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Topics in Game Theory

Emilio Bergroth

Department of Mathematical Sciences
Division of Mathematical Statistics
Chalmers University of Technology and University of Gothenburg
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Emilio Bergroth
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Department of Mathematical Sciences
Division of Mathematical Statistics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 1000

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Abstract

In this thesis we study different non-cooperative two-person games. First, we study a zero-sum game played by players I and II on a $n \times n$ random matrix, where the entries are iid standard normally distributed random variables. Given the realization of the matrix, player I chooses a row i and player II a column j. The entry at position (i, j) represents the winnings and losings of the players. Let $\mathbf{p} = [p_1, ..., p_n]^T$ denote the optimal strategy of player I. We show that $P\left(\max_{i \in [n]} p_i > \frac{c}{\sqrt{n}}\right) \to 0$ as $n \to \infty$ for any $c > 10\sqrt{\pi}\left(1 + \sqrt{2 \log 4}\right)\sqrt{\log 4}$.

The second game studied here is a spatial game in which each player, represented by a vertex in a given graph, plays the repeated prisoner's dilemma game against its neighbours. At time one, each player chooses a strategy at random independently of each other. At time $t = 2, 3, ...$, each player, looking at its neighbourhood (including the player itself), uses the strategy of the player that scored highest in the previous round. We study the game played on some deterministic graphs. For certain graphs and choices of the parameters of the game, we find the probability that a given player cooperates as time tends to infinity. We also analyse the iterated prisoner's dilemma played on the binomial random graph. In particular, we study the asymptotic distribution of cooperation when the number of players tends to infinity.

Finally, we analyse simultaneous zero-sum games played by player I and II. More precisely, we study how player I should choose strategy among the set of optimal strategies when taking into account the risk associated to the game. One measure of risk used here is the variance of the total winnings. In particular, we find the optimal strategy for player I which minimizes the maximum variance. In the same way, we find the optimal strategy for player I which maximizes the minimal variance.

Keywords: zero-sum game, optimal strategy, random matrix, iterated prisoner's dilemma, spatial game, cooperation, simultaneous games, minimal variance

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List of Papers

This thesis includes the following papers.

I. Bergroth, E. *The optimal strategy of a two-person zero-sum game.*
Manuscript, (2011).

II. Bergroth, E. *The iterated prisoner's dilemma on graphs.*
Manuscript, (2011).

III. Bergroth, E. and Jonasson, J. *How to play simultaneous zero-sum games.*
Manuscript, (2013).

Introduction

Game theory is a branch of applied mathematics originally related to economical and political problems. In its origins, John von Neumann and Oskar Morgenstern intended to study human behaviour when making strategic decisions. The assumption was that these decisions were based on rationality. Nowadays game theory is also related to other areas such as ecology and biology, in particular to evolution. In these areas, the individual's behaviour does not rely on rationality but on other aspects such as fitness and natural selection.
This thesis, as its name suggests, deals with some problems related to game theory. More precisely, it consists of the following papers:

I. Bergroth, E. *The optimal strategy of a two-person zero-sum game. Manuscript*, (2011).

II. Bergroth, E. *The iterated prisoner's dilemma on graphs. Manuscript*, (2011).

III. Bergroth, E. and Jonasson, J. *How to play simultaneous zero-sum games. Manuscript*, (2013).

In game theory, the decisions makers are called players [2]. Typical objects of study are so called two-person games. The players, player I and player II, have a choice to make, and each player's score depends on its own choice and the choice of the other player [8]. If the players' actions are independent, then the game is called non-cooperative. A formal description of a non-cooperative two-person game is given by $(S_I, S_{II}, A_I, A_{II})$, where

1. S_I is a nonempty set, the set of possible moves of player I;
2. S_{II} is a nonempty set, the set of possible moves of player II;
3. A_I and A_{II} are the score functions (real valued functions) defined on $S_I \times S_{II}$.

This way to describe the game is known as the "strategic form" [2]. The interpretation is as following: at the same time and without having information about the other player's choice, player I chooses $s_I \in S_I$ and player II chooses $s_{II} \in S_{II}$, resulting in that player I wins $A_I(s_I, s_{II})$ and player II wins $A_{II}(s_I, s_{II})$. When the total score to both players adds to zero, i.e. $A_I(s_I, s_{II}) = -A_{II}(s_I, s_{II})$ for all $s_I \in S_I$, $s_{II} \in S_{II}$, the game is called a zero-sum game. Two-person zero-sum games are also known as matrix games, since the scores can be represented by a matrix; if $S_I = \{s_{I,1}, ..., s_{I,n}\}$ and $S_{II} = \{s_{II,1}, ..., s_{II,m}\}$, then the matrix

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nm} \end{pmatrix}$$

is the payoff matrix of the game, where

$$A_I(s_{I,i}, s_{II,j}) = -A_{II}(s_{I,i}, s_{II,j}) = x_{ij}.$$

The matrix is assumed to be known to both players. Player I chooses a row, i $(s_{I,i})$, and player II chooses a column, j $(s_{I,i})$; then player II pays x_{ij} to player I if $x_{ij} > 0$ or player I pays $|x_{ij}|$ to player II if $x_{ij} < 0$. We distinguish between pure strategies and mixed strategies. Pure strategies for player I (II) are just deterministic choices of a move or element of S_I (S_{II}). In a mixed strategy $\mathbf{p} = [p_1, ..., p_n]^T$ $(\mathbf{q} = [q_1, ..., q_n]^T)$ for player I (II), each element p_i (q_j) is a probability, so that when I (II) makes its choice of move, it does so according to these probabilities: move $s_{I,i}$ $(s_{II,j})$ is chosen with probability p_i (q_j). Mixed strategies are also known as randomized strategies. According to the well-known Minimax Theorem of von Neumann and Morgenstern there exists a number

V, called the value of the game, and mixed strategies $\mathbf{p} = [p_1, ..., p_n]^T$ and $\mathbf{q} = [q_1, ..., q_n]^T$, called optimal strategies or minimax strategies, for both players respectively. These strategies have the following properties: when player I plays $\mathbf{p}$ then its expected winnings are at least V independently of what player II plays, and if player II plays $\mathbf{q}$ then its expected loses are at most V [5]. A central issue in game theory is the study of the optimal strategies. In this sense, Jonasson in [5] analysed the optimal strategies $\mathbf{p} = [p_1, ..., p_n]^T$ and $\mathbf{q} = [q_1, ..., q_n]^T$ of a two-person zero-sum game played on a random $n \times n$-matrix $\mathcal{X} = [X_{ij}]_{1 \le i,j \le n}$, where the X_{ij}'s are iid normally distributed random variables. More precisely, the author shows that if Z is the number of rows in the support of the optimal strategy for player I given the realization of the matrix, then there exists $a < \frac{1}{2}$ such that

$$P\left(\left(\frac{1}{2} - a\right) n < Z < \left(\frac{1}{2} + a\right) n\right) \quad \to \quad 1,$$

as $n \to \infty$. It is also shown that $\mathbb{E}Z = \left(\frac{1}{2} + o(1)\right) n$. In **Paper I** we study the same game, establishing a result about the maximal probability assigned to a row/column in the optimal strategy. More precisely, we find that for any $c > 10\sqrt{\pi}\left(1 + \sqrt{2\log 4}\right)\sqrt{\log 4}$,

$$P\left(\max_{i \in [n]} p_i > \frac{c}{\sqrt{n}}\right) \to 0,$$

as $n \to \infty$.

Paper II concerns a model introduced by Lindgren and Nordahl [6], in which the well-known "prisoner's dilemma" is studied in a spatial setting. Merrill Flood and Melvin Dresher stated the prisoner's dilemma for the first time in 1950. However, the name "prisoner's dilemma" was given by Albert W. Tucker who formalized the game using a payoff matrix to describe it [13]. The prisoner's dilemma is the following: two suspects are arrested by the police. The police offers the same deal to each prisoner. If one testifies against the other (defects) and the other remains silent (cooperates), the defector goes free and the cooperator gets a 5-year sentence. If both prisoners cooperate, each of them receive a 1-year sentence. If each prisoner testifies against the other, each receives a 3-years sentence. This means that there are only two actions for each prisoner, to defect (D) or to cooperate (C). What should the prisoners do? Assuming that each prisoner wants only to minimize his own time in prison, then the best action to take is to defect, whatever the other prisoner does. On the other hand, it is clear that if the two prisoners were to act for their common good, they should cooperate.

This need of choice between defection and cooperation is present in many social and biological contexts. In fact, the prisoner's dilemma is present at all levels, as explained by Nowak in [9]: "Replicating molecules had to cooperate to form the first cells. Single cells had to cooperate to form the first multicellular organisms. The soma cells of the body cooperate and help the cells of the germ line to reproduce. Animals cooperates to form social structures... Humans cooperate on large scale, giving raise to cities, states and countries. Cooperation allows specialization. Nobody needs to know everything. But cooperation is always vulnerable to exploitation by defectors."

Lindgren and Nordahl model the evolution of cooperation in a spatial setting: each player, associated with a vertex in a given graph, plays the prisoner's dilemma game against its neighbours. In their work, the authors made simulations in order to understand the behaviour of cooperation as the game is played repeatedly. Similar work was done, for example, in [1], [10], [11], [14], [9] and [3] (see [12] for more details). In this paper, we establish some rigorous results concerning the iterated prisoner's dilemma. Here the game is played with the following rules: (i) at time 0 each player chooses independently strategy C with probability p, and strategy D with probability $1-p=q$; (ii) at time $t=1,2,...$ each player plays the game against its neighbours; (iii) at time $t=2,3,...$ each player, looking at its own neighbourhood (the player itself and its neighbours), uses the same strategy as the player with highest score at time $t-1$. For each player, the payoff matrix of a single game is

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \end{array}$$

with $0 < a < 1$ and $1 < b < 2$. We are interested in the probability that cooperation survives as the game is played repeatedly, in particular that a given player i survives as a cooperator as $t \to \infty$. More formally, we study the limit $\pi_p(C) = \lim_{t\to\infty} P(s_t^i = C)$, where s_t^i stands for the strategy used by player i at time t. The structure of the population is determined by different graphs, characterized by vertices and edges, which represents players and interactions respectively. **Paper II** contains two parts. In the first part we study the iterated prisoner's dilemma played on some deterministic graphs, in which there are infinitely many players and the number of neighbours is equal for all players. Examples of such graphs are trees, d-dimensional lattices, etc. In all cases $\pi_p(C)$ depends on the parameters a and b. We study for each graph, as it is possible, the evolution of cooperation for different values of a and b. The simplest example analysed here is the one-dimensional lattice, where each player plays against its two neighbours. We show in this case that

$$\pi_p(\mathrm{C}) = \begin{cases} 0 & \text{if } a+b>2, \\ \left(q+p^2\right)^2 p^3\,(3-2p) & \text{if } a+b\leq 2. \end{cases}$$

This is a special case of the $n-1$-nary tree, in which each player plays the game against its n neighbours, for which given the conditions $a + (n-1)b \leq n$ and $(n-1)a + b > n-1$, we show that

$$\begin{aligned} \pi_p(\mathrm{C}) &= p^{n+1}x^{n(n-1)}\left(1-p^{n-1}x^{2-3n+n^2}\right)^n \\ &\quad -px^n\left(\left(1-p^n x^{(n-2)n}\right)^n - 1\right), \end{aligned}$$

where $x = p + q(1-p^{n-1})$.

In the second part of **Paper II**, we study the iterated prisoner's dilemma played on a random graph known as the *binomial random graph* or the *Erdös-Rényi random graph*. In this graph, denoted by $\mathbb{G}(n,r)$, there are n players which play against each other. This interaction is given by the result of $\binom{n}{2}$ independent coins flipping, each of them with probability of success equal to $r \in (0,1)$ [4]. We are interested in the behaviour of cooperation as n tends to infinity, with r as a function of n. Firstly, we find that too much interaction rules out cooperation. More precisely, we show that for all $0 < a < 1$ and $1 < b < 2$, the probability that cooperation survives in some part of the graph tends to zero if $r \geq \frac{1}{n^c}, 0 \leq c < 1$. Secondly, we analyse if limited interaction leaves room for cooperation to survive. It is well known that there is positive probability that, as n tends to infinity, there are isolated vertices if $r = \frac{\log n}{cn}$ and $c > 1$ [4].

Of course, this is not of so much interest since we are mainly interested in the probability that cooperation survives as a result of interaction between the players. In order to ensure interaction, we study the game played on the *largest* component with r big enough so that this component is *a.a.s.* unique. More precisely, we show that if $a+b \leq 2$, $r = \frac{c}{n}$ and $c > 1$, then

$$\lim_{n\to\infty} P(B_i) \quad > \quad 0,$$

where B_i is the event that an arbitrarily chosen player i survives as cooperator, as $t \to \infty$, in a component which has at least one more player than any other component. Finally, we find that cooperation can still survive in the largest component when r is of bigger order than $\frac{c}{n}$. To be more precise, we show that if $a+b \leq 2$, $r = \lambda\frac{\log(n)}{n}$ and $\lambda \leq \frac{1}{6}$ such that $\lambda \log n \to \infty$ as $n \to \infty$, then

$$\lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) \quad = \quad 1,$$

i.e. with this choice of a, b and r, the probability that cooperation survives in the biggest community of players tends to one as the numbers of players tends to infinity.

As mentioned previously, a central issue of game theory is the study of two-person zero-sum games. A very simple example of such games is the so called "envelope game": player I puts 20 dollars in one envelope and 40 dollars in a second envelope. Player II chooses one of the envelopes and guesses how much money the envelope contains. If the guess is correct, Player II gets the money contained in the envelope, otherwise 30 dollars are paid from player II to player I. The matrix of the game is then

$$\mathbf{M} = \begin{pmatrix} -20 & 30 \\ 30 & -40 \end{pmatrix}$$

In this case, the only optimal strategy for player I is $[\frac{7}{12}, \frac{5}{12}]^T$ and the value of the game is $r = \frac{5}{6}$. Nevertheless, if the game is played simultaneously two times for example, there is more than one optimal strategy.

One optimal strategy for player I could be to choose the first row in both games with probability 7/12 or the second row in both games with probability 5/12. Another optimal strategy is obtained by playing the strategy $[\frac{7}{12}, \frac{5}{12}]^T$ independently in each game. In both cases, player I's expected winnings are $2r = \frac{5}{3}$, no matter what the other players do. The question now is how player I should make the choice of strategy among the set of optimal strategies. In **Paper III**, we analyse the choice of strategy for player I when a slightly more general zero-sum game is played simultaneously several times (against several players or several times against the same player). More precisely, each game is typically played on a 2×2 matrix:

$$\mathbf{M} = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

with the property that there is no saddle point, i.e. $a \geq b$, $b < c$, $c > d$ and $d < a$ or $a < b$, $b > c$, $c < d$ and $d > a$ [2].

Inspired by a theory of finance called Modern portfolio theory [7], we take into account the risk associated to the game in order to make the choice of strategy. In particular, we analyse how two different kind of players should play: risk averse and risk seeking players. In our analysis, we use two different measures of risk. First, risk is measured by the variance of the total winnings. Assuming that player I is a risk averse player, we find the optimal strategy which minimizes the smallest number v, such that player I's variance is at most v independently of what the other players do. In the same way, we find the preferred strategy for player I in the risk seeking case, that is the optimal strategy that maximizes the largest number v such that player I's variance is at least v independently of what player II does. The second measure of risk used here is the probability that the total winnings are less than a given constant c. In this sense, player I would want to maximize the largest number s such that the probability that player I's total winnings are greater than or equal to c is at least s independently of what the other players do. In this case, we find the desired strategy when each game is played on the following matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

with the property that b is greater than or equal to the number of times the game is played.

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