equivalence class $[g] = \{gk \mid \forall k \in K\}$ where $g \in G$. We can then let $l \in G$ act on G/K with a left-action $l : [g] \mapsto [lg]$ which is well-defined since if $[g'] = [g]$, that is, there exists a $k \in K$ such that $g' = gk$ then $lg' = lgk \sim lg$ and $[lg'] = [lg]$.

3.2 The half plane

We shall see that the special linear group, $\mathrm{SL}(2,\mathbb{R})$, is tightly connected to the (upper) half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The group can be seen as a symmetry for the half plane, and can, in the spirit of the last section, also be used to describe the space itself.

The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$ which makes this possible is for $z \in \mathbb{H}$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})$

$$z \mapsto z' = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{3.4}$$

Since

$$\operatorname{Im} z' = \frac{\operatorname{Im}\big((az+b)(cz+d)^*\big)}{|cz+d|^2} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} \tag{3.5}$$

we have that $z' \in \mathbb{H} \iff z \in \mathbb{H}$.

The same action is used for $\mathrm{PSL}(2,\mathbb{R})$ which acts faithfully on $\mathbb{H}$ since

$$\gamma z = z \quad \forall z \in \mathbb{H} \implies cz^2 + (d-a)z - b = 0 \quad \forall z \in \mathbb{H} \implies b = c = 0,\, a = d \tag{3.6}$$

and

$$\begin{gathered} \det \gamma = 1 \implies a = \pm 1 \\ \gamma = \pm \mathbb{1} \sim e\,, \end{gathered} \tag{3.7}$$

where e is the identity element for $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathbb{1}\}$ since $+\mathbb{1} \sim -\mathbb{1}$.

We have seen that the $\mathrm{SL}(2,\mathbb{R})$ action is a symmetry of $\mathbb{H}$ and to describe the half plane using a coset space as described in the above section we will now show that $\mathrm{SO}(2,\mathbb{R})$ is the subgroup of $\mathrm{SL}(2,\mathbb{R})$ that keeps the point $i \in \mathbb{H}$ invariant.

Proposition 3.1. The stabilizer of the imaginary unit $i \in \mathbb{H}$, that is, $\mathrm{Stab}(i) := \{\gamma \in \mathrm{SL}(2,\mathbb{R}) \mid \gamma(i) = i\}$, is SO(2,$\mathbb{R}$).

Proof.
Let $\gamma \in \mathrm{SL}(2,\mathbb{R})$ such that it stabilizes $i \in \mathbb{H}$. Then

$$\frac{ai+b}{ci+d} = i \iff ai + b = -c + id \iff \begin{cases} a = d \\ b = -c \end{cases} . \tag{3.8}$$

We get that

$$\gamma = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with} \quad \det \gamma = a^2 + b^2 = 1\,. \tag{3.9}$$

We can then set $a = \cos\theta$ and $b = -\sin\theta$ with $\theta \in [0, 2\pi]$. Thus,

$$\gamma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2,\mathbb{R})\,. \tag{3.10}$$

□

This motivates that the following topological spaces are homeomorphic (topologically isomorphic, which means that they can be continuously deformed into each other)

$$\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cong \mathbb{H} \tag{3.11}$$

with the identification of $z = \gamma(i)$. The statement is more rigorously shown in [17], but can also be shown by factorizing $\mathrm{SL}(2,\mathbb{R})$ into its Iwasawa decomposition as discussed in the next section.

3.3 Iwasawa decomposition

We will now show how to uniquely decompose any element of $\mathrm{SL}(2,\mathbb{R})$ into a specific form of matrix factors called the Iwasawa decomposition following [18]. The arguments can be generalized to other connected semi-simple Lie groups.

Theorem 3.1 (Iwasawa decomposition)**.**

An element $g \in \mathrm{SL}(2,\mathbb{R})$ can be uniquely factorized into

$$g = n\,a\,k\,, \tag{3.12}$$

where n is an upper triangular 2×2 matrix with unit diagonal entries, a is a diagonal 2×2 matrix with positive eigenvalues and determinant 1, and $k \in \mathrm{SO}(2,\mathbb{R})$, that is,

$$n = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix},\ \xi \in \mathbb{R} \qquad a = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix},\ r > 0 \qquad k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2,\mathbb{R}) \tag{3.13}$$

Proof.

We start with the existence of a decomposition. Let g be a general element in $\mathrm{SL}(2,\mathbb{R})$ and $z = x + iy := g(i) \in \mathbb{H}$. We have that

$$\gamma_z = \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \implies \gamma_z(i) = \frac{yi + x}{0i + 1} = z\,. \tag{3.14}$$

Thus,

$$\gamma_z^{-1} g(i) = i \tag{3.15}$$

and $\gamma_z^{-1} g$ must therefore be an element of $\mathrm{Stab}(i) = \mathrm{SO}(2,\mathbb{R})$ by proposition 3.1. Let $k = \gamma_z^{-1} g \in \mathrm{SO}(2,\mathbb{R})$. We then have that

$$g = \gamma_z k \tag{3.16}$$

The matrix γ_z can be written as

$$\gamma_z = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} = n\,a\,, \tag{3.17}$$

with $r = \sqrt{y}$ and $\xi = x$ and then the factorization $g = n\,a\,k$ exists.

Now to the uniqueness of the factorization. Acting g on $i \in \mathbb{H}$ uniquely decides the parameters of a and n since

$$\begin{aligned} g(i) &= n\,a\,k(i) = n\,a(i) \\ &= \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} (i) \\ &= \begin{pmatrix} r & \xi/r \\ 0 & 1/r \end{pmatrix} (i) = \frac{ir + \xi/r}{1/r} = \xi + ir^2 \stackrel{!}{=} z = x + iy\,, \end{aligned} \tag{3.18}$$

where we recall that $r > 0$. k is then fixed by $k = g(an)^{-1}$.

$\square$

Note that we also have a unique factorization $g = k\,a\,n$ with the factors having the same form as before. This can be seen by factorizing $g^{-1} \in \mathrm{SL}(2,\mathbb{R})$ as $g^{-1} = n(\xi)a(r)k(\theta)$. Then $g = k(\theta)^{-1}\,a(r)^{-1}\,n(\xi)^{-1} = k(-\theta)\,a(1/r)\,n(-\xi)$ which is now of the required form and is unique by the same arguments as above.

The $g = n\,a\,k$ factorization can be used to uniquely specify elements in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ with the parameters ξ and r of n and a. Since $g(i) = na(i) = \xi + ir^2 = z$ this gives the explicit relation between cosets of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ and points in $\mathbb{H}$.

Chapter 4

Lie algebra

As mentioned above, the tangent space of a Lie group at the identity element gives the corresponding Lie algebra. In this chapter we will briefly introduce a few topics in this broad field that we will make use of in later chapters. Where not otherwise stated we will follow [19, 20, 21] and the underlying field will always be the reals.

4.1 Maurer-Cartan form

The Maurer-Cartan form $\omega = \omega_\mu \mathrm{d}x^\mu$ is a one-form on a Lie group G taking values in the Lie algebra of G, that is, it maps vectors in the tangent space around $g \in G$ to an element in tangent space around the identity — the Lie algebra $\mathfrak{g}$. Using a matrix representation the Maurer-Cartan form can be expressed as [22]

$$\omega = \omega_\mu \,\mathrm{d}x^\mu = g^{-1}(x)\partial_\mu g(x)\,\mathrm{d}x^\mu = g^{-1}(x)\mathrm{d}g(x) \tag{4.1}$$

where $\mathrm{d}g$ can be seen as taking the exterior derivative of every matrix element in g.

Since ω (when acting on vectors) takes values in the Lie algebra $\mathfrak{g}$ we have that the coefficients ω_μ also are in $\mathfrak{g}$ which we will later use in the construction of an action on G.

As a pure matrix manipulation we can write the very useful identity

$$0 = \partial_\mu(g^{-1}g) \implies g^{-1}\partial_\mu g = -(\partial_\mu g^{-1})g \tag{4.2}$$

4.2 Killing form

The Killing form is a product between elements of the Lie algebra. It is linear and symmetric but not positive definite. For A and B in some Lie algebra $\mathfrak{g}$ it is defined as

$$(A \mid B) := \mathrm{Tr}(\mathrm{Ad}_A \circ \mathrm{Ad}_B) \tag{4.3}$$

where $\mathrm{Ad}_X(Y) = [X, Y]$.

The Killing form is an invariant form in the following sense

$$([A, B] \mid C) = (A \mid [B, C]) \tag{4.4}$$

For a semi-simple Lie Algebra the Killing form simplifies to [21]

$$(A \mid B) = C_R \operatorname{Tr}(AB)\,, \tag{4.5}$$

where A and B are in some irreducible representation R of the Lie algebra and C_R is a constant depending on the representation.

4.3 Chevalley-Serre relations

In this section we will work with the field $\mathbb{C}$ and introduce what it actually means to restrict ourselves to the field $\mathbb{R}$.

By geometrically constructing all possible root systems satisfying certain properties directly derived from the theories of Lie algebras one can classify all simple (and thus also all semi-simple) Lie algebras as done in [19]. The geometry of the root system is encoded in a Cartan matrix A_{ij} which is defined by the different scalar products of the simple roots and we will now see how one can construct a general semi-simple Lie algebra from the Cartan matrix.

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ can be written in terms of the generators e, f and h

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.6}$$

as the sum of vector spaces $\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ with the commutation relations

$$[e, f] = h \qquad [h, e] = 2e \qquad [h, f] = -2f \tag{4.7}$$

In this basis the matrix representation consists of real (traceless) matrices, which means that it is possible to restrict ourselves to the so called real split form $\mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$.

To construct a general semi-simple Lie algebra of rank r (dimension of the Cartan subalgebra) we start by taking r copies of the $\mathfrak{sl}(2, \mathbb{C})$-generators (e_i, f_i, h_i), $i = 1 \dots r$ called a Chevalley basis and imposing the Chevalley commutations relations [6]

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j \\ [h_i, e_j] &= A_{ij} e_j \\ [h_i, f_j] &= -A_{ij} f_j \\ [h_i, h_j] &= 0 \end{aligned} \tag{4.8}$$

where A_{ij} is the Cartan matrix that completely defines the algebra we are constructing.

Together with the elements

$$\begin{aligned} &[e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{ik}] \cdots]] \\ &[f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{ik}] \cdots]] \end{aligned} \tag{4.9}$$

the above generators create an infinite-dimensional Lie algebra $\tilde{\mathfrak{g}}$ which can be written as the sum of vector spaces $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ where

$$\begin{aligned} \mathfrak{h} &= \sum_i \mathbb{C} h_i \\ \tilde{\mathfrak{n}}_+ &= \{[e_{i_1}, [e_{i_2}, \cdots, [e_{i_{k-1}}, e_{ik}] \cdots]]\} \\ \tilde{\mathfrak{n}}_- &= \{[f_{i_1}, [f_{i_2}, \cdots, [f_{i_{k-1}}, f_{ik}] \cdots]]\} \end{aligned} \tag{4.10}$$

This algebra is not semi-simple but has a maximal ideal $\mathfrak{i}$ which decomposes into the two ideals $\mathfrak{i}_-$ and $\mathfrak{i}_+$ for $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ respectively. The two ideals are generated by the elements in the two subsets [6]

$$\begin{aligned} S_+ &= \left\{ \left(\mathrm{ad}_{e_i} \right)^{1-A_{ij}} (e_j) \mid i \neq j \right\} \\ S_- &= \left\{ \left(\mathrm{ad}_{f_i} \right)^{1-A_{ij}} (f_j) \mid i \neq j \right\} \end{aligned} \tag{4.11}$$

where $\mathrm{ad}_X(Y) = [X, Y]$, $\left(\mathrm{ad}_X \right)^2 (Y) = [X, [X, Y]]$ etc.

The general semi-simple Lie algebra $\mathfrak{g}$, defined by the Cartan matrix A_{ij} is then [6]

$$\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{i} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \tag{4.12}$$

where $\mathfrak{n}_\pm := \tilde{\mathfrak{n}}_\pm / \mathfrak{i}_\pm$.

We will mostly consider algebras and their respective groups in their split real form which means that we only take real linear combinations of the elements above (similar to $\mathfrak{sl}(2, \mathbb{R})$) denoting the corresponding group as $G(\mathbb{R})$.

4.4 Cartan involution

The linear Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta^2 = \mathbb{1}$ is defined by how it acts on the Chevalley basis above with [6]

$$\theta(e_i) = -f_i \qquad \theta(f_i) = -e_i \qquad \theta(h_i) = -h_i \tag{4.13}$$

together with the property

$$\theta([X, Y]) = [\theta(X), \theta(Y)] \, . \tag{4.14}$$

The involution splits $\mathfrak{g}$ (as a vector space) into the eigenspaces [6]

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \theta(\mathfrak{k}) = \mathfrak{k} \qquad \theta(\mathfrak{p}) = -\mathfrak{p} \, . \tag{4.15}$$

where $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup K.

The commutation relations separate into

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \, , \tag{4.16}$$

which shows that only $\mathfrak{k}$ is a subalgebra. That $[\mathfrak{p}, \mathfrak{p}]$ does not contain elements of $\mathfrak{p}$ comes from the fact that G/K is a symmetric space. [6]

Using the Cartan involution, we will see that one can, starting with functions $g(x)$ taking values in the group G, construct objects that are globally invariant under the left-action of G itself

$$g(x) \mapsto hg(x) \qquad h \in G\,. \tag{4.17}$$

and that transform in a certain way under the local right-action of K (which we will call the gauge symmetry)

$$g(x) \mapsto g(x)k(x) \qquad k(x) \in K\,. \tag{4.18}$$

In section 5.2.1 we will use these objects to construct gauge invariant theories invariant under global G-transformations.

Let us consider the Maurer-Cartan form for elements $g(x) \in G$. Under a global (rigid) transformation $g(x) \mapsto hg(x)$ with $h \in G$ we have that

$$\omega_\mu \mapsto g^{-1}(x)h^{-1}\partial_\mu\Big(hg(x)\Big) = g^{-1}(x)\partial_\mu g(x) = \omega_\mu \tag{4.19}$$

but for a local (gauge) transformation $g(x) \mapsto g(x)k(x)$ with $k(x) \in K$

$$\omega_\mu \mapsto k^{-1}(x)g^{-1}(x)\partial_\mu\Big(g(x)k(x)\Big) = k^{-1}\partial_\mu k + k^{-1}\omega_\mu k \tag{4.20}$$

To see that the right hand side of (4.20) actually is in the Lie algebra $\mathfrak{g}$ we note that the first term is simply the Maurer-Cartan form of $k(x)$ and that the second can be rewritten using the Baker-Campbell-Hausdorff formula $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \ldots$ with $Y = \omega_\mu \in \mathfrak{g}$ and $k = e^{-X}$ where $X \in \mathfrak{k}$ and thus also lies in $\mathfrak{g}$. This is not the gauge transformation properties we are looking for though. We would like to find something that transforms as a gauge connection and as a field strength, which will enable us to use our knowledge of traditional gauge theories.

Since $\omega_\mu = g^{-1}(x)\partial_\mu\big(g(x)\big) \in \mathfrak{g}$ we can use the Cartan involution to decompose it into $\omega_\mu(x) = Q_\mu(x) + P_\mu(x)$ where $Q_\mu(x) \in \mathfrak{k}$ and $P_\mu(x) \in \mathfrak{p}$, that is, $\theta(Q_\mu) = Q_\mu$ and $\theta(P_\mu) = -P_\mu$. We get that

$$\begin{aligned} Q_\mu &= \frac{1}{2}\Big((Q_\mu + P_\mu) + (Q_\mu - P_\mu)\Big) = \frac{1}{2}\Big(\omega_\mu + \theta(\omega_\mu)\Big) \\ P_\mu &= \frac{1}{2}\Big((Q_\mu + P_\mu) - (Q_\mu - P_\mu)\Big) = \frac{1}{2}\Big(\omega_\mu - \theta(\omega_\mu)\Big) \end{aligned} \tag{4.21}$$

Since ω_μ is invariant under global left-transformation, so are also P_μ and Q_μ. Under a gauge transformation $g(x) \mapsto g(x)k(x)$ these transform as

$$\begin{aligned} Q_\mu \mapsto\ & \frac{1}{2}\Big(k^{-1}\partial_\mu k + k^{-1}\omega_\mu k + \theta(k^{-1}\partial_\mu k + k^{-1}\omega_\mu k)\Big) \\ &= \Big\{k(x) \in K \implies k^{-1}\partial_\mu k \in \mathfrak{k}\Big\} = \\ &= k^{-1}\partial_\mu k + \frac{1}{2}\Big(k^{-1}\omega_\mu k + \theta(k^{-1}\omega_\mu k)\Big) \\ &= k^{-1}\partial_\mu k + k^{-1}Q_\mu k\,, \end{aligned} \tag{4.22}$$

where we in the last step have used that $\theta(k^{-1}\omega_\mu k) = k^{-1}\theta(\omega_\mu)k$ which can be shown by splitting ω_μ into $Q_\mu + P_\mu$, using the Baker-Campbell-Hausdorff formula on each term and then applying the Cartan involution.

Similarly, under the same gauge transformation,

$$P_\mu \mapsto \frac{1}{2}\Big(k^{-1}\partial_\mu k + k^{-1}\omega_\mu k - \theta(k^{-1}\partial_\mu k + k^{-1}\omega_\mu k)\Big) = k^{-1}P_\mu k\,. \tag{4.23}$$

We note that Q_μ transforms as a gauge connection, while P_μ transforms covariantly, as a field strength [23] which are exactly the transformation properties we will need in the next chapter.

Chapter 5

Effective actions in string theory

The first part of this chapter will explain what it means that the type IIB supergravity action in 10 dimensions is an effective action of type IIB string theory. The remaining sections will be devoted to the description of the scalars in supergravity theories.

We saw in chapter 2 how the two scalar fields: the axion and the dilaton combined into one complex field living on the upper half plane or, equivalently, on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. When compactifying the ten-dimensional theory, something similar happens with other groups $G(\mathbb{R})$ as described in chapter 7. In the last sections of this chapter we will describe theories on such coset spaces.

5.1 Introduction to effective actions

In short, the tree-level amplitudes of an n-th order effective action are the same as the quantum corrected n-th order amplitude of the original action. This means that the effective action incorporates all the quantum corrections to a certain order of the theory in its classical couplings.

There are several different ways of computing the effective action and the methods differ between string theory and ordinary field theory. A universal approach is to plainly calculate the quantum corrected amplitudes using standard methods and create an interaction in the effective theory with a coupling constant corresponding to this amplitude.

The supergravity action in 10 dimensions of (2.3) is an effective action of type IIB string theory. The fields of the supergravity action enters as background fields in string theory and are not fluctuated. For example, the spacetime geometry that is described by string theory is a perturbative expansion around a fixed geometry that is decided beforehand.

Besides calculating scattering amplitudes for a given background, one can use the condition of conformal invariance on the worldsheet to obtain an effective action in string theory. At classical level, the invariance is readily checked, but the requirement that it should still hold at quantum level imposes conditions on the background fields. From these conditions one can reverse engineer an effective action whose equations of motion give exactly the required equations for the background fields.

In field theory, one can also find the effective action by directly using the partition function with classical sources $J(x)$ inserted. When adding sources, $J(x)$ is still a variable after doing the path integral and with different functions $J(x)$ one obtains different expectation values $\varphi(x) = \langle\phi(x)\rangle_J$. The partition function, which is the sum of the original exponentiated action over all possible fluctuations, incorporates the knowledge of all correlation functions. The effective action, as a functional of φ, can then be seen as the logarithm of the partition function [23].

5.2 Non-linear sigma models

In this section we will take a closer look at how the axion-dilaton part of the classical type IIB supergravity action was written in a manifestly $\mathrm{SL}(2,\mathbb{R})$-invariant way and also how this can be generalized to other groups $G(\mathbb{R})$.

Ultimately, we want to construct a theory in $\mathbb{R}^{1,9}$ where the fields are invariant under group transformations of a group G. This will be done in the next section, but let us first introduce some terminology.

A sigma model describes maps from a Riemannian base manifold X to another Riemannian manifold $\mathcal{M}$ called the target space. The space X of dimension D is equipped with a metric γ and coordinates x^μ, and $\mathcal{M}$ with a metric g and coordinates ϕ^α. In our case X is the Minkowski space $\mathbb{R}^{9,1}$ in which the supergravity theory lives, while $\mathcal{M}$ is the space $\mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ in which the axion-dilaton field τ takes its values.

The coordinates ϕ on the target space $\mathcal{M}$ are considered as functions of the base manifold X

$$\phi : X \to \mathcal{M} \qquad x \mapsto \phi(x) \tag{5.1}$$

The standard action for a sigma model is [6]

$$S = \int \mathrm{d}^D x \sqrt{\gamma}\gamma^{\mu\nu}(x)\partial_\mu\phi^\alpha(x)\partial_\nu\phi^\beta(x)g_{\alpha\beta}(\phi) \tag{5.2}$$

In string theory one considers a two dimensional base manifold, called the worldsheet, and the target space now as the Minkowski space.

The name sigma model is used for historical reasons. One say that the sigma model is linear if the metric $g_{\alpha\beta}$ is constant and non-linear otherwise.

5.2.1 Sigma models on coset spaces

To generalize the ten-dimensional case we will consider a target space of the form $\mathcal{M} = G/K$ where K is the maximal compact subgroup of the semi-simple group G and not only $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. We will construct sigma models that are invariant under global G-transformations using the tools introduced in chapter 4.

This will be accomplished by starting with fields $g(x)$ that take values on G as functions of the base manifold $X = \mathbb{R}^{9,1}$ and construct objects that are invariant under left-multiplication of a global element in G and under right-multiplication of a local (gauge) element $k(x)$ in K. The first requirement makes the theory G-invariant and the second that we effectively consider objects that take values in G/K and not only G.

The Maurer-Cartan form ω of g was, in section 4.4, shown to fulfill the first requirement, but not the second. We note however that the Cartan decomposition $\omega_\mu = Q_\mu + P_\mu$ provided us with objects that transformed in a promising way under gauge transformations.

Firstly, we recall that P_μ transformed just as a field strength.

$$g(x) \mapsto g(x)k(x) \qquad P_\mu \mapsto k^{-1}P_\mu k \tag{5.3}$$

Drawing inspiration from the standard Yang-Mills action with Lagrangian density $\mathcal{L} \propto \operatorname{Tr} F^2$ where F is the field strength, we consider $(P_\mu \mid P_\nu)$ which is invariant under both gauge transformations and global transformations since[1]

$$\begin{aligned}(P_\mu \mid P_\nu) \mapsto (k^{-1}P_\mu k \mid k^{-1}P_\nu k) = C_R \operatorname{Tr}(k^{-1}P_\mu k k^{-1} P_\nu k) = \\ = C_R \operatorname{Tr}(P_\mu P_\nu) = (P_\mu \mid P_\nu)\,,\end{aligned} \tag{5.4}$$

for some proportionality constant C_R depending on the choice of representation.

The full action for the sigma model on G/K has in this case the following form

$$S = \int \mathrm{d}^D x \sqrt{\gamma}\gamma^{\mu\nu}\big(P_\mu(x) \mid P_\nu(x)\big) \tag{5.5}$$

In the next section we will see how this is expressed on the familiar form of (5.2) for $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

We will now rewrite the action in a way that shows how Q_μ can be interpreted as a gauge connection.

The coefficients of the Maurer-Cartan form ω_μ are invariant under the global left-action of G as shown in (4.19) and using the Killing form we can also create a quadratic term that is invariant under a *global* right-action, that is, for a constant $K \ni k : g \mapsto gk$

$$\begin{aligned}(\omega_\mu \mid \omega_\nu) &= \big(g^{-1}\partial_\mu g \mid g^{-1}\partial_\nu g\big) \\ &\mapsto \big(k^{-1}g^{-1}(\partial_\mu g)k \mid k^{-1}g^{-1}(\partial_\nu g)k\big) \\ &= (\omega_\mu \mid \omega_\nu)\end{aligned} \tag{5.6}$$

To make a globally invariant expression also invariant under a local gauge transformation $K \ni k(x) : g \mapsto gk$, one usually introduces a covariant derivative $\partial_\mu \to D_\mu = \partial_\mu + Q_\mu$ where Q_μ is the gauge connection.

[1] This can be shown without going via the trace but instead using (4.4) directly.

Recalling that $\omega_\mu = g^{-1}\partial_\mu g = -(\partial_\mu g^{-1})g$ we note that with this definition of D_μ

$$\begin{aligned}(D_\mu g^{-1})g &= (\partial_\mu g^{-1})g + Q_\mu g^{-1}g = -g^{-1}\partial_\mu g + Q_\mu \\ &= -\omega_\mu + Q_\mu = -(Q_\mu + P_\mu) + Q_\mu = -P_\mu\end{aligned} \tag{5.7}$$

Thus, following the usual procedure for covariantization, we get that

$$(\omega_\mu \mid \omega_\nu) = \big((\partial_\mu g^{-1})g \mid (\partial_\nu g^{-1})g\big) \longrightarrow \big((D_\mu g^{-1})g \mid (D_\nu g^{-1})g\big) = (P_\mu \mid P_\nu) \tag{5.8}$$

The action is in this form

$$S = \int \mathrm{d}^D x \sqrt{\gamma}\gamma^{\mu\nu}\Big((D_\mu g^{-1})g \mid (D_\nu g^{-1})g\Big) \tag{5.9}$$

Finally, we will now show another form of the action which will prove to be very useful in actual calculations.

We define a generalized transpose from the Cartan involution above for $g = e^Y \in G$ with $Y \in \mathfrak{g}$ as

$$g^{\mathcal{T}} := \exp\big(-\theta(Y)\big) \tag{5.10}$$

This reduces to the ordinary transpose in the case of $\mathrm{SL}(2,\mathbb{R})$ where $\theta(e) = -f$, $\theta(f) = -e$ and $\theta(h) = -h$.

For $k = e^X \in K$ and $g = e^Y, h = e^Z \in G$ we have the following, familiar, properties for the generalized transpose

$$\begin{aligned}k^{\mathcal{T}} &= e^{-\theta(X)} = e^{-X} = k^{-1} \\ \big(g^{-1}\big)^{\mathcal{T}} &= \big(e^{-Y}\big)^{\mathcal{T}} = e^{\theta(Y)} = \big(e^{-\theta(Y)}\big)^{-1} = \big(g^{\mathcal{T}}\big)^{-1} \\ (gh)^{\mathcal{T}} &= h^{\mathcal{T}}g^{\mathcal{T}}\end{aligned} \tag{5.11}$$

where the last identity can be proven by using the Baker-Campbell-Hausdorff formula.

To write an action that is manifestly gauge invariant we construct the object

$$M(x) := g(x)g(x)^{\mathcal{T}} \in G\,, \tag{5.12}$$

which transforms under local and global transformations in the following manner

$$\begin{aligned}g(x) &\mapsto hg(x)k(x) \qquad h \in G, k(x) \in K \\ M &\mapsto (hgk)(hgk)^{\mathcal{T}} = hgkk^{-1}g^{\mathcal{T}}h^{\mathcal{T}} = hMh^{\mathcal{T}}\end{aligned} \tag{5.13}$$

We obtain a Lie algebra element by forming a Maurer-Cartan form of M and we now want to show that it is related to P_μ. To begin with, we have that

$$\begin{aligned}M^{-1}\partial_\mu M &= (gg^{\mathcal{T}})^{-1}\partial_\mu(gg^{\mathcal{T}}) = (g^{\mathcal{T}})^{-1}g^{-1}\Big((\partial_\mu g)g^{\mathcal{T}} + g\partial_\mu g^{\mathcal{T}}\Big) \\ &= (g^{\mathcal{T}})^{-1}\Big(\omega_\mu + (\partial_\mu g^{\mathcal{T}})(g^{\mathcal{T}})^{-1}\Big)g^{\mathcal{T}}\end{aligned} \tag{5.14}$$

We will rewrite the second term by comparing with ω_μ written in terms of $g(x) = \exp(Y(x))$. Using an integral expression for $\partial_\mu e^Y$ [24] together with the Baker-Campbell-Hausdorff formula one can show that

$$\omega_\mu = g^{-1}\partial_\mu g = e^{-Y}\partial_\mu e^Y = \partial_\mu Y - \frac{1}{2!}[Y, \partial_\mu Y] + \frac{1}{3!}[Y, [Y, \partial_\mu Y]] + \dots \tag{5.15}$$

Thus, for the second term

$$\begin{aligned}
(\partial_\mu g^{\mathcal{T}})(g^{\mathcal{T}})^{-1} &= -g^{\mathcal{T}}\Big(\partial_\mu (g^{\mathcal{T}})^{-1}\Big) = -e^{-\theta(Y)}\partial_\mu e^{\theta(Y)} \\
&= -\Big(\partial_\mu\theta(Y) - \frac{1}{2!}[\theta(Y), \partial_\mu\theta(Y)] + \frac{1}{3!}[\theta(Y), [\theta(Y), \partial_\mu\theta(Y)]] + \dots\Big) \\
&= -\theta\Big(\partial_\mu Y - \frac{1}{2!}[Y, \partial_\mu Y] + \frac{1}{3!}[Y, [Y, \partial_\mu Y]] + \dots\Big) = -\theta(\omega_\mu)
\end{aligned} \tag{5.16}$$

since $[\theta(A), \theta(B)] = \theta([A, B])$ and where one can show that $\partial_\mu\theta(Y) = \theta(\partial_\mu Y)$ by decomposing Y into $k_Y + p_Y$ with $k_Y \in \mathfrak{k}$ and $p_Y \in \mathfrak{p}$.

From (4.21) we then have that

$$M^{-1}\partial_\mu M = (g^{\mathcal{T}})^{-1}\Big(\omega_\mu - \theta(\omega_\mu)\Big)g^{\mathcal{T}} = 2(g^{\mathcal{T}})^{-1}P_\mu g^{\mathcal{T}} \tag{5.17}$$

The Killing form can now be used to isolate P_μ in a quadratic term, that is,

$$\frac{1}{4}(M^{-1}\partial_\mu M \mid M^{-1}\partial_\nu M) = ((g^{\mathcal{T}})^{-1}P_\mu g^{\mathcal{T}} \mid (g^{\mathcal{T}})^{-1}P_\nu g^{\mathcal{T}}) = (P_\mu \mid P_\nu) \tag{5.18}$$

The action is then

$$S = \frac{1}{4}\int \mathrm{d}^D x\sqrt{\gamma}\gamma^{\mu\nu}(M^{-1}\partial_\mu M \mid M^{-1}\partial_\nu M) \tag{5.19}$$

To summarize, we have now shown that the gauge invariant action can be written in three different forms using

$$(P_\mu \mid P_\nu) = \Big((D_\mu g^{-1})g \mid (D_\nu g^{-1})g\Big) = \frac{1}{4}(M^{-1}\partial_\mu M \mid M^{-1}\partial_\nu M) \tag{5.20}$$

5.2.2 The $\mathrm{SL}(2,\mathbb{R})$ sigma model

In the case of $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ as discussed in chapter 2 we have the Iwasawa decomposition for an element $g \in G$

$$g = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-\phi/2} & 0 \\ 0 & e^{\phi/2} \end{pmatrix} k\,, \tag{5.21}$$

where $k \in \mathrm{SO}(2)$. Note that $\tau = \chi + ie^{-\phi} = g(i)$ where i is the imaginary unit.

Then,

$$M = gg^{\mathcal{T}} = gg^T = \begin{pmatrix} e^{-\phi} + e^{\phi}\chi^2 & e^{\phi}\chi \\ e^{\phi}\chi & e^{\phi} \end{pmatrix} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} |\tau|^2 & \operatorname{Re}\tau \\ \operatorname{Re}\tau & 1 \end{pmatrix}\,, \tag{5.22}$$

which is exactly the M used in (2.9)

Using the construction in the above section, we can then find an action on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ with

$$\begin{aligned}\mathcal{L}_{\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)} &= -\frac{1}{4}(M^{-1}\partial_\mu M \mid M^{-1}\partial^\mu M) \propto -\frac{1}{4}\operatorname{Tr}(M^{-1}\partial_\mu M M^{-1}\partial^\mu M) \\ &= -\frac{1}{2}\big(\partial_\mu\phi\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\partial^\mu\chi\big)\end{aligned} \tag{5.23}$$

This is exactly the scalar part of the action (2.10) where

$$\mathcal{L}_{\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)} = -\frac{1}{4}\operatorname{Tr}(M^{-1}\partial_\mu M M^{-1}\partial^\mu M) = \frac{1}{4}\operatorname{Tr}(\partial_\mu M\partial^\mu M^{-1}) \tag{5.24}$$

Another frequently used form of the action is the one expressed only in τ (see for example [11])

$$\mathcal{L}_{\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)} = -\frac{1}{2(\operatorname{Im}\tau)^2}\partial_\mu\bar{\tau}\partial^\mu\tau \tag{5.25}$$

Chapter 6

Instantons

In this chapter we will first introduce instantons in ordinary field theories and describe how they contribute to an effective action. We then find the instantons solutions of the type IIB supergravity in ten dimensions, but since this is only a first order effective action from string theory we cannot treat it as a complete quantum theory; the field theory methods for finding the non-perturbative corrections to the effective action do not work. Instead we turn to the string theory description of instantons using D-branes and briefly discuss how to calculate their contributions to the effective action. We will find that the overall form of the corrections matches our anticipations from the field theory discussion.

6.1 Instantons in field theory

Instantons are solutions to the Euclidean equations of motion with a finite classical action [25]. The former means that they extremize the Euclidean action and thus also important for a saddle point approximation of the partition function.

Because the action is finite, the instanton solution cannot be invariant under translations in the Euclidean spacetime [26]. The finiteness also requires that the Lagrangian density vanishes at infinity meaning that the field is pure gauge at infinity or, in the simplest case, simply vanishes. In this sense the instanton is localized in both space and time giving rise to its name. This is consistent with string theory where instantons are intimately related to objects that are localized to points in spacetime as will be discussed later in this chapter.

After finding the Euclidean classical solution the quantum fluctuations are then done around this instanton background. For a field ϕ with ϕ_{cl} a solution to the equations of motions we have

$$\phi = \phi_{\mathrm{cl}} + \phi_{\mathrm{qu}} \tag{6.1}$$

with ϕ_{qu} taking the role of the usual field fluctuations in the path integral.

Expanding the action around the instanton background ϕ_{cl} in the quantum field ϕ_{qu} we get

$$S[\phi] = S_{\mathrm{inst}}[\phi_{\mathrm{cl}}] + \frac{1}{2}\phi_{\mathrm{qu}}^2 \left.\frac{\partial^2 S}{\partial\phi\partial\phi}\right|_{\phi=\phi_{\mathrm{cl}}} + \dots \tag{6.2}$$

where the linear term is zero since ϕ_{cl} satisfies the equations of motions.

This gives corrections to the Feynman propagators and vertices that are called instanton corrections. The field configurations of the instantons are very different from the vacuum of the theory, that is, the various instanton backgrounds cannot be reached from the vacuum by a series of small perturbations. For example, in gauge theory the instanton solutions are topologically different from the vacuum [26].

The instanton corrections are therefore non-perturbative meaning that they have vanishing expansions in, for example, the coupling constants.

Let us now examine how the inclusions of instantons corrects the effective action on a (very) schematic level. A more detailed discussion can be found in [26] for the quantum field picture.

From section 5.1 we know that the effective action is roughly the logarithm of the partition function

$$Z = \int \mathcal{D}[\phi] e^{-S[\phi]}\,, \tag{6.3}$$

in the Euclidean spacetime.

We see that the main contributions should indeed come from the finite minima of $S[\phi]$ (the saddle point approximation) and that we should sum over the possible instanton configurations. Note that we still have the ordinary perturbative corrections corresponding to an empty background without instantons. For a given instanton background the contribution to the partition function has the form [27]

$$Z_{\mathrm{inst}} \underset{\sim}{\propto} e^{-S_{\mathrm{inst}}} \left[\det \left(\frac{\partial^2 S}{\partial\phi\partial\phi} \right) \right]^{-1/2} \tag{6.4}$$

Let us consider a theory where we have a set of different instanton solutions labeled by a number N giving different values for the action $S_N := S_{\mathrm{inst}}^{(N)}$. We know that if we take an instanton solution to the equations of motion and translate it in spacetime we find a new solution with the same value for the action. This is because the *theory* is translation invariant, but the *instanton solutions* are not.

Because the instantons are localized we can then construct solutions by superimposing m solutions centered at well-separated points. The instanton term of the action ($S_{\mathrm{inst}}^{\mathrm{total}}$) in such a configuration is the sum over the individual instantons actions S_N.

Consider first individual instantons of a specific charge $N = 1$ with partition function contributions $a_1 e^{-S_1}$ where a_1 denotes the determinant and the rest of the quantum fluctuations in (6.4). The contribution for all possible configurations of charge $N = 1$ instantons is then the sum [26]

$$Z = \sum_m \frac{1}{m!} a_1^m e^{-mS_1} = \exp\left(a_1 e^{-S_1}\right) \tag{6.5}$$

where the m-factorial is needed as a symmetry factor for interchanging the identical instantons and where each instanton gives quantum fluctuations a_1.

With m_1 instantons of charge $N = 1$ and m_2 instantons of charge $N = 2$ we have a total action $S_{\mathrm{inst}}^{\mathrm{total}} = m_1 S_1 + m_2 S_2$. Summing over all possible superpositions we get the following contribution to the partition function (which includes the above for $m_2 = 0$)

$$Z = \sum_{m_1,m_2} \frac{a_1^{m_1} a_2^{m_2}}{m_1! m_2!} e^{-(m_1 S_1 + m_2 S_2)} = \exp\left(a_1 e^{-S_1}\right) \exp\left(a_2 e^{-S_2}\right) = \exp\left(a_1 e^{-S_1} + a_2 e^{-S_2}\right) \tag{6.6}$$

From these arguments we can see that the instanton contributions to the effective action should be of the form

$$S_{\text{eff}}^{\text{non-pert}} = \sum_N c_N e^{-S_N} \tag{6.7}$$

for some coefficients c_N that has to be calculated more carefully than above. In section 9.3 we will see in detail how this structure emerges from the Fourier expansion of the $\mathcal{R}^4$ coefficient.

6.2 Instantons in string theory

Now that we have had some introduction to instantons in field theory we will first find out if the supergravity action has any instanton solutions. But realizing that we cannot compute the instanton corrections to this action using only the tree-level effective action itself we need to find the corresponding description in string theory (which is the UV-completion of the supergravity theory) and find a method to calculate the contributions in this framework.

6.2.1 Instanton solutions in type IIB supergravity

In the previous chapter we obtained the action for the scalar fields in the type IIB SUGRA. Together with the curvature we have the action (in Lorentzian signature)

$$S = \int \mathrm{d}^{10}x \sqrt{-G} \Big(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} e^{2\phi} (\partial\chi)^2 \Big) \tag{6.8}$$

Upon Wick rotation with

$$\tau = it \qquad e^{iS} = e^{-S_E} \tag{6.9}$$

the axion field, being a pseudo scalar, transforms as $\chi \to i\chi$ [28] giving the Euclidean action [29, 30]

$$S_E = \int_{\mathcal{M}} \mathrm{d}^{10}x \sqrt{G} \Big(-R + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} e^{2\phi} (\partial\chi)^2 \Big) \tag{6.10}$$

with $\mathcal{M} = \mathbb{R}^{10}$.

The equations of motion are

$$\begin{aligned} g_{\mu\nu}: &\qquad R_{\mu\nu} - \frac{1}{2}(\partial_\mu\phi\partial_\nu\phi - e^{2\phi}\partial_\mu\chi\partial_\nu\phi) = 0 \\ \phi: &\qquad \nabla^2\phi + e^{2\phi}(\partial\chi)^2 = 0 \\ \chi: &\qquad \nabla_\mu(e^{2\phi}\partial^\mu\chi) = 0 \end{aligned} \tag{6.11}$$

We see that for a flat solution $g_{\mu\nu} = \delta_{\mu\nu}$ the trace of the first equation of motion requires

$$\partial_\mu \chi = \pm e^{-\phi} \partial_\mu \phi \tag{6.12}$$

where the upper and lower signs will correspond to instanton and anti-instanton solutions respectively [29].

This condition can also be seen as an ansatz that leads to the preservation of half the supersymmetries [30] (which agrees with the string picture of the instanton described below). Another view is to consider the dual theory, with a 9-form F_9 dual to $\mathrm{d}\chi$ in which (6.12) is the Bogomol'nyi bound saturation condition [29].

The remaining boundary conditions then reduce to

$$\begin{aligned} \phi: &\qquad \partial^2\phi + e^{2\phi}(\partial\chi)^2 = \partial^2\phi + (\partial\phi)^2 = e^{-\phi}\partial^2 e^{\phi} = 0 \\ \chi: &\qquad \partial_\mu(e^{2\phi}\partial^\mu\chi) = \pm\partial_\mu(e^{\phi}\partial^\mu\phi) = \pm\partial^2 e^{\phi} = 0 \end{aligned} \tag{6.13}$$

Letting ϕ be spherically symmetric we find the following general solution

$$e^{\phi} = e^{\phi_\infty} + \frac{c}{r^8} \tag{6.14}$$

where $\phi_\infty = \phi(r = \infty)$ with the string coupling being $g_s = e^{\phi_\infty}$ and c is a positive constant whose interpretation we will examine shortly. We note that the solution is localized at the origin in the Euclidean spacetime with a steep falloff.

From (6.12) we then have

$$\chi - \chi_\infty = \mp(e^{-\phi} - e^{-\phi_\infty}) \tag{6.15}$$

The action (6.10) has a symmetry $\chi \to \chi + b$ (with b a constant) that will be of interest to us. The Noether current and corresponding charge are [30]

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)} = e^{2\phi}\partial^\mu\chi = \pm e^{\phi}\partial^\mu\phi = \pm\partial^\mu e^{\phi} \\ Q &= \int\limits_{\partial\mathcal{M}=\mathrm{S}^9} J^\mu \mathrm{d}\Sigma_\mu \end{aligned} \tag{6.16}$$

where $\mathrm{d}\Sigma_\mu$ is a surface element normal to the sphere (pointing outwards). Letting the hypersurface be parametrized by u^i, $i = 1, \ldots, 9$ with the embedding $x^\mu(u^i)$ in spacetime the explicit expression is [31]

$$\mathrm{d}\Sigma_\mu = \frac{\sqrt{G}}{9!}\epsilon_{\mu\mu_1\cdots\mu_9}\frac{\partial(x^{\mu_1}, \ldots, x^{\mu_9})}{\partial(u^1, \ldots, u^9)}\,\mathrm{d}u^1\cdots\mathrm{d}u^9 \tag{6.17}$$

Choosing $x^0 = r$ and $x^i = u^i$ we get that

$$\begin{aligned} Q &= \int_{\mathrm{S}^9} \frac{\sqrt{G}}{9!} J^r \cdot 9! \cdot 1\,\mathrm{d}u^1\cdots\mathrm{d}u^9 \\ &= \int_{\mathrm{S}^9} J^r r^9 \mathrm{d}V_{\mathrm{S}^9} = \int_{\mathrm{S}^9} \Big(\mp\frac{8c}{r^9}\Big) r^9\,\mathrm{d}V_{\mathrm{S}^9} \\ &= \mp 8c\mathrm{Vol}(\mathrm{S}^9) \end{aligned} \tag{6.18}$$

Thus the constant of integration is related to the instanton charge Q. Note that Q is conserved in the meaning that it is invariant under deformation of the hypersurface above as long as we do not cross the origin where we will soon show that J^μ actually is not divergenceless.

We now want to find the value of the instanton action for a given c, ϕ_∞ and χ_∞ which would appear in the non-perturbative corrections to the effective action. Strangely though, we immediately notice that the action (6.10) (which we henceforth will call the bulk action) is zero because of the Bogomol'nyi bound (6.12).

Another sign that we have not quite taken everything into account can be seen by inserting our solution (6.14, 6.15) into the equations of motion (6.13)[1]

$$\partial^2 e^\phi = \frac{1}{r^9}\partial_r(r^9\partial_r e^\phi) = \frac{1}{r^9}\partial_r\left(r^9\partial_r\frac{c}{r^8}\right) = A\delta^{(10)}(x) \tag{6.19}$$

where

$$A = \int_{\mathcal{M}} \mathrm{d}^{10}x\, \partial^2 e^\phi = \int_{\partial\mathcal{M}} \partial^\mu e^\phi\, \mathrm{d}\Sigma_\mu = -8c\mathrm{Vol}(\mathrm{S}^9) = -|Q| \tag{6.20}$$

For this change of the equations of motion to be reflected in the action we need to add the source term

$$\begin{aligned} S &= S_{\text{bulk}} + S_{\text{source}} \\ &= \int_{\mathcal{M}} \mathrm{d}^{10}x\Big(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\Big) + \int_{\mathcal{M}} \mathrm{d}^{10}x\, |Q|\delta^{(10)}(x)\big(e^{-\phi}\pm\chi\big)\,. \end{aligned} \tag{6.21}$$

This gives the wanted equations of motion with a point-like source at the origin

$$\begin{aligned} \phi: &\qquad e^{-\phi}\partial^2 e^\phi = -|Q|e^{-\phi}\delta^{(10)}(x) \\ \chi: &\qquad \pm\,\partial^2 e^\phi = \mp|Q|\delta^{(10)}(x) \end{aligned} \tag{6.22}$$

There is another term that we have to add to the action before we are done and that is a boundary term (which does not affect the equations of motion) to restore the $\chi \to \chi + b$ symmetry to the new action in (6.21). It is also the same term that appears when starting from a theory with field strength $F^{(9)}$ and dualizing it to the axion action as seen in [29]. The result is

$$\begin{gathered} S = S_{\text{bulk}} + S_{\text{source}} + S_{\text{boundary}} \\ S_{\text{boundary}} = \int_{\mathcal{M}} \mathrm{d}^{10}x\, \partial_\mu\big(\chi e^{2\phi}\partial^\mu\chi\big) \end{gathered} \tag{6.23}$$

Then, under $\chi \to \chi + b$, where b is constant, we have

$$\begin{gathered} \delta S_{\text{bulk}} = 0 \qquad \delta S_{\text{source}} = \pm\int \mathrm{d}^{10}x\, |Q|\delta^{(10)}(x)b = \pm|Q|b \\ \delta S_{\text{boundary}} = \int \mathrm{d}^{10}x\, \partial_\mu\big(be^{2\phi}\partial^\mu\chi\big) \overset{(6.13)}{=} \pm b\int \mathrm{d}^{10}x\, \partial^2 e^\phi \overset{(6.22)}{=} \mp|Q|b \end{gathered} \tag{6.24}$$

[1]Note that this is also the divergence of the Noether current J^μ.

and, as wanted, $\delta S = 0$.

To summarize, the full action is

$$
\begin{aligned}
S = \int_{\mathcal{M}} \mathrm{d}^{10}x \, \Big(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\Big) + \int_{\mathcal{M}} \mathrm{d}^{10}x \, |Q|\delta^{(10)}(x)\big(e^{-\phi} \pm \chi\big) + \\
+ \int_{\mathcal{M}} \mathrm{d}^{10}x \, \partial_\mu\big(\chi e^{2\phi}\partial^\mu\chi\big)
\end{aligned}
\tag{6.25}
$$

Inserting our instanton solution (6.14, 6.15) to find the value of the action we get that

$$
\begin{aligned}
S_{\text{instanton}} &= 0 + \int_{\mathcal{M}} \mathrm{d}^{10}x \, |Q|\delta^{(10)}(x)\big(e^{-\phi} \pm \chi\big) + \int_{\mathcal{M}} \mathrm{d}^{10}x \, \partial_\mu\big(\chi e^{2\phi}\partial^\mu\chi\big) \\
&= \int_{\mathcal{M}} \mathrm{d}^{10}x \, |Q|\delta^{(10)}(x)\big(e^{-\phi} \pm \chi\big) + \int_{\mathcal{M}} \mathrm{d}^{10}x \, \Big[(\partial_\mu\chi)e^{2\phi}\partial^\mu\chi + \chi\partial_\mu(e^{2\phi}\partial^\mu\chi)\Big]
\end{aligned}
\tag{6.26}
$$

where $(\partial_\mu\chi)e^{2\phi}\partial^\mu\chi = e^{2\phi}(\partial\chi)^2 = (\partial\phi)^2$ and $\partial_\mu(e^{2\phi}\partial^\mu\chi) = \pm\partial^2 e^\phi = \mp|Q|\delta^{(10)}(x)$ which gives

$$
\begin{aligned}
S_{\text{instanton}} &= \int_{\mathcal{M}} \mathrm{d}^{10}x \, |Q|\delta^{(10)}(x)\big(e^{-\phi} \pm \chi\big) + \int_{\mathcal{M}} \mathrm{d}^{10}x \, \Big[(\partial\phi)^2 \mp \chi|Q|\delta^{(10)}(x)\Big] \\
&= \int_{\mathcal{M}} \mathrm{d}^{10}x \, |Q|\delta^{(10)}(x)e^{-\phi} + \int_{\mathcal{M}} \mathrm{d}^{10}x \, (\partial\phi)^2 \\
&= \int_{\mathcal{M}} \mathrm{d}^{10}x \, |Q|\delta^{(10)}(x)\frac{1}{e^{\phi_\infty} + \frac{c}{r^8}} + \int_{\mathcal{M}} \mathrm{d}^{10}x \, (\partial\phi)^2 \\
&= \int_{\mathcal{M}} \mathrm{d}^{10}x \, (\partial\phi)^2
\end{aligned}
\tag{6.27}
$$

Note the cancellation between source and boundary terms and that the remaining integral comes from the boundary term. We have that

$$
(\partial\phi)^2 = (e^{-\phi}\partial e^\phi)^2 = e^{-2\phi}(\partial_r e^\phi)^2 = e^{-2\phi}\left(-\frac{8c}{r^8}\right)^2 = \frac{1}{\big(e^{\phi_\infty} + \frac{c}{r^8}\big)^2}\frac{64c^2}{r^{18}}
\tag{6.28}
$$

which gives

$$
\begin{aligned}
S_{\text{instanton}} &= \int_{\mathcal{M}} \mathrm{d}^{10}x \, (\partial\phi)^2 = \int r^9 \, \mathrm{d}r \, \mathrm{d}V_{\mathrm{S}^9} \, (\partial\phi)^2 \\
&= 64c^2 \mathrm{Vol}(\mathrm{S}^9) \int \mathrm{d}r \, \frac{1}{r^9}\frac{1}{\big(e^{\phi_\infty} + \frac{c}{r^8}\big)^2} \\
&= 8c\mathrm{Vol}(\mathrm{S}^9)e^{-\phi_\infty} = |Q|e^{-\phi_\infty} = \frac{|Q|}{g_s}
\end{aligned}
\tag{6.29}
$$

As discussed in [29] there is also an imaginary contribution to $S_{\text{instanton}}$ in (6.29) resembling the θ-angle contributions for instantons in Yang-Mills theories. Adding this contribution is equivalent as adding another boundary term [4] which does not change the equations of motions and the classical instanton solution for ϕ and χ. In total, the actions for instantons and anti-instantons are [29, 4]

$$S_{\text{instanton}} = -i|Q|\tau_\infty \qquad S_{\text{anti-instanton}} = i|Q|\bar{\tau}_\infty \tag{6.30}$$

where $\tau_\infty = \chi_\infty + ie^{-\phi_\infty}$ and χ is the axion of the Lorentzian theory and not the Wick rotated one [29].

One can argue that Q becomes quantized by $Q = 2\pi n$ with $n \in \mathbb{Z}$ in the presence of seven-branes using the Dirac-Nepomechie-Teitelboim argument [30, 4]. We will instead use the string picture to see how the instanton is constructed from string theory and in this way get another interpretation of the instanton charge Q which will tell us that it has to be quantized.

6.2.2 D-instantons

In short, D-branes are extended objects in string theory on which open strings can end with Dirichlet boundary condition for the directions transversal to the brane and with Neumann boundary condition for the directions in the brane. The branes are often labeled by their number of space-like dimensions p, as in Dp-branes. An interesting property of (stable) D-branes is that they break half the supersymmetry [10].

These objects are connected to the classical solutions in supergravity theories that generalizes a black hole to p dimensions; called black p-branes. It is argued in [32] that the two are alternative representations of the same thing. In our case we are interested in D-instantons which are point-like in the Euclidean spacetime and thus labeled by $p = -1$ having Dirichlet boundary conditions in all directions.

Following [4, 6], we start with a type IIA string theory in nine dimensions where the Euclidean time, X^0, has been compactified on a circle of radius R. By T-duality we can relate the type IIA theory with type IIB mapping D-particles (D0-branes) into D-instantons (D(-1)-branes) where the world line of the particle is localized to a single point in spacetime.

The action for such a D-particle with n units of momentum along the circle and coupled to a one-form, $C^{(1)} = C_0^{(1)}\mathrm{d}X^0$, in the IIA theory is [4]

$$S = \int_0^{2\pi} \mathrm{d}t \left((\alpha')^{-\frac{1}{2}} |n| e^{-\phi} \sqrt{|g_{00}|} - inC_0^{(1)} \frac{\mathrm{d}X^0}{\mathrm{d}t} \right) \tag{6.31}$$

where

$$g_{00} = \frac{\mathrm{d}X^0}{\mathrm{d}t}\frac{\mathrm{d}X^0}{\mathrm{d}t} G_{00} = R^2 \left(\frac{\mathrm{d}X^0}{\mathrm{d}t} \right)^2 \tag{6.32}$$

If the wordline of the particle has a wrapping number m around the circle we have that

$$\int \mathrm{d}t\, \frac{\mathrm{d}X^0}{\mathrm{d}t} = 2\pi m, \qquad \int \mathrm{d}t\, \sqrt{|g_{00}|} = 2\pi |m| R \tag{6.33}$$

Keeping $C_0^{(1)}$ and $e^{-\phi}$ fixed, the value of the action is then

$$S = 2\pi |mn| \frac{R}{(\alpha')^{\frac{1}{2}}} e^{-\phi} - 2\pi imnC_0^{(1)} \tag{6.34}$$

which under T-duality, where $R(\alpha')^{-\frac{1}{2}}e^{-\phi} \to e^{-\phi}$ and $C_0^{(1)} \to C^{(0)} = \chi$ [4], becomes

$$S = -2\pi i(mn\chi + i|mn|e^{-\phi}) = \begin{cases} -2\pi i|mn|\tau & mn > 0, \text{ instanton} \\ 2\pi i|mn|\bar{\tau} & mn < 0, \text{ anti-instanton} \end{cases} \tag{6.35}$$

In the T-dual picture the type IIB string theory is compactified on a circle of radius α'/R and taking $R \to 0$ we are back in a ten dimensional, Euclidean spacetime.

Comparing with (6.30) we see that

$$|Q| = 2\pi|mn| \tag{6.36}$$

and must thus be quantized into $Q \in 2\pi\mathbb{Z}$. Notice also how Q is basically a factor of the two integers m and n which will be important when counting the number of instanton states for a given charge.

Let us now see how the stringy instantons contribute to the effective action, or rather how they contribute to scattering amplitudes. By looking at the perturbative expansion in g_s it was argued by Shenker in [14] that non-perturbative corrections on the form $\exp\big(-\mathcal{O}(1/g_s)\big)$ had to enter to remedy a divergent perturbation series.

Later in [7], Polchinski found that stringy non-perturbative effects arise from boundaries in the string worldsheet with Dirichlet boundary conditions. He found that the leading D-instanton contributions come from disconnected worldsheet disks with boundaries at a single spacetime point (which is later integrated over) and that the sum over such disks (together with a symmetry factor of $1/m!$) exponentiates giving the expected $\exp\big(-\mathcal{O}(1/g_s)\big)$. See also [33] for a detailed explanation.

In [4] Green and Gutperle calculates the single D(-1)-brane instanton correction to the $\mathcal{R}^4$ term in the effective supergravity action for a charge $N = 1$ instanton and finds that it is of the form $\exp(-S_1)$ with the shorthand notation $S_N = -2\pi i|N|\tau$. Arguing that the coefficient to the $\mathcal{R}^4$ term should be $\mathrm{SL}(2,\mathbb{Z})$ invariant (as discussed in the next chapter) they find that the contributions should be on the form

$$\sum_N a_N e^{-S_N}\Big(1 + \mathcal{O}(g_s)\Big) \tag{6.37}$$

which is expected from the field theory point of view and will be shown in section 9.3. Here the term $\mathcal{O}(g_s)$ comes from loop corrections in the D-instanton background. We note that this really gives us non-perturbative corrections since $S_N = -\mathcal{O}(1/g_s)$.

Chapter 7

Toroidal compactifications and U-dualities

In chapter 2 we showed that the classical type IIB supergravity theory was invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations of the two scalar fields ϕ and χ described as the complex field $\tau \in \mathbb{H} \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. When compactifying the ten-dimensional theory on a d-torus to $D = 10 - d$ dimensions we get more scalars and larger classical symmetries G as shown in table 7.1 [5].

These massless scalar fields, called moduli labeling the possible vacua in a theory, can be represented by a parameter in the classical moduli space G/K where K is the maximal compact subgroup of G. In section 5.2.1 we learned how one can construct an action on G/K that is manifestly invariant under left-multiplication of G. When going to the quantum theory, the classical symmetries G are conjectured to break to the discrete symmetries $G(\mathbb{Z})$ also shown in table 7.1. The quantum moduli space which label the different vacua is then $G(\mathbb{Z})\backslash G/K$.

The discrete symmetries $G(\mathbb{Z})$ are called U-dualities in string theory; dualities because they relate different string theories and U from unifying S- and T-dualities. For more information about U-dualities see [34].

The conjecture that the quantum theory breaks to $G(\mathbb{Z})$ has its origin in charge dualities in four dimensions which will be discussed below. There it is argued that the E_7 symmetry that is obtained when compactifying the type IIB supergravity in ten dimensions on a six-dimensional torus should be broken to $\mathrm{E}_7(\mathbb{Z})$. Since the fields parametrized by $\tau \in \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ in ten dimensions still are present after compactifying to four dimensions this should also tell us how the $\mathrm{SL}(2,\mathbb{R})$ symmetry is broken.

As we saw in chapter 2 this conjecture allows us to find the full coefficient of the $\mathcal{R}^4$ term in the ten-dimensional effective action. Several checks have been carried out to verify this result of the conjecture (see for example [4, 13]) as will be discussed in chapter 10 where we will also mention results for other dimensions and coefficients to other α' corrections.

Table 7.1: Moduli space symmetry groups G with their maximally compact groups K and discrete subgroups $G(\mathbb{Z})$ for type IIB SUGRA compactified on a $(10-D)$-dimensional torus to D dimensions. See [35, 1, 34] with results summarized in [5].

D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$\big(\mathrm{Spin}(5)\times\mathrm{Spin}(5)\big)/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

7.1 Charge duality in four dimensions

Following [1], we will now consider the bosonic part of a general four dimensional supergravity theory. All such actions without non-Abelian gauge fields and scalar potentials can be put on the following form

$$S=\int \mathrm{d}^4x\sqrt{-g}\Big(\frac{1}{4}R-\frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j-\frac{1}{4}m_{IJ}(\phi)F^{I,\mu\nu}F^J_{\mu\nu}-\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}a_{IJ}(\phi)F^I_{\mu\nu}F^J_{\rho\sigma}\Big) \tag{7.1}$$

where $\phi:\mathbb{R}^{3,1}\to\mathcal{M}=G/K$ and A^I_μ with $I=1,\dots,k$ are k Abelian gauge fields with field strengths $F^I_{\mu\nu}$.

In [1] they treat a more general set of supergravities in four dimensions, but here we are interested in the case of $\mathcal{N}=8$ with maximal supersymmetry since the number of supercharges are unchanged under torus compactifications [11]. Then we have that $G/K=\mathrm{E}_7(\mathbb{R})/\big(\mathrm{SU}(8)/\mathbb{Z}_2\big)$ as seen in table 7.1 and the number of vector bosons: $k=28$.

Let us define

$$G_{I,\mu\nu}=m_{IJ}*F^J_{\mu\nu}+a_{IJ}F^J_{\mu\nu} \tag{7.2}$$

where $*$ is the Hodge star operator.

We collect the field strengths F^I and G_I into a vector of length $2k=56$ transforming under the fundamental representation of $G=\mathrm{E}_7(\mathbb{R})$.

$$\mathcal{F}=\begin{pmatrix}F^I\\ G_I\end{pmatrix}. \tag{7.3}$$

The corresponding charges are found to be

$$\mathcal{Z} = \begin{pmatrix} p^I \\ q_I \end{pmatrix} = \oint_\Sigma \mathcal{F} \tag{7.4}$$

where p^I are the magnetic charges and q_I the Noether electric charges [1]. The surface of integration Σ is a two-sphere at spatial infinity.

The classical symmetry G is realized by the isometries on $\mathcal{M}$ for the scalars and by $\mathcal{F} \to \Lambda\mathcal{F}$ for the gauge fields where $\Lambda \in G$. Quantum mechanically though, we have seen in chapter 2 that the classical symmetry is broken.

What symmetry is left must conform to the quantization of charges. For one thing, it needs to leave the Dirac-Schwinger-Zwanziger quantization condition for two charges $\mathcal{Z}$ and $\mathcal{Z}'$ invariant [1]

$$\mathcal{Z}^T \Omega Z' := p^I q'_I - p'^I q_I = n \in \mathbb{Z} \tag{7.5}$$

where Ω is the $2k \times 2k$-matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{7.6}$$

This is seen to be the case since under a classical G-transformation,

$$\begin{aligned} &\mathcal{Z} \to \Lambda\mathcal{Z} \\ &\mathcal{Z}^T \Omega \mathcal{Z}' \to \mathcal{Z}^T \Lambda^T \Omega \Lambda \mathcal{Z}' = \mathcal{Z}^T \Omega \mathcal{Z}' \implies \Lambda \in \mathrm{Sp}(2k, \mathbb{R}) \,, \end{aligned} \tag{7.7}$$

but since $G \subseteq \mathrm{Sp}(2k, \mathbb{R})$ the quantization is trivially left invariant [1] and does not give us any extra information of the remaining symmetry.

If we assume that all types of charges exist the quantization condition implies that q_I must be in some discrete lattice $\Gamma \cong \mathbb{Z}^k$ and p_I in the dual lattice $\tilde{\Gamma}$ [1].

The remaining symmetry must then transform a vector $\mathcal{Z} \in \Gamma \oplus \tilde{\Gamma} \cong \mathbb{Z}^{2k}$ to another vector in the same discrete lattice. The subgroup of $\mathrm{Sp}(2k, \mathbb{R})$ that leaves the lattice invariant is $\mathrm{Sp}(2k, \mathbb{Z})$ which means that $G(\mathbb{Z}) \subseteq G \cap \mathrm{Sp}(2k, \mathbb{Z})$.

We conjecture that this is exactly the remaining symmetry $G(\mathbb{Z}) = G \cap \mathrm{Sp}(2k, \mathbb{Z})$, although it could in general have been broken even further.

Chapter 8

Automorphic forms

From the previous chapter we understand why the terms in the effective action should be functions on G/K invariant under $G(\mathbb{Z})$ transformations. Let us now discuss the theory of such functions and how to obtain physical information from them.

There are some slight differences in the literature when it comes to defining the term automorphic form. We are interested in automorphic forms for groups of arbitrary rank which are invariant under the action of $G(\mathbb{Z})$. In the case of $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}$, this means that we will focus on the non-holomorphic Eisenstein series instead of the holomorphic ones.

Let us try to sort out the terminology. For $G(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$, which is the most studied case, automorphic forms include the two notions: modular forms and Maass forms [36].

Modular forms of weight $2k \geq 0$ with $k \in \mathbb{N}$ are holomorphic functions $f(z)$ on $\mathbb{H}$ (and at ∞) that transform under $\gamma \in G(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$ with a weight factor $f\big(\gamma(z)\big) = (cz+d)^{2k} f(z)$ [36]. An example of a modular form is the holomorphic Eisenstein series described below.

Maass forms are smooth functions on $\mathbb{H}$ which are invariant under $G(\mathbb{Z})$ transformations, that is, $f\big(\gamma(z)\big) = f(z)$ [36]. Instead of being holomorphic they are eigenfunctions to the Laplacian $\Delta = y^2(\partial_y^2 + \partial_x^2)$ for $z = x + iy$. They should also satisfy the growth condition

$$|f(z)| \leq C|y|^n \text{ as } y \to \infty \tag{8.1}$$

for some integer $n > 0$ and constant C. An example of a Maass form is the non-holomorphic Eisenstein series.

Eisenstein series are a special type of automorphic forms by the way they are constructed (which will be discussed in the next section). The holomorphic Eisenstein series of weight $2k$ are [15]

$$\mathcal{G}_{2k}(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^{2k}} \qquad \mathcal{G}_{2k}(\gamma z) = (cz+d)^{2k}\mathcal{G}_{2k}(z) \tag{8.2}$$

The non-holomorphic Eisenstein series with eigenvalues $s(s-1)$ where $s \in \mathbb{C}$ and $\operatorname{Re} s > 1$ are [6]

$$\mathcal{E}(z;s) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|mz+n|^{2s}} \qquad \mathcal{E}(\gamma z;s) = \mathcal{E}(z;s) \tag{8.3}$$

We will in this thesis use the term automorphic form in the meaning of the Maass forms above, but extended to higher rank groups. Ultimately, we are interested in automorphic forms on G/K invariant under $G(\mathbb{Z})$ (often called an automorphic form on $G(\mathbb{Z})\backslash G/K$) but we already know how to make a function on G invariant under the right action of K using the Iwasawa decomposition. Let us now then consider functions on G that are invariant under the left action of an arithmetic subgroup $G(\mathbb{Z})$.

We define an automorphic form to be a function f on G that [37]

- is invariant under the action of some discrete subgroup $G(\mathbb{Z})$.
- is an eigenfunction to the G-invariant differential operators.
- satisfies some condition of moderate growth (that is a generalization of (8.1)).

8.1 Construction

In this section we will first discuss Poincaré series to create functions that are invariant under $G(\mathbb{Z})$-transformations in general, after which we will focus on a simple class of eigenfunctions to the Laplacian to construct the Eisenstein series.

Considering the simpler case when $G(\mathbb{Z})$ is a finite group it is enough to start with a function $f : G \to \mathbb{C}$ and average over $G(\mathbb{Z})$, that is, to construct

$$F(g) = \sum_{\gamma\in G(\mathbb{Z})} f(\gamma g)\,, \tag{8.4}$$

where F is invariant under the left action of $G(\mathbb{Z})$ since for $h \in G(\mathbb{Z})$

$$F(hg) = \sum_{\gamma\in G(\mathbb{Z})} f(\gamma h g) = \Big\{\gamma' = \gamma h \in G(\mathbb{Z})\Big\} = \sum_{\gamma'\in G(\mathbb{Z})} f(\gamma' g) = F(g) \tag{8.5}$$

On the other hand, when $G(\mathbb{Z})$ is an infinite group we need to worry about convergence. Firstly, if $f(\gamma g) = f(g)$ for γ in some infinite group $H(\mathbb{Z}) \subset G(\mathbb{Z})$, we should only sum over distinct terms, that is

$$\mathcal{F}(g) = \sum_{\gamma\in H(\mathbb{Z})\backslash G(\mathbb{Z})} f(\gamma g)\,. \tag{8.6}$$

Note that the terms $f(\gamma g)$ are well-defined for the coset γ because of the definition of $H(\mathbb{Z})$. This average over the cosets is called a Poincaré series [6].

Secondly, we will have constraints on the function $f(g)$ to make the remaining sum convergent. In the special case of the Eisenstein series, these will become constraints on the parameter s.

Let us now consider the coset space G/K again and let f be an eigenfunction to the invariant Casimir operator $\Delta_{G/K}$

$$\Delta_{G/K} f_s(g) = \lambda_s f_s(g) \tag{8.7}$$

That the Casimir operator (or Laplacian) is an invariant differential operator means that for any $h \in G$, $\Delta_{G/K} f_s(hg) = \lambda_s f_s(hg)$ [38]. Thus, the sum

$$\mathcal{F}_s(g) = \sum_{\gamma \in H(\mathbb{Z}) \backslash G(\mathbb{Z})} f_s(\gamma g) \tag{8.8}$$

is also an eigenfunction of $\Delta_{G/K}$ with eigenvalue λ_s and is called an Eisenstein series. Note that f_s is, by itself, not an automorphic form, but $\mathcal{F}_s$ is.

8.2 Fourier expansion

We can now construct automorphic forms, but to see their physical interpretation we need to expand them in Fourier series. Since the variable of the automorphic form is a group element in G, we have to discuss how one generalizes the ordinary Fourier expansion where the variables are usually in $\mathbb{R}^n$.

A Fourier expansion of a function f uses the fact that f is invariant under discrete translations in one or more of the variables, that is, it is periodic. This is generalized by replacing the translation operators with group elements in a discrete subgroup of G under which f is invariant. The automorphic forms constructed above are manifestly invariant under the left action of the discrete subgroup $G(\mathbb{Z})$.

Consider a group $G = G(\mathbb{R})$ in its split real form with a discrete subgroup Γ and a function f on $\Gamma\backslash G$, that is $f(\gamma g) = f(g)$ for all γ in Γ and g in G. Also, let U be a subgroup of G with respect to which we want to make a Fourier expansion — the discrete translations in the ordinary case. For now, we will assume that U is an Abelian subgroup of G.

Let us start with the easiest case of Fourier expanding $f(u)$ with $u \in U$, that is f restricted to $\Gamma \cap U\backslash U$, meaning that we want to expand $f(u)$ in a linear combination of $\phi_\chi \cdot \chi(u)$ where χ is a function on U which is trivial on $\Gamma \cap U$. Compare with Fourier coefficients and exponential functions $\exp(2\pi i u)$ that are trivial on $\mathbb{Z}$ for periodic functions on the real line.

The functions $\chi : U \to U(1)$, where $U(1)$ is the unitary group of degree one, are the multiplicative characters of U that are trivial on $\Gamma \cap U$ [39]. They satisfy the properties

$$\begin{gathered} \chi(u_1 u_2) = \chi(u_1)\chi(u_2) \qquad u_1, u_2 \in U \\ \chi(u^{-1}) = \chi(u)^{-1} = \overline{\chi(u)} \qquad u \in U \\ \chi(e) = 1 \end{gathered} \tag{8.9}$$

where e is the identity element.

One can show that f, as a function of the elements in the Abelian subgroup U, can be expanded in the following series [5, 39]

$$f(u) = \sum_{\chi} f_\chi \chi(u) \qquad f_\chi = \int\limits_{\Gamma\cap U\backslash U} f(u)\chi(u)^{-1}\,\mathrm{d}u \tag{8.10}$$

The same arguments can be made for $f(ug)$ as a function of u

$$f(ug) = \sum_{\chi} f_\chi(g)\chi(u) \qquad f_\chi(g) = \int\limits_{\Gamma\cap U\backslash U} f(ug)\chi(u)^{-1}\,\mathrm{d}u \tag{8.11}$$

and letting $u = e$

$$f(g) = \sum_{\chi} f_\chi(g) \tag{8.12}$$

which is called a character expansion of f. The term with a trivial $\chi(u) = 1$ is called the constant term.

Now, this might not look like a Fourier expansion at first sight, but we note that [39]

$$\begin{aligned} f_\chi(u'g) &= \int\limits_{\Gamma\cap U\backslash U} f(uu'g)\chi(u)^{-1}\,\mathrm{d}u \\ &= \chi(u') \int\limits_{\Gamma\cap U\backslash U} f(\tilde{u}g)\chi(\tilde{u})^{-1}\,\mathrm{d}\tilde{u} = \chi(u')f_\chi(g) \end{aligned} \tag{8.13}$$

where we in the second step used that the measure $\mathrm{d}u$ is invariant under the variable change $\tilde{u} = uu'$ and that $\chi(uu'^{-1}) = \chi(u)\chi(u')^{-1}$. Because of this property, the functions f_χ are then fixed by their restriction on $U\backslash G$ [39].

Thus, for $g = ug'$ we have that

$$f(g) = \sum_{\chi} f_\chi(g')\chi(u) \tag{8.14}$$

which is of a more familiar form.

Note that the Fourier coefficients $f_\chi(g')$ depend on the remaining parameters in G and that the ambiguity in factorizing g into ug' is reflected by the similar ambiguity of a passive coordinate translation for a periodic function on the real axis

$$\begin{gathered} f = \sum_n a_n e^{2\pi i x} = \sum_n b_n e^{2\pi i \xi} \\ x - \xi = \Delta = \text{constant} \qquad b_n = a_n e^{2\pi i \Delta} \end{gathered} \tag{8.15}$$

When U is not Abelian, the sum in (8.12) only captures parts of the function f, that is, the parts that transform trivially under the commutator subgroup $U' = [U, U]$ [5], and must be corrected with other terms. Concretely, [39]

$$\Pi_U f(g) := \int\limits_{\Gamma\cap U'\backslash U'} f(ug)\,\mathrm{d}u = \sum_{\chi} f_\chi(g) \tag{8.16}$$

where we have projected f on the Abelianization of U.

The Abelian coefficients f_χ in (8.11) contain a lot of information even though U is nonabelian as is discussed in [40, 41, 42].

All above can be generalized to functions not only on a group G but also on a coset space G/K and we will now consider such an example that will illustrate how the character expansion on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ reduces to ordinary Fourier expansion along the real axis for functions on $\mathbb{H}$.

Example 8.1.

Consider the coset space $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cong \mathbb{H}$ with $\Gamma = G(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$. We have shown that G has an Iwasawa decomposition $G = NAK$ where $N \cong \mathbb{R}$, corresponding the subgroup U above, is Abelian and parametrized by real translations ξ.

The characters of N that are trivial under $N(\mathbb{Z}) = \Gamma \cap N \cong \mathbb{Z}$ are $\chi_m(\xi) = \exp(2\pi i m\xi)$. For an automorphic form f on $G(\mathbb{Z})\backslash G/K$ with $u(\xi) \in N$ and $g(z) \in G/K$ we have that

$$f(g) = \sum_m f_m(g) \tag{8.17}$$

where

$$\begin{aligned} f_m(g) &= \int\limits_{N(\mathbb{Z})\backslash N} f(ug)\chi_m(u)^{-1}\,\mathrm{d}u = \int_0^1 f(z+\xi)e^{-2\pi i m\xi}\,\mathrm{d}\xi' \\ &= \int_0^1 f(x+iy+\xi)e^{-2\pi i m\xi}\,\mathrm{d}\xi = e^{2\pi i m x}\int_0^1 f(\xi'+iy)e^{-2\pi i m\xi'}\,\mathrm{d}\xi' \end{aligned} \tag{8.18}$$

where we in the last step have made the variable change $\xi' = x + \xi$. Note that we do not change the limits of the integral since the integrand is periodic on $[0,1]$. We then obtain the standard Fourier expansion

$$f(g) = \sum_m f_m(g) = \sum_m e^{2\pi i m x}\int_0^1 f(\xi'+iy)e^{-2\pi i m\xi'}\,\mathrm{d}\xi' \tag{8.19}$$

Chapter 9

Eisenstein series on $\mathrm{SL}(2,\mathbb{R})$

After this general introduction to automorphic forms, let us look at the example of automorphic forms on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ that were first introduced in the motivational chapter 2.

9.1 Construction

In the case of $G/K = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cong \mathbb{H}$ and $G(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$ we have for $z = x + iy \in \mathbb{H}$ that $\Delta = y^2(\partial_x^2 + \partial_y^2)$ and the function $f_s(z) = \mathrm{Im}(z)^s = y^s$ is an eigenfunction with $\lambda_s = s(s-1)$.

We have seen that for $\gamma \in G$

$$f_s(\gamma z) = \mathrm{Im}(\gamma z)^s = \frac{y^s}{|cz+d|^{2s}} \tag{9.1}$$

which shows that the cosets in $H(\mathbb{Z})\backslash G(\mathbb{Z})$ that parametrize distinct terms of the sum (8.6) should be more or less determined by the pair of integers (c, d).

Indeed, $f_s(z)$ is invariant under real translations and therefore $H(\mathbb{Z}) \supseteq N(\mathbb{Z}) := N \cap G(\mathbb{Z})$ where N is the nilpotent part of the Iwasawa decomposition of G.

$$N(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \;\middle|\; \xi \in \mathbb{Z} \right\}. \tag{9.2}$$

The right cosets of $N(\mathbb{Z})\backslash G(\mathbb{Z})$ with $\gamma \in G(\mathbb{Z})$ are then

$$N(\mathbb{Z})\gamma = \left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+\xi c & b+\xi d \\ c & d \end{pmatrix} \;\middle|\; \xi \in \mathbb{Z} \right\} \tag{9.3}$$

which can be parametrized by two coprime integers $(c, d) \neq (0, 0)$.

Bézout's lemma requires c and d to be coprime, that is, $\gcd(c, d) = 1$, for γ to satisfy the determinant condition. The reason why the mapping from coprime (c, d) to cosets in $N(\mathbb{Z})\backslash G(\mathbb{Z})$

is well-defined and surjects is that the determinant condition then gives unique integers a and b up to the equivalence relations $a \sim a + \xi c$ and $b \sim b + \xi d$, fully defining the coset.

This can be seen by considering the determinant condition $ad - bc = 1$ with a and b as unknowns. Letting (a, b) and (a', b') be two solutions to the above equation we see that $(a-a')d-(b-b')c = 0$. All solutions to the equation $xd - yc = 0$ with x and y real can be found as $(x, y) = t(c, d)$ for $t \in \mathbb{R}$. Restricting to the integers we get

$$(a - a', b - b') = (x, y)|_{\mathbb{Z}} = \frac{k}{\gcd(c, d)}(c, d) \qquad k \in \mathbb{Z}\,, \tag{9.4}$$

which coincides with the equivalence relations for $\gcd(c, d) = 1$. See also [43].

Note that, besides being invariant under $N(\mathbb{Z})$, $f_s(\gamma z)$ is also unchanged under $(c, d) \to (-c, -d)$. One can choose to include this factor of $\{\pm\mathbb{1}\}$ in $H(\mathbb{Z})$, but it is not necessary since it is of finite order and only changes the Poincaré sum by an overall factor of two.

The Eisenstein series are then defined as the following Poincaré series

$$E(z; s) := \sum_{\gamma\in N(\mathbb{Z})\backslash G(\mathbb{Z})} \mathrm{Im}(\gamma z)^s = \sum_{\gcd(c,d)=1}{}' \frac{y^s}{|cz + d|^{2s}}\,, \tag{9.5}$$

where the primed sum denotes $(c, d) \neq (0, 0)$.

We will find it useful to instead use the following form of the Eisenstein series

$$\mathcal{E}(z; s) := \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{y^s}{|mz + n|^{2s}}\,, \tag{9.6}$$

which is related to the above by making the substitution $m = kc$, $n = kd$, where $k = \gcd(m, n) = \gcd(|m|, |n|) \in \mathbb{N}$.

$$\begin{aligned}
\mathcal{E}(z; s) &= \sum_{\substack{(c,d)\in\mathbb{Z}^2\backslash\{(0,0)\}\\ \gcd(c,d)=1}} \sum_{k=1}^{\infty} \frac{y^s}{|k(cz + d)|^{2s}} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1}}{}' \frac{y^s}{|cz + d|^{2s}} = \zeta(2s) \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1}}{}' \frac{y^s}{|cz + d|^{2s}} \\
&= \zeta(2s)E(z; s)
\end{aligned} \tag{9.7}$$

where the primed sum leaves out the origin (and correspondingly for a single sum).

9.2 Fourier expansion

Since the Eisenstein series are invariant under $N(\mathbb{Z}) \subset G(\mathbb{Z})$ they are periodic in $x \to x + 1$. The Fourier expansion of the Eisenstein series have a very important physical interpretation that will be discussed in section 9.3. We will in this section first discuss the general form of the expansion decided up to some coefficients using the defining properties of the Eisenstein series.[1] This will be followed by an actual calculation of the coefficients using Poisson resummation. Alternatively, one could simply evaluate the integrals for the Fourier coefficients as done in [36].

9.2.1 General form

Since $\mathcal{E}(z+1;s) = \mathcal{E}(z;s)$ we can make a Fourier expansion

$$\mathcal{E}(z;s) = \sum_{N\in\mathbb{Z}} C_N(y) e^{2\pi i N x} \tag{9.8}$$

where the Fourier coefficients C_N still depend on y and can be obtained by integration over $[0,1]$ in the usual manner. The C_0 term is called the constant term (although it might still depend on y).

We will now use that the Eisenstein series, by construction, are eigenfunctions to the Laplace operator $\Delta = y^2(\partial_x^2 + \partial_y^2)$

$$\Delta\mathcal{E} = s(s-1)\mathcal{E}\,. \tag{9.9}$$

Since the Fourier modes are orthogonal on $[0,1]$ we get the following differential equations for the Fourier coefficients

$$\begin{cases} y^2 C_0''(y) = s(s-1)C_0(y) & \\ y^2\big(C_N''(y) - 4\pi^2 N^2 C_N(y)\big) = s(s-1)C_N(y) & \qquad N \neq 0 \end{cases} \tag{9.10}$$

The first one simply gives us

$$C_0(y) = A(s)y^2 + B(s)y^{1-s}\,. \tag{9.11}$$

In the second equation we substitute $C_N = \sqrt{y} f_N$ and $t = 2|N|\pi y$ to get

$$t^2 f_N''(t) + t f_N'(t) - \Big(t^2 + \big(s - \frac{1}{2}\big)^2\Big) f(t) = 0 \tag{9.12}$$

which is the modified Bessel equation of order $s - 1/2$. The solutions are linear combinations of the modified Bessel functions

$$f_N(t) = a_N(s) K_{s-1/2}(t) + b_N(s) I_{s-1/2}(t)\,, \tag{9.13}$$

but since we are summing over $N \neq 0$, and especially let $|N| \to \infty$, we will neglect the $I_{s-1/2}(t)$ solutions which are divergent in the limit $t \to \infty$.

Finally, we find that $\mathcal{E}(z;s)$ has a Fourier expansion on the form

$$\mathcal{E}(z;s) = A(s)y^s + B(s)y^{1-s} + \sqrt{y}\sum_{N\neq 0} a_N(s) K_{s-1/2}(2\pi|N|y) e^{2\pi i N x} \tag{9.14}$$

[1] In fact, the same arguments can be made for all automorphic forms.

9.2.2 Poisson resummation

We start with the expression from (9.6) and separate the $m = 0$ terms which trivially contribute to the constant term.

$$\begin{aligned}\mathcal{E}(z;s) &= \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{y^s}{|mz+n|^{2s}} = \overbrace{\sum_n{}' \frac{y^s}{|n|^{2s}}}^{m=0} + \overbrace{\sum_m{}' \sum_n \frac{y^s}{|mz+n|^{2s}}}^{m\neq 0} \\ &= 2\zeta(2s)y^s + \sum_m{}' \sum_n \frac{y^s}{|mz+n|^{2s}}\end{aligned} \tag{9.15}$$

Then, we rewrite the last part of the sum using an integral representation which is obtained from the Γ-function.

$$\begin{aligned}\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,\mathrm{d}x = \left\{x = \frac{\xi}{t}\right\} = \xi^s \int_0^\infty \frac{e^{-\frac{\xi}{t}}}{t^{s+1}}\,\mathrm{d}t \\ \frac{1}{\xi^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\frac{\xi}{t}}}{t^{s+1}}\,\mathrm{d}t\end{aligned} \tag{9.16}$$

Letting $\xi = \pi|mz+n|^2$ we get that (the reason for the extra π will become clear after the resummation)

$$\frac{1}{|mz+n|^{2s}} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{e^{-\frac{\pi}{t}|mz+n|^2}}{t^{s+1}}\,\mathrm{d}t \tag{9.17}$$

Thus,

$$\begin{aligned}\sum_n \frac{1}{|mz+n|^{2s}} &= \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{\mathrm{d}t}{t^{s+1}} \sum_n e^{-\frac{\pi}{t}|mz+n|^2} \\ &= \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{\mathrm{d}t}{t^{s+1}} e^{-\frac{\pi}{t}m^2y^2} \sum_n e^{-\frac{\pi}{t}(mx+n)^2}\end{aligned} \tag{9.18}$$

To simplify the remaining sum over n we will use Poisson resummation.

Theorem 9.1 (Poisson resummation)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwarz function (for convergence properties) with Fourier transform $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,\mathrm{d}x$. Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k) \tag{9.19}$$

Proof.

Let

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n) \tag{9.20}$$

which is periodic with $F(x+1) = F(x)$ and can thus be written as a Fourier series

$$F(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx} \tag{9.21}$$

with

$$\begin{aligned} c_k &= \int_0^1 F(x)e^{-2\pi ikx}\,\mathrm{d}x = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi ikx}\,\mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)e^{-2\pi ikx}\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x')e^{-2\pi ikx'}\,\mathrm{d}x' \\ &= \int_{-\infty}^{\infty} f(x')e^{-2\pi ikx'}\,\mathrm{d}x' = \hat{f}(k) \end{aligned} \tag{9.22}$$

where we have used that f is a Schwarz function when interchanging the sum and the integral, and where we have made the variable substitution $x' = x + n$.

Thus,

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx} \tag{9.23}$$

and letting $x = 0$ we obtain

$$\sum_{n\in\mathbb{Z}} f(n) = F(0) = \sum_{k\in\mathbb{Z}} \hat{f}(k) \tag{9.24}$$

□

Now let $f(n) = \exp(-\frac{\pi}{t}(mx+n)^2)$ with $\hat{f}(k) = \sqrt{t}\exp(-\pi k^2 t + 2\pi ikmx)$ which gives

$$\sum_n \exp(-\frac{\pi}{t}(mx+n)^2) = \sum_k \sqrt{t}\exp(-\pi k^2 t + 2\pi ikmx) \tag{9.25}$$

Using this in (9.18) we note that one term in the exponent is independent of t and can be factored out from the integral. This is where we notice the convenience of an extra factor of π which makes the separate exponential on the form of a Fourier expansion with period $x \to x+1$.

So far, we have that

$$\begin{aligned} \mathcal{E}(z;s) &= 2\zeta(2s)y^s + \frac{y^s\pi^s}{\Gamma(s)} \sum_m{}' \int_0^\infty \frac{\mathrm{d}t}{t^{s+1}} e^{-\frac{\pi}{t}m^2y^2} \sqrt{t} \sum_k e^{-\pi k^2 t + 2\pi ikmx} \\ &= 2\zeta(2s)y^s + \frac{y^s\pi^s}{\Gamma(s)} \left(\sum_k \sum_m{}' e^{2\pi ikmx} \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} e^{-\pi k^2 t - \frac{\pi}{t}m^2y^2} \right) \end{aligned} \tag{9.26}$$

Let us split the sums in parentheses into two terms with $k = 0$ and $k \neq 0$ and then determine the integrals separately.

$$\sum_k \sum_m{}' e^{2\pi ikmx} \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} e^{-\pi k^2 t - \frac{\pi}{t} m^2 y^2} =$$

$$= \sum_m{}' \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} e^{-\frac{\pi}{t} m^2 y^2} + \sum_k{}' \sum_m{}' e^{2\pi ikmx} \int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} e^{-\pi k^2 t - \frac{\pi}{t} m^2 y^2} \tag{9.27}$$

For $\mathrm{Re}(s) > 1/2$ the first integral can be found as

$$\int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} e^{-\frac{\pi}{t} m^2 y^2} = \left(\frac{1}{\pi m^2 y^2} \right)^{s-1/2} \Gamma(s - 1/2) \tag{9.28}$$

and the second integral

$$\int_0^\infty \frac{\mathrm{d}t}{t^{s+1/2}} e^{-\pi k^2 t - \frac{\pi}{t} m^2 y^2} = 2 \left| \frac{k}{my} \right|^{s-1/2} K_{s-1/2}(2\pi |km| y) \tag{9.29}$$

where $K_{s-1/2}$ is the modified Bessel function.

Thus, with $y > 0$ (that is, $z \in \mathbb{H}$)

$$\begin{aligned} \mathcal{E}(z; s) &= 2\zeta(2s) y^s + \frac{y^s \pi^s}{\Gamma(s)} \left(\sum_m{}' \left(\frac{1}{\pi m^2 y^2} \right)^{s-1/2} \Gamma(s - 1/2) + \right. \\ &\qquad \left. + 2 \sum_k{}' \sum_m{}' e^{2\pi ikmx} \left| \frac{k}{my} \right|^{s-1/2} K_{s-1/2}(2\pi |km| y) \right) \\ &= 2\zeta(2s) y^s + \sqrt{\pi} y^{1-s} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_m{}' \frac{1}{m^{2s-1}} + \\ &\qquad + 2\sqrt{y} \frac{\pi^s}{\Gamma(s)} \sum_k{}' \sum_m{}' \left| \frac{k}{m} \right|^{s-1/2} K_{s-1/2}(2\pi |km| y) e^{2\pi ikmx} \end{aligned} \tag{9.30}$$

We now want to make the substitution $N = km$ and sum over N and m to retrieve the form of (9.14). Since $k = N/m \in \mathbb{Z}$ we need that $m \mid N$, that is, that m is a divisor of N. We get that

$$\begin{aligned} \sum_{k\in\mathbb{Z}}{}' \sum_{m\in\mathbb{Z}}{}' \left| \frac{k}{m} \right|^{s-1/2} K_{s-1/2}(2\pi |km| y) e^{2\pi ikmx} &= \\ &= 2 \sum_{N\in\mathbb{Z}}{}' \sum_{\substack{m|N \\ m>0}} \left| \frac{N}{m^2} \right|^{s-1/2} K_{s-1/2}(2\pi |N| y) e^{2\pi iNx} \\ &= 2 \sum_N{}' |N|^{s-1/2} \mu_{1-2s}(N) K_{s-1/2}(2\pi |N| y) e^{2\pi iNx} \end{aligned} \tag{9.31}$$

where we have defined

$$\mu_s(N) := \sum_{\substack{m|N \\ m>0}} m^s \tag{9.32}$$

which is called the instanton measure [6].

Using the completed ζ-function $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ we have

$$\begin{aligned}
\mathcal{E}(z;s) &= 2\zeta(2s)y^s + 2\zeta(2s-1)\sqrt{\pi}y^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)} + \\
&\qquad + 4\sqrt{y}\frac{\pi^s}{\Gamma(s)}\sum_N{}' |N|^{s-1/2}\mu_{1-2s}(N)K_{s-1/2}(2\pi|N|y)e^{2\pi iNx} \\
&= 2\zeta(2s)\Bigg(y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s} + \\
&\qquad + 2\frac{\sqrt{y}}{\xi(2s)}\sum_N{}' |N|^{s-1/2}\mu_{1-2s}(N)K_{s-1/2}(2\pi|N|y)e^{2\pi iNx}\Bigg)
\end{aligned} \tag{9.33}$$

Comparing with (9.14) we see that

$$\begin{aligned}
A(s) &= 2\zeta(2s) \\
B(s) &= 2\zeta(2s)\frac{\xi(2s-1)}{\xi(2s)} \\
a_N(s) &= 4\frac{\zeta(2s)}{\xi(2s)}|N|^{s-1/2}\mu_{1-2s}(N)
\end{aligned} \tag{9.34}$$

9.3 Physical interpretation

In chapter 2 we stated that the coefficient for the $\mathcal{R}^4$ correction to the supergravity effective action is an automorphic form f on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ which was later motivated in chapter 7.

Using supersymmetry, one can show that f is an eigenfunction to the Laplacian with eigenvalue $3/4$ [44]

$$\nabla^2 f(\tau) = \frac{3}{4}f(\tau)\,. \tag{9.35}$$

That f should satisfy the above equation, be $\mathrm{SL}(2,\mathbb{Z})$ invariant and have the correct large τ_2 asymptotics as in (2.17) (that is, the right weak coupling expansion) gives the unique solution that f is, in fact, the Eisenstein series with $s = 3/2$ [44, 45]

$$f(\tau) = \mathcal{E}(\tau, 3/2) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^{3/2}}{|m\tau+n|^3} \tag{9.36}$$

which has the required eigenvalue $s(s-1) = 4/3$ and the correct asymptotics as can be seen from the Fourier expansion derived in the section above

$$f(\tau) = 2\zeta(3){\tau_2}^{3/2} + 4\zeta(2){\tau_2}^{-1/2} + 8\pi\sqrt{\tau_2}\sum_N{}' |N|\ \mu_{-2}(N)\ K_1(2\pi|N|\tau_2)e^{2\pi i N\tau_1} \tag{9.37}$$

Notice that the constant terms (with respect to $\tau_1 = \chi$) are exactly the two perturbative terms in the effective action (2.15) (and nothing more). We will now show that the remaining, non-constant, terms are non-perturbative.

For large x the Bessel function $K_1(x)$ can be expanded as

$$K_1(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\Big(1 + \mathcal{O}(x^{-1})\Big)\,. \tag{9.38}$$

Thus, for large τ_2

$$f(\tau) = 2\zeta(3){\tau_2}^{3/2} + 4\zeta(2){\tau_2}^{-1/2} + 4\pi\sum_N{}' \sqrt{|N|}\ \mu_{-2}(N)\ e^{2\pi(iN\tau_1 - |N|\tau_2)}\Big(1 + \mathcal{O}(\tau_2^{-1})\Big) \tag{9.39}$$

and hence the non-constant terms are non-perturbative in $g_s = \tau_2^{-1}$.

Splitting up the sum into positive and negative N and denoting as in section 6.2.1

$$\begin{aligned} S_{\text{inst}}(\tau, N) &= -2\pi i|N|\tau \\ S_{\text{anti}}(\tau, N) &= 2\pi i|N|\bar{\tau} \end{aligned} \tag{9.40}$$

we get

$$\begin{aligned} f(\tau) = {} & 2\zeta(3){\tau_2}^{3/2} + 4\zeta(2){\tau_2}^{-1/2} + \\ & + 4\pi\sum_{N>0}\sqrt{|N|}\ \mu_{-2}(N)\ \Big[e^{-S_{\text{inst}}(\tau,N)} + e^{-S_{\text{anti}}(\tau,N)}\Big]\Big(1 + \mathcal{O}(\tau_2^{-1})\Big)\,. \end{aligned} \tag{9.41}$$

We see that the non-perturbative corrections are really the contributions from the instantons and anti-instantons of section 6.2.1 with all charges N. The instanton measure $\mu_s(N)$ can now be understood as a counting of the number of states as seen in the string picture where $N = mn$, m being the wrapping number and n the units of momentum around the circle. That is, $\mu_s(N)$ roughly counts how many ways N can be split into two integers mn. More specifically it is a measure on the moduli space of charge N, single instantons [6]. The first non-perturbative corrections are then further corrected by loop amplitudes in the instanton background of $\mathcal{O}(\tau_2^{-1})$.

Chapter 10

Outlook and discussion

In this thesis we have discussed how dualities in string theory constrain quantum corrections to the effective action. We have shown that the coefficients are described by automorphic forms invariant under discrete U-duality groups and how these contain physical information about instanton states in the theory.

Specifically, we have shown that the $\mathcal{R}^4$ correction to the ten-dimensional type IIB supergravity action has a coefficient $f(\tau) = \mathcal{E}(\tau, 3/2)$ — where $\mathcal{E}(\tau, s)$ is the Eisenstein series with eigenvalue $s(s-1)$ — using the first known perturbative corrections together with supersymmetry and the conjectured $\mathrm{SL}(2,\mathbb{Z})$ invariance. In the previous chapter we showed how the Fourier expansion of $f(\tau)$ gives the non-perturbative instanton corrections and that $f(\tau)$ does not have any perturbative corrections beyond one loop in the string coupling constant.

By analysing these results we can check the conjectured invariance under the discrete U-duality group that we argued for in chapter 7. In [4], where the explicit form of $f(\tau)$ was first found, they computed the charge $N = 1$ single instanton correction up to a normalization constant and found that it matches the non-perturbative part of $f(\tau)$. In the same article they also gave a heuristic argument for why there should be no perturbative corrections beyond one loop in g_s and explained the degeneracy of instanton states as coming from the wrapping number and momentum in the T-dual picture which we discussed in sections 6.2.2 and 9.3.

Another check has been carried out in [13] where the $\mathcal{R}^4$ term obtained in [4] for the ten-dimensional type IIB supergravity was compactified on a circle and compared with expectations for the type IIA theory and from M-theory on a torus.

The method for finding the coefficient for the $\mathcal{R}^4$ term has also been applied to higher derivative terms, such as for $\partial^4\mathcal{R}^4$ and $\partial^6\mathcal{R}^4$ discussed in [46, 5]. The $\partial^6\mathcal{R}^4$ term is an example of where the coefficient is an automorphic form but not an Eisenstein series [5]. These are all four-graviton interactions — in the general case it is difficult to find precisely how the U-duality constrains the amplitudes.

When compactifying the theory on tori one still has the same kind of higher derivative corrections, but with different coefficients since the U-duality group is larger for lower dimensions as shown in table 7.1. In four dimensions we have, for example, the U-duality group $\mathrm{E}_7(\mathbb{Z})$ whose fundamental representation is 56-dimensional compared to the two dimensions of the fundamental representation of $\mathrm{SL}(2,\mathbb{Z})$. The $\mathcal{R}^4$ term in eight and seven dimensions with duality groups $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(5,\mathbb{Z})$ respectively was discussed in [47] just a few months after the ten-dimensional case [4]. A survey of other corrections in various dimensions can be found in [46, 5].

When going below three dimensions table 7.1 has been extended to include E_9, and the conjectured E_{10} and E_{11} which are infinite dimensional Kac-Moody groups. The Eisenstein series for these groups corresponding four-graviton scattering amplitudes have recently been studied in [48]. There it was shown that the Eisenstein series have a finite number of perturbative terms only for certain values of the parameter s including those for the physically occuring series.

Since the number of supercharges are unchanged under toroidal compactifications [11], it is rewarding to also consider different Calabi-Yau compactifications which break certain supersymmetries. To obtain $\mathcal{N} = 2$ supergravity in four dimensions for example, one compactifies the ten-dimensional theory on a Calabi-Yau threefold [10]. Recently, the automorphic forms for such theories has been the focus of [49, 40, 42].

Current efforts in the field include calculating non-abelian Fourier coefficients for higher rank groups. Recall that the Fourier expansion discussed in section 8.2 only captures parts of the function in question and has to be complemented by non-abelian terms. These terms capture additional physical information and have been studied in [6, 40, 41, 42]. For instance, in [42], the non-abelian terms describe NS5-instantons in four dimensions while the Abelian terms describe D-instantons. Besides being physically interesting, they have also been studied for their mathematical appeal in, for example, [39].

Additionally, recent endeavours have been made to understand how physical input used to find the unique automorphic form for a certain correction manifests in the representations of the underlying group. That the number of constant terms are constrained and that only certain instanton states are allowed to contribute imply that only specific automorphic forms are suitable and this can be described by representation theory. Automorphic representations have been examined in [8, 50, 5] as well as [37, 36] in the mathematical literature.

To conclude, the theory of automorphic forms is an active and exciting area of research at the interface between physics and mathematics showing a lot of promise at answering fundamental questions in both fields.

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