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ERROR DISTRIBUTIONS FOR RANDOM GRID APPROXIMATIONS OF MULTIDIMENSIONAL STOCHASTIC INTEGRALS[1]

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This paper proves joint convergence of the approximation error for several stochastic integrals with respect to local Brownian semimartingales, for nonequidistant and random grids. The conditions needed for convergence are that the Lebesgue integrals of the integrands tend uniformly to zero and that the squared variation and covariation processes converge. The paper also provides tools which simplify checking these conditions and which extend the range for the results. These results are used to prove an explicit limit theorem for random grid approximations of integrals based on solutions of multidimensional SDEs, and to find ways to "design" and optimize the distribution of the approximation error. As examples we briefly discuss strategies for discrete option hedging.

1. Introduction. The error in numerical approximations of stochastic integrals is a random variable, or, if one also is interested in the "time" development of the error, a stochastic process. Hence the most precise evaluation of the error, which is possible to obtain, is to derive the distribution of the error. The prototype example is the Euler method for the stochastic integral $\int_0^t f(B(s), s)\, dB(s)$, for a Brownian motion B. The Euler method approximates the integrand with a step-function which is constant between the "evaluation times" (or, in finance terminology, "intervention times") of the grid $i/n; i = 0, 1, \ldots$. This leads to the approximation $\int_0^t f \circ \eta_n\, dB(s)$, with $\eta_n(t) = i/n$ on the intervals $[i/n, (i+1)/n)$. In Rootzén (1980) it is shown that the approximation error $U^n = n^{1/2} \int_0^t (f - f \circ \eta_n)\, dB(s)$ converges stably in distribution,

$$U^n \Rightarrow_s \frac{1}{\sqrt{2}} \int_0^t f'(B(s), s)\, dW(s),$$

where W is a Brownian motion independent of B and $f'(x, y) = \frac{\partial f(x,y)}{\partial x}$, and where Rényi's quite useful concept of stable convergence means that U^n converges jointly with any sequence which converges in probability.

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The intuition behind this result is that "the small wiggles of a Brownian path are asymptotically independent of the global behavior of the path." The result has seen much further development, in particular, to the error in numerical solution schemes for SDEs, and has recently found significant application in measuring the risks associated with discrete hedging. A brief overview of some of this literature is given below.

The present paper generalizes this result in three ways: to joint convergence of the approximation error for several stochastic integrals, to local Brownian semimartingales instead of Brownian motions, and to nonequidistant and random evaluation times. The tools which help us quantify the intuition given above is Girsanov's theorem which shows how a multidimensional Brownian motion is affected by a change of measure, and Lévy's characterization of a multidimensional Brownian motion in terms of its square variation processes.

The conditions needed for convergence apply more generally than to approximation schemes. They are that the Lebesgue integrals of the integrands tend uniformly to zero in probability and that the square variation and covariation processes converge in probability. We additionally provide tools which simplify checking these conditions and which extend the range of the results. Further we apply these results to prove an explicit limit theorem for approximations of integrals based on solutions of multidimensional SDEs.

One center of interest for this paper is the possibility to improve approximation by using variable and random grids. In particular we study approximation schemes where the evaluation times i/n are replaced by time points given by the recursion $\tau_0^n = 0$ and

$$\tau_{k+1}^n = \tau_k^n + \frac{1}{n\theta(\tau_k^n)}$$

for a positive adapted process $\theta(t)$. We also study how the function θ can be chosen to design the approximation error so that it has desirable properties. For example, these could be homogeneous evolution of risk, or how to make the approximation error have minimal standard deviation.

A main motivation for writing this paper is to provide tools to study discrete hedging which uses random intervention times. We exemplify these possibilities by using the general results to exhibit a "no bad days" strategy and a minimum standard deviation strategy for the Black–Scholes model.

Weak convergence theory for approximations of stochastic integrals and solutions to stochastic differential equations is developed in Rootzén (1980), Kurtz and Protter (1991a, 1991b, 1996) and, in particular, an extensive study of the Euler method for SDEs is provided by Jacod and Protter (1998). This theory has been used and extended to solve and analyze various aspects of approximation and hedging error problems in mathematical finance. As examples we mention Duffie and Protter (1992), Bertsimas, Kogan and Lo (2000), Hayashi and Mykland (2005), Tankov and Voltchkova (2009), Brodén and Wiktorsson (2010) and

Fukasawa (2011). A Malliavin calculus approach to discrete hedging is used in Gobet and Temam (2001) and in a number of papers, which also consider variable but deterministic grids, by Geiss and coworkers; see Geiss and Toivola (2009) and the references therein. The main theoretical tool of Hayashi and Mykland (2005) is related to our general result, as discussed further below. The quite interesting paper Fukasawa (2011) also studies random grid approximations, for one-dimensional processes. The setting of Fukasawas paper is more or less in the middle between our Theorems 2.2 and 3.3. The conditions used by Fukasawa are rather different from ours, and there does not seem to be any simple relations between his results and ours.

Now a brief overview of the paper. The next section, Section 2, contains the basic general theorem on multidimensional convergence for stochastic integrals with respect to local multidimensional Brownian semimartingales, and the tools to check conditions and extend the result. In Section 3 we give the explicit result for random grid approximations of stochastic integrals based on the solution of a multidimensional SDE. Section 4 investigates ways to design and optimize approximation errors, and in Section 5 this is applied to discrete financial hedging.

2. General results. This section contains two main results. The first one gives a means to establish multidimensional convergence of the distribution of stochastic integrals with more and more rapidly varying integrands, and the second one shows how convergence of integrals with simple integrands can be extended to more general integrands. In addition, Lemma 2.8 provides tools to check the assumptions of the theorems. Our main aim is the error in approximations of stochastic integrals, but the results may in fact also have more general use.

Let $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ be the space of continuous $\mathbb{R}^d$-valued functions defined on $\mathbb{R}_+$, define $B_t = \{B_t^i\}_{1\le i\le d}$ by $B_t(\omega) = \omega(t)$, let $\mathbb{P}$ be the probability measure which makes B a Brownian motion starting at 0 and let $\mathcal{F}_t$ be the completion of the σ-algebra generated by $\{B_s; 0 \le s \le t\}$. Further write $\mathcal{F}$ for the smallest σ-algebra which contains all the $\mathcal{F}_t$. Until further notice is given all random variables we consider are defined on the filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$. Weak convergence will be for random variables (or "processes") with values in $C([0, T], \mathbb{R}^K)$, the space of continuous K-dimensional functions defined on the time interval $[0, T]$, and with respect to the uniform metric. Usually the dimension K of the processes will be clear from the context, and then we, for brevity, write $C[0, T]$, instead of $C([0, T], \mathbb{R}^K)$, and just write $\Rightarrow$ for weak convergence.

Weak convergence is stable (or "Rényi-stable") if it holds on any subset of $\mathcal{F}$, and the convergence is mixing (or "Rényi-mixing") if, in addition, the limit is the same on any subset. In the present setting this is specified by the definition which follows below. To appreciate part (ii) of the definition, recall that convergence in distribution often is written as $X^n \Rightarrow X$, but that in this notation X is not a random variable defined on some probability space. It is just a convenient notation for the limiting distribution of X_n. However, one can, of course, construct a random variable with this distribution, to give X a life of its own.

DEFINITION 2.1. (i) Let $(X^n)_{n\geq 1}$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in $C[0, T]$. Then $(X^n)_{n\geq 1}$ *converges stably* if $\mathbb{E}[Uf(X^n)]$ converges for any bounded continuous function $f : C[0, T] \to \mathbb{R}$ and any bounded measurable random variable U defined on $(\Omega, \mathcal{F}, \mathbb{P})$. If, in addition,

$$\lim_n \mathbb{E}[Uf(X^n)] = \mathbb{E}[U] \lim_n \mathbb{E}[f(X_n)], \tag{1}$$

then the convergence is *mixing*.

(ii) If $(X^n)_{n\geq 1}$ converges stably, then it is always possible to enlarge the probability space and construct a new random variable X on the enlarged probability space such that $\lim_n \mathbb{E}[Uf(X^n)] = \mathbb{E}[Uf(X)]$ for all bounded random variables U; see Aldous and Eagleson (1978). Thus, with this construction we can write stable convergence as $X^n \Rightarrow_s X$. If the convergence, in addition, is mixing, then X is independent of $\mathcal{F}$, and we write $X^n \Rightarrow_m X$.

It is straightforward to see that to establish stable or mixing convergence it is enough to prove convergence of $\mathbb{E}[Uf(X^n)]$ for strictly positive U with $EU = 1$. Further, see Aldous and Eagleson (1978), $X^n \Rightarrow_s X$ *if and only if* $(Y^n, X^n) \Rightarrow (Y, X^*)$ for any sequence of random variables $Y^n \to_p Y$ which converges in probability *if and only if* $X^n \Rightarrow X$ with respect to $\mathbb{P}(\cdot|A)$ for any set A with $\mathbb{P}(A) > 0$. (In the middle statement, convergence is with respect to the product topology.) Finally, if stability (or mixing) holds with respect to a sigma-algebra $\mathcal{F}$ and the sigma-algebra $\mathcal{F}'$ is independent of $\mathcal{F}$, then it also holds with respect to the sigma-algebra generated by $\mathcal{F}$ and $\mathcal{F}'$.

Let $X = (X_j, j = 1, \dots, d)$ be a continuous d-dimensional Brownian semimartingale defined on the space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ by

$$X_j(t) = \sum_{k=1}^{d} \int_0^t G_{j,k}(s)\, dB_k(s) + \int_0^t a_j(s)\, ds \tag{2}$$

with $G_{j,k}$ and a_j adapted, and with $\int_0^T G_{j,k}^2\, ds < \infty$ and $\int_0^T a_j^2\, ds < \infty$ *a.s.* for all j, k. Further let $\{H_{i,j}^n\} = \{H_{i,j}^n; 1 \leq i, j \leq d\}$ be a $d \times d$-dimensional array of $\mathcal{F}_t$-adapted processes such that $\int_0^T (H_{i,j}^n)^2\, dt < \infty$ *a.s.* for each i, j, and write

$$\begin{aligned}\{H_{i,j}^n \cdot X_j\} &= \{H_{i,j}^n \cdot X_j; 1 \leq i, j \leq d\}\\ &= \left\{\int_0^t H_{i,j}^n(s)\, dX_j(s); 1 \leq i, j \leq d\right\}_{0\leq t\leq T}.\end{aligned} \tag{3}$$

Thus $\{H_{i,j}^n \cdot X_j\}$ takes values in $C([0, T], \mathbb{R}^{d\times d})$. In the following we let $\to_p$ denote convergence in probability and take "positive" to mean the same as "nonnegative."

The form of the second condition, equation (5) of the following theorem requires some explanation. For simplicity of exposition suppressing the index k, it says that $\int_0^t H_{i,j}^n G_j H_{l,m}^n G_m\,ds$ converges in probability to some absolutely continuous limit, which we temporarily write as $\int_0^t C_{(i,j),(l,m)}\,ds$. Since limits of positive variable are positive, we further assume that for each t and ω the array $\{C_{(i,j),(l,m)}(t)\}$ is "positive definite," that is, equivalently, that it can be obtained as the covariances of some $d \times d$ array of random variables. The diagonal elements $C_{(i,j),(i,j)}(t)$ of the array are obtained from the limits of $\int_0^t (H_{i,j}^n)^2 G_j^2\,ds$ and hence it is natural to write them as $C_{(i,j),(i,j)}(t) = (H_{i,j})^2 G_j^2$. Further, taking positive square roots we may then more generally write $C_{(i,j),(l,m)}(t) = H_{i,j} G_j H_{l,m} G_m \rho_{(i,j),(l,m)}$. The array $\{\rho_{(i,j),(l,m)}\}$ then is the "correlation array" corresponding to the covariances $\{C_{(i,j),(l,m)}(t)\}$. This gives the formulation (5). (If some G_j is zero, we just set the corresponding $H_{i,j}$'s and off-diagonal elements of ρ to zero, and the diagonal elements to 1.)

Further, it is possible to find a "root" of $\{\rho_{(i,j),(l,m)}(t)\}$, that is, an array $\{\sigma_{(i,j),(l,m)}(t)\}$ such that $\rho_{(i,j),(l,m)}(t) = \sum_{1 \le r,s \le d} \sigma_{(i,j),(r,s)} \sigma_{(r,s),(l,m)}$. This can be seen by reordering the index set $\{(i,j); 1 \le r, s \le d\}$, linearly, say lexicographically, making the corresponding reordering of $\{\rho_{(i,j),(i,j)}\}$ into a matrix which then is positive definite, finding a root of this matrix, and then making the identification back to the array ordering.

THEOREM 2.2. *Suppose that* $\{H_{i,j}^n\}$ *satisfies*

$$\sup_{0 \le t \le T} \left| \int_0^t H_{i,j}^n\,ds \right| \to_p 0, \qquad n \to \infty, 1 \le i, j \le d, \tag{4}$$

and that for $k = 1, \dots, d$

$$\int_0^t H_{i,j}^n G_{j,k} H_{l,m}^n G_{m,k}\,ds \to_p \int_0^t H_{i,j} G_{j,k} H_{l,m} G_{m,k} \rho_{(i,j),(l,m)}^k\,ds \tag{5}$$

as $n \to \infty$, *for* $i, j, l, m = 1, \dots, d$, *and for some correlation array processes* $\rho^k = (\rho_{(i,j),(l,m)}^k; k = 1, \dots, d)$ *and processes* $\{H_{i,j} : 1 \le i, j \le d\}$ *such that all* $H_{i,j} G_j$ *are positive. Let* $\sigma^k(t)$ *be an arbitrary root of* $\rho^k(t)$; *see the discussion just before the theorem. Then, for* X *given by* (2),

$$\{H_{i,j}^n \cdot X_j\} \Rightarrow_s \left\{ \sum_{r,s,k=1}^d H_{i,j} G_{j,k} \sigma_{(i,j),(r,s)}^k \cdot W_{r,s,k} \right\} \tag{6}$$

as $n \to \infty$, *where* $W = (W_{r,s,k}; 1 \le r, s, k \le d)$ *is a* $d \times d \times d$*-dimensional Brownian motion which is independent of* $\mathcal{F}$.

This result simplifies in the special case when X is just a Brownian motion B; see the following corollary. The corollary is close to Theorem A.1 of Hayashi and Mykland (2005). Differences are that the corollary makes the basic condition (4) explicit, gives a more detailed description of the limit distribution and has the more powerful conclusion of stable convergence.

In Theorem 2.2 we, for simplicity of notation, considered a quadratic array $\{H_{i,j}^n \cdot X_j : 1 \le i, j \le d\}$. This does not involve any loss of generality, but still, for later use in the proof of Theorem 2.2, it is convenient to formulate the corollary for a rectangular array.

COROLLARY 2.3. *Suppose that* (4) *is satisfied for* $i = 1, \dots, d_1$, $j = 1, \dots, d_2$ *and that*

$$\int_0^t H_{i,k}^n H_{j,k}^n \, ds \to_p \int_0^t H_{i,k} H_{j,k} \rho_{i,j}^k \, ds, \qquad n \to \infty, \tag{7}$$

as $n \to \infty$, *for some correlation matrix processes* $\rho^k = \sigma^k (\sigma^k)'$, *where* $i, j = 1, \dots, d_1, k = 1, \dots, d_2$, *and positive processes* $\{H_{i,k} : i = 1, \dots, d_1, k = 1, \dots, d_2\}$, *and for* $0 \le t \le T$. *Then*

$$\{H_{i,k}^n \cdot B_k\} \Rightarrow_s \left\{ \sum_{j=1}^{d_1} H_{i,k} \sigma_{i,j}^k \cdot W_{j,k} \right\} \tag{8}$$

as $n \to \infty$, *where* $W = \{W_{j,k} : j = 1, \dots, d_1, k = 1, \dots, d_2\}$ *is a Brownian motion which is independent of* $\mathcal{F}$.

The following lemma plays an important role in the proofs.

LEMMA 2.4. *Suppose that* $\eta(t)$ *and* $H^n(t)$ *are real-valued random processes with* $\int_0^S \eta(t)^2 \, dt < \infty$ *a.s. and with* $\limsup_{n\to\infty} \int_0^S H^n(t)^2 \, dt < \infty$ *a.s. for some positive constant* $S \le \infty$. *Suppose further that*

$$\sup_{0 \le t \le S} \left| \int_0^t H^n \, ds \right| \to_p 0, \qquad n \to \infty.$$

Then

$$\sup_{0 \le t \le S} \left| \int_0^t H^n \eta \, ds \right| \to_p 0, \qquad n \to \infty. \tag{9}$$

PROOF. Suppose first that there exists a sequence $\{\eta_k\}$ of processes such that

$$\int_0^S \big(\eta(t) - \eta_k(t)\big)^2 \, dt \to_p 0 \qquad \text{as } k \to \infty,$$

$$\sup_{0 \le t \le S} \left| \int_0^t H^n \eta_k(s) \, ds \right| \to_p 0 \qquad \text{as } n \to \infty \qquad \text{for each } k.$$

Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\limsup_n \sup_{0\le t\le S}\left|\int_0^t H^n\eta\,ds\right| &\le \limsup_n \sup_{0\le t\le S}\left|\int_0^t H^n\eta_k\,ds\right| \\ &\quad+\limsup_n \sup_{0\le t\le S}\left|\int_0^t H^n(\eta-\eta_k)\,ds\right| \\ &\le 0+\sqrt{\limsup_n \int_0^S (H^n)^2\,dt}\sqrt{\int_0^S(\eta-\eta_k)^2\,dt},\end{aligned}$$

which tends to 0 as $k\to\infty$, so that (9) holds.

Thus the lemma follows if there exist a sequence $\{\eta_k\}$ which satisfies the two requirements above.

Now, for each k there exists a continuous process $\tilde\eta_k$, measurable in t and ω, such that $\mathbb{P}(\int_0^S(\eta(t)-\tilde\eta_k(t))^2\,dt>1/k)\le 1/k$. Briefly, to see this note that if $\eta(t)$ is approximated by convolving it with a sequence of "approximate δ-functions," for example, with a sequence of centered normal densities with variance parameters tending to 0, then the convolutions are measurable in t and ω and for almost all ω converge to $\eta[\cdot,\omega)$ in $L_2[0,S]$. The existence of the sequence $\tilde\eta_k$ follows at once from this, since convergence *a.s.* implies convergence in probability.

Next, with 1_A denoting the indicator function of a set A, for $\tilde\eta_{k,m}(t)=\sum_{i=0}^{[mS]}\tilde\eta_k(iS/m)1_{\{t\in[iS/m,(i+1)S/m)\}}$ it follows that

$$\int_0^S\big(\tilde\eta_k(t)-\tilde\eta_{k,m}(t)\big)^2\,dt\to_{\text{a.s.}}0\qquad\text{as } m\to\infty$$

and thus, choosing m_k suitably, $\eta_k=\sum_{i=0}^{[m_kS]}\tilde\eta_k(iS/m_k)1_{\{t\in[iS/m_k,(i+1)S/m_k)\}}$ satisfies the first one of the two relations above. Furthermore, the second one is easily seen to hold for η_k of this form. $\square$

PROOF OF THEOREM 2.2 AND COROLLARY 2.3. We do this in reverse order, and first prove Corollary 2.3. For simplicity of notation we only prove the corollary for a two-dimensional Brownian motion, that is, for the case $d=2$. The general case is the same.

By Rootzén [(1980), Theorem 1.2], each marginal process $\{H_{i,j}^n\cdot B_j(t),0\le t\le T\}$ is tight $C([0,T],\mathbb{R})$, and then also the entire $d\times d$-dimensional sequence $\{H_{i,j}^n\cdot B_j(t),0\le t\le T,1\le i,j\le d\}$ is tight $C([0,T],\mathbb{R}^{d\times d})$, so only stable finite-dimensional convergence remains to be proved. We prove this in two steps, where the first one follows along the lines of Rootzén (1980) and the second step uses the Cramér–Wold device. A final third step uses Corollary 2.3 to prove Theorem 2.2.

Step 1: Let $\{\psi_i^n; i=1,2\}$ be adapted processes such that, for $i=1,2$,

$$\sup_{0\le t\le T}\left|\int_0^t\psi_i^n\,ds\right|\to_p 0 \tag{10}$$

and such that

$$\int_0^t (\psi_i^n)^2\, ds \to_p \int_0^t (\psi_i)^2\, ds \tag{11}$$

for some $\psi_1, \psi_2 > 0$, $0 \le t \le T$. To make inverses well defined, we, without loss of generality, can assume that the $\psi_i^n(t)$ are defined also for $t > T$, and such that equations (10) and (11) hold with T replaced by S for any $S > 0$, and with $\psi_i(t) = 1$ for $t > T$ and $i = 1, 2$. This does not involve the result to be proved nor the assumptions, and hence can be done without loss of generality.

Let $C[0,\infty) = C([0,\infty), \mathbb{R})$ be the space of continuous real valued functions defined on $[0,\infty)$ and endowed with the topology of uniform convergence on compact sets; see Whitt (1970). Let the random variable $U > 0$ satisfy $\mathbb{E}U = 1$, and assume the functional $f : C[0,\infty) \to \mathbb{R}$ is bounded and continuous. Further, set $\tau_n(t) = \int_0^t (\psi_1^n)^2\, ds + \int_0^t (\psi_2^n)^2\, ds$, let $\tau(t) = \lim_{n\to\infty} \tau_n(t) = \int_0^t (\psi_1)^2\, ds + \int_0^t (\psi_2)^2\, ds$ and define τ_n^{-1} by $\tau_n^{-1}(t) = \inf\{s : \tau_n(s) > t\}$. Additionally let $\tilde{W}$ be a one-dimensional Brownian motion which is independent of $\mathcal{F}$. We first prove that

$$\mathbb{E}Uf\left(\int_0^{\tau_n^{-1}(\cdot)} \psi_1^n\, dB_1 + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n\, dB_2\right) \to \mathbb{E}f(\tilde{W}(\cdot)), \tag{12}$$

for each such U, so that $\int_0^{\tau_n^{-1}(\cdot)} \psi_1^n\, dB_1 + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n\, dB_2 \Rightarrow_m \tilde{W}$, on $C[0,\infty)$.

Now, define a new probability measure $\mathbb{Q}$ by $d\mathbb{Q}/d\mathbb{P} = U$, and write $\mathbb{E}_{\mathbb{Q}}$ for expectation taken with respect to $\mathbb{Q}$. Then, by Girsanov's theorem [Rogers and Williams (2000), Theorem IV 38.5] there exists an adapted square integrable process $c = (c_1, c_2)$ such that $(\tilde{B}(t) = (B_1(t) - \int_0^t c_1(s)\, ds, B_2(t) - \int_0^t c_2(s)\, ds)$ is a Brownian motion under $\mathbb{Q}$.

Hence,

$$\begin{aligned} &\mathbb{E}Uf\left(\int_0^{\tau_n^{-1}(\cdot)} \psi_1^n\, dB_1 + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n\, dB_2\right) \\ &\quad = \mathbb{E}_{\mathbb{Q}} f\left(\int_0^{\tau_n^{-1}(\cdot)} \psi_1^n\, d\tilde{B}_1 + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n\, d\tilde{B}_2 \right. \\ &\qquad\qquad \left. + \int_0^{\tau_n^{-1}(\cdot)} \psi_1^n c_1\, ds + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n c_2\, ds\right). \end{aligned} \tag{13}$$

Under $\mathbb{Q}$ the process $\int_0^{\tau_n^{-1}(\cdot)} \psi_1^n\, d\tilde{B}_1 + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n\, d\tilde{B}_2$ has the same distribution as $\tilde{W}$ [Rogers and Williams (2000), Theorem IV 34.1]. Further, by Lemma 2.4, we have that $\int_0^t \psi_1^n c_1\, ds + \int_0^t \psi_2^n c_2\, ds \to_p 0$ in $C[0, S]$, for any fixed S. Since f is bounded and continuous on $C[0,\infty)$, these two facts prove (12), and hence mixing convergence on $C[0,\infty)$.

It thus follows from $\tau_n \to_p \tau$ that $(\tau_n, \int_0^{\tau_n^{-1}(\cdot)} \psi_1^n \, dB_1 + \int_0^{\tau_n^{-1}(\cdot)} \psi_2^n \, dB_2)) \Rightarrow_s (\tau, \tilde{W})$, and hence, by composing τ_n^{-1} with τ_n [cf. Billingsley (1999), page 145], that

$$\int_0^t \psi_1^n \, dB_1 + \int_0^t \psi_2^n \, dB_2 \Rightarrow_s \tilde{W}(\tau(t)) \tag{14}$$

in $C[0, \infty)$, and hence, in particular, in $C[0, T]$.

Step 2: Finite-dimensional stable convergence now follows by standard but notationally complicated Cramér–Wold arguments. To lessen complications we here only consider two basic cases, and leave the general argument to the reader. Thus, first, let $\psi_i^n(s) = b_i 1_{\{0 \le s \le t_i\}} H_{1,i}^n(s)$ for $i = 1, 2$, with $0 < t_1, t_2 \le T$. Equation (7) implies that

$$\tau_n(t) \to_p \tau(t) = b_1^2 \int_0^{t \wedge t_1} (H_{1,1})^2 \, ds + b_2^2 \int_0^{t \wedge t_2} (H_{1,2})^2 \, ds$$

so that by (14),

$$\begin{aligned} b_1 \int_0^{t \wedge t_1} H_{1,1}^n \, dB_1 &+ b_2 \int_0^{t \wedge t_2} H_{1,2}^n \, dB_2 \\ &\Rightarrow_s \tilde{W}\left(b_1^2 \int_0^{t \wedge t_1} (H_{1,1})^2 \, ds + b_2^2 \int_0^{t \wedge t_2} (H_{1,2})^2 \, ds\right). \end{aligned}$$

Now, using elementary properties of Brownian motion together with Rogers and Williams [(2000), Theorem IV 34.1] we have that $\tilde{W}(b_1^2 \int_0^{t \wedge t_1} (H_{1,1})^2 \, ds + b_2^2 \int_0^{t \wedge t_2} (H_{1,2})^2 \, ds)$ has the same distribution, and the same dependency with any $\mathcal{F}$-measurable variable, as

$$b_1 \int_0^{t \wedge t_1} H_{1,1} \, dW_{1,1} + b_2 \int_0^{t \wedge t_2} H_{1,2} \, dW_{1,2}$$

for independent Brownian motions $W_{1,1}, W_{1,2}$, so that we by (14) have established that $b_1 \int_0^{t \wedge t_1} H_{1,1}^n \, dB_1 + b_2 \int_0^{t \wedge t_2} H_{1,2}^n \, dB_2 \Rightarrow_s b_1 \int_0^{t \wedge t_1} H_{1,1} \, dW_{1,1} + b_2 \int_0^{t \wedge t_2} H_{1,2} \, dW_{1,2}$, for any real numbers b_1, b_2. In particular stable two-dimensional convergence of $(H_{1,1}^n \cdot B_1(t_1), H_{1,2}^n \cdot B_2(t_2))$ to $(\int_0^{t_1} H_{1,1} \, dW_{1,1}, \int_0^{t_2} H_{1,2} \, dW_{1,2})$ follows by Cramér–Wold.

If we instead take $\psi_1^n = b_1 I_{\{0 \le s \le t_1\}} H_{1,1}^n(s) + b_2 I_{\{0 \le s \le t_2\}} H_{2,1}^n(s)$ and $\psi_2^n = 0$ then, by (7),

$$\begin{aligned} \tau_n(t) &\to_p \tau(t) \\ &= b_1^2 \int_0^{t \wedge t_1} (H_{1,1})^2 \, ds + 2 b_1 b_2 \int_0^{t \wedge t_1 \wedge t_2} H_{1,1} H_{2,1} \rho_{1,2}^1 \, ds \\ &\quad + b_2^2 \int_0^{t \wedge t_2} (H_{2,1})^2 \, ds. \end{aligned}$$

Furthermore, similarly as before and recalling that the matrix σ^1 is a root of the correlation matrix ρ^1, it can be seen that then $\tilde{W}(\tau(\cdot))$ has the same distribution, and the same dependency with any $\mathcal{F}$-measurable variable, as

$$b_1\left(\int_0^{t\wedge t_1} H_{1,1}\sigma^1_{1,1}\,dW_{1,1} + \int_0^{t\wedge t_1} H_{1,1}\sigma^1_{1,2}\,dW_{2,1}\right) + b_2\left(\int_0^{t\wedge t_2} H_{2,1}\sigma^1_{2,1}\,dW_{1,1} + \int_0^{t\wedge t_2} H_{2,1}\sigma^1_{2,2}\,dW_{2,1}\right).$$

Reasoning as above we get that

$$\begin{aligned} b_1\int_0^{t_1} H^n_{1,1}\,dB_1 + b_2\int_0^{t_2} H^n_{2,1}\,dB_1 \\ \Rightarrow_s b_1\int_0^{t_1} H_{1,1}\sigma^1_{1,1}\,dW_{1,1} + b_1\int_0^{t_1} H_{1,1}\sigma^1_{1,2}\,dW_{2,1} \\ + b_2\int_0^{t_2} H_{2,1}\sigma^1_{2,1}\,dW_{1,1} + b_2\int_0^{t_2} H_{2,1}\sigma^1_{2,2}\,dW_{2,1} \end{aligned}$$

for independent Brownian motions $W_{1,1}, W_{2,1}$. Since b_1 and b_2 are arbitrary, this proves stable two-dimensional convergence of $(H^n_{1,1}\cdot B_1(t_1), H^n_{2,1}\cdot B_1(t_2))$. A general proof of Corollary 2.3 is only notationally more complicated.

We next use Corollary 2.3 to obtain the conclusion of Theorem 2.2.

Step 3: By Lemma 2.4, if $H^n_{i,j}$ satisfies (4), then $\sup_{0\le t\le T}|\int_0^t H^n_{i,j}a_i\,ds| \to_p 0$, for all i, j, and hence the general result follows if we can prove that the result of the theorem holds for the case when all a_i are identically zero. Thus, to find the limit of $\{H^n_{i,j}\cdot X_j\}$ one only has to consider

$$\left\{\sum_{k=1}^d H^n_{i,j}G_{j,k}\cdot B_k\right\}.$$

Again by Lemma 2.4, if $H^n_{i,j}$ satisfies (4), then

$$\sup_{0\le t\le T}\left|\int_0^t H^n_{i,j}G_{j,k}\,ds\right| \to_p 0. \tag{15}$$

Now, making the definition $H^n_{(i,j),k} := H^n_{i,j}G_{j,k}$ and replacing the index i in (8) by the "multiindex" (i, j), convergence of the array $\{H^n_{i,j}G_{j,k}\cdot B_k\}$ follows from Corollary 2.3 with $d_1 = d^2, d_2 = d$. The result (6) then follows by summing over k and writing $W_{l,m,k}$ for $W_{(l,m),k}$. $\square$

We now change to a more general setup, from Brownian semimartingales to general processes (H^n, X^n) which are defined on filtered probability spaces $\Psi^n = (\Omega^n, \mathcal{F}^n, \mathbb{P}^n, (\mathcal{F}^n_t)_{0\le t<\infty})$. Here $\mathcal{F}^n$ is a $\mathbb{P}^n$-complete σ-algebra and $(\mathcal{F}^n_t)_{0\le t<\infty}$ is a filtration which satisfies the usual hypotheses (but which is not necessarily generated by a Brownian motion). The following definition is key to our goal. We give it for vector valued processes. The definition for matrix valued processes is analogous.

DEFINITION 2.5. Let $(X_n)_{n\geq 1}$ be a sequence of continuous $\mathbb{R}^d$-valued semimartingales defined on Ψ^n, $n \geq 1$ and assume that $X^n \Rightarrow X$. The sequence X^n is *good* if for any sequence of $\mathbb{R}^{d\times d}$-valued adapted càdlàg stochastic processes $(H^n)_{n\geq 1}$ defined on Ψ^n such that $(H^n, X^n) \Rightarrow (H, X)$, there exists a filtration $(\mathcal{G}_t)$ such that X is a semimartingale and H is an adapted càdlàg process, and $\{H^n_{i,j} \cdot X^n_j\} \Rightarrow \{H_{i,j} \cdot X_j\}$.

The following criterion is sufficient for goodness; see, for example, Theorem 2.2 in Kurtz and Protter (1991a).

DEFINITION 2.6. A sequence of continuous $\mathbb{R}^d$-valued semimartingales $(X^n)_{n\geq 1}$ is said to have *uniformly controlled variations* (UCV) if for each $n \geq 1$, there exist decompositions $X^n = M^n + A^n$ such that

$$\sup_n \mathbb{E}^n\left\{[M^n, M^n]_T + \int_0^T |dA^n_s|\right\} < \infty.$$

The next theorem combined with Theorem 2.3 will give the asymptotic distributions of approximation errors for stochastic integrals. If, in addition to the conditions of the theorem, f is bounded, then the result follows from Theorem 3.5 in Kurtz and Protter (1991b). However, in the present setting the result holds also without the boundedness condition, and it is further possible to give a quite simple proof. In the theorem, $0 = \tau^n_0 < \tau^n_1 < \cdots < \infty$ are $\{\mathcal{F}_t\}$-stopping times, and η_n is defined by $\eta_n(t) = \tau^n_k$, $\tau^n_k \leq t < \tau^n_{k+1}$.

THEOREM 2.7. *Let Y be a continuous $\mathbb{R}^d$-valued $\{\mathcal{F}_t\}$-semimartingale on $[0, T]$, and suppose that $f = (f_1, \dots, f_d)$ is continuously differentiable. Assume that $\eta_n(t)$ tends to the identity in probability for $t \in [0, T]$, and let $\{\lambda_n\}$ be a positive sequence converging to infinity. Further, set*

$$U^n = \lambda_n \int \big(f(Y) - f(Y \circ \eta_n)\big)\, dY$$

$$:= \lambda_n \sum_{i=1}^d \int \big(f_i(Y) - f_i(Y \circ \eta_n)\big)\, dY_i$$

and define

$$Z^n_{ij}(t) = \lambda_n \int_0^t \big(Y_i(s) - Y_i \circ \eta_n(s)\big)\, dY_j(s). \tag{16}$$

Suppose that $(Z^n)_{n\geq 1}$ *is good, and that* $(Z^n, Y) \Rightarrow (Z, Y)$. *Then* $U^n \Rightarrow U$ *on* $[0, T]$, *where*

$$U = \sum_{i,j=1}^{d} \int \frac{\partial f_j(Y)}{\partial y_i} \, dZ_{ij}.$$

Since η_n is nondecreasing, pointwise convergence in probability in $[0, T]$, as assumed in the theorem, is equivalent to uniform convergence in probability in $[0, T]$. Below we will use this without further comment.

PROOF OF THEOREM 2.7. For simplicity of exposition, we assume that $d = 1$. By the continuous mapping theorem we have that $(Z^n, Y, Y) \Rightarrow (Z, Y, Y)$. Since Y is continuous, and η_n converges uniformly in probability to the unity, this in turn can be seen to imply that $(Z^n, Y \circ \eta_n, Y) \Rightarrow (Z, Y, Y)$, for example, by using the Skorokhod translation of convergence in distribution to convergence *a.s.*

We now define

$$g(x, y) = \frac{f(x) - f(y)}{x - y},$$

where we make the continuous choice $g(x, x) = f'(x)$ when the denominator vanishes. The function g is uniformly continuous on $[0, T]^2$, so the continuous mapping theorem gives that $(Z^n, g(Y, Y \circ \eta_n)) \Rightarrow (Z, f'(Y))$. Now,

$$U^n = \lambda_n \int \big(f(Y) - f(Y \circ \eta_n)\big) \, dY = \int g(Y, Y \circ \eta_n) \, dZ^n.$$

But since $(Z^n)_{n\geq 1}$ is good, we have that

$$\int g(Y, Y \circ \eta_n) \, dZ^n \Rightarrow \int f'(Y) \, dZ,$$

which proves the theorem for $d = 1$. $\square$

The next lemma provides a tool for verification of criteria like (4) and (7). In the lemma we specialize to stopping times (cf. the Introduction) defined recursively by $\tau_0^n = 0$ and

$$\tau_{k+1}^n = \left(\tau_k^n + \frac{1}{n\theta(\tau_k^n)}\right) \wedge T \tag{17}$$

for some adapted stochastic process θ. As before, let

$$\eta_n(t) = \tau_k^n, \qquad \tau_k^n \leq t < \tau_{k+1}^n \qquad \text{for } k = 1, 2, \ldots \tag{18}$$

and write $E_p = \mathbb{E} \int_0^1 B(s)^p \, ds = \int_0^1 s^{p/2} \mathbb{E} B(1)^p \, ds = \mathbb{E} B(1)^p / (p/2 + 1)$ so that $E_1 = \mathbb{E} \int_0^1 B(s) \, ds = 0$ and $E_2 = \mathbb{E} \int_0^1 B(s)^2 \, ds = 1/2$.

In the lemma we will assume that the function $a(t); t \in [0, T]$ is *locally bounded*, that is, that to any $\varepsilon > 0$ there exists a localizing stopping time $\nu = \nu_\varepsilon$ such that $a(t \wedge \nu); t \in [0, T]$ is bounded, and such that $\mathbb{P}(\nu < T) < \varepsilon$. In particular, if a is continuous on $[0, T]$, then a is locally bounded.

LEMMA 2.8. *Assume that a and θ are adapted processes such that a is locally bounded, θ is strictly positive and $a(t)/\theta(t)^{p/2}$ is a.s. Riemann integrable over $[0, T]$, and let τ_k^n and η_n be defined by* (17) *and* (18). *Set*

$$\psi_n(t) = n^{p/2} \sum_{k=0}^{\infty} a(\tau_k^n)\big(B(t) - B(\tau_k^n)\big)^p 1_{\{\tau_k^n \le t < \tau_{k+1}^n\}}. \tag{19}$$

Further assume that η_n tends to the identity in probability. Then

$$\sup_{0 \le t \le T} \left| \int_0^t \psi_n(s)\, ds - E_p \int_0^t \frac{a(s)}{\theta(s)^{p/2}}\, ds \right| \to_p 0 \tag{20}$$

as $n \to \infty$, for $p = 1, 2$.

PROOF. If we prove the lemma under the additional restriction that a is bounded, then it follows in general, since it then holds for $a(t)$ replaced by $a(t \wedge \nu)$ for any localizing stopping time ν, and this in turn implies that (20) holds with probability greater than $1 - \varepsilon$, for arbitrary ε. Thus we assume in the rest of this proof that a is uniformly bounded, so that in particular the expectations exist.

To ease notation we below sometimes will write τ_k instead of τ_k^n and define $\bar{\mathcal{F}}_k = \mathcal{F}_{\tau_k}$. Clearly

$$\begin{aligned} n^{p/2}\mathbb{E}\left\{ \int_{\tau_k}^{\tau_{k+1}} a(\tau_k)\big(B(t) - B(\tau_k)\big)^p\, dt \middle| \bar{\mathcal{F}}_k \right\} &= n^{p/2} a(\tau_k) \int_0^{1/n\theta(\tau_k)} \mathbb{E}B(t)^p\, dt \\ &= E_p \frac{a(\tau_k)}{n\theta(\tau_k)^{p/2+1}}. \end{aligned}$$

Recalling the definition of η_k,

$$\sum_{k'=1}^{k-1} E_p \frac{a(\tau_{k'})}{n\theta(\tau_{k'})^{p/2+1}} = E_p \int_0^{\tau_k} \frac{a \circ \eta_n(s)}{\theta \circ \eta_n(s)^{p/2}}\, ds$$

and hence

$$X_k := \int_0^{\tau_k} \psi_n\, ds - E_p \int_0^{\tau_k} \frac{a \circ \eta_n(s)}{\theta \circ \eta_n(s)^{p/2}}\, ds$$

is a martingale with index set $\mathbb{Z}_+$.

In the following we show that $\sum_k \mathbb{E}((X_{k+1} - X_k)^2 | \bar{\mathcal{F}}_k) \to 0$. By the functional central limit theorem for martingales [see, e.g., Rootzén (1983), Theorem 3.5] this in turn implies that

$$\max_k |X_k| = \max_k \left| \int_0^{\tau_k^n} \psi_n\, ds - E_p \int_0^{\tau_k^n} \frac{a \circ \eta_n(s)}{\theta \circ \eta_n(s)^{p/2}}\, ds \right| \to_p 0 \tag{21}$$

as $n \to \infty$. Using the Cauchy–Schwarz inequality in the second step, elementary properties of Brownian motion in the third and that $(\tau_{k+1} - \tau_k) = 1/(n\theta(\tau_k))$ in the fourth step, we have that

$$
\begin{aligned}
&\sum_k \mathbb{E}[(X_{k+1} - X_k)^2 | \bar{\mathcal{F}}_k] \\
&\quad \le \sum_k \mathbb{E}\biggl[\biggl(\int_{\tau_k}^{\tau_{k+1}} \psi_n \, dt\biggr)^2 \bigg| \bar{\mathcal{F}}_k\biggr] \\
&\quad \le n^p \sum_k a(\tau_k)^2 (\tau_{k+1} - \tau_k) \int_{\tau_k}^{\tau_{k+1}} \mathbb{E}\bigl[\bigl(B(t) - B(\tau_k)\bigr)^{2p} | \bar{\mathcal{F}}_k\bigr] \, dt \\
&\quad = \frac{E_{2p}}{p+1} n^p \sum_k a(\tau_k)^2 (\tau_{k+1} - \tau_k)^{p+2} \\
&\quad \le \frac{E_{2p}}{p+1} \max_k \biggl(\frac{a(\tau_k)}{n\theta(\tau_k)^{p/2+1}}\biggr) \sum_{k : \tau_k < T} \biggl(\frac{a(\tau_k)}{n\theta(\tau_k)^{p/2+1}}\biggr).
\end{aligned}
$$

It follows from the Riemannn integrability of $a/\theta^{p/2}$ that in the last expression above the first factor tends to 0 and that the second tends to $\int_0^T a(s)/\theta(s)^{p/2} \, ds$, so that the product tends to zero. This completes the proof of (21).

The assumption that a is bounded and straightforward computation show that $\mathbb{E} \int_0^T \psi_n^2 \, ds$ is bounded in n, and since furthermore $\max_k \{\tau_{k+1}^n - \tau_k^n\} \to_p 0$, we can apply the Cauchy–Schwarz inequality, to see that

$$
\max_k \sup_{\tau_k^n \le t < \tau_{k+1}^n} \biggl| \int_{\tau_k^n}^t \psi_n \, ds \biggr| \le \biggl(\max_k \{\tau_{k+1}^n - \tau_k^n\} \int_0^T \psi_n^2 \, ds \biggr)^{1/2} \to_p 0
$$

for $n \to \infty$. Together with (21) this shows that

$$
\sup_{0 \le t \le T} \biggl| \int_0^t \psi_n \, ds - E_p \int_0^t \frac{a \circ \eta_n(s)}{\theta \circ \eta_n(s)^{p/2}} \, ds \biggr| \to_p 0. \tag{22}
$$

By assumption $a/\theta^{p/2}$ is Riemann integrable, and hence

$$
\sup_{0 \le t \le T} \biggl| \int_0^t \frac{a(s)}{\theta(s)^{p/2}} \, ds - \int_0^t \frac{a \circ \eta_n(s)}{\theta \circ \eta_n(s)^{p/2}} \, ds \biggr| \to_{\text{a.s.}} 0. \tag{23}
$$

The triangle inequality together with (22) and (23) completes the proof of the lemma. □

3. Approximation of stochastic integrals. We now use the results from the previous section to find the explicit form of the asymptotic distribution of the sum of the errors in approximating d stochastic integrals where the integrands are functions of the solution to a d-dimensional SDE and where the integrators are the same solutions to the SDE. The following condition is used in the theorem.

CONDITION 3.1. Let the measurable functions $\alpha(\cdot):\mathbb{R}^d \to \mathbb{R}^d$, $\beta(\cdot):\mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy

$$|\alpha(x)| + |\beta(x)| \le C(1 + |x|),$$

where $x \in \mathbb{R}^d$ for some constant C and

$$|\alpha(x) - \alpha(y)| + |\beta(x) - \beta(y)| \le D|x - y|,$$

where $x, y \in \mathbb{R}^d$ for some constant D.

This condition ensures that the SDE has an unique continuous solution. Further, we will need the following lemma, which is given as Lemma 2.5 in Rootzén (1983).

LEMMA 3.2. *Suppose* $\{Z^n\}_{n\ge1}$ *is a sequence of positive discrete time stochastic processes, adapted to their respective filtrations* $\{\mathcal{F}^n\}_{n\ge1}$ *and that* τ_n *is a stopping time with respect to* $\mathcal{F}^n$ *for each* n. *Then*

$$\sum_{j=1}^{\tau_n} \mathbb{E}(Z_j^n|\mathcal{F}_{j-1}^n) \to_p 0$$

implies that

$$\sum_{j=1}^{\tau_n} Z_j^n \to_p 0.$$

THEOREM 3.3. *Let* Y *be the solution of the SDE*

$$dY(t) = \alpha(Y(t))\,dt + \beta(Y(t))\,dB(t), \tag{24}$$

where B *is a* d-*dimensional Brownian motion*, α, β *satisfy Condition* 3.1 *and* $Y(0)$ *is independent of* B *and satisfies* $\mathbb{E}Y(0)^2 < \infty$. *Then the error in the Euler-type approximation scheme defined by*

$$\begin{aligned}U^n(t) &= n^{1/2}\int_0^t \bigl(f(Y(u)) - f\bigl(Y\circ\eta_n(u)\bigr)\bigr)\,dY(u)\\ &:= n^{1/2}\sum_{i=1}^d \int_0^t \bigl(f_i(Y(u)) - f_i\bigl(Y\circ\eta_n(u)\bigr)\bigr)\,dY_i(u),\end{aligned}$$

where $f = (f_1,\dots,f_d)$ *is continuously differentiable and the grid is given by* (17) *with* $\sup_{t\in[0,T]}\theta(t) < \infty$ *a.s. and* $1/\theta$ *a.s. Riemann integrable, satisfies*

$$U^n \Rightarrow \sum_{r,k=1}^d \int_0^t \Delta_{r,k}(u)\,dW_{r,k}(u)$$

on $[0, T]$, *where*

$$\Delta_{r,k}(t) = \frac{\sum_{i,j=1}^{d} (\partial f_j/\partial y_i)(Y(t))\beta_{i,r}(Y(t))\beta_{j,k}(Y(t))}{\sqrt{2\theta(t)}},$$

and W *is an* $d \times d$-*dimensional Brownian motion*, *independent of* B. *In particular*,

$$\sup_{0\le t\le T} |U^n(t)| \Rightarrow \sup_{0\le t\le T} \left| \sum_{r,k=1}^{d} \int_0^t \Delta_{r,k}(u)\, dW_{r,k}(u) \right|.$$

PROOF. For the convenience of the reader we begin by recalling that $\{X^n\}_{n\ge1}$ is $O_p(a_n)$ for some sequence a_n if

$$\lim_{c\to\infty} \limsup_{n\to\infty} \mathbb{P}[|X^n/a_n| \ge c] = 0$$

or, equivalently, if $\{X_n/a_n\}_{n\ge1}$ is *tight*. We first assume that the coefficients α and β are uniformly bounded, and prove that the result holds under this extra assumption. The general result for unbounded coefficients then follows by an easy localization argument which is given at the end of the proof. We again write $\bar{\mathcal{F}}_v$ instead of $\mathcal{F}_{\tau_v}$ and often suppress the explicit dependence on n and, for example, write τ_v instead of τ_v^n.

Since $1/\theta$ is Riemann integrable, and hence pathwise bounded *a.s.*, and $\sup_{t\in[0,T]} \theta(t) < \infty$ *a.s.*, it follows that η_n tends to t uniformly *a.s.* By Theorem 5.2.1 in Øksendal (2003) there exists a unique t-continuous solution Y to equation (24).

The first part of the proof consists of proving that

$$\{Z_{i,j}^n\} = \left\{ \sqrt{n} \int_0^t \big(Y_i(s) - Y_i \circ \eta_n(s)\big)\, dY_j(s) \right\}$$

converges jointly with Y. We do this by showing that the conditions of Theorem 2.2 are satisfied for the choices $H_{i,j}^n = \sqrt{n}(Y_i - Y_i \circ \eta_n)$ and $G_{j,k} = \beta_{j,k}$.

The bounded variation part of $Y_i - Y_i \circ \eta_n$ can be seen to give contributions which are $O_p(1/n)$, and thus, using the triangle inequality and writing $1_v(s) = 1_{\{\tau_v \le s < \tau_{v+1}\}}$, it can be seen that (4) follows if we show that

$$\sqrt{n} \sup_{t\in[0,T]} \left| \int_0^t \sum_v \int_{\tau_v}^s 1_v(s)\beta_{i,j}(u)\, dB_j(u)\, ds \right| \to_p 0 \tag{25}$$

for $1 \le i, j \le d$.

Now,

$$\begin{aligned} &\sqrt{n} \int_0^t \sum_v \int_{\tau_v}^s 1_v(s)\beta_{i,j}(u)\, dB_j(u)\, ds \\ &\quad = \sqrt{n} \int_0^t \sum_v \int_{\tau_v}^s 1_v(s)\big(\beta_{i,j}(u) - \beta_{i,j}(\tau_v)\big)\, dB_i(u)\, ds \qquad (26)\\ &\qquad + \sqrt{n} \int_0^t \sum_v 1_v(s)\beta_{i,j}(\tau_v)\big(B_i(u) - B_i(\tau_v)\big)\, ds. \end{aligned}$$

The last term tends to zero in probability by Lemma 2.8 with $p=1$, since Riemann integrability of $1/\sqrt{\theta}$ follows from Riemann integrability of $1/\theta$.

We next show that also the first term on the right-hand side is negligible. Let C denote a generic deterministic constant whose value may change from one appearance to the next. Since τ_{v+1} is measurable with respect to $\bar{\mathcal{F}}_v$ it follows from Condition 3.1, Itô's isometry, and the assumption that the constants in (24) are bounded that

$$\begin{aligned}\mathbb{E}\biggl[\int_{\tau_v}^{s}\bigl(\beta_{i,j}(u)-\beta_{i,j}(\tau_v)\bigr)^2\,du\Big|\bar{\mathcal{F}}_v\biggr] &\le C\int_{\tau_v}^{s}\mathbb{E}[|Y(u)-Y(\tau_v)|^2|\bar{\mathcal{F}}_v]\,du\\ &\le C\int_{\tau_v}^{s}(u-\tau_v)\,du\\ &\le C(\tau_{v+1}-\tau_v)^2.\end{aligned}\tag{27}$$

Define

$$\Delta_v(t)=\sqrt{n}\int_{\tau_v}^{t\wedge\tau_{v+1}}\int_{\tau_v}^{s\wedge\tau_{v+1}}\bigl(\beta_{i,j}(u)-\beta_{i,j}(\tau_v)\bigr)\,dB_i(u)\,ds,$$

so that the first term on the right-hand side of (26) equals $\sum_v\Delta_v(t)$. Using Doob's inequality together with the Cauchy–Schwarz inequality in the second step and (27) in the third step we have that

$$\begin{aligned}&\mathbb{E}\Bigl[\sup_{\tau_v\le t<\tau_{v+1}}|\Delta_v(t)|\big|\bar{\mathcal{F}}_v\Bigr]\\ &\quad\le\sqrt{n}(\tau_{v+1}-\tau_v)\mathbb{E}\biggl[\sup_{\tau_v\le s<\tau_{v+1}}\biggl|\int_{\tau_v}^{s}\bigl(\beta_{i,j}(u)-\beta_{i,j}(\tau_v)\bigr)\,dB_i(u)\biggr|\Big|\bar{\mathcal{F}}_v\biggr]\\ &\quad\le C\sqrt{n}(\tau_{v+1}-\tau_v)\mathbb{E}\biggl[\int_{\tau_v}^{\tau_{v+1}}\bigl(\beta_{i,j}(u)-\beta_{i,j}(\tau_v)\bigr)^2\,du\Big|\bar{\mathcal{F}}_v\biggr]^{1/2}\\ &\quad\le C\sqrt{n}(\tau_{v+1}-\tau_v)^2.\end{aligned}$$

Thus, by the definition (17),

$$\begin{aligned}\sum_v\mathbb{E}\Bigl[\sup_{\tau_v\le t<\tau_{v+1}}|\Delta_v(t)|\big|\bar{\mathcal{F}}_v\Bigr]&\le C\sqrt{n}\sum_v(\tau_{v+1}-\tau_v)^2\\ &\le C\sqrt{n}\frac{1}{n}T\sup_{0\le t\le T}\frac{1}{\theta(t)}\to_{\text{a.s.}}0.\end{aligned}$$

According to Lemma 3.2 it follows that $\sum_v\sup_{\tau_v\le t<\tau_{v+1}}|\Delta_v(t)|\to_p 0$. Hence,

$$\sup_{0\le t\le T}\Bigl|\sum_v\Delta_v(t)\Bigr|\le\sum_v\sup_{\tau_v\le t<\tau_{v+1}}|\Delta_v(t)|\to_p 0,$$

which completes the proof that the first term in the right-hand side of (26) tends uniformly to zero in probability.

Completely similar, but more complex computation show that for any indexes i, j, k, l, m, and using Lemma 2.8 with $p = 2$ for $j = m$ and computations similar to (but simpler than) the proof of Lemma 2.8 for $j \neq m$,

$$
\begin{aligned}
n \int_0^t \sum_v \int_{\tau_v}^s 1_v(s)\beta_{i,j}(u)\, dB_j(u) \int_{\tau_v}^s 1_v(s)\beta_{l,m}(z)\, dB_m(z)\beta_{j,k}(s)\beta_{m,k}(s)\, ds \\
= n \int_0^t \sum_v \beta_{i,j}(\tau_v)\beta_{j,k}(\tau_v)\beta_{l,m}(\tau_v)\beta_{m,k}(\tau_v) \\
\times \big(B_j(s) - B_j(\tau_v)\big)\big(B_m(s) - B_m(\tau_v)\big)1_v(s)\, ds + o_p(1) \\
\to_p \frac{1}{2}\int_0^t \beta_{i,j}(s)\beta_{j,k}(s)\beta_{l,m}(s)\beta_{m,k}(s)/\theta(s)\delta_{j,m}\, ds,
\end{aligned}
$$

where $\delta_{j,m}$ is 1 if $j = m$ and zero otherwise. Recalling that $G_{j,k} = \beta_{j,k}$, and as before approximating $H_{i,j}^n(s) = \sqrt{n}(Y_i(s) - Y_i \circ \eta_n(s))$ by $\sum_{k=1}^d \sum_v \int_{\tau_v}^s 1_v(s) \times \beta_{i,k}(u)\, dB_k(u)$ it follows that condition (5) of Theorem 2.2 holds as

$$
\int_0^t H_{i,j}^n G_{j,k} H_{l,m}^n G_{m,k}\, ds \to_p \frac{1}{2}\sum_{r=1}^d \int_0^t \beta_{i,r}\beta_{l,r}\beta_{j,k}\beta_{m,k}/\theta\, ds. \tag{28}
$$

Now we recognize that the choice $H_{i,j} G_{j,k}\sigma_{(i,j),(r,s)}^k = \delta_{r,s}\beta_{j,k}\beta_{i,r}/\sqrt{2\theta}$ satisfies equation (28). Hence,

$$
\{Z_{i,j}^n\} = \{H_{i,j}^n \cdot Y_j\} \Rightarrow_s \left\{\sum_{r,k=1}^d \frac{\beta_{j,k}\beta_{i,r}}{\sqrt{2\theta}} \cdot W_{r,k}\right\}. \tag{29}
$$

Arguments similar to those above show that $\{H_{i,j}^n \cdot Y_j\}$ has uniformly controlled variations and hence are good. Stable convergence implies that the left-hand side of (29) converges jointly with Y. The first conclusion of the theorem now follows from Theorem 2.7, for the case when the coefficients are bounded.

To remove the restriction that the coefficients are bounded, for general $\alpha_i, \beta_{i,j}$ define coefficients $\alpha_i^c = (-c) \vee \alpha_i \wedge c$ and $\beta_{i,j}^c = (-c) \vee \beta_{i,j} \wedge c$. Theorem 5.2.1 in Øksendal (2003) still yields unique t-continuous solution Y^c to (24) for these functions. Let $U^{n,c}$ be defined from $\alpha_i^c, \beta_{i,j}^c$ in the same way as U^n is defined from $\alpha_i, \beta_{i,j}$. With obvious notation, we have already proved that $U^{n,c} \Rightarrow U^c$, as $n \to \infty$ for each fixed c. Since $\mathbb{P}(\sup_{t\in[0,T]} |Y^c(t) - Y(t)| > 0) \to 0$, as $c \to \infty$ also $U^c \Rightarrow U$. Further,

$$
\begin{aligned}
\limsup_n \mathbb{P}\Big(\sup_{t\in[0,T]} |U^{n,c} - U^n| > 0\Big) \\
\le \mathbb{P}\big(\inf\{t : \max\{\max\{|\alpha_i(\bar{Y}_t)|\}, \max\{|\beta_{i,j}(\bar{Y}_t)|\}\} \ge c\} \le T\big) \to 0
\end{aligned}
$$

as $c \to \infty$. Hence, Theorem 3.2 in Billingsley (1999) gives that $U^n \Rightarrow U$, which proves that the first result of the theorem holds also for the general case.

The second conclusion follows from from the first by the continuous mapping theorem, since the supremum mapping is continuous. □

4. Designing the error in approximations of stochastic integrals. In deciding on which approximation scheme to use to compute a stochastic integral—or, to decide on a hedging strategy—one has to balance the error with the *number of intervention times* $N = N_n = \max\{k; \tau_k^n < T\}$. In this section we will investigate two such schemes. The first one could be called the "no bad days" strategy, and simply consists in choosing the stopping times $\{\tau_k\}$ where the stochastic integral is evaluated—or the times when the portfolio is rehedged—in such a way that the error is a Wiener process. In the second strategy we bound the expected number of evaluation times and minimize the asymptotic standard deviation of the approximation error under this restriction.

The setting of this section is the following: suppressing the superscript n the stopping times are given by (17), that is, $\tau_0 = 0$ and

$$\tau_{k+1} = \left(\tau_k + \frac{1}{n\theta(\tau_k)}\right) \wedge T \tag{30}$$

with θ adapted and positive, and the distribution of the approximation error $\varepsilon(t)$ satisfies

$$\sqrt{n}\varepsilon(t) \Rightarrow \int_0^t \frac{f(s)}{\sqrt{\theta(s)}}\, dW(t) \tag{31}$$

for some adapted process $f(s) \ge 0$ and Wiener process W which is independent of θ and f. Here it should be noted that (31) is more general than it looks at first; for example, the approximation error in Theorem 3.3 satisfies this for $f(t) = \sqrt{\frac{1}{2}\sum_{k,m=1}^d \Delta_{k,m}^2(t)}$.

It is straightforward to find the asymptotic number of evaluation times.

PROPOSITION 4.1. *Suppose that θ is Riemann integrable a.s. and that $\inf_{0\le t\le T}\theta(t) > 0$ a.s. Then*

$$\lim_{n\to\infty} \frac{N_n}{n} = \int_0^T \theta(t)\, dt \qquad a.s.$$

If, in addition, $\mathbb{E}[\sup_{0\le t\le T}\theta(t)] < \infty$, *then*

$$\lim_{n\to\infty} \mathbb{E}\frac{N_n}{n} = \int_0^T \mathbb{E}\theta(t)\, dt.$$

PROOF. Suppose first θ is of the form

$$\theta(t) = \sum_{i=0}^{k} \theta_i 1_{[a_i, a_{i+1})}(t) \tag{32}$$

for some random variables $\theta_i > 0$ and constants $0 = a_0 < a_1 < \cdots < a_k = T$, and with $1_{[a_i,a_{i+1})}$ the indicator function of the interval $[a_i, a_{i+1})$. For each ω, it is easily seen that the number of intervention times in the interval $[a_i, a_{i+1})$ is $n\theta_i(a_{i+1} - a_i) + O(1)$, and hence

$$\frac{N_n}{n} = \sum_{i=0}^{k} \theta_i(a_{i+1} - a_i) + O\left(\frac{1}{n}\right) = \int_0^T \theta(t)\,dt + O\left(\frac{1}{n}\right) \to \int_0^T \theta(t)\,dt$$

as $n \to \infty$. If $\tilde{\theta} \leq \theta$ and $\tilde{\theta}$ is of the form (32) then, with obvious notation, $N_n(\tilde{\theta}) \leq N_n(\theta) + O(1)$, and the corresponding bound with all the inequalities reversed is also true.

Now, by assumption θ is Riemann integrable, and hence can be approximated arbitrarily well from below and above by functions of the form (32). This proves the first assertion of the proposition.

Furthermore, $N_n/n \leq T \sup_{0\leq t\leq T} \theta(t) + 1/n$, and hence the second assertion follows from the first one by dominated convergence. □

In the rest of this section we assume that we "are in the asymptotic regime," that is, that n is so large that we, to the degree of approximation needed, may assume that the limits above can be replaced by equalities. Thus, below we will assume that

$$\mathbb{E}N = n\int_0^T \mathbb{E}\theta(t)\,dt, \qquad \varepsilon(t) = \frac{1}{\sqrt{n}}\int_0^t \frac{f(s)}{\sqrt{\theta(s)}}\,dW(t), \tag{33}$$

so that in particular $\mathbb{E}\varepsilon(t)^2 = \frac{1}{n}\int_0^t \mathbb{E}\frac{f(s)^2}{\theta(s)}\,ds$.

The no bad days strategy: It is at once seen, supposing that f^2 is Riemann integrable, that if we choose $\theta(t) = cf(t)^2$, for some constant c, then

$$\varepsilon(t) = \frac{1}{\sqrt{cn}} W(t)$$

and

$$\mathbb{E}N = cn\int_0^T \mathbb{E}f^2(s)\,ds.$$

Thus, in a financial setting, with this choice of θ, there are no "days" where the hedging error grows quicker than during other days, and hence a trader can sleep equally well (or equally badly!) each night.

Minimal standard deviation: We will now, supposing that f is Riemann integrable, show that the solution of the optimization problem

$$\inf_{\{\theta\,:\,\theta\geq 0,\text{adapted}\}} \left\{\sqrt{\mathbb{E}\varepsilon^2(T)} : \mathbb{E}N \leq nC\right\}$$

is given by $\theta(t) = Cf(t)/(\int_0^T \mathbb{E}f(s)\,ds)$. For this choice

$$\mathbb{E}N = nC, \qquad \varepsilon(t) = \sqrt{\frac{\int_0^T \mathbb{E}f\,ds}{nC}} \int_0^t \sqrt{f}\,dW.$$

Thus in particular, for the optimal strategy the standard deviation is $\sqrt{\mathbb{E}\varepsilon(T)^2} = \int_0^T Ef\,ds/\sqrt{nC}$.

Now, write $\tilde{\theta} = n\theta$. With this notation $\mathbb{E}\varepsilon(T)^2 = \mathbb{E}\int_0^T f^2/\tilde{\theta}\,ds$ and the restriction is $\mathbb{E}\int_0^T \tilde{\theta} \le nC$. Applying the Cauchy–Schwarz inequality twice, it follows that

$$\begin{aligned}\left(\mathbb{E}\int_0^T f\,ds\right)^2 &\le \left(\mathbb{E}\sqrt{\int_0^T f^2/\tilde{\theta}\,ds}\sqrt{\int_0^T \tilde{\theta}\,ds}\right)^2\\ &\le \mathbb{E}\left(\int_0^T f^2/\tilde{\theta}\,ds\right)\mathbb{E}\left(\int_0^T \tilde{\theta}\,ds\right)\end{aligned}$$

and hence

$$\mathbb{E}\varepsilon(t)^2 \ge \frac{(\mathbb{E}\int_0^T f\,ds)^2}{nC}.$$

However, above we have seen that $\theta = Cf/(\int_0^T \mathbb{E}f\,ds)$ achieves this bound, and hence is the optimal choice.

5. Application to hedging. An important application of the results in the previous section is to hedging of financial derivatives. Here we treat the simplest Black–Scholes model and only give a brief comment on more complicated problems. The limit distribution of the Black–Scholes hedging error for equidistant deterministic grids has been studied, for example, in Bertsimas, Kogan and Lo (2000) and Hayashi and Mykland (2005). [We have not been able to follow the proof of Theorem 1.b in Bertsimas, Kogan and Lo (2000); specifically, we could not understand the use of Lemma 5.1 from Duffie and Protter (1992).]

We distinguish between *complete* and *incomplete* financial markets. In complete markets, all derivatives can be replicated (hedged) perfectly by trading in a self-financing way in the underlying and a money market account. The approximation error distribution we analyze is here the total hedging error. In an incomplete market, an investor who hedges a contract will still choose a hedging portfolio which is, in some sense, optimal for her purposes. In this case, the error we obtain is relative to this optimal hedging portfolio. We give now an application of the results in the previous section to hedging in the complete Black–Scholes market.

We assume that a stock S follows the Black–Scholes model. In other words, we model the stock as a geometric Brownian motion, which has the dynamics

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t)$$

for $\mu, \sigma > 0$, where B is a Brownian motion, and $S(0) = s > 0$. Further, we have a risk-free money market account with dynamics

$$dR(t) = rR(t)\,dt$$

for $r > 0$, where $R(0) = 1$. It is well known that the price of a so-called *call option* with payoff $\max(S(T) - K, 0)$ at the deterministic terminal time T, for some strike price K, is at time t

$$\Pi(t) = \Phi(d_+)S(t) - Ke^{-(T-t)}\Phi(d_-),$$

where Φ denotes the standard normal cumulative distribution function and

$$d_\pm(t) = \frac{\log(S(t)/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Now, if we set

$$Y(t) = \begin{pmatrix} S(t) \\ R(t) \end{pmatrix}$$

and $f = (\Phi(d_+), -\Phi(d_-)Ke^{-rT})$, we get that

$$\Pi(t) = \int f(Y(t))\,dY(t)$$

gives the self-financing price process of the call option. This is of the form considered in Theorem 3.3, with $d = 2$ and $\beta_{1,1}(t) = \sigma S(t)$, and all other β-s equal to zero. Thus, using the stopping times (17), Theorem 3.3 gives that the hedging error satisfies

$$\begin{aligned}\sqrt{n}\big(\Pi(t) - \Pi \circ \eta_n(t)\big) &\Rightarrow \int_0^t \frac{df_1}{dx_1}(s)\sigma^2 S(s)^2/\sqrt{2\theta(s)}\,dW(s) \\ &= \int_0^t \frac{\phi(d_+(t))\sigma S(s)}{\sqrt{2\theta(s)(T-s)}}\,dW(s)\end{aligned}$$

with $\phi(t) = d\Phi(t)/dt$ the standard normal density function.

Consider now an investor who hedges a call option, but who only adjusts her hedge at some stopping times $\{\tau_k\}_{k\ge1}$ of her own choosing. If she wants to have a "uniform" increase of the error and make it approximately a Brownian motion, she should use the "no bad days" strategy from the previous section. This would mean that she would use the stopping times (30) with $\theta(t) = c\phi(d_+(t))^2\sigma^2 S(t)^2/(2(T - t))$. However, this leads to a (purely) technical difficulty: $\theta(t)$ tends to 0 as $t \to T$ if $S(T) \in \mathbb{R} \setminus K$ and to ∞ if $S(T) = K$. This means that the assumption of *a.s.* Riemann integrability of $1/\theta$ is not satisfied on $[0, T]$, nor is the assumption that $\sup_{t\in[0,T]} \theta(t) < \infty$ on $[0, T]$. A theoretical (and in fact also practical) solution is to instead only evaluate the hedging strategy up to a constant time $V < T$, with V close to T. Theorem 3.3 gives that the hedging error up until V for large cn then approximately is distributed as $W(t)/\sqrt{cn}$.

Alternatively, the minimum standard deviation strategy and the same reasoning as above lead to choosing

$$\theta(t) = \frac{C\phi(d_+(t))\sigma S(t)}{\sqrt{2(T-t)}}\Big/\left(n\int_0^V \mathbb{E}\left[\frac{\phi(d_+(s))\sigma S(s)}{\sqrt{2(T-s)}}\right]ds\right), \tag{34}$$

where C is the expected number of evaluation times. This yields the approximate distribution

$$\sqrt{\int_0^V \mathbb{E}\left[\frac{\phi(d_+(s))\sigma S(s)}{C\sqrt{2(T-s)}}\right]ds} \int_0^t \sqrt{\frac{\phi(d_+(s))\sigma S(s)}{\sqrt{2(T-s)}}}\, dW(s) \tag{35}$$

for the hedging error, for n large.

It is now completely straightforward to add one or more stocks to the portfolio and, using, for example, that $\int_0^t f_1/\sqrt{\theta}\, dW_1 + \int_0^t f_2/\sqrt{\theta}\, dW_2$ has the same distribution as $\int_0^t \sqrt{f_1^2 + f_1^2}/\sqrt{\theta}\, dW$, to find the optimal stopping times and the resulting error when the hedges for all of the stocks are adjusted at the same time points. This is how portfolio hedging is done in practice. We leave these calculations to the reader.

An alternative and equally interesting application of our results is to the field of portfolio optimization. For example, in managing a large equity portfolio a tracking error arises due to that it is expensive, or otherwise infeasible, to rebalance the portfolio back to its optimal state too frequently. Since the optimal portfolio to be held by the investor is always known, we are exactly in the setting of the present paper. Here, too, we leave the calculations to the reader.

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