THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

The N-Player War of Attrition in the Limit of Infinitely Many Players

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Abstract

The topic of this thesis is a selected problem in game theory, namely the N-player *War of Attrition*. The War of Attrition is a well established game theoretic model that was first introduced in the 2-player case by John Maynard Smith. Although the original idea was to describe certain animal behaviour in, for instance, territorial competition the interest in the model increased and has found interesting applications also in economic theory. Following the results of Maynard Smith, John Haigh and Chris Cannings generalised the War of Attrition to allow for more than 2 players. Their work resulted in two separate models, in this thesis called the dynamic- and the static model, both reducing to the 2-player case when $N = 2$.

In the paper we study the asymptotic behavior of the N-player models as the number of players tend to infinity. By a thorough analysis of the dynamic model we find a connection to the more difficult static one in the infinite regime. This connection is then confirmed by approaching the limit of infinitely many players also in the static model. Finally, by using the limit results as a source of inspiration for the finite case, we manage to prove new results concerning existence and non-existence of an equilibrium strategy in the N-player static case.

Acknowledgement

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Contents

Introduction

Game theory, when defined in its broadest sense, could be thought of as a collection of models formulated to study situations of conflict and cooperation. By analysing these models game theorists try to find answers to questions concerning best actions for individual decision making, but also to what mechanisms that could underlie social behaviour. This introduction aims in giving a concise survey of some fundamental parts of the subject, and present the most central results. We start off by a quick journey in history.

1.1 Brief History

When writing a text on the history of a specific scientific subject it is always an inevitable fact that the story will not be complete, no matter how hard you try. Game theory is not an exception. It is even difficult to say for sure when and where it began since, in a wide perspective, strategic thinking has of course always been around and hence the foundation of what game theory is today rests in hundreds, and even thousands, of years of history. The very first contribution to mathematical game theory is, however, often credited to the French mathematician Antoine Augustin Cournot (1801-1877) for his book *Recherches sur les principes mathématiques de la théorie des richesses* from 1838. Cournot introduces mathematical tools (functions, probabilities, etc.) in the context of economic analysis and, most importantly, constructs a theory of oligopolistic firms and analyses oligopolistic competition. Thirty years after its publication these ideas were to have a strong influential impact on, what was to become, modern national economy.

One of the first contributions to what we recognise as pure classical game theory today was made in a series of papers and notes by Émile Borel (1871-1956) during the period 1921-1927. Borel studied finite symmetric 2-player games at an abstract and more general level without having any particular application in mind. In his work he introduced the concept of "*méthode de* *jeu*" (method of game, strategy) which he used to pose the fundamental question of whether it was possible to determine a "*méthode de jeu meilleure*" (best method of game) or not, even if it was not properly defined what a "*méthode de jeu meilleure*" actually would be. Strange as it is though, the work done by Borel was never really recognised until years after the breakthrough by von Neumann that was to come. In 1953, while translating Borel's work to English, the French mathematician Maurice Fréchet is quoted saying "... in reading these notes of Borel's I discovered that in this domain [game theory], as in so many others, Borel had been an initiator." (see [4]).

The first huge impact in the mathematical theory of games came in year 1928 through the works of John von Neumann in [12]; *Zur Theorie der Gesellschaftsspiele*. In this paper he gives a complete proof of the classical "minimax theorem" for 2-player zero-sum games, basically saying that Borel's "best method of game" indeed always exists in the zero-sum case. This result is probably the most influential in the history of game theory and it would not be an overstatement to claim that the subject was born in 1928 through the paper of von Neumann. Apart from the minimax theorem von Neumann was the first to clearly explain the passing from extensive-form games to the more useful notion of normal-form games. The normal-form was to become of great importance not only for game theory, but also for the shaping of modern economic theory.

In their famous book *Theory of games and economic behavior* from 1944 von Neumann and Morgenstern present the state of the art theory available at the time, including cooperative games with definitions of *TU-games* and the solution concept of *stable sets*. A short survey of John von Neumann's contribution to game theory is given in e.g. [3].

During the 30's and 40's much of the research done in game theory was focused on cooperative games in which players are engaged into coalitions. Even though this analysis was (and still is) both interesting and important it was somewhat leading away from other interesting questions of games where negotiation and individual decision making based on personal information is present. This direction gained momentum in the 1950's thanks to the contributions of John F. Nash to non-cooperative game theory and his famous equilibrium theorem. In 1951 Nash published the paper *Non-Cooperative Games* in which he defined and proved the general existence of a "best play" *equilibrium strategy*, or *Nash-equilibrium*[1], valid in all finite normal-form games. This re-

[1]The concept was actually formulated already by Cournot, but in a much less general setting. Some scholars have suggested the name Cournot-Nash-equilibrium, or even Cournot-equilibrium, but most people do agree on that the depth of the definition is due to Nash.

sult was the ultimate generalization of von Neumann's minimax theorem, but it was also the igniting spark for future research in non-cooperative games (even though the impact of the result initially spread slowly).

Among many significant contributions by Nash (that range even outside game theory) he gave strong arguments for why any theory of games should be reducible to equilibrium analysis of a normal-form game and he gave a beautiful (axiomatic) argument to solve the so called two-person *bargaining problem*. For a good further reading of the importance that Nash's work have had on game theory and economy I recommend [10].

By the mid 50's and onward game theory had become a well established area within the mathematical community and persons like John Harsanyi, Reinhard Selten, Robert Aumann and Lloyd Shapley, just to mention a few, made important and astonishing contributions to the subject.

Considering applications the spectrum was initially rather narrow, mainly concentrated in economic theory. It would take until the 1970's for this to change by the works of two mathematical biologists, John Maynard Smith and George R. Price, and their definition of *evolutionary stable strategy*, or *ESS*, in "*The logic of animal conflict*" (see [9]). What Maynard Smith and Price realised was that ideas from game theory could be used to formulate a related dynamical theory, potentially useful for describing population dynamics. The big difference was that the players in this model were not assumed to act in a rational manner. This was the starting point of what today is known as *evolutionary game theory* which, apart from its original intention of being a tool in theoretical biology, has found applications in economy, social science and philosophy.

Ever since its first major breakthrough in 1928 game theory has continued to expand both in terms of applications and theoretical development. Today there is a wide range of ongoing research of all kinds in subjects from classical and evolutionary game theory to stochastic and differential games. As late as in 2006 a new theory (related to differential games) was being developed independently in works by P. L. Lions, J. M. Lasry and P. E. Caines, M. Huang, R. Malhamé, called *mean field games*, or just *MFG*. MFG has attracted lots of attention by opening doors to many potential applications, but the theory is still somewhat under construction (there is not yet even a text book in the subject).

1.2 Extensive-Form and Normal-Form games

Games of various kinds have been present in our society for ages in connection to economy and gambling, in strategic decision making and conflict scenarios, but quite often also just for the sake of fun. One of the most prominent of those games is undoubtably chess which has diverted mankind for centuries. The rules are so simple that one can learn about how to play in only a few minutes yet, not even a lifetime of practise is enough to fully master the complexity of the game. The difficulty of playing chess originates from the enormous number of moves a player can make during a play. Indeed, if we were to reduce the number of pieces for each player from 16 to 8 the game would become rather poor. Apart from the matter of "size", theoretically speaking, chess is actually very simple. Its general structure is common for a wide range of other 2-player games such as for instance Othello, Nim, Go etc., which are all typical examples of games that can be represented in a so called *extensive-form*. Loosely speaking an extensive-form 2-player game is a finite directed tree where each node represents a player in some position in a play of the game. There are three different types of nodes; one having outgoing edges but no incoming called the *root*, nodes having both outgoing and incoming edges are *intermediate*, and nodes having an incoming edge but no outgoing are called *terminal*. An edge connecting two nodes represent a move that the player at the first node can use to get to the second node. The root of the tree is the starting point of the game. In chess for instance the white player is in the root from which there are 20 different moves leading to an intermediate node for the black player. The game is played via intermediate nodes until it finally reaches a terminal node where the game ends (checkmate or draw).

The outcome of a play is measured by means of a *payoff function* which assigns a 2-vector to each of the terminal nodes. The elements in this vector represent the payoffs given to each of the players respectively. If we again use chess as a game of reference we could assign values to each possible outcome as "win = 1", "draw = 0" and "loose = -1", and hence get $\{(1,-1),(-1,1),(0,0)\}$ as the set of possible values of the payoff function.

At this point we have all that is needed to describe the simplest kinds of 2-player games in extensive-form namely; a game tree, on which the game is being played, and a pay-off function, measuring the outcome of a play. This is indeed a good start, but in order to find a complete theory of games it is obvious that we should be looking for a more general description. Take for instance a game of poker in 4 players. We then face two new features that our simple description can not yet meet. Firstly that the number of players is greater than two and, secondly, that the players are unaware of how the opponents are playing. One could also think of games having "chance moves", i.e. random moves that does not connect to any particular player. The following definition of a game in extensive-form covers all of the above features and can be found in [14]:

Definition 1.2.1. By an n-player game in extensive form is meant

1. a topological tree Γ with a distinguished node A called the *starting point* of Γ
2. a function, called the *pay-off function*, which assigns an n-vector to each terminal node of Γ
3. a partition of the intermediate nodes of Γ into $n+1$ sets $S_0, S_1, ..., S_n$, called the player sets
4. a probability distribution, defined at each node of S_0, among the immediate followers of this node
5. for each $i = 1, ..., n$ a subpartition of S_i into subsets S_i^j, called *information sets*, such that two nodes in the same information set have the same number of immediate followers and no node can follow another node in the same information set
6. for each S_i^j there is an index set I_i^j together with a 1-1 mapping of I_i^j onto the set of immediate followers of each node in S_i^j.

As mentioned earlier condition (1) and (2) suffice to describe the simplest games in extensive form like e.g. chess. Condition (3) sets the stage for the n-player generalization where S_i, $i \neq 0$, should be thought of as the collection of nodes in Γ from which player i makes a move. The set S_0 differs from $S_1, ..., S_n$ in that it contains nodes from which the game proceeds at random (without any player making a move) to an immediately following node, i.e. S_0 is the collection of chance nodes. Conditions (5) and (6) open up the possibility of having a "lack of knowledge" in the game. For $i \neq 0$ one should think of the nodes in S_i^j as different positions in the play of player i that, however, are indistinguishable to him. In poker for instance every player move, that is not terminal, is followed by a chance move (drawing a card) and the only information available to a given player i is what cards he has at the moment and what cards he has decided to discard. No information of the opponents hands is available so, for a fixed hand, all the possible nodes of player i in a round j for which the same cards have been discarded by i (in any order) is indistinguishable to him. Thus there is a natural partition of S_i, for each $i = 1, ..., n$, where each S_i^j in the partition contains several nodes of Γ. A game in which $|S_i^j| = 1$ for all i and j, i.e. all nodes are always distinguishable, is said to have *perfect information*. Chess is a typical game of perfect information.

Using the terminology of Definition 1.2.1 we are now ready to introduce the fundamental concept of *strategy*.

Definition 1.2.2. Let Γ be an n-player game in extensive form and let S_i^j, for $1 \leq i \leq n$, be the information sets of a player i. A *(pure) strategy* of player i is defined as a function σ_i from each S_i^j to any of the edges which follow a representative node of S_i^j. The set of all strategies available to player i is denoted by Σ_i. An element in the product space $\Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n$ is called a *profile* of strategies.

The above definition captures the intuitive idea of what a strategy "should be", namely; a strategy is a complete plan of how to play in any given situation. There is, however, also a drawback with Definition 1.2.2 in that it somewhat assumes the player to have decided about how to play even before the game has started. One may of course argue that this is unreasonable in many situations. Again chess is a good example since you (in practice) would not be able to make up a plan on what to do in more than a few moves ahead. Indeed this is a practical limitation that we will have to overlook. It will not make the theory less interesting.

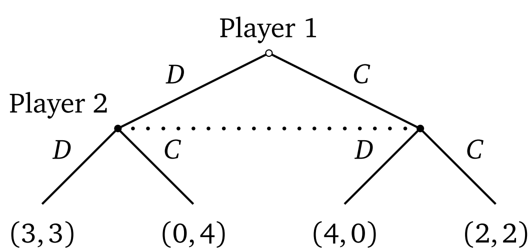
The introduction of strategies is of course of fundamental importance since they are representing the basic elements of what game theory means to study. We want find out if there is a best way of choosing strategy in a given situation. From a player point of view this would be to pick a strategy that maximises the personal payoff. Given that the opponents play according to some profile of strategies you would pick your own strategy so that your final position gets as good as possible. It is time to introduce some further notation. We can make the notion of payoff-function in an n -player game precise by declaring it as a function $\mathscr{J} : \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n \to \mathbb{R}^n$ where

$$\mathscr{J}(\sigma_1, ..., \sigma_n) = (\mathscr{J}_1(\sigma_1, ..., \sigma_n), ..., \mathscr{J}_n(\sigma_1, ..., \sigma_n))$$

and $\mathscr{J}_i$ is the payoff-function of player i. Note that since Γ may consist of chance moves one should, in general, interpret $\mathscr{J}$ as an expectation.

In many situations, both in theory and practise, given that every player have chosen a strategy we are only interested in the values of each individual payoff-function. In principle, given the product space $\Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n$ and the payoff-function $\mathscr{J}$ we have a characterisation of the game that is in many ways sufficient for our needs. Describing a game in this way via the payoff-function is commonly known as a *normal-form* representation. Every game in extensive form can canonically be represented in normal form, but not in the converse order. The reason is that a game in extensive form contains information of the game tree Γ which is lost in the normal form representation. If the number of strategies in each Σ_i is finite (which we have assumed) the normal-form representation is simply given by an n-dimensional array of n-vectors. In the special case of a 2-player game this array reduces to a bimatrix.

Example 1.2.3. One of the most famous examples of games in game theory is undoubtedly *The Prisoner's Dilemma*, first introduced in 1950 by Merrill Flood and Melvin Dresher working at the RAND Corporation. Two members of a criminal gang are arrested by the police and imprisoned in two different rooms without being able to communicate with each other. Both of them can act in either of the two following ways: either choose to *cooperate*, i.e. keep quite to the police during the interrogation, or else they *defect* and choose to testify against the other prisoner. We denote cooperation by C and defection by D. If the prisoners cooperate they will both go to jail for two years and if they defect they will get three years in prison. The catch with the game is that if one of the prisoners choose to cooperate while the other defects the latter will be released while the cooperative prisoner will get four years behind the bars. Denoting the prisoners by Player 1 and Player 2 this game can easily be represented in both extensive- and normal form as in Fig. 1.1.



	C	D
C	2,2	4,0
D	0,4	3,3

Figure 1.1: The Prisoner's Dilemma represented in extensive- and normal form.

The dashed line in the extensive form representation of the prisoner's dilemma indicates that the nodes belong to the same information set. This information is not included in the normal-form representation.

We are now ready to introduce the notion of Nash-equilibrium.

Definition 1.2.4. Given an n-player game Γ we say that an n-tuple of strategies $(\sigma_1^*, \ldots, \sigma_n^*) \in \Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_n$ is a *(pure) Nash-equilibrium* if and only if for any $i = 1, ..., n$ and $\sigma_i \in \Sigma_i$,

$$\mathscr{J}_i(\sigma_1^*, ..., \sigma_n^*) \geq \mathscr{J}_i(\sigma_1^*, ..., \sigma_{i-1}^*, \sigma_i, \sigma_{i+1}^*, ..., \sigma_n^*).$$

Nash-equilibrium is one of the most celebrated definitions in non cooperative game theory and serves as the major solution concept. The intuitive meaning is clear. Given that all players in the game play according to their Nash-equilibrium strategy, none of them can get a higher (expected) payoff by changing to another strategy. Despite its great importance Nash-equilibrium is far from being the only solution concept present in the litterateur. Other important solution concepts are for instance *subgame perfect equilibrium* and *evolutionary stable strategy*.

At this point an important question naturally rises; given a finite game Γ, does it always exist a Nash-equilibrium? A moment's thought will revile this to be false. Consider for instance the 2-player normal-form game:

$$\begin{pmatrix} (1,-1) & (0,0) \\ (0,0) & (1,-1) \end{pmatrix} \tag{1.2.1}$$

where, at each entry, either Player 1 or Player 2 can do better by changing to another strategy. There is, however, more to be said about this matter as we will see shortly.

We conclude this section with the important notion of symmetric games.

Definition 1.2.5. Let Γ be an n-player game given in normal form. We say that Γ is *symmetric* if and only if for every $i = 1, ..., n$ and permutation π it holds that

$$\mathscr{J}_i(\sigma_1, \sigma_2, ..., \sigma_n) = \mathscr{J}_{\pi(i)}(\sigma_{\pi(1)}, \sigma_{\pi(2)}, ..., \sigma_{\pi(n)}),$$

for all $(\sigma_1, \sigma_2, ..., \sigma_n) \in \Sigma_1 \times \Sigma_2 \times ... \times \Sigma_n$.

1.3 Zero-Sum Games

The simplest possible games to study are the so called *zero-sum games*. They are characterised by the fact that the sum of the elements of the payoff-vector at any terminal node always equals to zero. Thus, in a zero-sum game the winnings of one player has to be paid, in some way or another, by the other players. If only 2-player zero-sum games are considered, things simplify even further since the elements of the payoff-vector then have to be additive inverses of each other. Hence it suffices to represent it with the payoff of only one player, say Player 1. The normal-form bimatrix representation can therefore be reduced to a matrix representation. For this reason zero-sum games are sometimes also referred to as *matrix games*. The game in (1.2.1) is a zero-sum game and in the reduced notation of the first player's payoff we get the game matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3.1}$$

The easy payoff-matrix representation makes the 2-player zero-sum games tractable for closer analysis like, for instance, existence of solution strategies. We know that (1.3.1) does not have a pure Nash-equilibrium but maybe we can quantify those matrix game that have? Consider a general zero-sum game with a pay-off function $\mathscr{J}$ corresponding to a payoff-matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n\times m}$. The strategy sets of Player 1 and Player 2 are Σ_1 and Σ_2 respectively and both are assumed to be finite. The pair $(\sigma_1^*, \sigma_2^*) \in \Sigma_1 \times \Sigma_2$ is a pure Nash-equilibrium if and only if both $a_{i^*j^*} = \max_i a_{ij^*}$ and $a_{i^*j^*} = \min_j a_{i^*j}$, where $\mathscr{J}(\sigma_1^*, \sigma_2^*) = a_{i^*j^*}$. Such an element, if it exists, is called a *saddle point* of $\mathbf{A}$. If $\mathbf{A}$ lack saddle points the game lacks pure Nash-equilibria. What would happen if we were playing such a game? The goal of Player 1 is to win as much as possible while minimising the risk of loosing too much. Thus, in each row two elements are of interest; the greatest (maximal gain) and the least (maximal loss). A rational choice of strategy for Player 1 would be to pick a strategy corresponding to the row in $\mathbf{A}$ in which the least possible win is maximised. An analogue argument also holds for Player 2 who preferably would choose to play a strategy corresponding to the column in which the greatest loss is minimised. We define

$$\begin{aligned} \underline{v} &:= \max_{\sigma_1\in\Sigma_1} \min_{\sigma_2\in\Sigma_2} \mathscr{J}(\sigma_1, \sigma_2) \\ \overline{v} &:= \min_{\sigma_2\in\Sigma_2} \max_{\sigma_1\in\Sigma_1} \mathscr{J}(\sigma_1, \sigma_2) \end{aligned}$$

and call $\underline{v}$ the *gain-floor* and $\overline{v}$ the *loss-ceiling*. By construction we have the inequality

$$\underline{v} \leq \overline{v}, \tag{1.3.2}$$

and if "=" in (1.3.2) we say that the game has *value* $v = \underline{v} = \overline{v}$. Hence, if a zero-sum game has value the payoff-matrix has a saddle point and there exists a pure Nash-equilibrium. If not, then the gain-floor and the loss-ceiling only represent the best possible win and loss each of the players can hope for.

However, there is a way of playing in games without saddle points so that both players can gain from it. That is the concept of *mixed strategies*, first introduced by Émile Borel:

Definition 1.3.1. Let Γ be an n-player game in normal form with strategy spaces $\Sigma_1, \Sigma_2, ..., \Sigma_n$ and payoff-function $\mathscr{J}$. We say that μ_i is a *mixed strategy* of player i if $\mu \in \mathscr{M}_1\left(\Sigma_i\right)$, where $\mathscr{M}_1\left(\Sigma_i\right)$ is the set of probability measures over the space Σ_i.

Note that in mixed strategies the payoff-function $\mathscr{J}$ turns into an expected payoff-function given by

$$\int_{\Sigma_1 \times \Sigma_2} \mathscr{J}(\sigma_1, \sigma_2)\mu_1(d\sigma_1)\mu_2(d\sigma_2),$$

for $(\mu_1, \mu_2) \in \mathscr{M}_1(\Sigma_1) \times \mathscr{M}_1(\Sigma_2)$. For simplicity though we will stick to the same notation as for the ordinary payoff-function.

To play a mixed strategy is indeed a bit odd. Practically it means that instead of using rational reasoning to find a good strategy one would draw a strategy at random according to some probability distribution. By the inclusion $\Sigma_i \subset \mathscr{M}_1(\Sigma_i)$, there is however good reason to believe that the mixed strategies enable us to find values in a wider class of games. The following result is due to von Neumann in [12] (1928) and can be considered as the fundamental theorem of game theory.

Theorem 1.3.2 (The minimax theorem). *Let Γ be a 2-player zero-sum game with finite strategy spaces Σ_1 and Σ_2 and payoff-function $\mathscr{J}$. Then there exists at least one Nash-equilibrium in mixed strategies $(\mu_1, \mu_2) \in \mathscr{M}_1(\Sigma_1) \times \mathscr{M}_1(\Sigma_2)$ and the game has value, i.e.*

$$\max_{\mu_1 \in \mathscr{M}_1(\Sigma_1)} \min_{\mu_2 \in \mathscr{M}_1(\Sigma_2)} \mathscr{J}(\mu_1, \mu_2) = \min_{\mu_2 \in \mathscr{M}_1(\Sigma_2)} \max_{\mu_1 \in \mathscr{M}_1(\Sigma_1)} \mathscr{J}(\mu_1, \mu_2).$$

The minimax theorem was the first major breakthrough in what was to become the theory of games and it has later been generalized by several authors. John von Neumann him self was quoted as saying "As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the minimax theorem was proved" (see [1]).

Since we have assumed the number of strategies available to each player to be finite (n say) any mixed strategy may be represented by a vector $\mathbf{x} \in \mathbb{R}^n$ such that all $x_i > 0$ and $\sum_{i=0}^n x_i = 1$. Each element x_i is the probability of getting the pure strategy indexed by i when playing $\mathbf{x}$. The payoff-function can be written as

$$\mathscr{J}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{A} \mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m.$$

The minimax theorem is a pure existence result, but there is also another interesting result saying that equilibrium strategies in symmetric 2-player zero-sum games can be derived as solutions to a certain ode-system. The following theorem can be found in [2] and is due to von Neumann:

Theorem 1.3.3. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times m}$ be the payoff-matrix in a 2-player zero-sum game and define the functions $u_i(\mathbf{y}) = \mathbf{e}_i^T \boldsymbol{A}\mathbf{y}$, for $i = 1, 2, ..., n$, $\phi(a) = \max(0, a)$ and $\Phi(\mathbf{y}) = \sum_{i=1}^n \phi(u_i(\mathbf{y}))$. For any mixed strategy $\mathbf{y}^0$ of Player 2 consider the following problem:*

$$\begin{cases} y_j'(t) = \phi\left(u_j(\mathbf{y}(t))\right) - \Phi\left(\mathbf{y}(t)\right) y_j(t) \\ y_j(0) = y_j^0 \end{cases}$$

Then, for any positive monotone sequence $\{t_k\}$ growing to infinity any limit point of $\{\mathbf{y}(t_k)\}$ is an equilibrium strategy of Player 2 and, furthermore, there is a constant C such that $\mathbf{e}_i^T \boldsymbol{A}\mathbf{y} \leq \sqrt{n}/(C + t_k)$.

It should be mentioned that, even though interesting, finding the equilibrium solution by means of ordinary differential equations is not very efficient. Much faster solution algorithms have been developed using methods from linear programming.

The literature on 2-player zero-sum games is huge, both from a theoretical- and applied point of view, and the topic serves as the foundation for what is called *classical game theory*. Apart from the 2-player setup there are also results available on n-player zero-sum games, but not as extensive. A substantial part of the classical text *Theory of games and economic behavior* by von Neumann and Morgenstern (see [13]) is though devoted to these types of problems.

1.4 Nash's Theorem

Following the footsteps of von Neumann the young Princeton mathematician John F. Nash was to give game theory its next major breakthrough by a beautiful generalisation of the minimax theorem. Based on Nash's work in his doctoral thesis the following result can be found in [11]:

Theorem 1.4.1. *Every normal-form n-player game with finitely many strategies has at least one Nash-equilibrium in mixed strategies.*

The proof of Nash's theorem builds upon a fixed point argument, preferably using Brouwer's or Kakutani's fixed point theorem, and is surprisingly elegant.

The statement is true even for more general normal-form games having infinite strategy sets. Ever since its publication in 1951 the result has generated much attention in research and has by now created a whole avenue of interesting research in n-player non-cooperative games.

1.5 Evolutionary Game Theory

Classical non-cooperative game theory typically deals with questions concerning equilibrium analysis. Problems like; Is there a Nash-equilibrium? Is it unique? What is the expected payoff and what is the risk when playing a Nash-equilibrium? etc. are fundamental. From a game theorists point of view these questions are of course very natural to ask, but are they equally natural for the economist or the evolutionary biologist when trying to understand actual social behaviour? Do agents really play Nash-equilibrium in a given situation and, if they do, which one do they chose if several exist? These are some of the basic problems of interest in what is called *evolutionary game theory*. In contrast to the classical theory, evolutionary game theory does not assume players to act rationally, and there is very good reason for this. In every day life we are all being exposed to new situations which, in principle, could be analysed game theoretically. The problem is that we most often do not tend to think of those situations in strategic terms. It would therefore be absurd to simply assume our behavior to be reflected by what is game theoretically rational since, what we probably really use are just simple strategies, imitation of others, experience and rules of thumb. The best one can hope for is that rational behaviour, as described by Nash-equilibrium, is reached over time as agents eventually learn how to play. In evolutionary game theory the basic setup is a large population of players who repeatedly engage in strategic interaction. Changes in the behaviour in these populations are driven on an individual level by features such as for instance imitating more successful behaviours. For a thorough discussion of the passing from classical to evolutionary game theory we recommend [8].

We are now going to present the basic model. Consider a finite population of N players and a normal form game admitting a finite set of (pure) strategies $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$. For simplicity we are going to assume all the individual strategy sets to be the same and equal to Σ. A *population state* x is a point in the n-dimensional unit simplex X and describes the proportion of each strategy being used within the population, i.e. x_i is the proportion of players using strategy σ_i. Note that for $N < \infty$ the values of x are in the grid $\mathscr{X}_N := X \cap \frac{1}{N}\mathbb{Z}^n$, embedded in X. A *population game* is a continuous vector-valued payoff-function $F : X \to \mathbb{R}^n$ for which each $F_i(x)$ is the (expected) payoff to a σ_i-player given a population state x. In this context a Nash-equilibrium is a population state x^* satisfying the implication

$$x_i^* > 0 \;\; \Rightarrow \;\; i \in \operatorname*{argmax}_{1 \le j \le n} F_j(x),$$

see e.g. [17]. It is easy to show that the above definition coincides with the classical Nash-equilibrium if the population game is a finite symmetric normal-form game.

As we mentioned earlier the goal of evolutionary game theory is to understand the possible mechanisms driving the strategic behaviour within a population. How and why do players switch from one strategy to another and how do we model it? To deal with this we introduce *revision protocols*. Formally, a revision protocol is a map $\rho : \mathbb{R}^n \times X \to \mathbb{R}_+^{n \times n}$ taking payoff vectors π (π_i is the expected payoff to a σ_i-player given x) and population states x as arguments and returns a square matrix having positive elements. The idea is to consider a population of N individuals, each of them being equipped identical "alarm clocks". The time durations between two consecutive rings of a clock are independent and exponentially distributed with exponential parameter R. As soon a a clock rings the player carrying it gets a chance to switch to another strategy. The switching process is random and related to ρ in a way so that the probability of changing from strategy σ_i to σ_j, $i \neq j$, is given by ρ_{ij}/R. When a switch occurs at a time t the population state vector also reacts by jumping to a neighbouring point in $\mathscr{X}_N$. This jump-process can be described as a continuous time Markov-chain which we denote by $X_N(t)$. The aim is to study the time evolution of $X_N(t)$ and especially its asymptotic behaviour when $t \to \infty$. At this point there are two possible routs to take; either we study $X_N(t)$ directly, the so called *stochastic dynamics*, or we study the expectation $x_t := \mathbb{E}[X_N(t)]$, the *mean dynamics*. In this text we are going to focus on the latter in the case of population games.

Given a revision protocol ρ and a population game F there should be an equation for x_t. To find it we consider the expected differential of $X_N(t)$ over a small time interval $[t, t + dt]$. In dt units of time the expected number of revision opportunities for each player is Rdt. Thus, given a population state x at time t, there will be on average Nx_iRdt revision opportunities within the group of σ_i-players. Therefore, since the probability of changing strategy is ρ_{ij}/R, we expect $Nx_i\rho_{ij}dt$ of the players in this group to switch to σ_j in the time interval $[t, t + dt]$. Adding up the expected number of immigrants and emigrants to and from strategy σ_i one finds

$$Ndx_i = N\left(\sum_{j=1,j\neq i}^{j=n} x_j\rho_{ji}(F(x),x) - x_i\sum_{j=1,j\neq i}^{j=n}\rho_{ij}(F(x),x)\right)dt$$

which yields following differential equation:

$$\dot{x}_i = \sum_{j=1,j\neq i}^{j=n} x_j\rho_{ji}(F(x),x) - x_i\sum_{j=1,j\neq i}^{j=n}\rho_{ij}(F(x),x) =: V^F(x), \tag{1.5.1}$$

forming a system of n ordinary differential equations called *the mean dynamic*. A population state x such that $V^F(x) = 0$ is called a *stationary point*.

So far nothing particular has been said about the revision protocol and its properties. The explicit form of ρ depends on what problem one would like to study and on what application one has in mind. It must therefore be constructed on a case to case basis. There is, however, a handful of models in the literature of certain interest. In the context of evolutionary biology the most common model by far is the so called *replicator dynamics* (first introduced in [19] by Taylor and Jonker) which is generated from (1.5.1) by choosing $\rho_{ij}(\pi,x) = x_j[\pi_j - \pi_i]_+$. The basic idea behind this choice is simple; the probability of switching from strategy i to strategy j should be proportional to the proportion of players using σ_j (imitation) and to the advantage in payoff of playing σ_j instead of σ_i (payoff-advantage). If, given x, σ_j-players do worse than σ_i-player the probability of a switch is zero. Inserting this protocol in (1.5.1) we get the replicator dynamics:

$$\dot{x}_i = x_i\left(F_i(x) - \sum_{i=1}^{n} x_i F_i(x)\right). \tag{1.5.2}$$

Note that the form of (1.5.2) makes it impossible for strategies that are not present in the initial population to emerge later on.

In the case of a linear population game, i.e. $F(x) = \mathbf{A}x$ for some matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, the replicator equation can be written

$$\dot{x}_i = x_i\left((\mathbf{A}x_t)_i - x_t^T\mathbf{A}x_t\right). \tag{1.5.3}$$

The system created from (1.5.3) satisfy the following properties (see [5]):

1. if x is a Nash-equilibrium, then $V^F(x) = 0$

2. if x is a strict Nash-equilibrium, then it is asymptotically stable

3. if $V^F(x) = 0$ and x is the limit of an orbit in the interior of the simplex X as $t \to \infty$, then x is a Nash-equilibrium

4. if $V^F(x) = 0$ and x is stable[2], then it is a Nash-equilibrium.

Note, however, that the converse implications of (1) - (4) are all false. Thus, the replicator dynamics does not guarantee convergence of solutions to a Nash-equilibrium. The following is a simple example of such a situation.

Example 1.5.1 (Rock-Paper-Scissor). In the classic game of rock-paper-scissor there are obviously three pure strategies to chose from. We identify each of them by the unit vectors in $\mathbb{R}^3$: e_1, e_2 and e_3. The game is characterised by the fact that e_1 wins against e_2, e_2 wins against e_3 and e_3 wins against e_1. In the general setup the individual payoff of playing e_i against e_j can be written like $\mathscr{I}(e_i, e_j) = e_i^T \mathbf{A} e_j$ where

$$\mathbf{A} = \begin{pmatrix} 0 & -a_2 & b_3 \\ b_1 & 0 & -a_3 \\ -a_1 & b_2 & 0 \end{pmatrix}$$

for any $a_1, a_2, a_3, b_1, b_2, b_3 > 0$. This game has a unique Nash-equilibrium x^* in the interior of the unit simplex X (if for instance $a_1 = a_2 = a_3$ and $b_1 = b_2 = b_3$ it is $(1/3, 1/3, 1/3)$) which is asymptotically stable if and only if $\det \mathbf{A} > 0$. In the case $\det \mathbf{A} < 0$ the solutions of the replicator dynamics, when starting at any interior state (not equal to x^*), will spiral to the boundary of X and never settle at the equilibrium state. This is illustrated in Fig. 1.2.

The inability of the replicator dynamics to ensure convergence to a Nash-equilibrium should not be considered as a flaw. It is merely an indication that it takes more than imitation of success to reach game theoretic rationality, which is an interesting observation in it self.

The replicator equation (1.5.2) is said to be *permanent* if there is a compact set $K \subset \operatorname{int} X$ such that for all $x_0 \in \operatorname{int} X$ there is a $T > 0$ such that for all $t > T$ one has $x_t \in K$. For such a dynamics (having F linear) we have the following (see [5]):

Theorem 1.5.2. *If (1.5.3) is permanent, then there exists a unique stationary point $z \in \operatorname{int} X$. The time averages along each internal orbit converge to z:*

$$\frac{1}{T}\int_0^T x_i(t)dt \xrightarrow{T\to\infty} z_i, \quad \text{for } i = 1, 2, \ldots, n.$$

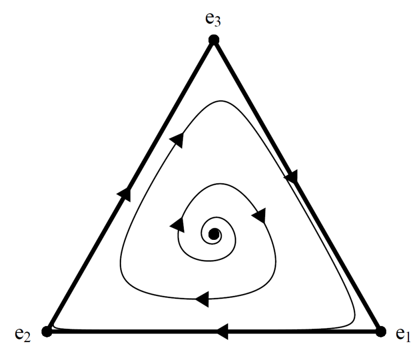


Figure 1.2: Replicator dynamics for the rock-paper-scissor game with $a_i = 1$ and $b_i = 0{,}55$ for all $i = 1, 2, 3$.

The concept of permanence means, roughly, that if all strategies are present in the population at time zero, then they will not go extinct. Theorem 1.5.2 says that the time average of the solution curves of a permanent replicator dynamics equals to its unique interior stationary point.

We are now going to introduce the basic solution concept of evolutionary game theory, namely that of *evolutionary stable strategy* or, more commonly, *ESS*.

Definition 1.5.3. Consider a 2-player symmetric normal form game with strategy set Σ and payoff-function $\mathscr{J} : \Sigma \times \Sigma \to \mathbb{R}$. A mixed strategy $\mu^* \in \mathscr{M}_1(\Sigma)$ is an ESS if either

$$\mathscr{J}(\mu^*, \mu^*) > \mathscr{J}(\mu, \mu^*)$$

for all $\mu \in \mathscr{M}_1(\Sigma) \setminus \{\mu^*\}$ or else, if equality in the above for some $\hat{\mu}$,

$$\mathscr{J}(\mu^*, \hat{\mu}) > \mathscr{J}(\hat{\mu}, \hat{\mu}).$$

The notion of ESS was first introduced in [9] by Maynard Smith and Price as an alternative to Nash-equilibrium when trying to apply game theory to problems in evolutionary biology. As concept the ESS is slightly weaker than strict Nash-equilibrium, but nevertheless it is always an equilibrium in the usual sense. Intuitively, for a strategy to qualify as evolutionary stable it should, if

[2] A point $z \in X$ is *stable* if for every neighbourhood U of z there exists another neighbourhood V of z such that if $x \in V$ then $x(t) \in U$ for all $t \geq 0$. Moreover, a state is *asymptotically stable* if it is a stable attractor.

played by all agents in a population, be resistent to attempts of invasion by any other strategy. In connection to the replicator dynamics an ESS is always an asymptotically stable stationary point and moreover, if it is an interior point of X it is even globally stable.

As we mentioned earlier another interesting path to follow in evolutionary games is that of stochastic evolutionary dynamics, being a "high-resolution" version of the mean dynamics. Indeed, according to Kurtz's theorem (see [7]), we have that

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| X_N(t) - x_t \right| < \varepsilon \right) = 1$$

for any positive $T < \infty$ and $\varepsilon > 0$. Apart from questions related to convergence of population states the stochastic dynamical approach is also well suited to address problems of equilibrium selection in games with multiple locally stable equilibria. Even though interesting we will not bring up any of these results in this text, but readily refer to [16] for a short survey and to [18] for a more thorough discussion.

Bibliography

[1] CASTI, J.L. (1996), *Five golden rules: great theories of 20th-century mathematics - and why they matter*, New York: Wiley-Interscience

[2] FORGÓ, F (2006), *Contribution of Hungarian Mathematicians to Game Theory*, Bolyai Society Mathematical Studies, **14**, 537-548.

[3] FORGÓ, F (2004), *John von Neumann's Contribution to Modern Game Theory*, Acta Oeconomica, **54**, 73-84.

[4] FRÉCHET, M (1953), *Emile Borel, Initiator of the Theory of Psychological Games and its Application*, Econometrica, **21**, 95-96.

[5] HOFBAUER, J.; SIGMUND, K. (2003), *Evolutionary Game Dynamics*, Bulletin of the American Math. Soc., **40**, No. 4, 479-519.

[6] KJELDSEN, T.H. (2001), *John von Neumann's Conception of the Minimax Theorem: A Journey Through Different Mathematical Contexts*, Arch. Hist. Exact Sci., **56**, 39-68.

[7] KURTZ, T.G. (1970), *Solutions of ordinary differential equations as limits of pure jump Markov processes.*, J. of Appl. Prob., **7**, 49-58.

[8] MAILATH, G.J. (1998), *Do People Play Nash Equilibrium? Lessons From Evolutionary Game Theory.*, J. of Econ. Lit., **36**, 1347-1374.

[9] MAYNARD SMITH, J.; PRICE, G.R. (1973), *The logic of animal conflict.*, Nature, **246** No. 5427, 15-18.

[10] MYERSON, R.B. (1999), *Nash Equilibrium and the History of Economic Theory.*, J. of Econ. Lit., **37**, No. 3, 1067-1082.

[11] NASH, J.F. (1951), *Non-Cooperative Games.*, Ann. of Math., **54** No. 2, 286-295.

[12] von Neumann, J. (1928), *Zur Theorie der Gesellschaftsspiele.*, Math. Annalen, **100**, 295-320.

[13] von Neumann, J.; Morgenstern, O. (1944), *Theory of games and economic behavior*, Princeton University Press

[14] Owen, G. (1995), *Game Theory, 3rd edition*, Academic Press Inc.

[15] Pier, J.-P.; Sorin, S. (2000), *Development of Mathematics 1950-2000*, Birkhäuser Verlag

[16] Sandholm, W.H. (2007), *Evolutionary Game Theory*, Lecture Notes

[17] Sandholm, W.H. (2009), *Pairwise Comparison Dynamics and Evolutionary Foundation for Nash Equilibrium.*, Games, No. 1, 3-17.

[18] Sandholm, W.H. (2011), *Stochastic Evolutionary Game Dynamics: Foundations, Deterministic Approximation, and Equilibrium Selection.*, Proc. of Symp. in Appl. Math., **69**.

[19] Taylor, P.D.; Jonker, L. (1978), *Evolutionarily stable strategies and game dynamics*, Math. Biosciences, **40**, 145-156.

Summary of Paper

The paper of this thesis treats asymptotic properties of two different N-player models of the classical game *The War of Attrition*, introduced by Haigh and Cannings in [3], as the number of players grows to infinity. By analysing the models in the limit regime we gain insights of the large scale characteristics of the games that are used to establish new results for one of the finite models.
The War of Attrition was first introduced by John Maynard Smith in 1974 in the well known paper *Theory of games and the evolution of animal contests* (see [4]). The game considers two identical players competing for one prize V of positive value by observing each other and waiting, which is connected to a running cost. The first player to quit the competition looses and agrees to leave the prize to the remaining opponent. In the classical setup the cost of waiting is modeled to be linear in time, i.e. by waiting t units of time the player gets the possibility of winning the prize $V > 0$, but he will also be obligated to pay $-t$ units in time cost. If Player X and Player Y choose waiting times τ_x and τ_y respectively, the payoff function of Player X can be written:

$$\mathscr{J}_x(\tau_x, \tau_y) := \begin{cases} V - \tau_y, & \text{if } \tau_x > \tau_y \\ V/2 - \tau_x, & \text{if } \tau_x = \tau_y \\ -\tau_x, & \text{if } \tau_x < \tau_y. \end{cases}$$

Note that the winning player only pays the time cost of the loosing player since he can observe his opponent leaving the game.

Equilibrium analysis of the 2-player game was done by Bishop and Cannings in 1976 who proved that the War of Attrition admits a unique ESS in mixed strategies given by an exponential distribution having mean V (see [2]).

In 1989 Haigh and Cannings constructed two canonical N-player generalisations of The War of Attrition; *the dynamic model* and *the static model*. The dynamic model is an N-player repetitive game having a sequence of prizes $\{V_k\}_{k=1}^N \subset \mathbb{R}_+$ at stake and is played in $N-1$ rounds. The first round begins by letting all players choose individual waiting times (independently of each other). The player having the least waiting time wins the prize V_1, pays his time cost (still linear in time) and leaves the game. The players remaining pay the same time cost as the player leaving and enter the second round which proceeds just as the first, but having the prize V_2 at stake. The $(N-1)$'th round thus becomes a normal 2-player War of Attrition. For increasing prize sequences, i.e. $V_1 < V_2 < ... < V_N$, it is proven in [3] that the dynamic model, just as the 2-player War of Attrition, in each round k admits a unique mixed ESS given by an exponential distribution, but with mean $(N-k)(V_{k+1}-V_k)$. Also the general case of arbitrary sequences is analysed, still having the existence of a unique ESS as a result (though not as explicit as in the case of increasing sequences).

The static model differs from the dynamic model only in being a one-shot game rather than a repetitive, that is, the game finishes in one turn. Just as in the dynamic model the static model starts by letting all participating players pick a waiting time (independently). The results are then presented and prizes are handed out in the natural order, i.e. the player with the least waiting time receives V_1, the player with the second least receives V_2 and so forth. All players pay their individual time cost except for the "last" player who pays the time cost of the second last player so that, for $N = 2$, we get back to the original 2-player War of Attrition.

The equilibrium analysis of the static model is a bit more intricate than in the dynamic model since all players are competing for all prizes in $\{V_k\}_{k=1}^N$ (and not only one) at once. In [3] it is proven that the static model admits a unique ESS for prize sequences such that $V_{k+1} - V_k = c > 0$. For more general sequences though, the question of existence and uniqueness is unclear. For instance, the 3-player game generated by the prize sequence $\{1,4,6\}$ admits a unique mixed Nash-equilibrium that, however, is not an ESS. There are even games in the static model that lack Nash-equilibria, like for example the game generated by $\{1,2,1\}$. The goal of this paper is to study asymptotic behaviors in the dynamic- and the static model of the N-player War of Attrition as $N \to \infty$.

In Section 2 of the paper presents a heuristic approach to analyse the limiting behaviour of the dynamic model as the number of players tend to infinity. To maintain regularity in the limit we introduce the concept of *prize function* $V(x)$, defined on the compact unit interval, to replace prize sequences by making the assumption that $V_k = V(k/N)$, $1 \leq k \leq N$. We also assume $V(x)$ to be increasing and in $\mathscr{C}^1[0,1]$. We find that, in the limit, the fraction of players q that has left the game at time t (after game start at $t = 0$) is given by the equation $V(q(t)) = t - V(0)$ and, in particular, if $V(0) = 0$ then $q(t) = V^{-1}(t)$.

In Section 3 we investigate the results from Section 2 rigourously and proceed with the asymptotic analysis of the dynamic model. Considering N players we introduce the continuous time Markov chain $X(t)$ (suppressing the index N) counting the fraction of players that have left the game at time t. Given a prize function $V(x)$ such that $V(0) = 0$ it is, by the results from Section 2, natural to believe that "$X \to V^{-1}$" in some sense or another. Indeed, by Theorem 3.2 we manage to prove in Corollary 3.3 that $\lim_{N\to\infty} \mathbb{E}[X(t)] = V^{-1}(t)$ and $\lim_{N\to\infty} \mathrm{Var}(X(t)) = 0$ on the time interval $t \in [0, V(1))$. In addition to the result on convergence in mean of $X(t)$ we also manage to prove in Theorem 3.6 that, in a certain sense, in the limit when $N = \infty$ the dynamic model behaves like a static model in which the players use the mixed strategy $\dot{q}(t) = d/dt(V^{-1})(t)$. We therefore have good reason to proceed by analysing the asymptotic properties of the static model.

Section 4 is devoted to convergence properties in the static model of the War of Attrition. In [3] one can find necessary condition for a given smooth probability density to be an ESS in the static model, stated as a nonlinear autonomous ode-problem of its cdf. For a general prize sequence $\{V_k\}_{k=1}^N$ this ode might be singular, but under the assumption of monotonicity (increasing) the right hand side is always well defined. By well known properties of asymptotic stability in autonomous equations we give an argument (valid for any increasing prize sequence) for existence and uniqueness of a solution and for why the solution is the cdf of a probability density. Furthermore, by introducing a prize function $V(x)$ just as we did in Section 2 and Section 3, we are able to prove Proposition 4.2 saying that the solution converges uniformly to $q(t) = V^{-1}(t)$ on $t \in [0, V(1))$ as $N \to \infty$. Hence the dynamic- and the static model "coincide" in the limit of infinitely many players.

Section 4 explained to us that the only candidate ESS in the N-player limit of the static model is given by the density function $\dot{q}(t) = d/dt(V^{-1}(t))$. In Section 5 we analyse wether $\dot{q}(t)$ is an ESS or not and establish new results for the static model having finitely many players. By introducing theory of normal form games with a continuum of players (according to [1]) we start by defining the notion of ESS. Assuming the prize function to be in $\mathscr{C}^2[0,1]$ rather than in $\mathscr{C}^1[0,1]$ (still increasing and normalised so that $V(0) = 0$) it is an easy task to prove by direct calculation that $\dot{q}(t)$ is an ESS in the continuum model if and only if $V(x)$ is strictly convex. Moreover, if instead $V(x)$ is concave the $\dot{q}$-strategy is not an ESS. The importance of convexity/concavity of $\{V_k\}_{k=1}^N$ is not obvious in the N-player static model, but it is reasonable to believe that the conclusions made in the continuum limit also hold in the finite case if N is large enough. Surprisingly enough we can prove as a corollary to what is called Theorem 5.3 that the static model admits a unique ESS, not only for N large enough, but for for all $N \geq 2$ if the prize sequence (not necessarily connected to a prize function) is convex. The concave case turn out a bit more difficult to handel, but in Theorem 5.5 we manage to prove that if $V(x) := x^{\alpha}$, for any $0 < \alpha < 1$, (hence V is concave) and $V_k = V(k/N)$, then for any N large enough the static model lacks an ESS.

Bibliography

[1] BALDER, E.J. (1995), *A unifying approach to existence of Nash equilibria.*, International Journal of Game Theory, **24**, No.1, 79-94.

[2] BISHOP, D.T., CANNINGS, C. (1976), *Models of animal conflict.*, Adv. Appl. Prob., **8**, 616-621.

[3] HAIGH, J., CANNINGS, C. (1989), *The n-Person War of Attrition.*, Acta Appl. Math., **14**, 59-74.

[4] MAYNARD SMITH, J. (1974), *The theory of games and the evolution of animal conflicts.*, J. Theoret. Biol., **47**, 209-221.