Schmidt decompositions of parametric processes I: Basic theory and simple examples

Schmidt decompositions of parametric processes I: Basic theory and simple examples

C. J. McKinstrie[1,*] and M. Karlsson[2]

[1]*Bell Laboratories, Alcatel–Lucent, Holmdel, New Jersey 07733, USA*
[2]*Department of Microtechnology and Nanoscience, Chalmers University of Technology, SE-41296 Gothenburg, Sweden*

[*]*mckinstrie@alcatel-lucent.com*

Abstract: Parametric devices based on four-wave mixing in fibers perform many signal-processing functions required by optical communication systems. In these devices, strong pumps drive weak signal and idler sidebands, which can have one or two polarization components, and one or many frequency components. The evolution of these components (modes) is governed by a system of coupled-mode equations. Schmidt decompositions of the associated transfer matrices determine the natural input and output mode vectors of such systems, and facilitate the optimization of device performance. In this paper, the basic properties of Schmidt decompositions are derived from first principles and are illustrated by two simple examples (one- and two-mode parametric amplification). In a forthcoming paper, several nontrivial examples relevant to current research (including four-mode parametric amplification) will be discussed.

OCIS codes: (060.2320) Fiber optics amplifiers and oscillators; (190.4380) Nonlinear optics, four-wave mixing; (270.6570) Squeezed states.

References and links

1. M. E. Marhic, *Fiber Optical Parametric Amplifiers, Oscillators and Related Devices* (Cambridge, 2007).
2. J. Hansryd, P. A. Andrekson, M. Westlund, J. Li and P. O. Hedekvist, "Fiber-based optical parametric amplifiers and their applications," IEEE J. Sel. Top. Quantum Electron. **8**, 506–520 (2002).
3. J. H. Lee, "All-optical signal processing devices based on holey fiber," IEICE Trans. Electron. **E88-C**, 327–334 (2005).
4. S. Radic and C. J. McKinstrie, "Optical amplification and signal processing in highly nonlinear optical fiber," IEICE Trans. Electron. **E88-C**, 859–869 (2005).
5. P. A. Andrekson and M. Westlund, "Nonlinear optical fiber based high resolution all-optical waveform sampling," Laser Photon. Rev. **1**, 231–248 (2007).
6. S. Radic, "Parametric signal processing," IEEE J. Sel. Top. Quantum Electron. **18**, 670–680 (2012).
7. R. Loudon, *The Quantum Theory of Light, 3rd Ed.* (Oxford, 2000).
8. M. Fiorentino, P. L. Voss, J. E. Sharping and P. Kumar, "All-fiber photon-pair source for quantum communications," IEEE Photon. Technol. Lett. **14**, 983–985 (2002).
9. M. Halder, J. Fulconis, B. Cemlyn, A. Clark, C. Xiong, W. J. Wadsworth and J. G. Rarity, "Nonclassical 2-photon interference with separate intrinsically narrowband fibre sources," Opt. Express **17**, 4670–4676 (2009).
10. O. Cohen, J. S. Lundeen, B. J. Smith, G. Puentes, P. J. Mosley and I. A. Walmsley, "Tailored photon-pair generation in optical fibers," Phys. Rev. Lett. **102**, 123603 (2009).
11. K. Inoue, "Polarization effect on four-wave mixing efficiency in a single-mode fiber," IEEE J. Quantum Electron. **28**, 883–894 (1992).
12. C. J. McKinstrie, H. Kogelnik, R. M. Jopson, S. Radic and A. V. Kanaev, "Four-wave mixing in fibers with random birefringence," Opt. Express **12**, 2033–2055 (2004).

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

13. M. E. Marhic, K. K. Y. Wong and L. G. Kazovsky, "Fiber optic parametric amplifiers with lineary or circularly polarized waves," J. Opt. Soc. Am. B **20**, 2425–2433 (2003).
14. C. J. McKinstrie, H. Kogelnik and L. Schenato, "Four-wave mixing in a rapidly-spun fiber," Opt. Express **14**, 8516–8534 (2006).
15. G. W. Stewart, "On the early history of the singular value decomposition," SIAM Rev. **35**, 551–566 (1993).
16. G. J. Gbur, *Mathematical Methods for Optical Physics and Engineering* (Cambridge, 2011), Sec. 5.4.
17. A. K. Ekert and P. L. Knight, "Relationship between semiclassical and quantum-mechanical input-output theories of optical response," Phys. Rev. A **43**, 3934–3938 (1991).
18. S. L. Braunstein, "Squeezing as an irreducible resource," Phys. Rev. A **71**, 055801 (2005).
19. H. P. Yuen, "Two-photon states of the radiation field," Phys. Rev. A **13**, 2226–2243 (1976).
20. C. M. Caves, "Quantum limits on noise in linear amplifiers," Phys. Rev. D **26**, 1817–1839 (1982).
21. C. K. Law, I. A. Walmsley and J. H. Eberly, "Continuous frequency entanglement: Effective finite Hilbert space and entropy control," Phys. Rev. Lett. **84**, 5304–5307 (2000).
22. W. P. Grice, A. B. U'Ren and I. A. Walmsley, "Eliminating frequency and space-time correlations in multiphoton states," Phys. Rev. A **64**, 063815 (2001).
23. M. G. Raymer, S. J. van Enk, C. J. McKinstrie and H. J. McGuinness, "Interference of two photons of different color," Opt. Commun. **283**, 747–752 (2010).
24. C. J. McKinstrie, L. Mejling, M. G. Raymer and K. Rottwitt, "Quantum-state-preserving optical pulse reshaping and multiplexing by four-wave mixing in fibers," Phys. Rev. A **85**, 053829 (2012).
25. C. J. McKinstrie, M. Yu, M. G. Raymer and S. Radic, "Quantum noise properties of parametric processes," Opt. Express **13**, 4986–5012 (2005).
26. Z. Tong, C. Lundström, P. A. Andrekson, M. Karlsson and A. Bogris, "Ultralow noise, broadband phase-sensitive optical amplifiers and their applications," IEEE J. Sel. Top. Quantum Electron. **18**, 1016–1032 (2012).
27. E. Telatar, "Capacity of multi-antenna Gaussian channels," Euro. Trans. Telecom. **10**, 585–595 (1999).
28. C. J. McKinstrie and N. Alic, "Information efficiencies of parametric devices," J. Sel. Top. Quantum Electron. **18**, 794–811 (2012).
29. C. J. McKinstrie, "Unitary and singular value decompositions of parametric processes in fibers," Opt. Commun. **282**, 583–593 (2009).
30. D. A. Edwards, J. D. Fehribach, R. O. Moore and C. J. McKinstrie, "An application of matrix theory to the evolution of coupled modes," to appear in SIAM Rev.
31. C. J. McKinstrie and S. Radic, "Phase-sensitive amplification in a fiber," Opt. Express **12**, 4973–4979 (2004).
32. K. Croussore and G. Li, "Phase and amplitude regeneration of differential phase-shift keyed signals using phase-sensitive amplification," IEEE J. Sel. Top. Quantum Electron. **14**, 648–658 (2008).
33. J. M. Manley and H. E. Rowe, "Some general properties of nonlinear elements—Part I. General energy relations," Proc. IRE **44**, 904–913 (1956).
34. M. T. Weiss, "Quantum derivation of energy relations analogous to those for nonlinear reactances," Proc. IRE **45**, 1012–1013 (1957).
35. A. I. Lvovsky, W. Wasilewski and K. Banaszek, "Decomposing a pulsed optical parametric amplifier into independent squeezers," J. Mod. Opt. **54**, 721–733 (2007).
36. C. J. McKinstrie, M. G. Raymer and H. J. McGuinness, "Spatial-temporal evolution of asymmetrically-pumped phase conjugation I: General formalism," Alcatel-Lucent ITD-09-48636Q, available upon request.
37. H. Goldstein, *Classical Mechanics, 2nd Ed.* (Addison-Wesley, 1980).
38. V. I. Arnold, *Mathematical Methods of Classical Mechanics, 2nd Ed.* (Springer, 2000).
39. D. H. Sattinger and O. L. Weaver, *Lie groups and Algebras with Applications to Physics, Geometry and Mechanics* (Springer, 1986).
40. M. Hamermesh, *Group Theory and its Application to Physical Problems* (Dover, 1989).
41. H. Takenaka, "A unified formalism for polarization optics by using group theory," Nou. Rev. Opt. **4**, 37–41 (1973).
42. Y. S. Kim and M. E. Noz, "Illustrative examples of the symplectic group," Am. J. Phys. **51**, 368–375 (1983).
43. A. Mufti, H. A. Schmitt and M. Sargent, "Finite-dimensional matrix representations as calculational tools in quantum optics," Am. J. Phys. **61**, 729–733 (1993).
44. C. C. Gerry, "Remarks on the use of group theory in quantum optics," Opt. Express **8**, 76–85 (2001).

---

1. Introduction

Parametric devices based on four-wave mixing (FWM) in fibers can amplify, frequency convert, phase conjugate, regenerate and sample optical signals in classical communication systems [1–6]. They can also generate photon pairs for quantum information experiments [7–10]. Three different types of FWM are illustrated in Fig. 1. Modulation interaction (MI) is the degenerate process in which two photons from the same pump are destroyed, and signal and idler (sideband) photons are created ($2\pi_p \rightarrow \pi_s + \pi_i$, where π_j represents a photon with frequency ω_j). Inverse MI is the degenerate process in which two photons from different pumps are destroyed and two signal photons are created ($\pi_p + \pi_q \rightarrow 2\pi_s$). Phase conjugation (PC) is the nondegenerate process in which two different pump photons are destroyed and two different sideband photons are created ($\pi_p + \pi_q \rightarrow \pi_s + \pi_i$). The polarization properties of these processes are reviewed in [11–14].

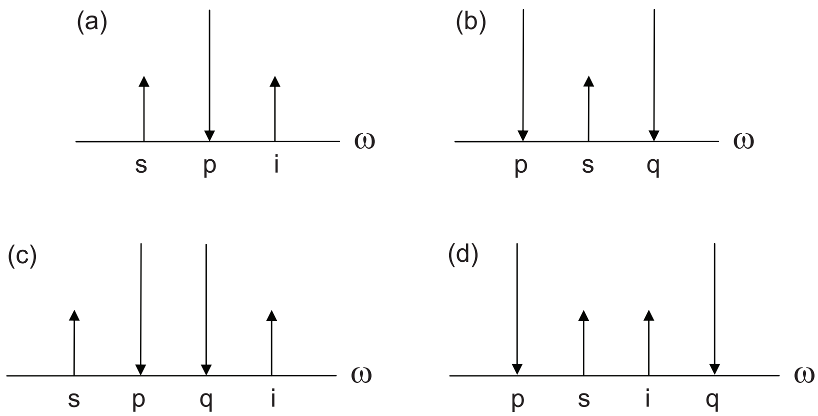

Fig. 1. Frequency diagrams for (*a*) modulation interaction, (*b*) inverse modulation interaction, and (*c*) outer-band and (*d*) inner-band phase conjugation. Long arrows denote pumps (*p* and *q*), whereas short arrows denote sidebands (*s* and *i*). Downward arrows denote modes that lose photons, whereas upward arrows denote modes that gain photons.

Parametric interactions of weak sidebands, driven by strong pumps, are governed by coupled-mode equations (CMEs) of the form

$$d_z X = AX + BX^*, \tag{1}$$

where z is distance, $d_z = d/dz$, $X = [x_j]$ is the vector of sideband amplitudes (modes), $A = [\alpha_{jk}]$ and $B = [\beta_{jk}]$ are coefficient matrices, and $*$ denotes a complex conjugate. The entries of the amplitude vector could be the amplitudes of distinct monochromatic sidebands (continuous waves), or different frequency components of multichromatic sidebands (pulses), with one or two polarization components. For uniform fibers (media) the coupling coefficients are constants, whereas for nonuniform media they vary with distance. Because Eq. (1) is linear in the amplitude vector and its conjugate, the (explicit or implicit) solution of Eq. (1) can be written in the input–output (IO) form

$$X(z) = M(z)X(0) + N(z)X^*(0), \tag{2}$$

where $M = [\mu_{jk}]$ and $N = [\nu_{jk}]$ are transfer (Green) matrices.

For the aforementioned one- and two-mode interactions (scalar MI and PC of continuous waves), it is easy to solve the CMEs and interpret the IO relations. However, for multiple-mode interactions, the CMEs and IO relations are complicated and two related questions arise: Under what conditions can one solve the CMEs explicitly and how should one interpret the IO relations physically?

Recall that every complex matrix M has the singular value (Schmidt) decomposition $M = VDU^\dagger$, where U and V are unitary and D is diagonal [15, 16]. The columns of U are the eigenvectors of $M^\dagger M$, the columns of V are the eigenvectors of $MM^\dagger$, and the entries of D are the (common) non-negative eigenvalues of $M^\dagger M$ and $MM^\dagger$. In the context of parametric interactions, the laws of Hamiltonian mechanics impose constraints on the transfer matrices M and N, which ensure that they have the simultaneous (related) decompositions $M = VD_\mu U^\dagger$ and $N = VD_\nu U^t$, where $D_\mu = \mathrm{diag}(\mu_j)$, $D_\nu = \mathrm{diag}(\nu_j)$ and their entries (Schmidt coefficients) satisfy the auxiliary equations $\mu_j^2 - \nu_j^2 = 1$ [17,18]. Hence, Eq. (2) can be rewritten in the form

$$X(z) = VD_\mu U^\dagger X(0) + VD_\nu U^t X^*(0), \tag{3}$$

where U, V, D_μ and D_ν depend implicitly on z. It follows from Eq. (3) that the columns of U define input (Schmidt) mode vectors, the columns of V define output vectors, and the mode amplitudes $\overline{x}_j(0)$ and $\overline{x}_j(z)$, which are the components of X relative to the input and output Schmidt bases, respectively, satisfy the one-mode squeezing equations

$$\overline{x}_j(z) = \mu_j(z)\overline{x}_j(0) + \nu_j(z)\overline{x}_j^*(0). \tag{4}$$

Equations (3) and (4) are remarkable. They imply that every parametric process, no matter how complicated, can be decomposed into independent squeezing processes, the properties of which are known (in-phase signal quadratures are stretched, whereas out-of-phase quadratures are squeezed) [7,19,20]. This decomposition is the answer to the second of the aforementioned questions. The first question is harder to answer, but in a forthcoming paper it will be shown that the (known) form of the decomposition sometimes provides guidance that facilitates the solution of the CMEs.

The simultaneous Schmidt decomposition is more than an elegant mathematical result. Determining the Schmidt modes and coefficients of a parametric process, and the device based upon it, facilitates the optimization of device performance. For example, in photon pair generation by pulsed pumps, the Schmidt modes are the (temporal) wave-packets of the signal and idler photons, and the squares of the Schmidt coefficients are the probabilities with which photon pairs are produced [21,22]. If one designs the system so that only one coefficient is nonzero, the output state is pure, as required for a variety of quantum information experiments. In photon frequency conversion by pulsed pumps, the Schmidt modes are the natural input and output wave-packets of the signal and idler photons, and the squares of the Schmidt coefficients are the conversion probabilities [23, 24]. One can optimize two-photon interference (or single-photon conversion) experiments by designing the system so that at least one squared coefficient is 0.5 (or 1.0). It is well known that one-mode squeezing and stretching processes dilate the coherent and incoherent parts of input signals by the same amounts. The former processes are of interest in quantum optics, because the out-of-phase quadratures have smaller fluctuations than vacuum quadratures [19, 20], whereas the latter are of interest in optical communications, because they can amplify in-phase signals without degrading their signal-to-noise ratios [25, 26]. To minimize the effects of noise in a communication system, one should encode information in the Schmidt modes of the system (superpositions of frequency or polarization components), not the physical modes (individual components). Encoding information in this way also maximizes the information capacity of the system [27,28].

This paper is organized as follows: In Sec. 2, one- and two-mode parametric amplification are analyzed in detail. The transfer matrices for these processes are determined explicitly by solving the CMEs analytically. These matrices are shown to have several interesting and useful properties, which are not accidental. In Sec. 3, the Schmidt decomposition theorem is proved constructively (from first principles) and the aforementioned properties are established. Previous derivations of these results were based on the laws of quantum optics [17, 18], in which context they are standard [21–24]. In contrast, the present derivation is based solely on the laws of classical mechanics, which are less fundamental, but sufficient. Readers who are interested in Schmidt decompositions and their applications can learn about them easily, unburdened by

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

the complexities of quantum optics. In Sec. 4 the specific results derived in Sec. 3 are related to general properties of Hamiltonian systems. Finally, in Sec. 5 the main results of this paper are summarized.

Aspects of Schmidt decompositions were also discussed in [29, 30]. In these articles, the mathematical properties of Schmidt decompositions, and the adjoint decompositions to which they are related, were described in detail, and some simple examples (including one- and two-mode amplification) were mentioned briefly. In this paper, the physical consequences of Schmidt decompositions are emphasized, the aforementioned simple examples are discussed in detail and key results are explained in the context of Hamiltonian dynamics. In a forthcoming paper, new solutions of the CMEs are obtained and used to discuss several nontrivial examples of current interest (including four-mode parametric amplification).

2. Simple examples of Schmidt decompositions

In this section, two simple parametric processes are considered, for which the Schmidt decompositions are easy to determine. These examples provide useful checks and illustrations of the general results derived in the next section.

2.1. One-mode amplification

Consider a one-mode parametric process (inverse MI), which is governed by the equation

$$d_z x = i\delta x + i\gamma x^*, \tag{5}$$

where x is the mode amplitude, δ is the (real) mismatch coefficient and γ is the (complex) coupling coefficient [31, 32]. Equation (5) depends linearly on x and x^*, so its solution can be written in the input–output (IO) form

$$x(z) = \mu(z)x(0) + \nu(z)x^*(0). \tag{6}$$

For the common case in which δ and γ are constants, the transfer (Green) functions

$$\mu(z) = \cos(kz) + i\delta\sin(kz)/k, \quad \nu(z) = i\gamma\sin(kz)/k, \tag{7}$$

and the characteristic wavenumber $k = (\delta^2 - |\gamma|^2)^{1/2}$. If coupling is stronger than mismatch ($|\gamma| > \delta$), the system is unstable. The transfer functions (7) satisfy the auxiliary equation

$$|\mu(z)|^2 - |\nu(z)|^2 = 1. \tag{8}$$

They also have the interesting properties

$$\mu(-z) = \mu^*(z), \quad \nu(-z) = -\nu(z). \tag{9}$$

Furthermore, if γ is real, then $\nu^*(z) = -\nu(z)$. These properties are not accidental.

Because the transfer functions usually have different phases, some analysis is required to determine the consequences of Eq. (6). Let $\mu = |\mu|e^{i\phi_\mu}$ and $\nu = |\nu|e^{i\phi_\nu}$, and define the sum and difference phases $\phi_s = (\phi_\mu + \phi_\nu)/2$ and $\phi_d = (\phi_\nu - \phi_\mu)/2$, respectively. Then Eq. (6) can be rewritten in the form

$$x(z) = v|\mu|u^* x(0) + v|\nu|u x^*(0), \tag{10}$$

where $u = e^{i\phi_d}$ and $v = e^{i\phi_s}$ are phase factors (input and output phase references). If the signal phase $\phi_x = \phi_u = \phi_d$, the terms on the right side of Eq. (10) add constructively: The signal is said to be in-phase and is amplified (stretched) by the factor $|\mu| + |\nu|$. Conversely, if $\phi_x = \phi_d + \pi/2$, the terms on the right side of Eq. (10) add destructively: The signal is said to be out-of-phase

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

and is attenuated (squeezed) by the factor $|\mu|+|\nu| = 1/(|\mu|-|\nu|)$. If one were to measure the phase of the input signal relative to the aforementioned reference phase, one would say that the real quadrature is amplified and the imaginary quadrature is attenuated. Notice that Eq. (10) has the canonical form of Eq. (3).

It is instructive to formalize the method of solution. Equation (5) and its conjugate can be written in the augmented matrix form

$$d_z Y = iL_y Y, \tag{11}$$

where the 2×1 mode vector $Y = [x, x^*]^t$ and the 2×2 coefficient matrix

$$L_y = \begin{bmatrix} \delta & \gamma \\ -\gamma^* & -\delta \end{bmatrix}. \tag{12}$$

Notice that L_y is specified by three real parameters (δ, γ_r and γ_i). The solution of Eq. (11) can be written in the IO form

$$Y(z) = T_y(z)Y(0), \tag{13}$$

where the transfer (Green) matrix

$$T_y(z) = \exp(iL_y z). \tag{14}$$

Two important results follow from Eqs. (12) and (14). First, $\mathrm{tr}(L_y) = 0$, so $\det(T_y) = 1$, and second, $T_y(-z) = T_y^{-1}(z)$. Because Eqs. (11) and (13) describe two copies of the same process (the original and its conjugate), the transfer matrix can be written in the form

$$T_y(z) = \begin{bmatrix} \mu(z) & \nu(z) \\ \nu^*(z) & \mu^*(z) \end{bmatrix}. \tag{15}$$

Notice that $\det(T_y) = |\mu|^2 - |\nu|^2 = 1$, so T_y is defined by three real parameters ($|\nu|$, ϕ_μ and ϕ_ν), the same number that specified L_y. Notice also that

$$T_y(-z) = \begin{bmatrix} \mu(-z) & \nu(-z) \\ \nu^*(-z) & \mu^*(-z) \end{bmatrix} = T_y^{-1}(z) = \begin{bmatrix} \mu^*(z) & -\nu(z) \\ -\nu^*(z) & \mu(z) \end{bmatrix}. \tag{16}$$

Hence, $\mu(-z) = \mu^*(z)$ and $\nu(-z) = -\nu(z)$, as stated in Eqs. (9).

It was stated in Sec. 1 that the Schmidt vectors of T_y are the eigenvectors of $T_y^\dagger T_y$ and $T_y T_y^\dagger$, and the Schmidt coefficients are the square roots of the (common) eigenvalues of these matrices. One can determine these eigensystems explicitly, or simply verify that

$$T_y(z) = \frac{1}{2^{1/2}} \begin{bmatrix} e^{i\phi_s} & e^{i\phi_s} \\ e^{-i\phi_s} & -e^{-i\phi_s} \end{bmatrix} \begin{bmatrix} |\mu|+|\nu| & 0 \\ 0 & |\mu|-|\nu| \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} e^{-i\phi_d} & e^{i\phi_d} \\ e^{-i\phi_d} & -e^{i\phi_d} \end{bmatrix}, \tag{17}$$

where ϕ_s and ϕ_d were defined before Eq. (10). All three matrices in Eq. (17) depend on z. The evolution equation (11) governs x and x^* simultaneously, so it is natural that the associated transfer matrix describes stretching and squeezing simultaneously. Specifically, Eq. (17) shows that the stretching condition is $2\phi_x = 2\phi_d$, whereas the squeezing condition is $2\phi_x = 2\phi_d + \pi$, and the associated Schmidt coefficients are reciprocals. The input Schmidt vectors, which are the natural inputs for one-mode amplification, correspond to in-phase and out-of-phase signals. These results are consistent with the discussion that follows Eq. (10).

Equations (15) and (16) show that there is a simple relation between the transfer matrix and its inverse. The replacements $\mu \to \mu^*$ and $\nu \to -\nu$ (which do not affect the moduli of the transfer functions) are equivalent to $\phi_\mu \to -\phi_\mu$ and $\phi_\nu \to \phi_\nu + \pi$ and, ultimately, to $e^{i\phi_s} \to ie^{i\phi_d}$

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

and $e^{i\phi_d} \to ie^{i\phi_s}$. By making these replacements in decomposition (17), one obtains the inverse decomposition

$$T_y^{-1}(z) = \frac{1}{2^{1/2}} \begin{bmatrix} ie^{i\phi_d} & ie^{i\phi_d} \\ -ie^{-i\phi_d} & ie^{-i\phi_d} \end{bmatrix} \begin{bmatrix} |\mu|+|\nu| & 0 \\ 0 & |\mu|-|\nu| \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} -ie^{-i\phi_s} & ie^{i\phi_s} \\ -ie^{-i\phi_s} & -ie^{i\phi_s} \end{bmatrix}. \tag{18}$$

For reference, neither the forward decomposition (17), nor the backward decomposition (18), is unique.

2.2. Two-mode amplification

Now consider a two-mode parametric process (MI or PC), which is governed by the CMEs

$$d_z x_1 = i\delta_1 x_1 + i\gamma x_2^*, \;\; d_z x_2 = i\delta_2 x_2 + i\gamma x_1^*, \tag{19}$$

where x_j is a mode amplitude and δ_j is a (real) mismatch coefficient [1, 2]. We will refer to mode 1 as the signal and mode 2 as the idler. The solutions of Eqs. (19) can be written in the IO forms

$$x_1(z) = \mu_{11}(z)x_1(0) + \nu_{12}(z)x_2^*(0), \;\; x_2(z) = \mu_{22}(z)x_2(0) + \nu_{21}(z)x_1^*(0). \tag{20}$$

If δ_j and γ are constants, the two-mode transfer functions $\mu_{11}(z) = e(z)\mu(z)$, $\nu_{12}(z) = e(z)\nu(z)$, $\mu_{22}(z) = e^*(z)\mu(z)$ and $\nu_{21}(z) = e^*(z)\nu(z)$, where the one-mode transfer functions $\mu(z)$ and $\nu(z)$ are defined by Eqs. (7), with the mismatch coefficient δ replaced by $(\delta_1+\delta_2)/2$, and the phase factor $e(z) = \exp[i(\delta_1-\delta_2)z/2] = \exp(i\phi_\delta)$. The two-mode transfer functions satisfy the auxiliary equations

$$|\mu_{11}(z)|^2 - |\nu_{12}(z)|^2 = 1, \;\; |\mu_{11}(z)|^2 - |\nu_{21}(z)|^2 = 1, \tag{21}$$

and have the interesting properties

$$\mu_{11}(-z) = \mu_{11}^*(z), \;\; \nu_{12}(-z) = -\nu_{21}(z). \tag{22}$$

Furthermore, if γ is real, then $\nu_{12}^*(z) = -\nu_{21}(z)$. One obtains additional constraints and properties by interchanging the subscripts 1 and 2. For the special case in which $\delta_1 = \delta_2$, solutions (20) reduce to solution (6).

In the first of Eqs. (20), the transfer functions have the common factor $e(z)$, which affects the output signal phase, but does not affect the interference conditions. Hence, if $\phi_1 + \phi_2 = \phi_\nu - \phi_\mu = 2\phi_d$, the terms in the first of Eqs. (20) add constructively: The sidebands are said to be in-phase and (if their input amplitudes are equal) are stretched by the factor $|\mu| + |\nu|$. Conversely, if $\phi_1 + \phi_2 = \phi_\nu - \phi_\mu + \pi$, the terms in the first of Eqs. (20) add destructively: The sidebands are said to be out-of-phase and (if their input amplitudes are equal) are squeezed by the factor $|\mu| + |\nu| = 1/(|\mu| - |\nu|)$. The same interference conditions apply to the second of Eqs. (20).

Equations (19) can be written in the standard matrix form

$$d_z X = iL_x X, \tag{23}$$

where the 2×1 mode vector $X = [x_1, x_2^*]^t$ and the 2×2 coefficient matrix

$$L_x = \begin{bmatrix} \delta_1 & \gamma \\ -\gamma^* & -\delta_2 \end{bmatrix}. \tag{24}$$

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

Notice that L_x is specified by four real parameters (δ_1, δ_2, γ_r and γ_i). Notice also that Eq. (23) is closed (involves only x_1 and x_2^*), so no augmentation is necessary. (The equation for $[x_1^*, x_2]^t$ is simply the conjugate of the stated equation and contains no extra information.) The solution of Eq. (23) can be written in the IO form

$$X(z) = T_x(z)X(0), \tag{25}$$

where the transfer matrix

$$T_x(z) = \exp(iL_x z). \tag{26}$$

The properties of L_x and T_x [Eqs. (24) and (26)] differ only slightly from those of L_y and T_y [Eqs. (12) and (14)]. Because $\mathrm{tr}(L_x) = \delta_1 - \delta_2 \neq 0$, $\det(T_x) = \exp[i(\delta_1 - \delta_2)z] \neq 1$. Nonetheless, $T_x(-z) = T^{-1}(z)$.

It follows from Eqs. (20) that the transfer matrix

$$T_x(z) = \begin{bmatrix} \mu_{11}(z) & \nu_{12}(z) \\ \nu_{21}^*(z) & \mu_{22}^*(z) \end{bmatrix} = e(z) \begin{bmatrix} \mu(z) & \nu(z) \\ \nu^*(z) & \mu^*(z) \end{bmatrix}. \tag{27}$$

Notice that T_x is determined by four real parameters ($|\nu|$, ϕ_μ, ϕ_ν and ϕ_δ), the same number that specified L_x. Notice also that

$$T_x(-z) = e(-z) \begin{bmatrix} \mu(-z) & \nu(-z) \\ \nu^*(-z) & \mu^*(-z) \end{bmatrix} = T_x^{-1}(z) = e^*(z) \begin{bmatrix} \mu^*(z) & -\nu(z) \\ -\nu^*(z) & \mu(z) \end{bmatrix}. \tag{28}$$

Hence, $\mu_{11}(-z) = \mu_{11}^*(z)$ and $\nu_{12}(-z) = -\nu_{21}(z)$, as stated in Eqs. (22).

The only difference between Eqs. (15) and (27) is the phase factor $e(z) = e^{i\phi_\delta}$, so it follows from Eq. (17) that the forward transfer matrix has the Schmidt decomposition

$$T_x(z) = \frac{e^{i\phi_\delta}}{2^{1/2}} \begin{bmatrix} e^{i\phi_s} & e^{i\phi_s} \\ e^{-i\phi_s} & -e^{-i\phi_s} \end{bmatrix} \begin{bmatrix} |\mu|+|\nu| & 0 \\ 0 & |\mu|-|\nu| \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} e^{-i\phi_d} & e^{i\phi_d} \\ e^{-i\phi_d} & -e^{i\phi_d} \end{bmatrix}, \tag{29}$$

where ϕ_s and ϕ_d were defined after Eq. (9), and ϕ_δ was defined after Eq. (20). Suppose that the input sideband phases are measured relative to the reference phase ϕ_d. Then decomposition (29) implies that the combination $x_1 + x_2^*$ is stretched, whereas the combination $x_1 - x_2^*$ is squeezed. For stretching, the optimal phase condition is $\phi_1 + \phi_2 = 0$, whereas for squeezing, the optimal phase condition is $\phi_1 + \phi_2 = \pi$. These results are consistent with the results stated after Eq. (22). It follows from Eq. (18) that the backward transfer matrix has the Schmidt decomposition

$$T_x^{-1}(z) = \frac{1}{2^{1/2}} \begin{bmatrix} ie^{i\phi_d} & ie^{i\phi_d} \\ -ie^{-i\phi_d} & ie^{-i\phi_d} \end{bmatrix} \begin{bmatrix} |\mu|+|\nu| & 0 \\ 0 & |\mu|-|\nu| \end{bmatrix} \frac{e^{-i\phi_\delta}}{2^{1/2}} \begin{bmatrix} -ie^{-i\phi_s} & ie^{i\phi_s} \\ -ie^{-i\phi_s} & -ie^{i\phi_s} \end{bmatrix}. \tag{30}$$

One can derive Eq. (30) from Eq. (29) by making the replacements $e^{i\phi_s} \to ie^{i\phi_d}$, $e^{i\phi_d} \to ie^{i\phi_s}$ and $e^{i\phi_\delta} \to e^{-i\phi_\delta}$.

It only remains to rewrite Eq. (25) in the canonical form of Eq. (3). By combining Eqs. (25) and (27), one finds that the transfer matrices

$$M = \begin{bmatrix} \mu_{11} & 0 \\ 0 & \mu_{22} \end{bmatrix} = \begin{bmatrix} e\mu & 0 \\ 0 & e^*\mu \end{bmatrix}, \tag{31}$$

$$N = \begin{bmatrix} 0 & \nu_{12} \\ \nu_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e\nu \\ e^*\nu & 0 \end{bmatrix}. \tag{32}$$

These matrices have the Schmidt-like decompositions

$$M = \frac{1}{2^{1/2}} \begin{bmatrix} e^{i\phi_\delta} & ie^{i\phi_\delta} \\ e^{-i\phi_\delta} & -ie^{-i\phi_\delta} \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad (33)$$

$$N = \frac{1}{2^{1/2}} \begin{bmatrix} e^{i\phi_\delta} & ie^{i\phi_\delta} \\ e^{-i\phi_\delta} & -ie^{-i\phi_\delta} \end{bmatrix} \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad (34)$$

It is the difference between the last matrices in Eqs. (33) and (34) that allows M to be diagonal and N to be off-diagonal. Decompositions (33) and (34) are not quite in canonical form, because the elements of the diagonal matrices (μ and ν) are complex. However, by generalizing the derivation of Eq. (10), one obtains the Schmidt decompositions

$$M = \frac{e^{i\phi_s}}{2^{1/2}} \begin{bmatrix} e^{i\phi_\delta} & ie^{i\phi_\delta} \\ e^{-i\phi_\delta} & -ie^{-i\phi_\delta} \end{bmatrix} \begin{bmatrix} |\mu| & 0 \\ 0 & |\mu| \end{bmatrix} \frac{e^{-i\phi_d}}{2^{1/2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad (35)$$

$$N = \frac{e^{i\phi_s}}{2^{1/2}} \begin{bmatrix} e^{i\phi_\delta} & ie^{i\phi_\delta} \\ e^{-i\phi_\delta} & -ie^{-i\phi_\delta} \end{bmatrix} \begin{bmatrix} |\nu| & 0 \\ 0 & |\nu| \end{bmatrix} \frac{e^{i\phi_d}}{2^{1/2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad (36)$$

For reference, decompositions (29), (30), (35) and (36) are not unique.

At this point, it is instructive to introduce the superposition modes

$$x_\pm = (x_1 \pm x_2)/2^{1/2}. \quad (37)$$

By combining Eqs. (19), one obtains the superposition-mode equations

$$d_z x_\pm = i\delta_s x_\pm + i\delta_d x_\mp \pm i\gamma x_\pm^*, \quad (38)$$

where the mismatch coefficients $\delta_s = (\delta_1 + \delta_2)/2$ and $\delta_d = (\delta_1 - \delta_2)/2$. For the special case in which $\delta_d = 0$, the sum $(+)$ and difference $(-)$ modes evolve independently: Each mode undergoes a one-mode parametric process that is governed by Eq. (4) or (6). Because the coupling coefficients in Eqs. (38) differ by a factor of -1, the input phases required for stretching differ by $\pi/2$, as do the input phases required for squeezing. This method of analysis fails for the general case in which $\delta_d \neq 0$. However, the concept of a superposition mode remains useful.

Decompositions (35) and (36), which are valid for arbitrary values of δ_d, show that it is appropriate to measure the input sideband phases relative to the (common) reference phase ϕ_d, as did decomposition (29). With this convention, the real quadrature of the sum mode and the imaginary quadrature of the difference mode are stretched. It follows from Eq. (4), in which the transfer functions are non-negative by construction, that the imaginary sum quadrature and the real difference quadrature are squeezed.

In summary, the Schmidt decompositions (17), (29), (35) and (36) reveal automatically the input-phase conditions required for stretching and squeezing [26]. The decompositions of the augmented and standard transfer matrices feature stretched and squeezed modes, whereas the decompositions of the canonical matrices feature only stretched modes (from which the squeezed modes can be deduced). These decompositions also lead to the concept of superposition modes: The Schmidt modes of a system are the natural superposition modes of that system.

3. Basic theory of Schmidt decompositions

In the preceding section, the Schmidt decompositions of two simple parametric processes were determined explicitly, and were shown to have interesting and useful properties. In this section,

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

the basic properties of such decompositions are established for arbitrary parametric processes. The evolution of a conservative system of m coupled modes is governed by the Hamiltonian

$$H = X^\dagger J X + X^\dagger K X^* + X^t K^* X, \tag{39}$$

where X is an $m \times 1$ mode-amplitude vector, and J and K are $m \times m$ coefficient matrices. In order for the first term on the right side of Eq. (39) to be real, J must be Hermitian. The sum of the second and third terms is real by construction, and one can always write these terms in such a way that K is symmetric. For reference, J is specified by (up to) m^2 real parameters, whereas K is specified by $m(m+1)$ real parameters. By applying the (complex) Hamilton equation

$$d_z X = i\partial H/\partial X^\dagger \tag{40}$$

to Hamiltonian (39), one obtains the CME

$$d_z X = iJX + iKX^*. \tag{41}$$

(The complex Hamiltonian formalism is reviewed in the Appendix.) Equation (41) depends linearly on X and X^*, so its solution can be written in the IO form

$$X(z) = M(z)X(0) + N(z)X^*(0), \tag{42}$$

where M and N are $m \times m$ transfer matrices. [Equation (41) is just Eq. (1), with $A = iJ$ and $B = iK$, and Eq. (42) is just Eq. (2), repeated for convenience.] For the special case in which $m = 1$, $J = \delta$ and $K = \gamma$, and Eq. (41) reduces to Eq. (5).

Alternatively, one can write Eq. (41) and its conjugate as the augmented matrix equation

$$d_z Y = iL_y Y, \tag{43}$$

where the $2m \times 1$ mode vector and $2m \times 2m$ coefficient matrix are

$$Y = \begin{bmatrix} X \\ X^* \end{bmatrix}, \quad L_y = \begin{bmatrix} J & K \\ -K^* & -J^* \end{bmatrix}, \tag{44}$$

respectively. The solution of Eq. (43) can be written in the IO form

$$Y(z) = T_y(z)Y(0), \tag{45}$$

where T_y is the $2m \times 2m$ transmission matrix. If L_y is a constant matrix, then

$$T_y(z) = \exp(iL_y z). \tag{46}$$

Because Eq. (43) describes two copies of the same process (the original and its conjugate), the $m \times m$ blocks of T_y are the transfer matrices M and N, which appeared in Eq. (42), and their conjugates [see Eq. (15)]. Clearly, Eqs. (43)–(46) are generalizations of Eqs. (11)–(14).

In general, each component of X is coupled to every component of X and X^*. However, there are many important systems in which a subset of the components of X (denoted by X_1 and called the signal vector) is coupled to itself and a different subset of X^* (denoted by X_2^* and called the idler vector). Such systems, which include the MI- and PC-based systems described in the introduction, are governed by the Hamiltonian

$$H = X_1^\dagger J_1 X_1 + X_2^\dagger J_2 X_2 + X_1^\dagger K X_2^* + X_1^t K^* X_2, \tag{47}$$

where J_1, J_2 and K are coefficient matrices. J_1 and J_2 are Hermitian, whereas K is arbitrary. For definiteness, suppose that X_1 and X_2 are $n \times 1$ vectors, where $2n \leq m$, so J_1, J_2 and K are

$n \times n$ matrices. Then J_1 and J_2 are each specified by (up to) n^2 real parameters, whereas K is specified by $2n^2$ real parameters. By applying the Hamilton equations

$$d_z X_j = i\partial H/\partial X_j^\dagger \tag{48}$$

to Hamiltonian (47), one obtains the CMEs

$$d_z X_1 = iJ_1 X_1 + iKX_2^*, \;\; d_z X_2 = iJ_2 X_2 + iK^t X_1^*. \tag{49}$$

The solutions of Eqs. (49) can be written in the IO forms

$$X_1(z) = M_{11}(z)X_1(0) + N_{12}(z)X_2^*(0), \;\; X_2(z) = M_{22}(z)X_2(0) + N_{21}(z)X_1^*(0), \tag{50}$$

where M_{11}, N_{12}, M_{22} and N_{21} are $n \times n$ transfer matrices. For the special case in which $n = 1$, $J_1 = \delta_1$, $J_2 = \delta_2$ and $K = \gamma$, and Eqs. (49) reduce to Eqs. (19).

Alternatively, Eqs. (49) can be rewritten as the standard matrix equation

$$d_z X = iL_x X, \tag{51}$$

where the $2n \times 1$ mode vector and $2n \times 2n$ coefficient matrix are

$$X = \begin{bmatrix} X_1 \\ X_2^* \end{bmatrix}, \;\; L_x = \begin{bmatrix} J_1 & K \\ -K^\dagger & -J_2^* \end{bmatrix}, \tag{52}$$

respectively. The solution of Eq. (51) can be written in the IO form

$$X(z) = T_x(z)X(0), \tag{53}$$

where T_x is the $2n \times 2n$ transmission matrix. If L_x is a constant matrix, then

$$T_x(z) = \exp(iL_x z). \tag{54}$$

The $n \times n$ blocks of T_x are M_{11}, N_{12}, N_{21}^* and M_{22}^* [see Eq. (27)]. Clearly, Eqs. (51)–(54) are generalizations of Eqs. (23)–(26).

Although the special system described by Eq. (51) is of lower order that the general system described by Eq. (43), its mathematical structure is more complicated, because the diagonal blocks of L_x are not necessarily equal and the off-diagonal blocks are not necessarily symmetric, as are the corresponding blocks of L_y [Eqs. (44) and (52)]. Hence, we will derive the properties of the special system and deduce the corresponding properties of the general system.

The special system is defined by Eqs. (51) and (52). Although L_x is not Hermitian, it is closely related to Hermitian matrices. Define the (spin-like) matrix

$$S = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{55}$$

Then it is easy to show that $L_x = SH_1 = H_2 S$, where H_1 and H_2 are different Hermitian matrices. Because $S = S^\dagger = S^{-1}$, the first of these equations implies that

$$SL_x = L_x^\dagger S. \tag{56}$$

Equation (56) has several important (mathematical and physical) consequences.

In two-mode amplification, the Manley-Rowe-Weiss (MRW) variable $c = |x_1|^2 - |x_2|^2$ is conserved [33, 34]. The classical (quantal) interpretation of this result is that the difference between the action (photon) fluxes of the signal and idler is conserved (sideband photons are

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

produced in pairs). In the present context, the MRW variable $C = X_1^\dagger X_1 - X_2^t X_2^* = X^\dagger S X$. By combining this definition with Eq. (51), one finds that

$$d_z C = iX^\dagger(SL_x - L_x^\dagger S)X. \tag{57}$$

Thus, Eq. (56) is both a necessary and sufficient condition for the conservation of action (photon) flux.

Starting from Eq. (56), one can prove by induction that

$$SL_x^n = (L_x^\dagger)^n S. \tag{58}$$

By combining Eqs. (54) and (58), one finds that

$$ST_x(-z) = T_x^\dagger(z)S. \tag{59}$$

Equation (54) implies that

$$T_x(-z) = T_x^{-1}(z), \tag{60}$$

independent of the properties of L_x. By using this result, one can show that Eq. (59) is equivalent to the equations $T_x^{-1}(z) = ST_x^\dagger(z)S$ and $T_x(z)ST_x^\dagger(z) = S$. The first equation provides a useful formula for the inverse transfer matrix (if the forward matrix is known, so also is the backward matrix), whereas the second provides a link to the theory of Hamiltonian systems (which will be described in Sec. 4).

The Schmidt decomposition theorem [16] states that the transfer matrix $T_x = VDU^\dagger$, where U and V are unitary, and D is positive and diagonal. (Positivity is required because T_x is invertible.) The columns of U (input Schmidt vectors) are the eigenvectors of $T_x^\dagger T_x$, the columns of V (output Schmidt vectors) are the eigenvectors of $T_x T_x^\dagger$ and the entries of D (Schmidt coefficients) are the square roots of the (common) eigenvalues of $T_x^\dagger T_x$ and $T_x T_x^\dagger$. The stated decomposition is not unique: One can multiply the columns of U and V by the same set of phase factors, and one can permute (reorder) the columns of U and V in the same way, without invalidating the decomposition. Notice that the input vectors of $T_x^\dagger$ are the eigenvectors of $T_x T_x^\dagger$ and the output vectors of $T_x^\dagger$ are the eigenvectors of $T_x^\dagger T_x$. Hence, the input (output) vectors of $T_x^\dagger$ are the output (input) vectors of T_x and the Schmidt coefficients of $T_x^\dagger$ equal those of T_x.

Before determining the Schmidt decomposition in its entirety, we pause to prove an important result about the Schmidt coefficients. For any matrix A,

$$\det(\lambda I - A) = \det[S(\lambda I - A)S] = \det(\lambda I - SAS). \tag{61}$$

Hence, pre- and post-multiplying a matrix by S does not change its eigenvalues. (Furthermore, if E is an eigenvector of A, then SE is an eigenvector of SAS.) For any invertible matrix A and any other matrix B,

$$\det(\lambda I - AB) = \det[A^{-1}(\lambda I - AB)A] = \det(\lambda I - BA). \tag{62}$$

Hence, interchanging two matrices does not change the eigenvalues of their product. Equation (62) implies that $T_x^\dagger T_x$ and $T_x T_x^\dagger$ have the same (positive) eigenvalues, and the identity $T_x^{-1} = ST_x^\dagger S$ implies that

$$(T_x^\dagger T_x)^{-1} = T_x^{-1}(T_x^\dagger)^{-1} = (ST_x^\dagger S)(ST_x S) = S(T_x^\dagger T_x)S, \tag{63}$$

$$(T_x T_x^\dagger)^{-1} = (T_x^\dagger)^{-1} T_x^{-1} = (ST_x S)(ST_x^\dagger S) = S(T_x T_x^\dagger)S. \tag{64}$$

Hence, if λ is an eigenvalue of $T_x^\dagger T_x$ or $T_x T_x^\dagger$, so also is λ^{-1}. (Furthermore, if E is the eigenvector associated with λ, then SE is the eigenvector associated with λ^{-1}.) Because the Schmidt

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

coefficients of T_x are the square roots of these eigenvalues, they always occur in reciprocal pairs, as they did in Eqs. (17) and (29).

We now return to the Schmidt decomposition. Equations (59) and (60) impose several constraints on the block matrices M_{11}, N_{12}, N_{21} and M_{22}, which allow the decomposition to be determined. The former equation implies that

$$T_x(-z) = \begin{bmatrix} M_{11}(-z) & N_{12}(-z) \\ N_{21}^*(-z) & M_{22}^*(-z) \end{bmatrix} = ST_x^\dagger(z)S = \begin{bmatrix} M_{11}^\dagger(z) & -N_{21}^t(z) \\ -N_{12}^\dagger(z) & M_{22}^t(z) \end{bmatrix}. \tag{65}$$

By equating the blocks in Eq. (65), one finds that

$$\begin{aligned} M_{11}(-z) = M_{11}^\dagger(z), \quad N_{12}(-z) = -N_{21}^t(z), \\ M_{22}(-z) = M_{22}^\dagger(z), \quad N_{21}(-z) = -N_{12}^t(z). \end{aligned} \tag{66}$$

Notice that Eqs. (66) reduce to Eqs. (9) and (22) in the appropriate limits. The latter equation implies that

$$\begin{aligned} T_x^{-1}(z)T_x(z) &= \begin{bmatrix} M_{11}^\dagger & -N_{21}^t \\ -N_{12}^\dagger & M_{22}^t \end{bmatrix} \begin{bmatrix} M_{11} & N_{12} \\ N_{21}^* & M_{22}^* \end{bmatrix} \\ &= \begin{bmatrix} M_{11}^\dagger M_{11} - N_{21}^t N_{21}^* & M_{11}^\dagger N_{12} - N_{21}^t M_{22}^* \\ M_{22}^t N_{21}^* - N_{12}^\dagger M_{11} & M_{22}^t M_{22}^* - N_{12}^\dagger N_{12} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \end{aligned} \tag{67}$$

$$\begin{aligned} T_x(z)T_x^{-1}(z) &= \begin{bmatrix} M_{11} & N_{12} \\ N_{21}^* & M_{22}^* \end{bmatrix} \begin{bmatrix} M_{11}^\dagger & -N_{21}^t \\ -N_{12}^\dagger & M_{22}^t \end{bmatrix} \\ &= \begin{bmatrix} M_{11}M_{11}^\dagger - N_{12}N_{12}^\dagger & N_{12}M_{22}^t - M_{11}N_{21}^t \\ N_{21}^*M_{11}^\dagger - M_{22}^*N_{12}^\dagger & M_{22}^*M_{22}^t - N_{21}^*N_{21}^t \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{68}$$

The block matrices have the Schmidt decompositions $M_{11} = V_{11}D_{11}U_{11}^\dagger$, $N_{12} = V_{12}D_{12}U_{12}^\dagger$, $N_{21} = V_{21}D_{21}U_{21}^\dagger$ and $M_{22} = V_{22}D_{22}U_{22}^\dagger$. By substituting these decompositions in the diagonal terms in Eqs. (67) and (68), one finds that

$$U_{11}D_{11}^2U_{11}^\dagger - U_{21}^*D_{21}^2U_{21}^t = I, \tag{69}$$

$$U_{22}^*D_{22}^2U_{11}^t - U_{12}D_{12}^2U_{12}^\dagger = I, \tag{70}$$

$$V_{11}D_{11}^2V_{11}^\dagger - V_{12}D_{12}^2V_{12}^\dagger = I, \tag{71}$$

$$V_{22}^*D_{22}^2V_{22}^t - V_{21}^*D_{21}^2V_{21}^t = I. \tag{72}$$

Equation (69) can be rewritten as $U_{11}(D_{11}^2 - I)U_{11}^\dagger = U_{21}^*D_{21}^2U_{21}^t$. Because the unitary decomposition of a Hermitian matrix is unique (apart from phase shifts and ordering), $U_{21}^* = U_{11} = U_1$ and $D_{11}^2 - D_{21}^2 = I$. By continuing in this way, one finds that $U_{12} = U_{22}^* = U_2^*$ and $D_{22}^2 - D_{12}^2 = I$, $V_{12} = V_{11} = V_1$ and $D_{11}^2 - D_{12}^2 = I$, and $V_{21}^* = V_{22}^* = V_2^*$ and $D_{22}^2 - D_{21}^2 = I$. Hence, $D_{11} = D_{22} = D_\mu$ and $D_{12} = D_{21} = D_\nu$, where $D_\mu^2 - D_\nu^2 = I$. The equations associated with the off-diagonal terms in Eqs. (67) and (68) are satisfied identically. By assembling the preceding results, one finds that [35, 36]

$$T_x(z) = \begin{bmatrix} V_1D_\mu U_1^\dagger & V_1D_\nu U_2^t \\ V_2^*D_\nu U_1^\dagger & V_2^*D_\mu U_2^t \end{bmatrix}. \tag{73}$$

Notice that Eq. (73) does not define the overall phases of the blocks uniquely. If one were to replace U_j and V_j by $U_je^{i\phi_j}$ and $V_je^{i\phi_j}$, respectively, the diagonal blocks of the transfer matrix would be unaltered, whereas the off-diagonal blocks would be multiplied by $e^{i(\phi_1+\phi_2)}$ and

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

$e^{-i(\phi_1+\phi_2)}$. One can exploit this non-uniqueness to write some decompositions in particularly simple ways.

For each of the n sets of Schmidt modes in decomposition (73), there are two Schmidt coefficients ($|\mu|$ and $|\nu|$) and four phase combinations ($\phi_{v1}-\phi_{u1}$, $\phi_{v1}+\phi_{u2}$, $-\phi_{v2}-\phi_{u1}$ and $-\phi_{v2}+\phi_{u2}$). However, only one of the coefficients is independent ($|\nu|$) and only three of the combinations are independent. If one defines the reference phase $(\phi_{v1}-\phi_{v2}-\phi_{u1}+\phi_{u2})/2$, then the first and fourth combinations have the relative phases $\pm(\phi_{v1}+\phi_{v2}-\phi_{u1}-\phi_{u2})/2$, whereas the second and third combinations have the relative phases $\pm(\phi_{v1}+\phi_{v2}+\phi_{u1}+\phi_{u2})/2$. Thus, if the Schmidt modes are known, only $4n$ real parameters are required to specify the transfer matrix. [The other $4n(n-1)$ parameters in the coefficient matrix specify the Schmidt modes.] For the special case in which $J_1 = J_2$ and $K = K^t$, the signal and idler equations are identical, so $U_1 = U_2$, $V_1 = V_2$ and the reference phase is 0. Thus, if the Schmidt modes are known, only $3n$ real parameters are required to specify the transfer matrix. [The other $2n(n-1)$ parameters in the coefficient matrix specify the Schmidt modes.] These results are consistent with Eqs. (15) and (27), which apply to cases in which $n = 1$.

According to the Schmidt decomposition theorem, the Schmidt coefficients and input Schmidt modes are determined by the eigenvalues and eigenvectors of the Hermitian matrix

$$T_x^\dagger T_x = \begin{bmatrix} U_1(D_\nu^2+D_\nu^2)U_1^\dagger & U_1(2D_\mu D_\nu)U_2^t \\ U_2^*(2D_\mu D_\nu)U_1^\dagger & U_2^*(D_\mu^2+D_\nu^2)U_2^t \end{bmatrix}. \tag{74}$$

Short calculations show that this matrix has the eigensystem (written in block form)

$$D_\pm = (D_\mu \pm D_\nu)^2, \;\; E_\pm = [U_1^t, \pm U_2^\dagger]^t/2^{1/2}. \tag{75}$$

Similarly, the coefficients and output modes are determined by the eigenvalues and eigenvectors of the Hermitian matrix

$$T_x T_x^\dagger = \begin{bmatrix} V_1(D_\nu^2+D_\nu^2)V_1^\dagger & V_1(2D_\mu D_\nu)V_2^t \\ V_2^*(2D_\mu D_\nu)V_1^\dagger & V_2^*(D_\mu^2+D_\nu^2)V_2^t \end{bmatrix}. \tag{76}$$

Short calculations show that this matrix has the eigensystem

$$D_\pm = (D_\mu \pm D_\nu)^2, \;\; E_\pm = [V_1^t, \pm V_2^\dagger]^t/2^{1/2}. \tag{77}$$

Notice that in both eigensystems $E_\mp = SE_\pm$, as was predicted after Eq. (64). Hence, if $E_\pm$ are the eigenvectors of $T_x^\dagger T_x$ or $T_x T_x^\dagger$, then $SE_\pm = E_\mp$ are the eigenvectors of $ST_x^\dagger T_x S$ or $ST_x T_x^\dagger S$. It follows from Eqs. (75) and (77) that the transfer matrix (73) has the Schmidt decomposition

$$T_x(z) = \frac{1}{2^{1/2}} \begin{bmatrix} V_1 & V_1 \\ V_2^* & -V_2^* \end{bmatrix} \begin{bmatrix} D_\mu + D_\nu & 0 \\ 0 & D_\mu - D_\nu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} U_1^\dagger & U_2^t \\ U_1^\dagger & -U_2^t \end{bmatrix}, \tag{78}$$

where U_j, V_j, D_μ and D_ν depend implicitly on z. Let $X_1 = U_1\overline{X}_1$ and $X_2 = U_2\overline{X}_2$, where $\overline{X}_1 = [\overline{x}_{1j}]^t$ and $\overline{X}_2 = [\overline{x}_{2j}]^t$. Then decomposition (78) implies that the (input) combinations $\overline{x}_{1j} + \overline{x}_{2j}^*$ are stretched, whereas the combinations $\overline{x}_{1j} - \overline{x}_{2j}^*$ are squeezed. Notice that Eq. (78) reduces to Eqs. (17) and (29) in the appropriate limits.

If the forward transfer matrix can be written in the form (73), the backward transfer matrix can be written in the form

$$T_x^{-1}(z) = \begin{bmatrix} U_1 D_\mu V_1^\dagger & -U_1 D_\nu V_2^t \\ -U_2^* D_\nu V_1^\dagger & U_2^* D_\mu V_2^t \end{bmatrix}. \tag{79}$$

One could obtain Eq. (79) from Eq. (73) by interchanging the input and output modes, and changing the sign of D_ν. However, D_ν is non-negative by construction, so this empirical rule is not canonical.

It is instructive to consider the decomposition of the backward matrix, which can be calculated in (at least) three ways. First, $T_x = VDU^\dagger$, so the laws of matrix algebra require that $T_x^{-1} = UD^{-1}V^\dagger$. Hence,

$$T_x^{-1}(z) = \frac{1}{2^{1/2}} \begin{bmatrix} U_1 & U_1 \\ U_2^* & -U_2^* \end{bmatrix} \begin{bmatrix} D_\mu - D_\nu & 0 \\ 0 & D_\mu + D_\nu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} V_1^\dagger & V_2^t \\ V_1^\dagger & -V_2^t \end{bmatrix}. \tag{80}$$

In decomposition (80), which is consistent with Eq. (79), the input and output modes are interchanged, as are the associated stretching and squeezing factors.

Second, the identity $T_x^{-1} = ST_x^\dagger S$ requires that

$$T_x^{-1}(z) = \frac{1}{2^{1/2}} \begin{bmatrix} U_1 & U_1 \\ -U_2^* & U_2^* \end{bmatrix} \begin{bmatrix} D_\mu + D_\nu & 0 \\ 0 & D_\mu - D_\nu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} V_1^\dagger & -V_2^t \\ V_1^\dagger & V_2^t \end{bmatrix}. \tag{81}$$

In decomposition (81), which is also consistent with Eq. (79), the input and output modes are interchanged, as are the associated stretched and squeezed modes, but the stretching and squeezing factors are unaltered.

Decompositions (80) and (81) are equivalent only because Schmidt decompositions are not unique! To explore this issue, it is useful to define the permutation matrix

$$P = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{82}$$

When P acts to the right on a matrix, it interchanges the row blocks of that matrix, and when P acts to the left, it interchanges the column blocks. It is easy to verify that

$$D^{-1} = PDP. \tag{83}$$

Decomposition (80) works by permuting the Schmidt coefficients so that stretched modes are squeezed and *vice versa*. It is also easy to verify that

$$SU = UP, \;\; V^\dagger S = PV^\dagger. \tag{84}$$

Decomposition (81) works by permuting the Schmidt vectors so that squeezed modes are stretched and *vice versa*. These actions are equivalent ways to obtain the same result: In the (common) inversion formula $T_x^{-1} = UPDPV^\dagger$, the P matrices can act to the middle, or to the outsides.

Third, the identity $T_x(-z) = T_x^{-1}(z)$ implies that there are simple relations between the forward and backward Schmidt coefficients and vectors. Equations (73) and (79) do not determine these relations uniquely. However, by applying the transformations

$$D_\mu(-z) \to D_\mu(z), \;\; D_\nu(-z) \to D_\nu(z), \;\; U_j(-z) \to iV_j(z), \;\; V_j(-z) \to iU_j(z) \tag{85}$$

to Eq. (73), one does obtain Eq. (79). Transformations (85) allow D_μ and D_ν to remain positive. By applying them twice, one finds that $U_j(z) \to iV_j(-z) \to i^2U_j(z)$ and $V_j(z) \to iU_j(-z) \to i^2V_j(z)$. These results are acceptable in the context of a Schmidt decomposition, because the signs (phases) of the Schmidt modes are not unique. By applying transformations (85) to the forward decomposition (78), one obtains a backward decomposition that is similar to decomposition (81): The first unitary matrix is multiplied by i and the second is multiplied by $-i$, so the

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

decompositions are equivalent. Notice that transformations (85) are consistent with Eqs. (17) and (18), and Eqs. (29) and (30). In the former case $u_1 = u_2 = e^{i\phi_d}$ and $v_1 = v_2 = e^{i\phi_s}$, whereas in the latter case $u_1 = u_2 = e^{i\phi_d}$, $v_1 = e^{i\phi_s + i\phi_\delta}$ and $v_2 = e^{i\phi_s - i\phi_\delta}$.

It only remains to rewrite Eq. (73) in the canonical form of Eq. (3). The transfer matrices are

$$M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}, \tag{86}$$

where M_{11} and N_{12} are the upper blocks of T_x and N_{21} and M_{22} are the conjugates of the lower blocks. It is easy to verify that

$$M = \frac{1}{2^{1/2}} \begin{bmatrix} V_1 & iV_1 \\ V_2 & -iV_2 \end{bmatrix} \begin{bmatrix} D_\mu & 0 \\ 0 & D_\mu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} U_1^\dagger & U_2^\dagger \\ -iU_1^\dagger & iU_2^\dagger \end{bmatrix}, \tag{87}$$

$$N = \frac{1}{2^{1/2}} \begin{bmatrix} V_1 & iV_1 \\ V_2 & -iV_2 \end{bmatrix} \begin{bmatrix} D_\nu & 0 \\ 0 & D_\nu \end{bmatrix} \frac{1}{2^{1/2}} \begin{bmatrix} U_1^t & U_2^t \\ iU_1^t & -iU_2^t \end{bmatrix}. \tag{88}$$

Relative to the basis vectors contained in U_1 and U_2, the real quadratures of the sum modes and the imaginary quadratures of the difference modes are stretched. It follows from Eq. (4), in which the transfer functions are non-negative by construction, that the imaginary sum quadratures and the real difference quadratures are squeezed. These results are valid for the general case in which $J_1 \neq J_2$ and $K \neq K^t$. Notice that Eqs. (87) and (88) reduce to Eqs. (35) and (36) in the appropriate limit.

4. Unifying principles

The specific results derived in the preceding section are closely connected to the general properties of Hamiltonian dynamical systems [37,38]. To make these connections, one must rewrite the CMEs (49) in the simplest form possible. By using the fact that J_2 is Hermitian, one can rewrite Hamiltonian (47) in the alternative form

$$H = X_1^\dagger J_1 X_1 + X_2^t J_2^* X_2^* + X_1^\dagger K X_2^* + X_2^t K^\dagger X_1, \tag{89}$$

and by applying the alternative Hamilton equations

$$d_z X_1 = i\partial H/\partial X_1^\dagger, \;\; d_z X_2^* = -i\partial H/\partial X_2^t \tag{90}$$

to Hamiltonian (89), one can reproduce the aforementioned CMEs. Equations (89) and (90) are equivalent to the Hamiltonian

$$H = X^\dagger G X \tag{91}$$

and the single Hamilton equation

$$d_z X = iS\partial H/\partial X^\dagger, \tag{92}$$

where the mode vector and coefficient matrix are

$$X = \begin{bmatrix} X_1 \\ X_2^* \end{bmatrix}, \quad G = \begin{bmatrix} J_1 & K \\ K^\dagger & J_2^* \end{bmatrix}, \tag{93}$$

respectively, and the spin matrix S was defined in Eq. (55). Notice that G is Hermitian. (Consequently, if $L = SG$, then $SL = L^\dagger S$.) Equations (91) and (92) are said to be in canonical form.

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

Suppose that $X' = TX$, where T is an arbitrary transformation (change-of-variables) matrix. Then, in component form,

$$\begin{aligned} d_z x_i' &= \sum_j T_{ij} d_z x_j \\ &= i\sum_j \sum_k T_{ij} S_{jk} \partial H / \partial x_k^* \\ &= i\sum_j \sum_k \sum_l T_{ij} S_{jk} T_{lk}^* \partial H / \partial x_l'^* . \end{aligned} \quad (94)$$

A transformation is said to be canonical (symplectic) if the equation for x_i' has the same Hamiltonian form as the equation for x_i. Equation (94) shows that T is symplectic if and only if

$$\sum_j \sum_k \sum_l T_{ij} S_{jk} T_{lk}^* = S_{il}. \quad (95)$$

Condition (95) can be rewritten in the matrix form

$$TST^\dagger = S. \quad (96)$$

If condition (96) is satisfied, then $T^{-1} = ST^\dagger S$ and $(T^\dagger)^{-1} = STS$. For reference, the set of (nonsingular) matrices that satisfy condition (96) form a group with respect to multiplication.

Now suppose that $T(z)$ is the transfer matrix for the system, which satisfies the evolution equation

$$d_z T = iSGT, \quad (97)$$

together with the input condition $T(0) = I$. Then

$$d_z(T^\dagger ST) = i(T^\dagger S^2 GT - T^\dagger G^\dagger S^2 T) = 0, \quad (98)$$

because G is Hermitian. Hence,

$$T^\dagger ST = S. \quad (99)$$

Equation (99) implies that

$$(TX)^\dagger S(TX) = X^\dagger (T^\dagger ST)X = X^\dagger SX. \quad (100)$$

Hence, the MRW variable $X^\dagger SX$ is conserved. Equation (99) also implies that $T^{-1} = ST^\dagger S$ and $(T^\dagger)^{-1} = STS$.

The transfer matrix must also satisfy the symplectic condition (96), because the Hamilton equation (92) retains its form as X evolves. One can prove this statement directly. Alternatively, by multiplying the identity $T^\dagger ST = S$ by $S(T^\dagger)^{-1}$ on the left and $T^{-1}S$ on the right, one can show that $S = (TS)S(ST^\dagger) = TST^\dagger$. Hence, the transfer matrix satisfies the symplectic condition, which is equivalent to the MRW condition (99). Notice that the proofs of the preceding results were based on the assumption that T is a linear transformation, but not on the assumption that G is a constant: The results remain valid when G is a function of z.

The key results of the preceding section are consequences of the identity $T^{-1} = ST^\dagger S$. For example, if $(T^\dagger T)E = \lambda E$, where $\lambda \neq 0$, then

$$(T^\dagger T)SE = S(ST^\dagger S)(STS)E = S(T^\dagger T)^{-1}E = \lambda^{-1}SE. \quad (101)$$

Thus, not only do the Schmidt coefficients occur in reciprocal pairs, but there is also a simple relation between the associated Schmidt vectors. The Schmidt decomposition (73) owes its form to the constraints imposed on the blocks of the transfer matrix [Eqs. (67) and (68)], which are just $(ST^\dagger S)T = I$ and $T(ST^\dagger S) = I$. Decompositions (78), (79), (81), (87) and (88) all follow directly from decomposition (73). Furthermore, if $T = VDU^\dagger$, then

$$T^{-1} = S(UDV^\dagger)S = (SU)D(SV)^\dagger. \quad (102)$$

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

It is always true that the input (output) vectors of $T^\dagger$ are the output (input) vectors of T. Equations (101) and (102) show that the stretched modes for the forward transformation are the squeezed modes for the backward transformation. These relationships guarantee that the combined transformation is the identity transformation.

The preceding results show that the Schmidt decomposition of the transfer matrix owes its (extremely useful) form to the symplectic properties of the associated evolution equation, which is Hamiltonian. There are many physical processes for which knowledge of the underlying mathematical (algebraic) structure facilitates the derivation of important physical results. Consequently, it is worthwhile to review some definitions and make some specific connections.

Nonsingular matrices form a variety of groups with respect to multiplication. GL(m,C) is the general linear group, whose members are $m \times m$ complex matrices. SL(m,C) is the special linear group of degree m, whose members are unimodular (have determinant 1).

U(m) is the unitary group, whose members are $m \times m$ unitary matrices (which are complex by definition). The actions of these matrices preserve the quadratic form $X^\dagger X$, where X is an arbitrary $m \times 1$ vector. SU(m) is the special unitary group of degree m, whose members have determinant 1. It is sometimes called the unimodular unitary group. These groups occur in models of conservative phenomena, in which $X^\dagger X$ is the total power (or energy). For example, $U(2)$ and $SU(2)$ underly polarization rotation, beam splitting, directional coupling and (stable) frequency conversion.

U(n,n) is the pseudo-unitary group, whose members are $2n \times 2n$ complex matrices. The actions of these matrices preserve the quadratic form $X^\dagger SX$, where X is an arbitrary $2n \times 1$ vector. (This group has a subgroup of diagonal matrices that are unitary, but most of its members are nonunitary.) In the context of parametric amplification, $X^\dagger SX$ is the MRW variable and T_x is a member of U(n,n). SU(n,n) is the special pseudo-unitary group of degree $2n$, whose members have determinant 1. (This group also has a unitary subgroup.) In the aforementioned context, T_y is a member of SU(n,n). T_y is also a member of the symplectic group Sp($2n$), whose members satisfy condition (96) and have determinant 1. (In the Appendix, it is shown that this definition of the symplectic group is equivalent to the standard definition, which involves a different auxiliary matrix.) The mathematical properties of continuous groups are described in [39, 40] and several simple examples (including those mentioned above) are described in [41–44].

5. Summary

Parametric devices based on four-wave mixing in fibers perform many signal-processing functions required by optical communication systems. In these devices, strong pumps drive weak signal and idler sidebands, which can have one or two polarization components, and one or many frequency components. The evolutions of these components (modes) are governed by systems of coupled-mode equations [Eq. (1)], the solutions of which are specified by transfer matrices [Eq. (2)]. Schmidt decompositions of these transfer matrices [Eq. (3)] determine the natural input and output mode vectors of such systems, and the effects of the systems on the associated mode amplitudes [Eqs. (4)].

In Sec. 2, two simple examples were considered: one- and two-mode parametric amplification. The transfer matrices for these processes were determined explicitly, as were their Schmidt decompositions. In the first process, different quadratures of the same mode are stretched and squeezed (dilated), whereas in the second process, different combinations (superpositions) of the signal and idler modes are dilated. These superpositions are the Schmidt modes. For every mode (quadrature) that is stretched, there is another mode (quadrature) that is squeezed by the same amount. Furthermore, simple relations exist between the transfer matrices for the forward and backward processes [Eqs. (27) and (28)], and their Schmidt decompositions [Eqs. (29) and (30)]. These properties turn out to be generic.

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

In Sec. 3, the properties of Schmidt decompositions were derived from first principles, for parametric processes that involve $2n$ modes (n is arbitrary). The coefficient matrix that appears in the coupled-mode equation (51) has a symmetry property [Eq. (56)] that constrains the associated transfer matrix [Eq. (59)]. This constraint links the decompositions of the signal and idler blocks, and allows the decomposition of the forward transfer matrix to be determined [Eqs. (73) and (78)]. The transfer matrix involves $4n$ mode vectors, for the input and output signal and idler, which are equivalent to $2n$ input and output Schmidt vectors (combinations of the signal and idler vectors). In addition to these vectors, the transfer matrix involves (up to) $4n$ real parameters (n Schmidt coefficients and $3n$ phase factors). This number of parameters is much smaller than the number required to specify the aforementioned coefficient matrix, which is of order n^2. If the forward matrix and decomposition are known, so also are the backward matrix and decomposition [Eqs. (79) and (80)]: One obtains the latter entities from the former by interchanging the input and output vectors, and interchanging the stretching and squeezing factors.

In Sec. 4, the specific properties established for parametric processes were shown to be general properties of Hamiltonian dynamical systems. The symplectic identity [Eq. (96)] was shown to be equivalent to the Manley-Rowe-Weiss identity [Eq. (99)]. Either identity (together with the laws of matrix algebra) is sufficient to establish the main properties of Schmidt decompositions: For every Schmidt mode that is stretched in a forward (or backward) transformation, there is another, closely related, mode that is squeezed by the same amount. Furthermore, the forward and backward transformations have the same Schmidt coefficients, and the input squeezed (stretched) modes for the backward transformation are the output stretched (squeezed) modes for the forward transformation. These relationships guarantee that the combined transformation is the identity transformation.

Not only do Schmidt decompositions provide physical insight into complicated parametric processes, they also provide the mathematical means to optimize the performance of devices based on these processes. In a forthcoming paper, several nontrivial examples relevant to current research (including four-mode parametric amplification) will be discussed in detail.

Appendix: Real and complex Hamiltonian systems

Consider a real dynamical system with Hamiltonian $H(q,p)$, where q and p are conjugate variables. The Hamilton equations (for z-evolution) are

$$d_z p = \partial H/\partial q, \quad d_z q = -\partial H/\partial p. \tag{103}$$

By defining the vector variable $X = [x_1, x_2]^t = [p,q]^t$, one can rewrite Eqs. (103) in the matrix form

$$d_z X = J \partial H/\partial X, \tag{104}$$

where the auxiliary matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{105}$$

and the derivative of H is taken componentwise. Notice that $J^2 = -I$.

Now consider the quadratic Hamiltonian

$$H = \bar{\alpha} p^2/2 + \bar{\beta} pq + \bar{\gamma} q^2/2, \tag{106}$$

where $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are real constants (parameters). By applying the Hamilton equations (103) to Hamiltonian (106), one obtains the linear evolution equations

$$d_z p = \bar{\beta} p + \bar{\gamma} q, \quad d_z q = -\bar{\alpha} p - \bar{\beta} q. \tag{107}$$

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013

Alternatively, one can rewrite Hamiltonian (106) in the compact form

$$H = X^t G X/2, \tag{108}$$

where the coefficient matrix

$$G = \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\beta} & \bar{\gamma} \end{bmatrix}. \tag{109}$$

Notice that G is symmetric. By applying the Hamilton equation (104) to Hamiltonian (108), one obtains the matrix evolution equation

$$d_z X = JGX, \tag{110}$$

which is equivalent to the component equations (107).

Suppose that $X' = TX$, where T is an arbitrary transformation (change-of-variables) matrix. Then, in component form,

$$\begin{aligned} d_z x_i' &= \textstyle\sum_j T_{ij} d_z x_j \\ &= \textstyle\sum_j \sum_k T_{ij} J_{jk} \partial H/\partial x_k \\ &= \textstyle\sum_j \sum_k \sum_l T_{ij} J_{jk} T_{lk} \partial H/\partial x_l'. \end{aligned} \tag{111}$$

Hence, the transformation is symplectic if and only if

$$\textstyle\sum_j \sum_k \sum_l T_{ij} J_{jk} T_{lk} = J_{il}. \tag{112}$$

This condition can be rewritten in the matrix form

$$TJT^t = J. \tag{113}$$

If condition (113) is satisfied, then $T^{-1} = -JT^tJ$ and $(T^t)^{-1} = -JTJ$.

Now suppose that $T(z)$ is the transfer matrix for the system, which satisfies the evolution equation

$$d_z T = JGT, \tag{114}$$

together with the input condition $T(0) = I$. Then

$$d_z(T^t JT) = T^t J^2 GT + T^t G^t J^t JT = 0, \tag{115}$$

because $J^t = -J$, $J^2 = -I$ and $G = G^t$. Hence,

$$T^t JT = J. \tag{116}$$

Conditions (113) and (116) are equivalent, and require that $\det(T) = \pm 1$. However, G is symmetric, so $\mathrm{tr}(JG) = 0$ and $\det(T) = 1$. Hence, T is a member of Sp(2,R), the three-parameter group whose members are symplectic 2×2 matrices with determinant 1. [This group is isomororphic to SL(2,R) and SU(1,1).]

To establish the connection between real and complex dynamical systems, one defines the complex variables

$$a = (q + ip)/2^{1/2}, \;\; a^* = (q - ip)/2^{1/2}, \tag{117}$$

in which case

$$p = i(a^* - a)/2^{1/2}, \;\; q = (a^* + a)/2^{1/2}. \tag{118}$$

The Hamiltonian $H(a,a^*) = H[q(a,a^*), p(a,a^*)]$, where a and a^* are treated as independent variables. By combining Eqs. (103), (117) and (118), one obtains the complex Hamilton equations

$$d_z a = i\partial H/\partial a^*, \;\; d_z a^* = -i\partial H/\partial a. \tag{119}$$

By defining the vector variable $X = [a, a^*]^t$, one can rewrite Eqs. (119) in the matrix form

$$d_z X = iS\partial H/\partial X^*, \tag{120}$$

where the auxiliary matrix S was defined in Eq. (55) and the derivative of H is taken componentwise. Equation (120) is equivalent to Eq. (104).

Now consider the quadratic Hamiltonian

$$H = \delta|a|^2 + \gamma(a^*)^2/2 + \gamma^* a^2/2, \tag{121}$$

which also involves three (real) parameters (δ, γ_r and γ_i). By applying the Hamilton equations (119) to Hamiltonian (121), one obtains the linear evolution equations

$$d_z a = i\delta a + i\gamma a^*, \;\; d_z a^* = -i\delta a^* - i\gamma^* a. \tag{122}$$

The first of Eqs. (122) is just Eq. (5), which describes one-mode squeezing. One can reconcile the real and complex descriptions of this process by setting

$$\bar{\alpha} = \delta - \gamma_r, \;\; \bar{\beta} = \gamma_i, \;\; \bar{\gamma} = \delta + \gamma_r. \tag{123}$$

Alternatively, one can rewrite Hamiltonian (121) in the compact form

$$H = X^\dagger G X/2, \tag{124}$$

where the coefficient matrix

$$G = \begin{bmatrix} \delta & \gamma \\ \gamma^* & \delta \end{bmatrix}. \tag{125}$$

Notice that G is Hermitian. By applying the Hamilton equation (120) to Hamiltonian (124) componentwise, and reassembling the results, one obtains the matrix evolution equation

$$d_z X = iSGX. \tag{126}$$

However, it is easier to rewrite Hamiltonian (124) without the factor of 2, and differentiate it with respect to $X^\dagger$, treated as a single (vector) variable. This approach was taken in the main text.

Because Eqs. (110) and (126) are equivalent descriptions of the same system, their consequences, prime among which are the symplectic identities, must also be equivalent. To prove this result explicitly, define $X_r = [p,q]^t$ and $X_c = [a,a^*]^t$. Then $X_c = UX_r$, where the unitary matrix

$$U = \frac{1}{2^{1/2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}. \tag{127}$$

It follows from these definitions that if $X_r(z) = T_r(z)X_r(0)$, then $X_c(z) = UT_r(z)U^\dagger X_c(0)$, so $T_c(z) = UT_r(z)U^\dagger$. By multiplying Eq. (113) by U on the left and $U^\dagger$ on the right, one finds that

$$U(T_r J T_r^t)U^\dagger = (UT_rU^\dagger)(UJU^\dagger)(UT_r^tU^\dagger) = UJU^\dagger. \tag{128}$$

But $UJU^\dagger = iS$, so Eq. (128) is just Eq. (96). Thus, the real and complex symplectic identities are equivalent, so the groups formed by T_r and T_c [Sp(2,R) and SU(1,1)] are isomorphic. This equivalence extends to systems of $2n$ variables.

#175921 - $15.00 USD Received 11 Sep 2012; revised 5 Dec 2012; accepted 30 Dec 2012; published 14 Jan 2013