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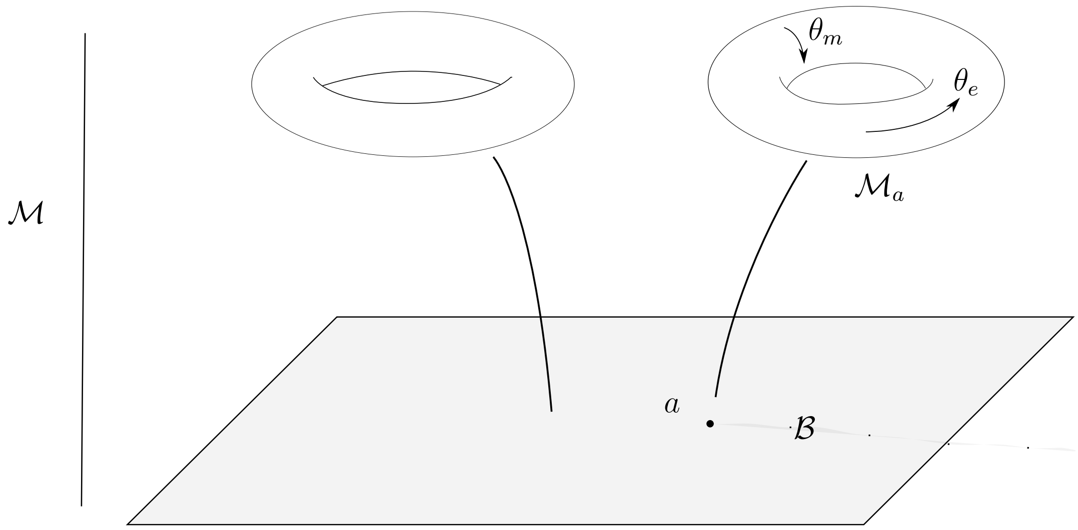

Supersymmetric Gauge Theory, Wall-Crossing and Hyperkähler Geometry

Thesis for the degree Master of Science in Fundamental Physics

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Gothenburg, Sweden 2013

Abstract

In this thesis we study moduli spaces of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. We focus on the vector multiplet moduli space and describe how the rigid special geometry of the Coulomb branch determines the couplings in the effective Lagrangian. Compactification to three dimensions gives rise to an $\mathcal{N} = 4$ theory whose moduli space is hyperkähler. The twistor space construction of this hyperkähler metric is presented and put in the context of Gaiotto, Moore and Neitzke's physical interpretation of the solution by Kontsevich and Soibelman of the wall-crossing problem.

Acknowledgements

I want to thank my supervisor Daniel Persson for all help and valuable discussions while introducing me to the subjects of this thesis. I'm also grateful to all the faculty members and students of the Department of Fundamental Physics for a good and friendly environment during my thesis work. I especially mention Bengt E W Nilsson for good advise and taking time for any questions. Thank you Henrik Gustafsson for proof-reading and giving comments before presentation of my work. Last and not the least, thank you Denis Karateev, Fredrik Strannerdahl, Joakim Edlund and Tobias Wenger for the help, the discussions and the nice company.

Oskar Till, Göteborg 2013-01-17

Contents

1 Introduction

The study of supersymmetric gauge theories has been intimately related to complex geometry since the birth of the first models in the seventies. There are a multitude of complex manifolds parametrised by the scalar fields. Together the dimensionality, the amount of supersymmetry and the gauge symmetry restricts the scalar geometries. The possibilities ranges from Riemannian manifolds to highly restricted complex manifolds with very specific geometric structures. Starting from a theory in high space-time dimension many of these may be obtained by compactification on compact spaces preserving different amounts of supersymmetry.

In this thesis we present the Kähler geometry of the vector multiplet moduli space of an $\mathcal{N} = 2$ gauge theory in four dimensions. At generic points of this space the gauge group is broken to a maximal Abelian symmetry making the moduli space a rigid special Kähler manifold. When compactifying to three dimensions an effective $\mathcal{N} = 4$ theory is obtained whose moduli space is hyperkähler.

The representation theory of the $\mathcal{N} = 2$ superalgebra contains representations which in a precise sense are 'smaller' than a generic representation. These are called BPS representations or short representations. In general the spectrum of the theory changes over the moduli space but the BPS spectrum is generically stable. However, at certain loci splitting or fusion of BPS states may occur, dividing the moduli space into subregions of constant spectrum. There is an index, the second helicity supertrace, that count (with signs) BPS state degeneracies and hence it is locally constant. The wall-crossing problem is about relating this index between different regions of the moduli space, defining a global invariant.

The solution of the wall-crossing problem by Kontsevich and Soibelman (KS) in [18] was physically interpreted by Gaiotto, Moore and Neitzke in [13]. They make use of the twistor space description of a hyperkähler manifold in terms of Darboux coordinates. The main idea of their paper is that the metric is only continuous globally if the BPS-index satisifies the KS wall-crossing formula.

The BPS states in four dimensions may wrap the compactified direction and in the compactification limit these field excitations occur as instantons in three dimensions. The tower of wrapped states contributes to the moduli space metric and the instanton corrected geometry may be explicitly constructed in terms of Darboux coordinate functions over the twistor space. It is in terms of these functions the wall-crossing problem was adressed.

This thesis is organised as follows. In section 2 the $\mathcal{N} = 2$ field theory in four dimensions is introduced, aiming at the scalar geometry. Moduli spaces are then introduced in general terms and the constraints from supersymmetry and gauge symmetry are analysed. The implication of magnetic-electric duality on the geometry is presented, leading to the definition of rigid special geometry.

Section 3 is concerned with the representation theory of the $\mathcal{N} = 2$ superalgebra. The BPS bound is derived and the form of the general BPS representations are obtained. We aim the presentation towards the hypermultiplet and the vector multiplet since they contain the scalars that constitute the moduli space.

In section 4 the BPS index is introduced to separate the BPS representations from the non-BPS ones. We analyse under which conditions the spectrum may jump giving discontinuities to the BPS-index. Here the KS solution is introduced and the wall-crossing formula is presented and its interpretation with respect to the BPS spectrum is discussed.

Compactification to three dimensions are performed in section 5. We derive explicitly how the bosonic field content changes and how the hyperkähler moduli space is described as a torus fibration over the Coulomb branch. We introduce and verify the Darboux coordinate ansatz for the metric in the semi-flat case, when no instanton contributions are present. Then the instanton corrected metric is constructed in the case when only electric charges are present. This is the Ooguri-Vafa metric. Finally we describe how the general wall-crossing problem is formulated and how the KS formula in the BPS-index is necessary for the continuity of the moduli space metric.

In appendix A some background on hyperkähler manifolds and their twistor spaces are presented. Appendix B contains a brief discussion on rigid special Kähler manifolds in the context of Riemann surface moduli spaces.

2 $\mathcal{N} = 2$ Gauge Theories in Four Dimensions

In this section we review some features of global $\mathcal{N} = 2$ supersymmetric gauge theories in $d = 4$. The discussion is inclined towards the scalar field geometry and how it is encoded in the Lagrangian. We restrict to color gauge symmetries and omit any possible flavour symmetry. The field theory has a large parameter space of vacua and we see how gauge invariance and supersymmetry puts restrictions on what theories may be constructed. We will see that these two concepts become closely intertwined and result in a fascinating piece of geometry. The moduli space metric determines all couplings in the effective Lagrangian and may be expressed solely by the *prepotential*, a holomorphic function of the scalar fields. In [23] Seiberg and Witten solved the $SU(2)$ $\mathcal{N} = 2$ gauge theory by giving the prepotential explicitly.

2.1 The Field Theory

In this section we discuss the bosonic degrees of freedom of the theory and some of its implications. The notation we set here will come back throughout the following chapters and some of the objects which are briefly introduced here will be clarified later on. The treatment in this section and the following is close to the one in [3].

The theory we are interested in is the low energy limit of a theory with gauge group G. The bosonic fields are complex scalar fields a^i, $i = 1, \dots, r$ and gauge fields A^I, all in the adjoint representation. We assume that all scalars have a non-zero vacuum expectation value, breaking the rank r gauge group to $U(1)^r$ at low energies. The general form of the bosonic Lagrangian in two derivatives is

$$\mathcal{L} = -\frac{1}{2} g_{i\bar{j}}(a) Da^i \wedge *D\bar{a}^j - \frac{1}{8\pi} \Im \left[\tau_{IJ}(a) \mathcal{F}^I \wedge *\mathcal{F}^J\right] + V(a) \tag{2.1}$$

where $V(a)$ is some gauge invariant real potential in the scalar fields. g is a real, symmetric and positive definite tensor field. Under an infinitesimal $U(1)$ transformation of the scalars $a^i \to a^i + \xi^i_I(a)$ we introduce D_μ as the covariant derivative with respect to the $U(1)$ gauge field 1-forms $A^I = A^I_\mu \mathrm{d}x^\mu$. The covariant exterior derivative acts on the scalars as

$$Da^i = \mathrm{d}a^i + A^I \xi^i_I \tag{2.2}$$

treating the generators of the gauge transformation as Killing vectors of isometries of the parameter space. The field strength $F = \mathrm{d}A$ build up a generalised complex field strength

$$\mathcal{F}^I = F^I - i * F^I \tag{2.3}$$

where $*$ is the Hodge operator in the Minkowski metric on $\mathbb{R}^{1,3}$. The matrix τ is a complex symmetric function of the scalars which is conventionally splitted as

$$\tau_{IJ} = \frac{\theta_{IJ}}{2\pi} + i\frac{4\pi}{(e^2)_{IJ}} \tag{2.4}$$

and if we write out the gauge field part of (2.1) we have

$$\begin{aligned}\mathcal{L}_{U(1)^r} &= -\frac{1}{(e^2)_{IJ}} F^I \wedge *F^J + \frac{\theta_{IJ}}{8\pi^2} F^I \wedge F^J \\ &= \frac{1}{4\pi}\big[- \Im\,\tau_{IJ} F^I \wedge *F^J + \Re\,\tau_{IJ} F^I \wedge F^J \big] ,\end{aligned} \tag{2.5}$$

motivating the generalised field strength (2.3). The kinetic term coefficient function $\Im\,\tau(a)$ must be positive definite for unitarity and we may identify $(e^2)_{IJ}(a)$ as the electromagnetic couplings and $\theta_{IJ}(a)$ as the theta angles of the topological term $F \wedge F$. For the path integral to be well defined this angle has to be periodic since a topological term can only contribute an integer multiple of 2π to the action. Physically the θ-term counts the instanton number of the quantised field configuration which is an integer, giving the θ-angles the periodicity $\theta = \theta + 2\pi$. This is equivalent to

$$\tau = \tau + 1 \ . \tag{2.6}$$

2.2 The Moduli Space

The vacuum of the field theory is obtained as the low energy limit where all field excitations tends to zero leaving only the vacuum expectation value of the field. The set of all possible such values parametrize the Riemannian manifold $\mathcal{M}_0$ with metric g. For notational simplicity we denote the coordinates (vevs) of $\mathcal{M}_0$ as the corresponding fields i.e a^i. The potential function V may be defined such that its minimum is $V = 0$. Assuming that V attains its minima we see that the scalar contribution is diffeomorphism invariant under field redefinitions $a^i \to \tilde{a}^i(a)$, which is consistent with the picture of $\mathcal{M}_0$ being a manifold when g transforms as a metric tensor [2].

Generically $V \neq 0$, potentially destroying the diffeomorphism invariance of $\mathcal{M}_0$. The quotient space

$$\mathcal{M}_V = \mathcal{M}_0 / \{V = 0\} \tag{2.7}$$

gives however the manifold of vacua of the theory. The set $\{V = 0\}$ might define varieties in $\mathcal{M}_0$ of different dimension at different loci. Hence the total manifold need not to be a manifold but may have discontinuities where such varieties meet.

Including the $U(1)$ gauge fields in the picture we need to identify points of $\mathcal{M}_V$ that are related by a gauge transformation. The moduli space is then

$$\mathcal{M} = \mathcal{M}_V / U(1)^r \tag{2.8}$$

which leads to the definition of the metric g' of $\mathcal{M}$ by

$$g'_{ij}\mathrm{d}a^i \wedge *\mathrm{d}\bar{a}^j = g_{ij}Da^i \wedge *D\bar{a}^j \tag{2.9}$$

which is gauge covariant by construction. The 1-forms $\mathrm{d}a^i$ and A^I are valued in the cotangent space of $\mathcal{M}_0$. The moduli space coordinates must correspond to neutral scalar fields. To see this suppose the contrary i.e a scalar charged under $U(1)$. If this scalar gets a non-vanishing vacuum expectation value it will effectively give a mass term to the gauge field which breaks the gauge invariance and introduces a scalar potential. By changing the vev one leaves the moduli space since diffeomorphism invariance breaks with the introduction of a potential. From this consideration the Lagrangian simplifies to

$$\mathcal{L} = -\frac{1}{2}g_{i\bar{j}}(a)\mathrm{d}a^i \wedge *\mathrm{d}\bar{a}^j - \frac{1}{8\pi}\Im\left[\tau_{IJ}(a)\mathcal{F}^I \wedge *\mathcal{F}^J\right] \tag{2.10}$$

since the covariant exterior derivative D is just d on neutral scalars.

2.2.1 The Coulomb Branch and Supersymmetric Constraints

So far we have treated the field theory and the parameter space in general. In this section we specialise to the moduli space of the $\mathcal{N} = 2$ gauge theory in four dimensions. We describe the splitting of the moduli space with respect to the representations of the algebra and the restrictions emerging from invariance under supersymmetry [3] [11].

As will be treated in more detail in section 3 there are two representations of the $\mathcal{N} = 2$ superalgebra that contain scalars, called the hypermultiplet and the vector multiplet. The supersymmetry transformations for these two representations implies that no invariant Lagrangian containing kinetic cross terms of the hypermultiplet and vector multiplet scalars may be constructed. This implies that the metric is block diagonal and that the moduli space decomposes as

$$\mathcal{M} = \mathcal{M}_H \times \mathcal{M}_V \ . \tag{2.11}$$

If the vector multiplet parameter space is trivial we have $\mathcal{M} = \mathcal{M}_H$, called the Higgs branch and the other case; $\mathcal{M} = \mathcal{M}_V$ is called the Coulomb branch. This space is the object of study in this thesis and is denoted $\mathcal{B}$. Recall that at all points of $\mathcal{B}$ the gauge group is broken down to its maximal torus of $U(1)^r$ gauge symmetry. The vector multiplet contains one complex scalar for each gauge vector and hence we may now enumerate all bosonic fields in (2.10) by the same set of indices $I = 1, \ldots, r$.

The invariance under supersymmetry transformations of the Lagrangian put restrictions on the moduli space geometry. These are mild for few supersymmetry generators and more severe as the number of generators increase. As an example any σ-model in two dimensions with a Riemannian target space metric allows for one supersymmetry. Considering the group of holonomy transformations on $\mathcal{M}$ it is shown that the set of matrices that commute with the holonomy group is a division algebra over $\mathbb{R}$ [1]. In the superalgebra this corresponds to having the center equal to the reals, the complex numbers or the quaternions corresponding to Riemann, Kähler and hyperkähler geometry respectively (the octonions are not present due to Bott periodicity) [21].

In four dimensions already $\mathcal{N}=1$ supersymmetry restricts the metric to be a Kählerian one. We argue for this by considering the simplest case in four dimensions with minimal supersymmetry. This is the so called chiral multiplet, of one complex scalar ϕ and one $(\frac{1}{2},0)$ Weyl spinor χ. All extensions of this model, both in terms of field content and number of supersymmetry generators, inherits the basic geometric structure from this example. For the scalar kinetic term to be invariant under complex conjugation of the fields i.e $g(X,\bar{Y})=g(\bar{X},Y)$ the metric is to be hermitian which ensures the positive definiteness [20].

We consider the schematic Lagrangian

$$\mathcal{L}=g_{\alpha\bar{\beta}}\big(-\partial_\mu\phi^\alpha\partial^\mu\bar{\phi}^{\bar{\beta}}-\frac{1}{2}\bar{\chi}^\alpha\nabla\!\!\!/\chi^{\bar{\beta}}-\frac{1}{2}\bar{\chi}^{\bar{\beta}}\nabla\!\!\!/\chi^\alpha\big)+\frac{1}{4}R_{\alpha\bar{\gamma}\beta\bar{\delta}}\bar{\chi}^\alpha\chi^\beta\bar{\chi}^{\bar{\gamma}}\chi^{\bar{\delta}} \tag{2.12}$$

up to two bosonic dimensions where the left- and right projectors are omitted for simplicity, see e.g [11]. The covariant derivatives act on the spinors as

$$\begin{aligned}\nabla_\mu\chi^\alpha &= \partial_\mu\chi^\alpha+\Gamma^\alpha_{\beta\gamma}\chi^\gamma\partial_\mu\phi^\beta\\ \nabla_\mu\chi^\alpha &= \partial_\mu\chi^{\bar{\alpha}}+\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}\chi^{\bar{\gamma}}\partial_\mu\phi^{\bar{\beta}}\,.\end{aligned} \tag{2.13}$$

As an ansatz $R(\phi)$ and $\Gamma(\phi)$ are taken as any functions of the scalar field. The index symmetry of R coincide with that of the Riemann curvature tensor to match the anticommuting spinors. The supersymmetry of the chiral multiplet is realised through the action

$$\begin{aligned} Q\phi &= \chi \qquad \bar{Q}\phi=0\\ Q\chi &= 0 \qquad \bar{Q}\chi=\partial\!\!\!/\phi\end{aligned} \tag{2.14}$$

of the supercharges, along with their complex conjugate counterparts. The invariance under supersymmetry requires $Q\mathcal{L}=0$ up to a total derivative. The action of Q on the quartic term is

$$Q\mathcal{L}_{\chi^4}\sim R_{\alpha\bar{\gamma}\beta\bar{\delta}}\bar{\chi}^\alpha\bar{\chi}^{\bar{\gamma}}\partial\!\!\!/\phi^{(\beta}\chi^{\bar{\delta})}+\dots \tag{2.15}$$

which may only be cancelled by the last term of the variation of the covariant derivative

$$Q\mathcal{L}_{\bar{\phi}\chi^2} \sim Q\big[\,\Gamma_{\alpha\bar{\gamma}\bar{\delta}}\bar{\chi}^{\alpha}\chi^{\bar{\gamma}}\partial\!\!\!/\bar{\phi}^{\bar{\delta}}\,\big] = \Gamma_{\alpha\bar{\gamma}\bar{\delta}}\partial\!\!\!/\bar{\phi}^{\alpha}\chi^{\bar{\gamma}}\partial\!\!\!/\bar{\phi}^{\bar{\delta}} + \frac{\partial}{\partial\bar{\phi}^{\bar{\epsilon}}}\Gamma_{\alpha\bar{\gamma}\bar{\delta}}\bar{\chi}^{\alpha}\chi^{\bar{\gamma}}\bar{\chi}^{\bar{\epsilon}}\partial\!\!\!/\phi^{\bar{\delta}}\,. \tag{2.16}$$

This cancelling requires the imposing of index symmetry $\Gamma_{\alpha\bar{\gamma}\bar{\delta}} = \Gamma_{\alpha\bar{\delta}\bar{\gamma}}$. It is derivable from the Lagrangian that R and and Γ actually is the Riemann tensor and the Christoffel symbol, a derivation which we leave out of this treatment focusing on the metric. The corresponding expression of (2.16) for the conjugated supercharge $\bar{Q}$ contains the term

$$\Gamma_{\alpha\bar{\gamma}\bar{\delta}}\bar{\chi}^{\bar{\gamma}}\partial\!\!\!/\phi^{\alpha}\partial\!\!\!/\bar{\phi}^{\bar{\delta}} \tag{2.17}$$

and by the same reasoning as above we find that the only possible term to cancel this one is the one from the scalar kinetic term

$$\bar{Q}\big[\,g_{\alpha\bar{\beta}}\partial\!\!\!/\phi^{\alpha}\partial\!\!\!/\bar{\phi}^{\bar{\beta}}\,\big] = \frac{\partial}{\partial\bar{\phi}^{\bar{\gamma}}}g_{\alpha\bar{\beta}}\bar{\chi}^{\bar{\gamma}}\partial\!\!\!/\phi^{\alpha}\partial\!\!\!/\bar{\phi}^{\bar{\beta}} + \dots \equiv \partial_{\bar{\gamma}}g_{\alpha\bar{\beta}}\bar{\chi}^{\bar{\gamma}}\partial\!\!\!/\phi^{\alpha}\partial\!\!\!/\bar{\phi}^{\bar{\beta}} + \dots \tag{2.18}$$

The symmetry of the Γ-indicies then gives the condition (and its conjugate)

$$\partial_{\bar{\gamma}}g_{\alpha\bar{\beta}} = \partial_{\bar{\beta}}g_{\alpha\bar{\gamma}} \qquad \partial_{\gamma}g_{\bar{\alpha}\beta} = \partial_{\beta}g_{\bar{\alpha}\gamma} \tag{2.19}$$

which locally has the Kähler potential solution

$$g_{\alpha\bar{\beta}} = \partial_{\alpha}\bar{\partial}_{\bar{\beta}}\mathcal{K}(\phi,\bar{\phi})\,. \tag{2.20}$$

A manifold equipped with a hermitian metric and a Kähler potential is a Kähler manifold, which implies the existence of an integrable almost complex structure compatible with the metric. Thus there is a complex structure I in which ϕ is a complex coordinate for a chart on $\mathcal{M}$. Adding more multiplets gives higher even-dimensional manifolds. The $U(1)$ gauge field may also be incorporated in the gauge multiplet $(\lambda, \mathcal{F})$ of a Weyl spinor and the 2-form field strength defined in section 2.1. The supercharge action on the gauge term of the Lagrangian, schematically written as proportional to $\Im\,\tau(\phi,\bar{\phi})\mathcal{F}^2$, necessarily gives the contribution

$$\bar{Q}[\,\Im\,(\tau)\mathcal{F}^2] = \partial_{\bar{\phi}}\,\tau\,\bar{\chi}\,\mathcal{F}^2 + \dots \tag{2.21}$$

This term has no counterpart in any other term, and hence it is required to vanish, making $\tau(\phi)$ holomorphic.

Passing on to more supersymmetry there is even more structure to the scalar field geometry. In the case of a $\mathcal{N}=2$ sigma model in four dimensions the second generator of supersymmetry gives rise to another covariantly constant complex structure

J. This is the case for the hypermultiplet scalars parametrising $\mathcal{M}_H$ of (2.11). It turns out that the matrix product $K = IJ$ also constitutes a complex structure and that I, J and K satisfy a quaternion algebra giving a hyperkähler structure to the target space [5]. The hypermultiplet has a complex scalar doublet, making the manifold $4n$-dimensional. Hyperkähler geometry is introduced in appendix A and will be the center of attention when compactifying to three dimensions in section 5.

The geometry of the vector multiplet moduli space $\mathcal{M}_V$ is not a hyperkähler manifold but a rigid special Kähler manifold, one in a class of geometries arising from extended supersymmetry and supergravity theories. The $\mathcal{N} = 1$ chiral and gauge multiplets are combined to the $\mathcal{N} = 2$ vector multiplet $(a, \chi_n, \mathcal{F})$ where $\chi_n = (\chi, \lambda)$. In the schematic notation the action is

$$\begin{aligned} Q_n a &= \chi_n & \bar{Q}_n &= 0 \\ Q_n \chi_m &= \varepsilon_{nm}\mathcal{F} & \bar{Q}_n \chi_m &= \delta_{mn} \not{\partial} a \\ Q_n \mathcal{F} &= \varepsilon_{nm} \not{\partial} \bar{\chi}_m & \bar{Q}_n \mathcal{F} &= -\varepsilon_{nm} \not{\partial} \chi_m \end{aligned} \tag{2.22}$$

of the supercharges on the multiplet fields. Now, for supersymmetry invariance of the gauge field kinetic term $\tau_{IJ}\mathcal{F}^I \cdot \mathcal{F}^I$ there must be a cancelling of the variation

$$\tau_{IJ}\mathcal{F}^I \cdot Q(\mathcal{F}^I) \sim \tau_{IJ}\mathcal{F}^I \cdot \not{\partial}\bar{\chi}^J \tag{2.23}$$

which implies that there has to be a kinetic fermion term proportional to $\tau_{IJ}\chi^I \not{\partial}\bar{\chi}^J$. The conjugate variation of this term gives

$$\bar{Q}(\tau_{IJ}\chi^I \not{\partial}\bar{\chi}^J) \sim \tau_{IJ} \not{\partial} a^I \not{\partial}\bar{\chi}^J + \ldots \tag{2.24}$$

and to finally cancel this contribution requires a scalar kinetic term

$$\tau_{IJ} \not{\partial} a^I \not{\partial} \bar{a}^J = \tau_{IJ} \partial_\mu a^I \partial^\mu \bar{a}^J \,. \tag{2.25}$$

Considering also the complex conjugate term one conclude that the supersymmetry transformation of the gauge field kinetic term $\tau\mathcal{F} \wedge *\mathcal{F}$ may only be cancelled if the metric is

$$g(a) = \Im\, \tau(a) \tag{2.26}$$

This is striking - the full theory is encoded in the electro-magnetic couplings and instanton numbers. In the following sections we will see that this is not the end and that the full matrix τ_{IJ} is obtained from a single holomorphic function. The Lagrangian may now be brought to the form

$$\mathcal{L} = \frac{1}{4\pi}\big[-\Im\, \tau_{IJ}(\mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J + F^I \wedge *F^J) + \Re\, \tau_{IJ} F^I \wedge F^J \big] \tag{2.27}$$

with the normalisation of the scalar fields chosen consistently with [13]. In the following the special geometry arising from electromagnetic duality is presented in detail.

2.3 Electric and Magnetic Charges

The Maxwell equations of motion with a field source of magnetic charge q_m, electric charge q_e and electric coupling e are

$$\frac{1}{e}\mathrm{d} * F = eq_e\delta^{(3)} \qquad \frac{1}{e}\mathrm{d}F = \frac{4\pi}{e}q_m\delta^{(3)} \tag{2.28}$$

which are invariant under redefinitions of what we call magnetic and electric quantities. Transforming

$$F \to *F \qquad q_m \to q_e \qquad e \to \frac{4\pi}{e} \tag{2.29}$$

leaves the content of the equations unchanged. If the transformation is applied twice we get additional minus signs in the first two transformations since $** = -1$ on two-forms in four dimensional Minkowski space. With no sources we have the Bianchi identity $\mathrm{d}F = 0$ as the second case in (2.28). This may be used in a constraint term $S_{con} = \int \tilde{A}_I \wedge \mathrm{d}F^I$ in the path integral. Performing the integral over F^I transforms the gauge term of (2.1) into a new one in the gauge field $\tilde{A}^I$ with the corresponding field strength $\tilde{F} = \mathrm{d}\tilde{A}$. The generalised field strength is $\tilde{\mathcal{F}} = \tilde{F} + i * \tilde{F}$ as above and the gauge field Lagrangian takes the form

$$\tilde{\mathcal{L}}_{U(1)} = -\frac{1}{8\pi}\Im\left[(-\tau^{IJ}(a))\tilde{\mathcal{F}}_I \wedge *\tilde{\mathcal{F}}_J\right] \tag{2.30}$$

where $\tau^{IJ}\tau_{JK} = \delta^I_K$. We are thus left with an equivalent theory in a redefined one-form gauge field $\tilde{A}$ and the couplings $-\tau^{-1}$. The shift invariance $T : \tau \mapsto \tau + 1$ in (2.6) and the 'duality' map $S : \tau \mapsto -\tau^{-1}$ generates together the transformation

$$\tau \to (A\tau + B)(C\tau + D)^{-1} \tag{2.31}$$

with the matrices A, B, C and D are such that the block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.32}$$

is an element of $Sp(2r, \mathbb{Z})$. In this context this is referred to as the magnetic-electric duality group. Note that for rank one $Sp(2, \mathbb{Z}) \approx SL(2, \mathbb{Z})$ and the τ parameter may be interpreted as the complex structure of a torus. This will be important when compactifying the theory on a circle. Arranging the charges in a $2r$-dimensional vector $(q^I_m, q_{e,I})$ they transform as

$$(q^I_m, q_{e,I}) \to (q^I_m, q_{e,I})M^{-1} \tag{2.33}$$

The magnetic and electric charges of matter fields satisfy the Dirac-Zwanziger quantisation condition. In other words, the electric and magnetic charges takes value in a lattice Γ which for a given theory is isomorphic to $\mathbb{Z}^{2r}$. The quantisation condition equip the lattice with the pairing

$$\langle\, ,\, \rangle : \Gamma \to \mathbb{Z} \tag{2.34}$$

which is a nondegenerate symplectic form. A basis $\{\alpha_I, \beta^I\}$, $I = 1 \dots r$ for a symplectic lattice is called a Darboux basis, or a duality frame. In the case where the lattice originates from an Abelian gauge theory the basis elements are referred to as magnetic and electric respectively and they obey

$$\begin{aligned} \langle \alpha_I, \alpha_J \rangle &= 0 \\ \langle \beta^I, \beta^J \rangle &= 0 \\ \langle \alpha_I, \beta^J \rangle &= -\langle \beta^J, \alpha_I \rangle = \delta_I^J \, . \end{aligned} \tag{2.35}$$

An element γ of the lattice, here collectively called a 'charge', is a linear combination

$$\gamma = q_m^I \alpha_I + q_{e,I} \beta^I \tag{2.36}$$

of magnetic and electric charges which in the case $r = 1$ also is written as a vector $\gamma = (q_m, q_e)$. The quantisation condition expressed in the symplectic form is

$$\langle \gamma, \gamma' \rangle = q_m^I q'_{e,I} - {q'}_m^I q_{e,I} \in \mathbb{Z} \tag{2.37}$$

which for rank one takes the usual form of Dirac and Zwanziger.

2.4 Rigid Special Kähler Geometry

Extending the construction in the previous section to any point in the Coulomb branch we find a particular charge lattice Γ_a for each gauge theory, as τ varies over $\mathcal{B}$. For each lattice fiber Γ_a we may construct the symplectic vector space $\Gamma_a \otimes \mathbb{C}^*$. Together they form a fibration $\Gamma_a \otimes \mathbb{C}^* \to E \xrightarrow{\pi} \mathcal{B}$ over the Coulomb branch *i.e* a symplectic vector bundle of rank $2r$. As was realised in [26] this makes the geometry of the configuration space *rigid special Kähler*. The case of *local* special Kähler manifolds is related to local supersymmetry i.e the corresponding supergravity model which we do not treat here. This work is reviewed in [28].

We describe here the mathematical construction of such geometries. Take $\mathcal{L}$ a flat line bundle over the r-dimensional Kähler manifold $\mathcal{B}$ and let $E \to \mathcal{B}$ be a flat, holomorphic symplectic vector bundle. The holomorphicity of E refers to the holomorphic projection $\pi : E \to \mathcal{B}$. The symplectic form on the vector space fiber of E is denoted $\langle\, ,\, \rangle$. The manifold $\mathcal{B}$ is *rigid special Kähler* if there is a section $Z \in \Gamma(E \otimes \mathcal{L}, \mathcal{B})$ such that the pairing of its differentials vanish

$$\langle \mathrm{d}Z, \mathrm{d}Z \rangle = 0\ . \tag{2.38}$$

Note that the wedge product of the one-forms together with the symplectic form gives a symmetric condition, and is thus not by default zero. This will be seen when we expand this condition in equation (2.41). The Kähler form on $\mathcal{B}$ is given by

$$\omega = \frac{i}{2\pi} \partial\bar{\partial} \langle Z, \bar{Z} \rangle \tag{2.39}$$

defining the Kähler potential as $\mathcal{K}(a, \bar{a}) = \langle Z(a), \bar{Z}(\bar{a}) \rangle$. By expanding the section in the symplectic basis as

$$Z = X^I \alpha_I - F_I \beta^I \tag{2.40}$$

we see that the coordinates are extracted as $X^I = \langle Z, \beta^I \rangle$. The Kähler form (2.39) is non-degenerate which implies that locally the X^I, $I = 1, \ldots, r$ is a set of complex coordinates on $\mathcal{B}$. These are called *special* coordinates and will henceforth be denoted a^I in accordance with the notation for the scalar fields. These are, by the symplectic pairing, dual to the coordinates $\tilde{a}_I = F_I$, which depend holomorphically on a^I.

Applying the condition (2.38) to the expanded form of Z gives

$$0 = -2\mathrm{d}X^I \wedge \mathrm{d}F_I = -2\frac{\partial F_I}{\partial X^J} \mathrm{d}X^I \wedge \mathrm{d}X^J \;\Rightarrow\; \partial_I F_J - \partial_J F_I = 0 \tag{2.41}$$

as an integrability condition. From this it follows that the dual coordinates $F_I = \langle Z, \alpha_I \rangle$ locally may be expressed as

$$F_I = \frac{\partial \mathcal{F}(X)}{\partial X^I} \tag{2.42}$$

for some holomorphic function $\mathcal{F}(a^I)$ called the *prepotential.* Knowing this function locally specifies the coordinates of $\mathcal{B}$. Since the prepotential determines the Kähler potential it also determines the couplings of the Lagrangian (2.27). The gauge coupling matrix τ is given in terms of the prepotential as

$$\tau_{IJ} \equiv \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \tag{2.43}$$

and is named the *period matrix* in this context, see e.g appendix B. This is symmetric by construction and all gauge theory data for a given choice of special coordinates is encoded in the prepotential.

The occurrence of the line bundle $\mathcal{L}$ is due to the redundancy in definition of the Kähler potential. The metric does not change under the Kähler transformation

$$\mathcal{K} \to \mathcal{K} + f + \bar{f} \tag{2.44}$$

under which the section transforms as $Z \to e^f Z$ for a holomorphic function f. This invariance under holomorphic scaling states that Z besides being a section of E it is also a section of $\mathcal{L}$. Furthermore it can be shown that the line bundle is determined by the Kähler form as $c_1(L) = [\omega]$ i.e the first Chern class is the cohomology class of which the Kähler form is a representative [7]. This makes the manifold $\mathcal{B}$ a Hodge-Kähler manifold. An equivalent statement is that the Kähler class is an integer cohomology class [14].

The Coulomb branch of $d = 4$, $\mathcal{N} = 2$ gauge theory is not the only case where rigid special Kähler manifolds are encountered. The same structure arises in the analysis of Calabi-Yau and Riemann surface moduli spaces. The latter case is briefly presented in appendix B.

2.5 The Central Charge

The central charge of an $\mathcal{N} = 2$ supersymmetric theory is an operator whose eigenvalues are complex scalars. It is in the center of the superalgebra which will be encountered again in section 3. Hence the bracket

$$[Z, g\} = 0 \tag{2.45}$$

for all field operators in the superalgebra. Seen from the moduli space one may view the contiuum of eigenvalues of Z as defining a complex function $Z(a)$, holomorphic in a over $\mathcal{B}$. This is however not the full story. The magnetic-electric charge γ is to be specified for a complete description of the theory in question. Hence there is a fibration of charge lattices Γ_a over the Coulomb branch. γ is a section of this fibration and for each point $a \in \mathcal{B}$ the fiber Γ_a is isomorphic to $\mathbb{Z}^{2r}$. The physical charge that we denote γ is thus the value of this section at a particular fiber. From the previous section we know that for a chosen duality frame, the geometry and coordinates of $\mathcal{B}$ is obtained from the prepotential $\mathcal{F}$, which in turn determines the section Z of the symplectic vector bundle.

Given a charge $\gamma \in \Gamma_a$ we get a complex scalar, holomorphic over $\mathcal{B}$, by the symplectic pairing. This is the central charge

$$Z_\gamma(a) = \langle Z(a), \gamma \rangle \tag{2.46}$$

which by the bilinearity of the symplectic form ensures that the map is linear over Γ i.e

$$Z_{\gamma_1}(a) + Z_{\gamma_2}(a) = Z_{\gamma_1+\gamma_2}(a)\,. \tag{2.47}$$

$Z(a)$ is a vector of the fiber Γ_a for each fixed a. Via the symplectic pairing this vector defines a map

$$\begin{aligned} Z : \Gamma_a &\to \mathbb{C} \\ \gamma &\mapsto Z_\gamma(a)\,, \quad \gamma \in \Gamma_a \end{aligned} \tag{2.48}$$

and for $\gamma = q_m^I \alpha_I + qe,_I \beta^I$ we have

$$Z_\gamma = q_m^I F_I + q_{e,I} X^I = q_m^I \partial_{a^I} \mathcal{F}(a) + q_{e,I} a^I\,. \tag{2.49}$$

The central charge is thus an intertwined object constructed out of the dyonic electromagnetic charges and expressed in the chosen coordinate frame of the special geometry.

3 BPS-states

In this section we focus on the $\mathcal{N} = 2$ superalgebra and its representations. We will derive the short representations, and in particular the vector multiplet whose moduli space is our main interest. The treatment follows roughly the one in [19] and [12].

3.1 The Supersymmetry Algebra

The $\mathcal{N} = 2$ extended Poincaré algebra

$$\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1 \tag{3.1}$$

is a graded algebra with an even part $\mathfrak{s}^0$ and an odd part $\mathfrak{s}^1$. The transformations generated by the supercharges transforms even elements to odd and *vice versa.* Hence they belong to the same representation and by taking the odd part of the algebra $\mathfrak{s}^0 \oplus \mathfrak{s}^1$ to be a representation of the even part we get a superalgebra. The even algebra

$$\mathfrak{s}^0 = \mathfrak{poin}(1{,}3) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C} \tag{3.2}$$

is composed of the Poincaré algebra, the R-symmetry algebra and the center respectively. The R-symmetry algebra $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ reflects the ambiguity in rotating the supercharges into each other, and in addition, the phase-shift symmetry giving the Q's electrical charge. The central charge Z is the representation of the center $\mathbb{C}$ of the superalgebra and is identified with $Z_\gamma(a)$ by the eigenvalue relation

$$Z\,|a,\gamma\rangle = Z_\gamma(a)|a,\gamma\rangle \tag{3.3}$$

for an eigenstate of charge γ in the one particle Hilbert space $\mathcal{H}_{a,\gamma}$ at the point $a \in \mathcal{B}$.

The Poincaré algebra is composed of space-time translations and Lorentz transformations in $\mathbb{R}^{1,3}$. By algebra isomorphism one may view the Lorentz algebra $\mathfrak{so}(1,3)$ as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and then write the odd part as a representation of $\mathfrak{s}^0$

$$\mathfrak{s}^1 = (\mathbf{2},\mathbf{1};\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2};\mathbf{2})_- \ . \tag{3.4}$$

The first two representations in the parantheses refers to a singlet or a fundamental of the Lorentz $\mathfrak{su}(2)$'s, the third entry refers to the fundamental of the R-symmetry $\mathfrak{su}(2)_R$ and the sign to the $\mathfrak{u}(1)_R$.

The two conserved supercharges Q^A_α, $A = 1{,}2$ are two-component Weyl spinors with chiral spinor component indices denoted α and $\dot\alpha$. The condition

$$Q^{A\dagger}_\alpha = \bar{Q}_{\dot\alpha A} \equiv \varepsilon_{AB}\bar{Q}^B_{\dot\alpha} \tag{3.5}$$

is a reality condition and the lowering of indicies by the Levi-Civita symbol is introduced here. The anticommutators of the odd generators are

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}_{\dot\beta B}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^A_B \\ \{Q^A_\alpha, Q^B_\beta\} &= 2\varepsilon^{AB}\varepsilon_{\alpha\beta}\bar{Z} \\ \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\} &= -2\varepsilon_{AB}\varepsilon_{\dot\alpha\dot\beta} Z \end{aligned} \tag{3.6}$$

where μ is the spacetime index, P the energy-momentum Lorentz vector and $\varepsilon_{AB}Z$ the central charge matrix which do only occur in $\mathcal{N} > 1$ theories due to its antisymmetry [6]. We will refer to just Z as the central charge.

3.2 Representations

We restrict ourselves to massive representations of the superalgebra. We may then consider the particle in its rest frame and describe the representations of the little superalgebra. At the end one may get the full representations by Lorentz boosting to an arbitrary frame. For a particle at rest we have

$$P^\mu|\psi\rangle = M\delta_0^\mu|\psi\rangle \tag{3.7}$$

and the Casimir of the representation is the quadratic form $P^2 = -M^2$. The bosonic part of the little superalgebra is

$$\mathfrak{s}_l^0 = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \tag{3.8}$$

where the $\mathfrak{so}(3)$ is the little algebra of the Lorentz algebra $\mathfrak{so}(1{,}3)$ and corresponds to rotations of the particle at rest. A particle in its restframe is invariant under spatial parity and we denote this operator by $\mathcal{P}$. The parity transformation of the supercharges are

$$\begin{aligned} \mathcal{P}[Q_\alpha^A] &= \sigma^0_{\alpha\dot\beta}\bar{Q}^{\dot\beta A} \\ \mathcal{P}[\bar{Q}^{\dot\beta A}] &= \sigma^{0\,\dot\beta\alpha}Q_{\alpha A}\,. \end{aligned} \tag{3.9}$$

In addition we are free to use the $\mathfrak{u}(1)_R$-symmetry to multiply the supercharges by some complex phase ζ, this operation we denote $\mathcal{U}$ with

$$\mathcal{U}Q = \zeta Q \qquad \mathcal{U}\bar{Q} = \zeta^{-1}\bar{Q}\ . \tag{3.10}$$

We may then define the linear combinations

$$\begin{aligned} \mathcal{R}_\alpha^A &= \zeta^{1/2}Q_\alpha^A + \zeta^{-1/2}\sigma^0_{\alpha\dot\gamma}\bar{Q}^{\dot\gamma A} \\ \mathcal{T}_\alpha^A &= \zeta^{1/2}Q_\alpha^A - \zeta^{-1/2}\sigma^0_{\alpha\dot\gamma}\bar{Q}^{\dot\gamma A} \end{aligned} \tag{3.11}$$

with eigenvalue $+1$ and -1 respectively under the combined transformation $\mathcal{U}\circ\mathcal{P}$.

The reason for these supercharge combinations is to make a splitting of the algebra in two mutually invariant parts. This will make the representations easy to find and is used in the subsequent sections. The commutation relation of the $\mathcal{R}$-spinors are derived by using $\sigma^0 = -I_{2\times 2}$ and spinor indices are raised/lowered by the Levi-Civita symbol $\varepsilon_{21} = \varepsilon^{12} = 1$. The restframe energy-momentum operator is as above $P^\mu = M\delta^\mu_0$ and the $\mathcal{R}$-anticommutator becomes

$$
\begin{aligned}
\{\mathcal{R}^A_\alpha,\mathcal{R}^B_\beta\} &= \{\zeta^{1/2}Q^A_\alpha + \zeta^{-1/2}\sigma^0_{\alpha\dot\gamma}\bar{Q}^{\dot\gamma A}, \zeta^{1/2}Q^B_\beta + \zeta^{-1/2}\sigma^0_{\beta\dot\delta}\bar{Q}^{\dot\delta B}\}\\
&= \zeta\{Q^A_\alpha,Q^B_\beta\} + \zeta^{-1}\sigma^0_{\alpha\dot\gamma}\sigma^0_{\beta\dot\delta}\{\bar{Q}^{\dot\gamma A},\bar{Q}^{\dot\delta B}\} + \sigma^0_{\alpha\dot\gamma}\{Q^B_\beta,\bar{Q}^{\dot\gamma A}\} + \sigma^0_{\beta\dot\delta}\{Q^A_\alpha,\bar{Q}^{\dot\delta B}\}\\
&= 2\zeta\bar{Z}\varepsilon_{\alpha\beta}\varepsilon^{AB} + \zeta^{-1}\sigma^0_{\alpha\dot\gamma}\sigma^0_{\beta\dot\delta}\varepsilon^{AE}\varepsilon^{BF}\varepsilon^{\dot\gamma i}\varepsilon^{\dot\delta\dot\kappa}\{\bar{Q}_{iE},\bar{Q}_{\dot\kappa F}\}\\
&\qquad + \sigma^0_{\alpha\dot\gamma}\varepsilon^{\dot\gamma i}\varepsilon^{AE}\{Q^B_\beta,\bar{Q}_{iE}\} + \sigma^0_{\beta\dot\delta}\varepsilon^{\dot\delta\dot\kappa}\varepsilon^{BF}\{Q^A_\alpha,\bar{Q}_{\dot\kappa F}\}\\
&= 2\zeta\bar{Z}\varepsilon_{\alpha\beta}\varepsilon^{AB} - 2\zeta^{-1}Z\sigma^0_{\alpha\dot\gamma}\sigma^0_{\beta\dot\delta}\varepsilon^{AE}\varepsilon^{BF}\varepsilon^{\dot\gamma i}\varepsilon^{\dot\delta\dot\kappa}\varepsilon_{i\dot\kappa}\varepsilon_{EF}\\
&\qquad + 2M\sigma^0_{\alpha\dot\gamma}\varepsilon^{\dot\gamma i}\varepsilon^{AE}\sigma^0_{\beta i}\delta^B_E + 2M\sigma^0_{\beta\dot\delta}\varepsilon^{\dot\delta\dot\kappa}\varepsilon^{BF}\sigma^0_{\alpha\dot\kappa}\delta^A_F\\
&= \big(2\zeta\bar{Z}+2\zeta^{-1}Z + 4M\big)\varepsilon_{\alpha\beta}\varepsilon^{AB} = 4(M + \Re(Z/\zeta))
\end{aligned}
$$

where we in the last equality use that $\bar\zeta = \zeta^{-1}$. Likewise the $\mathcal{T}$ commutation relations is derived and in total we have that

$$
\begin{aligned}
\{\mathcal{R}^A_\alpha,\mathcal{R}^B_\beta\} &= 4(M + \Re(Z/\zeta))\\
\{\mathcal{T}^A_\alpha,\mathcal{T}^B_\beta\} &= 4(-M + \Re(Z/\zeta))\\
\{\mathcal{R}^A_\alpha,\mathcal{T}^B_\beta\} &= 0
\end{aligned}
\tag{3.12}
$$

splitting the odd algebra into blocks $\mathfrak{s}^1 = \mathfrak{s}^1_+ \oplus \mathfrak{s}^1_-$. In the following sections we use this block diagonalisation to find the representations one at the time.

By investigating the Hermitian properties of $\mathcal{R}$ we will be able to derive an important bound on the particle mass. The conjugated operator is derived as

$$
\begin{aligned}
\mathcal{R}^1_1 &= \zeta^{1/2}Q^1_1 + \zeta^{-1/2}\sigma^0_{1\dot1}\bar{Q}^{\dot1 1} = \zeta^{1/2}Q^1_1 - \zeta^{-1/2}\bar{Q}^{\dot1 1} \quad \Rightarrow\\
(\mathcal{R}^1_1)^\dagger &= \zeta^{-1/2}(Q^1_1)^\dagger - \zeta^{1/2}(\varepsilon^{\dot1\dot2}\varepsilon^{12}\bar{Q}_{\dot22})^\dagger = \zeta^{-1/2}\bar{Q}_{\dot11} - \zeta^{1/2}Q^2_2\\
&= -\zeta^{1/2}Q^2_2 - \zeta^{-1/2}\sigma^0_{2\dot2}\varepsilon_{\dot1\dot2}\varepsilon_{12}\bar{Q}^{\dot22} = -\mathcal{R}^2_2
\end{aligned}
$$

where we remember that $\sigma^0 = -I_{2\times2}$.

By construction the operator $\mathcal{R}^1_1 + (\mathcal{R}^1_1)^\dagger$ is Hermitian and since a Hermitian operator squared is positive semidefinite we have

$$
\{\mathcal{R}^1_1 + (\mathcal{R}^1_1)^\dagger,\mathcal{R}^1_1 + (\mathcal{R}^1_1)^\dagger\} = -2\{\mathcal{R}^1_1,\mathcal{R}^2_2\} = 8(M + \Re(Z/\zeta)) \geq 0\ . \tag{3.13}
$$

By decomposition of the central charge as $Z = e^{i\alpha}|Z|$ for some real phase α we see that the strongest bound

$$
M \geq |Z| \tag{3.14}
$$

is obtained by choosing the $\mathfrak{u}(1)_R$ gauge $\zeta = -e^{i\alpha}$. This bound is called the *BPS bound*. Under this particular choice of phase the algebra simplifies to

$$\begin{aligned} \{\mathcal{R}^A_\alpha, \mathcal{R}^B_\beta\} &= 4\,\varepsilon_{\alpha\beta}\varepsilon^{AB}(M - |Z|) \\ \{\mathcal{T}^A_\alpha, \mathcal{T}^B_\beta\} &= -4\,\varepsilon_{\alpha\beta}\varepsilon^{AB}(M + |Z|) \\ \{\mathcal{R}^A_\alpha, \mathcal{T}^B_\beta\} &= 0 \end{aligned} \tag{3.15}$$

which defines a representation of the subalgebra $\mathfrak{s}_l^{0\,new} = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$ where the $\mathfrak{u}(1)_R$-part is absent due to the gauge fixing.

3.2.1 Long Representations

In the case that $M > |Z|$ we have two disjoint, nontrivial algebras, and we can look for the representations separately and then combine them to get the full rest frame representation. We present the $\mathcal{R}$-case here and the treatment of the other case is completely analogous. Written out explicitly equation (3.15) is

$$\begin{array}{ll} \{\mathcal{R}^1_1, \mathcal{R}^1_1\} = \{\mathcal{R}^2_2, \mathcal{R}^2_2\} = 0 \qquad & \{\mathcal{R}^1_1, \mathcal{R}^2_2\} = 4\,(M - |Z|) \\ \{\mathcal{R}^2_1, \mathcal{R}^2_1\} = \{\mathcal{R}^1_2, \mathcal{R}^1_2\} = 0 \qquad & \{\mathcal{R}^2_1, \mathcal{R}^1_2\} = -4\,(M - |Z|)\ . \end{array} \tag{3.16}$$

Each of the two lines above takes the form of a Clifford algebra of creation and annihilation operators. One may choose the $\mathcal{R}^A_1$ to be the annihilation operators

$$\mathcal{R}^A_1|\Omega\rangle = 0 \tag{3.17}$$

which definines the vacuum state $|\Omega\rangle$. A basis for the Hilbert space of the $\mathcal{R}$-algebra is then

$$\{|\Omega\rangle,\ \mathcal{R}^1_2|\Omega\rangle,\ \mathcal{R}^2_2|\Omega\rangle,\ \mathcal{R}^1_2\mathcal{R}^2_2|\Omega\rangle\} \tag{3.18}$$

which is denoted ρ_{hh} (for half-hypermultiplet). As mentioned in section 3.1 we want the fermionic part as a representation of the bosonic algebra. Moreover the (little) superalgebra is a direct sum, meaning that the odd part is invariant under the bosonic part. ρ_{hh} must then occur as a representation of

$$\mathfrak{s}_l^{0\,new} = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R\,. \tag{3.19}$$

The basis (3.18) is four dimensional and thus we need a four dimensional representation. This is accomplished by letting the Clifford vacuum $|\Omega\rangle$ be the highest weight state of the $(\frac{1}{2}; 0)$ of (3.19) and identifying $\{|\Omega\rangle, \mathcal{R}^1_2\mathcal{R}^2_2|\Omega\rangle\}$ as the two states of the **2** of $\mathfrak{so}(3)$.

The two other states $\{\mathcal{R}_2^1|\Omega\rangle, \mathcal{R}_2^2|\Omega\rangle\}$ are spin zero, or singlets under $\mathfrak{so}(3)$ and are identified with the $\mathbf{2}$ of $\mathfrak{su}(2)_R$. Together they give a four dimensional representation and the half-hypermultiplet is thus

$$\rho_{hh} = (0; \frac{1}{2}) \oplus (\frac{1}{2}; 0) \tag{3.20}$$

as a representation of $\mathfrak{s}_l^{0\,new}$. These two blocks are representations of the bosonic parts left of the superalgebra. The fermionic generators $\mathcal{R}$ of $\mathfrak{s}_+^1$ are the links between these two blocks of $\mathfrak{s}_l^0$. By a result in [27] a general representation of $\mathfrak{s}_l^{0\,new} \oplus \mathfrak{s}_{even}^1$ is

$$\rho_{hh} \otimes \mathfrak{h} \tag{3.21}$$

where $\mathfrak{h}$ is an arbitrary representation of $\mathfrak{s}_l^{0new} = \mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$. Including the $\mathcal{T}$-generator Clifford algebra as well gives the long representation

$$t_L = \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h} \tag{3.22}$$

of the full superalgebra with the first two contributions from the parity odd and even algebras. Recall that this is the rest frame representation and that a general representation is obtained by a Lorentz boost.

3.2.2 BPS Representations

If the BPS bound is saturated, $M = |Z|$, we see from (3.15) that the $\mathcal{R}$-part of the algebra becomes trivial, and hence that the representation is the trivial one. Hence there is only one contribution from the fermionic generators to the full representation

$$t_{BPS} = \rho_{hh} \otimes \mathfrak{h} \tag{3.23}$$

which is named either a short representation or a BPS representation. As the moduli space geometry is central in this thesis we are interested in the representations which contains scalar fields, whose vevs parametrize the moduli space. The two cases are called the *hypermultiplet* and the *vector multiplet*.

The hypermultiplet are obtained by taking $\mathfrak{h}$ to be the trivial representation $(0; 0)$ which leaves just ρ_{hh}. It contains a spinor that is a singlet under $\mathfrak{su}(2)_R$ and a scalar in the 2-dimensional representation of $\mathfrak{su}(2)_R$, hence a doublet of complex scalars.

By taking $\mathfrak{h} = (\frac{1}{2}; 0)$ the vector multiplet is obtained as

$$\begin{aligned}\rho_{hh} \otimes (\frac{1}{2}; 0) &= \big[(\frac{1}{2}; 0) \otimes (\frac{1}{2}; 0)\big] \oplus \big[(0; \frac{1}{2}) \otimes (\frac{1}{2}; 0)\big] \\ &= (\frac{1}{2}; \frac{1}{2}) \oplus (1; 0) \oplus (0; 0)\end{aligned} \tag{3.24}$$

and consists of a spinor doublet of $\mathfrak{su}(2)_R$, a vector and a complex scalar. It is possible to get more specific representations containing scalars by choosing other $\mathfrak{su}(2)_R$-representations in $\mathfrak{h}$, as

$$\begin{aligned}\mathfrak{h} = (\frac{1}{2}, \frac{1}{2}) \quad &\Rightarrow \\ \rho_{hh} \otimes (\frac{1}{2}, \frac{1}{2}) = \big[(\frac{1}{2}; 0) \oplus (0; \frac{1}{2})\big] \otimes (\frac{1}{2}, \frac{1}{2}) &= (\frac{1}{2}; 0) \oplus (\frac{1}{2}; 1) \oplus (1; \frac{1}{2}) \oplus (0; \frac{1}{2})\end{aligned}$$

finding the scalar doublet as the last term but this refinement is not interesting for our purposes. BPS-representations are specified by the mass and are hence defined by the little algebra which make the choice of representation of the R-symmetry algebra redundant. This will be treated again in section 4.1.

The space of BPS states

$$\mathcal{H}^{BPS} = \big\{|\psi\rangle \in \mathcal{H} : H|\psi\rangle = |Z||\psi\rangle\big\} \tag{3.25}$$

is the subspace of the one-particle Hilbert state space $\mathcal{H}$ for which the total energy of the state equals the central charge. The single particle Hilbert space $\mathcal{H}$ for any state is graded by the magnetic-electric charge lattice

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \ . \tag{3.26}$$

A BPS state may be supported by any charge γ and therefore the BPS Hilbert space inherits the grading by the charge lattice so that for some point $a \in \mathcal{B}$ we have

$$\begin{aligned}\mathcal{H}^{BPS}_a &= \bigoplus_{\gamma \in \Gamma} \mathcal{H}^{BPS}_{a,\gamma} \\ \mathcal{H}^{BPS}_{a,\gamma} &= \big\{|\psi\rangle \in \mathcal{H}_{a,\gamma} : H|\psi\rangle = |Z_\gamma(a)||\psi\rangle\big\}\end{aligned} \tag{3.27}$$

4 Wall-Crossing

In this section we introduce the BPS index that 'count' BPS representations and discuss its global properties. In general the spectrum of the supersymmetric gauge theory will vary over the moduli space. Considering only the BPS representations one expect them to be stable over $\mathcal{B}$ since there are no 'smaller' representations to which they can decay. We will see that this is not true in general and that there are loci in the Coulomb branch where fusing and decay of BPS states do occur. These loci form real codimensional one hypersurfaces in the moduli space named walls of marginal stability. The wall-crossing problem is about determining the change of the BPS spectrum, or the BPS index, when crossing such a wall.

4.1 The BPS Index

The subject of wall-crossing, as treated in [13], is about determining the BPS spectrum of a theory at different loci in the moduli space. Hence one need some measure, a way to count the number of BPS states. The 'counting' of BPS states is realised through an index which maps a representation to an integer. It should be zero for any representation that is not BPS and non-zero for the BPS representations. This distinction contains a difficulty, namely that some non-BPS representations might, as sums of BPS states, 'look like' BPS states. In the following we omit the $\mathfrak{su}(2)_R$-part of the bosonic algebra since we will distinguish the representations exclusively through data from the Poincaré algebra. In this case, the half-hypermultiplet is rewritten as

$$\rho_{hh} = (\frac{1}{2};0) \oplus (0;\frac{1}{2}) \quad \longrightarrow \quad \rho_{hh} = 2[0] \oplus [\frac{1}{2}] \tag{4.1}$$

highlighting the field content; two scalars and one spinor. As an example we take one BPS representation S_j and one long representation L_j ;

$$\begin{aligned} L_j &= (2[0] \oplus [\frac{1}{2}]) \otimes (2[0] \oplus [\frac{1}{2}]) \otimes [j] = ([1] \oplus 4[\frac{1}{2}] \oplus 5[0]) \otimes [j] \\ S_j &= (2[0] \oplus [\frac{1}{2}]) \otimes [j] \end{aligned} \tag{4.2}$$

with $[j]$ an arbitrary representation of the little algebra $\mathfrak{so}(3)$ and j the eigenvalue of the generator J_3 of the little algebra. In this case we may form the sum

$$L_0 = 2S_0 \oplus S_{\frac{1}{2}} \tag{4.3}$$

and it is in this sense L_0 'looks like' a BPS state and is called a fake-BPS state. The index should distinguish between fake and true representations.

The *second helicity supertrace*, here called the *BPS-index*, is defined as

$$\Omega(a,\gamma) \equiv -Tr_{\mathcal{H}^{BPS}_{a,\gamma}}(-1)^{2J_3}(2J_3)^2 \tag{4.4}$$

for a state of charge γ. This type of index was introduced for $\mathcal{N}=2$ theories in [8] and is unique up to a normalisation factor. By diagonalisation the trace reduces to the sum of its eigenvalues over the space of BPS states. Computing the BPS index of the long representation L_0 and the short representations S_0 and $S_{\frac{1}{2}}$ (the hypermultiplet and the vector multiplet) gives

$$\begin{aligned}
-\mathrm{Tr}_{L_0}(-1)^{2J_3}(2J_3)^2 &= -\left\{(-1)^2\cdot 2^2+4\cdot(-1)^1\cdot 1^2+5\cdot(-1)^0\cdot 0\right\}=0\\
-\mathrm{Tr}_{S_0}(-1)^{2J_3}(2J_3)^2 &= -\left\{2((-1)^0\cdot 0)+(-1)^1\cdot 1\right\}=1\\
-\mathrm{Tr}_{S_{\frac{1}{2}}}(-1)^{2J_3}(2J_3)^2 &= -\left\{(-1)^2\cdot 2^2+2\cdot(-1)^1\cdot 1^2+0\right\}=-2\ .
\end{aligned} \tag{4.5}$$

This illustrates the fact that the index takes nonzero values for BPS representations and otherwise zero. Note that the indices sum up as $\Omega_{L_0}=2\Omega_{S_0}+\Omega_{\frac{1}{2}}$ in accordance with the construction of the long representation L_0.

4.2 The Wall-Crossing Formula

The BPS index is, as we will see, a piecewise constant function over $\mathcal{B}$. On the moduli space the index will then locally be an invariant. A full description of the spectrum for any point in $\mathcal{B}$ require a method of determining the spectrum in any region of the moduli space. This may contain regions of strong coupling where methods of finding the spectrum is either very hard or non-existing. This problem may also be adressed in finding the exact BPS-index discontinuities when passing the walls of marginal stability. Then the knowledge of the spectrum at some region of the moduli space may be used to determine the global spectrum. This is the wall-crossing problem and the main issue of the paper [13].

A corresponding problem is solved in algebraic geometry where an analogous phenomenon occur for generalised Donaldson-Thomas invariants, a topological invariant related to Calabi-Yau geometries. The solution by Kontsevich and Soibelman (KS) [18] make use of a Lie algebra of operators $\mathcal{K}_\gamma$ acting on a torus. By forming operator products, with the Donaldson-Thomas invariants as exponents, they managed to account for the discontinuities by setting equal different orderings of the products on the two sides of the wall of marginal stability. The discontinuities are accounted for as contributions from the commutation relations when rearranging the operator product.

In the following we describe under what circumstances the BPS spectrum may change and how this splits the moduli space in regions separated by codimension one walls. Then we present the Lie algebra used by KS to formulate the wall-crossing formula. Note that we present the KS transformations and wall-crossing formula in terms of the BPS-index, not in the context of the Donaldson-Thomas invariants. In section 6 the connection between the particle spectrum and the moduli space will be unraveled, and the formulation of this relation will be expressed in the BPS-index.

The BPS spectrum is defined by the central charge. At some locus $a_0 \in \mathcal{B}$ the central charges for two or more BPS-states may align, sharing the same complex argument and then the triangle inequality

$$|Z_{\gamma^1}(a_0)| + |Z_{\gamma^2}(a_0)| \geq |Z_{\gamma^1+\gamma^2}(a_0)| \tag{4.6}$$

saturates. The binding energy E of a bound BPS-state is generically negative

$$E = |Z_{\gamma^1+\gamma^2}(a_0)| - |Z_{\gamma^1}(a_0)| - |Z_{\gamma^2}(a_0)| \leq 0 \tag{4.7}$$

and a splitting of the bound state is only possible when equality holds. The saturated case is a complex argument condition and hence the points a_0 for which this holds constitutes a $\mathrm{codim}_{\mathbb{R}} = 1$ loci $\Xi \subset \mathcal{B}$. At these loci BPS-states may split or fuse in agreement with (4.6). In a neighbourhood of a generic point in the Coulomb branch this possibility does not exist and hence a BPS-state is locally stable. The moduli space may therefore be divided by one ore more codimension one *walls of marginal stability*, away from which the BPS-spectrum is fixed or 'constant'. The wall-crossing phenomenon is thus the jump in the BPS-spectrum when passing such a wall in the parameter space.

Consider the symplectic vector space $\Gamma_a \otimes \mathbb{C}^*$ at some point a on the Coulomb branch. This space is a complexified torus T_a for each $a \in \mathcal{B}$. Let X_γ be functions on this torus obeying $X_\gamma X_\eta = X_{\gamma+\eta}$. Moreover we chose these functions such that if the set $\{\gamma^i\}$ is a basis for Γ_a the corresponding set of functions $\{X_{\gamma^i}\}$ is a coordinate basis for T_a. If we take the rank one case as an example there are two charges spanning the charge lattice, and they correspond to two coordinates for the torus T^2.

Since the space is symplectic any function serves as a Hamiltonian function and we will construct a Lie algebra of the transformations generated by X_γ as Hamiltonian functions. The symplectic 2-form on T_a is defined by

$$\vartheta_{T_a} = \frac{1}{2}\langle \gamma^i, \gamma^j \rangle^{-1} \mathrm{d}\log X_{\gamma^i} \wedge \mathrm{d}\log X_{\gamma^i} \tag{4.8}$$

and is closed by construction. Let the operator f_γ be the infinitesimal symplectomorphism of the torus generated by the Hamiltonian X_γ. The Poisson bracket is defined through the Poisson structure which is the inverse of the symplectic form.

For two functions f and g we have

$$\{f,g\} = \vartheta_{T_a}^{-1}(\mathrm{d}f,\mathrm{d}g) \tag{4.9}$$

and thus the action of the Lie algebra generators is

$$f_\gamma X_\eta = \{X_\gamma, X_\eta\} = \vartheta_{T_a}^{-1}(\mathrm{d}X_\gamma, \mathrm{d}X_\eta) = \langle\gamma,\eta\rangle X_\gamma X_\eta = \langle\gamma,\eta\rangle X_{\gamma+\eta}\ . \tag{4.10}$$

In the following we motivate how to arrive at the KS Lie algebra by studying the f_γ operators. The bracket acting on the torus functions is by the Poisson bracket Jacobi identity

$$[f_\gamma,\ f_\eta]X_\rho = \{\{X_\gamma, X_\eta\}, X_\rho\} \tag{4.11}$$

and thus by acting with the bracket on X_ρ one gets a multiple of the generator $f_{\gamma+\eta}$ as

$$[f_{\gamma_1}, f_{\gamma_2}] = \langle\gamma_1,\gamma_2\rangle f_{\gamma_1+\gamma_2} \tag{4.12}$$

with the proportionality factor given by the symplectic pairing. This defines the algebra used in [18] up to a sign, which may be picked up by introducing a map $\sigma : \Gamma \to \mathbb{Z}_2$. Letting the multiplication rule for σ be

$$\sigma(\gamma_1)\sigma(\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle}\sigma(\gamma_1+\gamma_2) \tag{4.13}$$

and defining e_γ as the symplectomorphisms generated by $\sigma(\gamma)X_\gamma$ the bracket (4.12) modifies just by a sign. One realisation of the σ-map is given by $\sigma(\gamma) = (-1)^{pq}$ for $\gamma = (p,q)$. The action on X_η and the commutation relation is

$$\begin{aligned} e_\gamma X_\eta &= \langle\gamma,\eta\rangle\sigma(\gamma)X_{\gamma+\eta} \\ [e_{\gamma_1}, e_{\gamma_2}] &= (-1)^{\langle\gamma_1,\gamma_2\rangle}\langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2} \end{aligned} \tag{4.14}$$

for e_γ. The Lie algebra of this bracket is the foundation for the mathematicians solution of the wall crossing problem. The symplectomorphisms

$$\mathcal{K}_\gamma \equiv \exp\sum_{n=1}^{\infty}\frac{1}{n^2}e_{n\gamma} \tag{4.15}$$

are the ones used by KS in the operator products in the vicinity of the wall.

For each BPS-state of charge γ there is a ray l_γ in the complex ζ-plane. For $a \in \mathcal{B}$

$$l_\gamma = \{\zeta\ :\ Z_\gamma(a)/\zeta \in \mathbb{R}_-\} \tag{4.16}$$

and these rays rotate in their plane as $Z_\gamma(a)$ varies over the Coulomb branch. The ordering of the rays are constant in the vicinity of generic points of $\mathcal{B}$ and so are the number of rays. If a reaches a wall of marginal stability a set of central charges becomes aligned, sharing the same complex argument. The rays then come together, and when passing the wall the order and the number of rays are changed as in figure 1. Let γ_1 and γ_2 be primitive vectors of Γ. At some point a_0 on the wall a general splitting of the central charge is

$$Z_\gamma(a_0) = mZ_{\gamma_1}(a_0) + nZ_{\gamma_2}(a_0) \qquad m,\, n > 0 \tag{4.17}$$

and the set of charges for which the central charges of the BPS-states line up may thus be expressed as $m\gamma_1 + n\gamma_2$ for positive integers m and n. Forming the composition of the $\mathcal{K}_\gamma$ on one side of the wall

$$\overset{\frown}{A}(u) \equiv \overset{\curvearrowright}{\prod_{\gamma=m\gamma_1+n\gamma_2}} \mathcal{K}_\gamma^{\Omega(\gamma,a)} \qquad m,\, n > 0 \tag{4.18}$$

with the ordering such that the rays l_γ line up clockwise in the $\zeta-$plane. Let $a_\pm$ be points separated by a single wall. The wall-crossing formula can now be stated as follows:

$$\overset{\curvearrowright}{A}(a_+) = \overset{\curvearrowleft}{A}(a_-)\ . \tag{4.19}$$

When crossing the wall the order of $\mathcal{K}$-factors is reversed and the condition that the product $\overset{\curvearrowright}{A}$ is unchanged means that the contributions from commuting all symplectomorphism generators must be balanced by discrete changes in the index Ω. In the physical context this is the information needed to compute the change

$$\Delta\Omega = \Omega(\gamma, a_+) - \Omega(\gamma, a_-) \tag{4.20}$$

in the BPS-index across the wall. To make use of this formula one notices that the only generators that occur in $\overset{\curvearrowright}{A}$ are $e_{m\gamma_1+n\gamma_2}$ from which it follows that the set $\{e_\gamma\}$ with $m, n \geq M$ is an ideal of the Lie algebra (4.14) [25]. Hence one may construct quotient algebras by ideals with any positive M. The infinite product in $\overset{\curvearrowright}{A}$ then reduces to finite products accessible for determining the degeneracies $\Omega(\gamma)$.

The explicit action of $\mathcal{K}_\gamma$ on the torus coordinates is obtained by first noting that $e_{n\gamma}X_\eta = n\langle\gamma,\eta\rangle(\sigma(\gamma)X_\gamma)^n X_\eta$ and then considering the action of the operator $W_\gamma = \log \mathcal{K}_\gamma$ on the function X_η

$$\begin{aligned} W_\gamma X_\eta &= \sum_{n=1}^{\infty} \frac{1}{n^2} e_{n\gamma} X_\eta = \langle\gamma,\eta\rangle \sum_{n=1}^{\infty} \frac{1}{n}(\sigma(\gamma)X_\gamma)^n X_\eta \\ &= \log\left[(1-\sigma(\gamma)X_\gamma)^{\langle\eta,\gamma\rangle}\right] X_\eta \end{aligned} \tag{4.21}$$

concluding that $\mathcal{K}_\gamma X_\eta = (1-\sigma(\gamma)X_\gamma)^{\langle\eta,\gamma\rangle}\, X_\eta$.

A few examples are in order to see how wall-crossing formulae behave and the interpretation of them. Since the X_η are multiplicative it suffices to investigate the transformation of the unit magnetic and electric functions respectively. By denoting $X_{(1,0)} = x$ and $X_{(0,1)} = y$ the KS symplectomorphism is

$$K_\gamma : (x, y) \mapsto \Big((1 - (-1)^{pq} x^p y^q)^q\, x, (1 - (-1)^{pq} x^p y^q)^{-p}\, y\Big) \tag{4.22}$$

for any charge $\gamma = (p,q)$. By straightforward algebra one verifies the operator equality

$$\mathcal{K}_{(1,0)}\mathcal{K}_{(0,1)} = \mathcal{K}_{(0,1)}\mathcal{K}_{(1,1)}\mathcal{K}_{(1,0)} \tag{4.23}$$

which realises the general formula (4.19) when quoting the algebra by the ideal

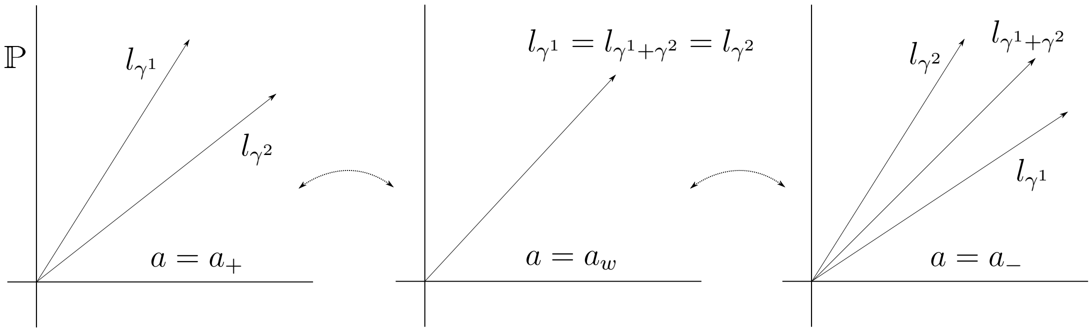

Figure 1: *When $a \in \mathcal{B}$ crosses the wall the rays l_{γ^i} corresponding to BPS states of charge γ^i rotate towards each other and their mutual ordering is eventually reverted. a_w belongs to the wall of marginal stability and at this locus all rays coalesce to a single ray. This example illustrates the behaviour of equation* (4.23) *where a bound state of charge $\gamma^1 + \gamma^2$ is formed when $a \in \mathcal{B}$ follows a path $a_+ \to a_-$ that crosses the wall.*

given by $m,n > 1$. This is illustrated in figure 1. The physical interpretation of this formula is two BPS particles of unit magnetic and electric charge respectively coming together at a wall. When passing the wall one new dyonic charged particle is created. This is in accordance with results from supergravity investigations [10] where this extra bound state is found. A simililar result for the $\mathcal{N} = 2$ theory with gauge group $SU(3)$ was analysed in [4]. For the wall-crossing formula (4.23) the BPS index discontinuities $\Delta\Omega(\gamma) = \Omega(a^+, \gamma) - \Omega(a^-, \gamma)$ are

$$\begin{aligned} \Delta\Omega(1,0) &= 0 \\ \Delta\Omega(1,1) &= -1 \\ \Delta\Omega(0,1) &= 0\ . \end{aligned} \tag{4.24}$$

In $SU(2)$ Seiberg-Witten theory there is one wall of marginal stability separating the parameter space in a strong coupling and a weak coupling region. The wall-crossing phenomena is captured in the formula

$$\mathcal{K}_{(2,-1)}\mathcal{K}_{(0,1)} = (\mathcal{K}_{(0,1)}\mathcal{K}_{(2,1)}\mathcal{K}_{(4,1)}\dots)\mathcal{K}_{(2,0)}^{-2}(\dots\mathcal{K}_{(6,-1)}\mathcal{K}_{(4,-1)}\mathcal{K}_{(2,-1)}) \qquad (4.25)$$

where the left hand side reflects the strong coupling spectrum - one monopole and one dyonic state. The exponents of both KS factors are 1 and hence these belong to the hypermultiplet. The right hand side represents the weak coupling side of the wall, where an infinite sequence of hypermultiplet states of dyonic charges reside. The 'middle' factor $\mathcal{K}_{(2,0)}^{-2}$ corresponds to a state in the vector multiplet since $\Omega = -2$, recall equation (4.5). This state has charge (2,0) and is thus a vector boson since the vector multiplet scalars are all neutral.

5 Compactification to Three Dimensions

In this section we compactify one direction of the four dimensional theory on a circle, keeping the circle radius as a parameter. An effective three dimensional theory is obtained in the small radius limit. We describe how the dimensional reduction is performed reaching a $\mathcal{N} = 4$ sigma model for the scalars. The vector field is dualised to two scalars in three dimensions and the moduli space geometry is restricted to be a hyperkähler manifold $\mathcal{M}$. The rest of this section is devoted to the explicit description of the geometry, starting from the Gibbons-Hawking ansatz. With a compactified direction there is a tower of field excitations on the circle which contributes to the metric. It is described how these instanton corrections are accounted for in a twistor space construction of the metric. This approach is the one used in [13] to formulate a Riemann-Hilbert problem for the Darboux coordinates on $\mathcal{M}$. The continous solution for this problem is then shown to be in one-to-one correspondence with a KS wall-crossing formula for the BPS-index.

5.1 The Bosonic Field Content

Compactifying the theory on S^1 will give an effective theory in three dimensions with the circle radius R as a parameter. The gauge field A^I splits to a three dimensional 1-form $A^I_{(3)}$ and a scalar A^I_4 upon compactification. The two-form field strength in three dimensions is Hodge dual to a 1-form field strength of a real scalar field. Thus each gauge field will give rise to two real scalar fields and since there are one gauge field to each complex scalar a^I in the vector multiplet this doubles the dimension of the moduli space.

By imposing that the fields of (2.27) not depend on the fourth coordinate x^4 we get the splitting

$$\mathrm{d}_4 a^I \to \mathrm{d}a^I(x^i) \qquad A^I \to A^I_{(3)} + A^I_4(x^i)\mathrm{d}x^4 \qquad F^I \to F^I_{(3)} + \mathrm{d}A^I_4 \wedge \mathrm{d}x^4 \tag{5.1}$$

where all exterior derivatives on the right hand side are three dimensional, a convention which is used from now on. The metric on $\mathbb{R}^{1,2} \times S^1$ is $g = \mathrm{d}x^i \otimes \mathrm{d}x_i + R^2\mathrm{d}x^4 \otimes \mathrm{d}x^4$ with Lorentzian signature and $i = 0{,}1{,}2$. Under the splitting (5.1) the kinetic terms become

$$\begin{aligned} \mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J &\to R\mathrm{d}x^4 \wedge \mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J \\ F^I \wedge *F^J &\to R\mathrm{d}x^4 \wedge F^I_{(3)} \wedge *F^J_{(3)} + \frac{1}{R}\mathrm{d}x^4 \wedge \mathrm{d}A^I_4 \wedge *\mathrm{d}A^J_4 \end{aligned} \tag{5.2}$$

and all quantities on the right hand side are three dimensional. We also note here that $** = -1$ on two-forms in three dimensional Minkowski space which will be used in the following calculations.

Inserting the splitted fields in the Lagrangian (2.27) and integrating out the S^1 direction gives

$$\begin{aligned} \mathcal{L}_{(3)} &= -\frac{1}{4\pi}\int_{S^1}\Big[\mathrm{d}x^4 \wedge \Im\,\tau_{IJ}(R\,\mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J + \frac{1}{R}\mathrm{d}A^I_4 \wedge *\mathrm{d}A^J_4 \\ &\qquad + R\,F^I_{(3)} \wedge *F^J_{(3)}) + 2\mathrm{d}x^4 \wedge \Re\,\tau_{IJ}F^I_{(3)} \wedge \mathrm{d}A^J_4\Big] \\ &= -\frac{1}{2}\Big[\Im\,\tau_{IJ}(R\,\mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J + \frac{1}{R}\mathrm{d}A^I_4 \wedge *\mathrm{d}A^J_4 \\ &\qquad + R\,F^I_{(3)} \wedge *F^J_{(3)}) + 2\,\Re\,\tau_{IJ}F^I_{(3)} \wedge \mathrm{d}A^J_4\Big] \end{aligned} \tag{5.3}$$

since the x^4 periodicity is 2π. The gauge field $A_{(3)}$ is dualised to a magnetic scalar Λ_I by adding a Lagrange multiplier term $F^I_{(3)} \wedge \mathrm{d}\Lambda_I$. This term is exact and will not change the equations of motion. Then the Euler-Lagrange equations

$$\begin{aligned} 0 = \delta_F\mathcal{L} &= \delta F^I_{(3)} \wedge \big(-R\,\Im\tau_{IJ} * F^J_{(3)} - \Re\,\tau_{IJ}\mathrm{d}A^J_4 + \mathrm{d}\Lambda_I\big) \\ \Rightarrow \quad *F^J_{(3)} &= \frac{1}{R}\,(\Im\,\tau)^{-1,IJ}\,(\mathrm{d}\Lambda_I - \Re\,\tau_{IJ}\mathrm{d}A^J_4) \end{aligned} \tag{5.4}$$

allow us to rephrase the Lagrangian in terms of Λ instead of $A_{(3)}$. By replacing all field strengths $F^I_{(3)} = -*(*F^I_{(3)})$ in (5.3) eliminates the field strength in favor of the introduced scalar field.

To be consistent with the notation of [13] we introduce the magnetic and electric coordinates $\theta_{m,I} = \Lambda_I/2\pi$ and $\theta_e^I = A_4^I/2\pi$ of periodicity 1. These are the 'electric' and 'magnetic' Wilson lines over the circle dimension

$$\theta_e^I = \int_{S^1} A_4^I \, dx^4 \qquad\qquad \theta_{m,I} = \int_{S^1} (A^*_{(3)})_I dx^4 \tag{5.5}$$

where $(A^*_{(3)})_I$ are the scalars dual to the three-dimensional gauge fields. This is not the scalars Λ_I, which are obtained after truncating the dependence on x^4. After some simplification the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_{(3)} = -\frac{1}{2}\Big\{ & R\,\Im\,\tau_{IJ}\mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J \\ & + \frac{1}{4\pi^2 R}(\Im\,\tau^{-1})^{IJ}(\mathrm{d}\theta_{m,I} - \tau_{IK}\mathrm{d}\theta_e^K) \wedge *(\mathrm{d}\theta_{m,J} - \bar{\tau}_{JL}\mathrm{d}\theta_e^L)\Big\}\ .\end{aligned} \tag{5.6}$$

This is a sigma model of maps from $\mathbb{R}^{1,2}$ into the target space $\mathcal{M}$ and the metric is that of a fibration of $2r$-tori over $\mathcal{B}$ pictured in figure 2. By introducing the coordinates z_I by $\mathrm{d}z_I = \mathrm{d}\theta_{m,I} - \tau_{IJ}\mathrm{d}\theta_e^I$ we get the metric

$$g^{sf} = R\,\Im\,\tau|\mathrm{d}a|^2 + \frac{1}{4\pi^2 R}\Im\,\tau^{-1}|\mathrm{d}z|^2 \tag{5.7}$$

adopting the index free notation $\tau|\mathrm{d}a|^2 \equiv \tau_{IJ}\mathrm{d}a^I \wedge *\mathrm{d}\bar{a}^J$. Note that $\mathrm{d}z_I$ is closed only for each fixed point $a \in \mathcal{B}$ i.e on each fixed torus fiber $\mathcal{M}_u$. The *sf* label stands for semiflat and reflects that the torus fiber is flat, which makes up half of the dimension of the manifold. This last form of the metric states that the Kähler metric g^{sf} is compatible with a complex structure in which $\mathrm{d}a^I$ and $\mathrm{d}z_I$ is a basis for the holomorphic one-forms on $\mathcal{M}$.

5.2 Dimensional Reduction of the Supercharges

In four dimensions and $\mathcal{N} = 2$ extended supersymmetry there is 8 real supersymmetries. These are organised in the two supersymmetry generating Weyl spinors Q^A of $SO(1,3)$. When compactifying on a circle and performing the dimensional reduction the supercharges must occur as representations of $\mathfrak{so}(1,3) \to \mathfrak{so}(1,2)$. Compactification on S^1 preserves all supersymmetries and the irreducible spinor in three dimensions is Majorana with $2^{(3-1)/2} = 2$ components [11]. This yields four 2-component conserved spinor charges $\tilde{Q}^a$ in three dimensions i.e an $\mathcal{N} = 4$ theory.

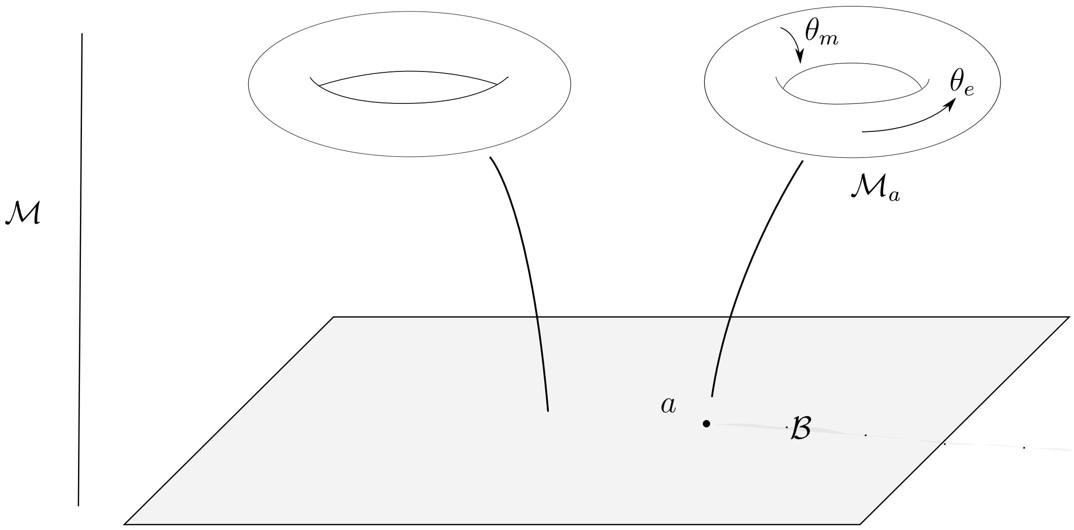

Figure 2: *The hyperkähler manifold $\mathcal{M}$ as a fibration of tori $\mathcal{M}_a$ over the Coulomb branch $\mathcal{B}$.*

The R-symmetry is enhanced under the dimensional reduction. To see this we start from the $\mathcal{N} = 1$ gauge multiplet in six dimensions as in [24]. The fields are the six dimensional gauge field A and Weyl fermions ψ, both in the adjoint representation. The fermion fields come in a doublet of $\mathfrak{su}(2)_R$, so do the supercharges. Imposing independence of the last three coordinates x^3, x^4, x^5 gives three scalar fields from the corresponding components of the gauge field. These transform in the **3** of the rotations $\mathfrak{so}(3) = \mathfrak{su}(2)_N$.

When reducing the six dimensional theory to four dimensions one gets the $\mathcal{N} = 2$ theory studied in this thesis. The six dimensional gauge field gives rise to two real scalars i.e the complex scalar we already encountered and the symmetry is the $\mathfrak{u}(1)_R$ rotations of the x^4, x^5-plane.

The $\mathfrak{u}(1)_R \to \mathfrak{su}(2)_N$ enhancement when going down to three dimensions acts on the scalars, which in turn constitute the moduli space geometry. In this context the $\mathfrak{su}(2)_N$ rotates the three complex structures of $\mathcal{M}$. This is the full S^2 of complex structures defining the hyperkähler moduli space of the $\mathcal{N} = 4$ field theory with bosonic Lagrangian (5.6).

5.3 Describing the Hyperkähler Geometry

The following subsections are devoted to reproducing the metric (5.7) from a hyperkähler metric ansatz. First we consider the classical, or semi-flat, metric valid in the large radius limit. Then we describe how corrections to this metric due to instantonic excitations along the compact direction is accounted for. Some background on hyperkähler geometry is reviewed in appendix A.

5.3.1 The Semiflat Metric

We now describe how the semiflat hyperkähler geometry with metric g^{sf} may be obtained using the twistor space construction which is introduced in appendix A. The twistor space $\mathcal{Z} = \mathcal{M} \times \mathbb{P}$ is topologically the trivial fibration of the hyperkähler manifold $\mathcal{M}$ over the complex projective line $\mathbb{P}$. The projection $\pi : \mathcal{Z} \to \mathbb{P}$ is the map $\pi(u, \zeta) = \zeta$ for $u \in \mathcal{M}$ and $\zeta \in \mathbb{P}$ is the twistor parameter.

In the twistor approach Darboux coordinates ξ on the twistor space $\mathcal{Z}$ are searched for, such that

$$\vartheta = \frac{1}{4\pi^2 R} \mathrm{d}\xi_m \wedge \mathrm{d}\xi_e \tag{5.8}$$

is a section of $\Omega^2_{\mathcal{M}_\zeta} \otimes \mathcal{O}_{\mathbb{P}}(2)$. The prefactor $(4\pi^2 R)^{-1}$ is due to the chosen notation. This $\vartheta(\zeta)$ is a holomorphic symplectic form for each fiber $\mathcal{M}_\zeta = \pi^{-1}(\zeta)$, that is, $\vartheta(\zeta)$ is a (2,0)-form in the complex structure J^ζ for each fixed $\zeta \in \mathbb{P}$. As a section of the line bundle $\mathcal{O}_{\mathbb{P}}(2)$ ϑ is a section defined as a second degree polynomial in ζ (with the proper transition functions between the patches of $\mathbb{P}$). The three Kähler forms on $\mathcal{M}$ are $\omega_\pm = \omega_1 \pm i\omega_2$ and ω_3 and the symplectic form on $\mathcal{Z}$ is expressed as

$$\vartheta(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i\zeta}{2}\omega_- \ . \tag{5.9}$$

At $\zeta = 0$ and $\zeta = \infty$ this form is to be multiplied with the $\mathcal{O}_{\mathbb{P}}$ transition functions ζ and ζ^{-1} respectively.

The semiflat Darboux coordinates have a neat realisation in terms of the central charge Z and the gauge field Wilson lines θ. When no instanton contributions are considered they are given by

$$\xi_\gamma = \pi R \zeta^{-1} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma \tag{5.10}$$

and in the following we will see that they reproduce the semiflat hyperkähler geometry (5.7). In the work of [13] the concept of 'Darboux functions' $\mathcal{X}$ are introduced as the exponentiations

$$\mathcal{X}_\gamma = \exp(\xi_\gamma) \tag{5.11}$$

of the Darboux coordinates which turns out to be convenient in the wall-crossing context, treated in section 6. The holomorphicity of ξ, $\mathcal{X}$ or ϑ for each fixed ζ may be phrased in terms of the Cauchy-Riemann equations over $\mathcal{M}$ [13]. In appendix A the holomorphicity of ϑ over $\mathcal{M}$ in complex structure J^ζ is shown explicitly. The semiflat moduli space metric of the compactified theory has a simple realisation in terms of the Darboux functions. Collecting the magnetic and electric coordinates as $\theta = \theta_e^I \alpha_I - \theta_{m,I}\beta^I$ the Darboux functions

$$\mathcal{X}_\gamma^{sf}(\zeta) = \exp[\pi R \zeta^{-1} Z_\gamma + i\theta_\gamma + \pi R\zeta \bar{Z}_\gamma] \tag{5.12}$$

are defined. Computing the holomorphic symplectic form using the special Kähler condition (2.38) gives

$$\begin{aligned}\vartheta^{sf}(\zeta) &= \frac{1}{8\pi^2 R}\langle \gamma^i, \gamma^j\rangle^{-1} \mathrm{d}\log \mathcal{X}_{\gamma^i}^{sf}(\zeta) \wedge \mathrm{d}\log \mathcal{X}_{\gamma^j}^{sf}(\zeta) \\ &= \frac{1}{4\pi}\left[i\zeta^{-1}\langle \mathrm{d}Z, \mathrm{d}\theta\rangle + \left(\pi R\langle \mathrm{d}Z, \mathrm{d}\bar{Z}\rangle - \frac{1}{2\pi R}\langle \mathrm{d}\theta, \mathrm{d}\theta\rangle\right) + i\zeta\langle \mathrm{d}\bar{Z}, \mathrm{d}\theta\rangle\right] .\end{aligned} \tag{5.13}$$

Comparing this expression with (5.9) the Kähler form ω_+ is extracted as

$$\begin{aligned}\omega_+ &= -\frac{1}{2\pi}\langle \mathrm{d}Z, \mathrm{d}\theta\rangle = -\frac{1}{2\pi}\langle \mathrm{d}X^I\alpha_I - \mathrm{d}F_I\beta^I, \mathrm{d}\theta_e^J\alpha_J - \mathrm{d}\theta_{m,J}\beta^J\rangle \\ &= \frac{1}{2\pi}(\mathrm{d}a^I \wedge \mathrm{d}\theta_{m,I} - \frac{\partial^2 \mathcal{F}(a)}{\partial a^I \partial a^J}\mathrm{d}a^I \wedge \mathrm{d}\theta_e^J) = \frac{1}{2\pi}\mathrm{d}a^I \wedge \mathrm{d}z_I\end{aligned} \tag{5.14}$$

by choosing a duality frame. In appendix A it is shown how the complex structure J_3^{sf} may be calculated from this Kähler form. The terms independent of ζ in (5.13) constitute the Kähler form

$$\omega_3^{sf} = \frac{R}{4}\langle \mathrm{d}Z, \mathrm{d}\bar{Z}\rangle - \frac{1}{8\pi^2 R}\langle \mathrm{d}\theta, \mathrm{d}\theta\rangle \tag{5.15}$$

in the third complex structure. Again, by choosing a duality frame this Kähler form is rewritten in the coordinates $\{a^I, z_I\}$ as

$$\omega_3^{sf} = \frac{i}{2}\Big(R(\Im\,\tau)_{IJ}\mathrm{d}a^I \wedge \mathrm{d}\bar{a}^J + \frac{1}{4\pi^2 R}(\Im\,\tau)^{-1,IJ}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J\Big) , \tag{5.16}$$

which states that in J_3^{sf} the metric g^{sf} is Kähler with the Kähler form given above. It follows that this *is* the hyperkähler metric as obtained from the compactification on S^1 (5.6).

5.3.2 The Ooguri-Vafa Metric

When compactifying on a circle one introduces the possibility for field excitations around the compact direction which are referred to as instanton contributions. BPS states of mass $M = |Z|$ wrapping a circle of radius R are exponentially supressed by $e^{-|Z|R}$ as the radius goes to infinity. Their action is $\sim |Z|R$ and hence the partition function gets the exponential supression. In this limit there are no contribution from winding states and the semiflat metric is the classical one. The opposite limit, when $R \to 0$ was identified as the Atiyah-Hitchin manifold in the case of a $SU(2)$ theory [24].

In the following we are interested in the region where R is not small. In this case the contributions from all the BPS states in the four-dimensional theory must be accounted for and this will lead to the wall-crossing phenomenon. For arbitrary finite R one may expect contributions from the wrapped excitations to the metric. In the following we describe how to take this into account in a simplified case. Given a number of charged particles that are all mutually local it is possible to choose a duality frame where all particles are electrically charged [22]. To start with we restrict the treatment to the case of one particle of electric charge $\gamma = (0, q)$.

Recall that the moduli space $\mathcal{M}$ is a torus fibration over the Coulomb branch $\mathcal{B}$. Without coupling to some charged particles the magnetic and electric coordinates of the torus in are invariant under a $U(1)_m \times U(1)_e$ action of shift symmetry along the torus directions. When introducing a electrically charged particle the isometry in the electrical coordinate are broken while shifts of the magnetic coordinates still are isometries.

The Gibbons-Hawking ansatz describes a family of hyperkähler metrics with a $U(1)$ isometry. In this first example we consider the rank 1 case and we take coordinates (x^1, x^2, x^3) with $a = x^1 + ix^2$ and $\theta_e = 2\pi R x^3$. For the isometry Killing vector ∂_{θ_m} the Gibbons-Hawking form of the metric is

$$g = V^{-1}(x)(\frac{\mathrm{d}\theta_m}{2\pi} + A(x))^2 + V(x)\mathrm{d}x_i \otimes \mathrm{d}x^i \tag{5.17}$$

for A the $U(1)$ connection and V obeying

$$\mathrm{d}A = *\mathrm{d}V \tag{5.18}$$

making V the dual scalar potential to the connection. Given this form of the metric three Kähler forms are obtained as

$$\omega_a = \mathrm{d}x^a \wedge (\mathrm{d}\theta_m + A) + \frac{V}{2}\varepsilon^{abc}\mathrm{d}x^b \wedge \mathrm{d}x^c \tag{5.19}$$

giving the hyperkähler structure to the geometry on $\mathcal{M}$ specified by g. These are closed due to the condition (5.18) and it follows that V is harmonic, [15]. To give an explicit expression for the Gibbons-Hawking form of the metric some physical constraints are needed.

First of all the metric should reduce to the semiflat metric when instanton contributions are suppressed and $R \to \infty$. The singular point in this example is $a = 0$ where the BPS-states become massless and the full gauge group is restored. Hence for $|a| \to \infty$ the limit of g must be g^{sf}. The wanted metric is periodic in θ_e and have no continuous shift invariance along this direction. Also, the semiflat metric is invariant under rotations of a which suggests that the full metric is just a function of $|a|$ and the torus coordinates. As concluded in [22] these constrains have a unique solution (up to a possible integration constant) which in our notation takes the form

$$V(a, \theta_e) = \frac{q^2 R}{4\pi} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{q^2 R^2 |a|^2 + (\frac{q\theta_e}{2\pi} + n)^2}} - b_n \tag{5.20}$$

with some b_n of order $1/|n|$ introduced for convergence.

This potential contains all contributions to the metric, with the semiflat metric as the zero mode. In extracting this mode it is convenient to perform a Poisson resummation of this expression. Starting from the identity

$$\frac{1}{|x|} = \int_0^\infty \frac{dt}{t^{3/2}} \exp[-\frac{\pi}{t}|x|^2] \tag{5.21}$$

and for simplicity defining the denominator expression

$$\sqrt{q^2 R^2 |a|^2 + (\frac{q\theta_e}{2\pi} + n)^2} \equiv |y|, \qquad y = y_1 + i(y_2 + n) \tag{5.22}$$

the potential is rewritten

$$V = \int_0^\infty \frac{dt}{t^{3/2}} \exp[-\frac{\pi}{t} y_1^2] \sum_{n=-\infty}^{\infty} \exp[-\frac{\pi}{t}(y_2 + n)^2] \ . \tag{5.23}$$

By employing the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) \quad \text{for} \quad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \tag{5.24}$$

V may be rewritten as

$$V = \int_0^\infty \frac{dt}{t^{3/2}} \exp[-\frac{\pi}{t} y_1^2] \sqrt{t} \sum_{n=-\infty}^{\infty} \exp[-\pi t n^2 - 2\pi i y_2 n] \tag{5.25}$$

and the occurence of the factor 2π in the resummation formula is due to the choice of Fourier transform convention. Extracting the oscillating term in the equation above, which is independent of the integration parameter, gives the Bessel integral

$$V = \sum_{n=-\infty}^{\infty} \exp[-2\pi i y_2 n] \int_0^\infty \frac{dt}{t} \exp[-\frac{\pi}{t} y_1^2 - \pi t n^2] = \tag{5.26}$$
$$= 2 \sum_{n=-\infty}^{\infty} \exp[-2\pi i y_2 n] K_0(2\pi |y_1 n|) \ .$$

The modified Bessel function $K_0(x)$ has the asymptotic behaviour $K_0(x) \sim -\log(x/c)$ for small x and for some constant c. The zero mode may be viewed as the $|a| \to 0$ behaviour of K_0 for some non-zero n. The choice of regularization b_n will contribute to this constant and they are treated collectively as the constant Λ. Inserting the expressions for y_1, y_2 and the prefactor gives the final expression

$$V = V^{sf} + V^{inst} = -\frac{q^2 R}{4\pi} \log(\frac{a\bar{a}}{\Lambda\bar{\Lambda}}) + \frac{q^2 R}{2\pi} \sum_{n\neq 0} \exp[iq\theta_e n] K_0(2\pi R|qan|) \tag{5.27}$$

and the Λ is to be interpreted as the energy scale. The Bessel function in (5.27) has a large argument expansion in which the leading behaviour is $K_0(2\pi|a|R) \sim e^{-2\pi R|a|}$ for large R and a, as expected for instanton contributions. The classical metric is obtained in the region far away from the singular point $|a| = 0$ as well as in the decompactification limit. The exponential in θ_e breaks the translational invariance from $\mathbb{R}$ to $\mathbb{Z}$ as expected in the presence of an electric charge. The connection dual to the potential V in (5.18) is given by $A = A^{sf} + A^{inst}$ with

$$A^{sf} = \frac{iq^2}{8\pi^2}\big(\log\frac{a}{\Lambda} - \log\frac{\bar{a}}{\bar{\Lambda}}\big)\mathrm{d}\theta_e \tag{5.28}$$
$$A^{inst} = -\frac{q^2 R}{4\pi}\big(\frac{\mathrm{d}a}{a} - \frac{\mathrm{d}\bar{a}}{\bar{a}}\big) \sum_{n\neq 0} \mathrm{sgn}(n) \exp[iq\theta_e n] |a| K_1(2\pi R|qan|) \ .$$

The semiflat contribution is singular at $a = 0$ and taking the corrections into account we see that the possible singularities is $a = 0, \theta_e = 2\pi n/q$. The second condition has q solutions while θ_e is 2π-periodic. The four dimensional space with coordinates (a, θ_m, θ_e) is locally $\mathbb{C}^2$, and subject to the identification $\theta_e \sim \theta_e + 2\pi n/q$. The space in the vicinity of $a = 0$ is then the quotient space $\mathbb{C}^2/\mathbb{Z}_q$ i.e an A_{q-1} singularity. In the special case of $q = 1$ there is no singularity and the space is perfectly smooth.

5.3.3 Twistor Space Description

In this section we make connection with section 5.3.1 and describe how to extend the twistor construction of the semiflat case to the instanton corrected metric. The holomorphic symplectic form ϑ over the twistor space $\mathcal{M} \times \mathbb{P}$ is obtained as

$$\vartheta(\zeta) = \frac{1}{4\pi^2 R} \mathrm{d}\xi_m \wedge \mathrm{d}\xi_e \tag{5.29}$$

for $\xi_{m,e}$ the magnetic and electric Darboux coordinates whose differentials are

$$\begin{aligned} \mathrm{d}\xi_m &= i\mathrm{d}\theta_m + 2\pi i A(x) + i\pi V(x)\big(\zeta^{-1}\mathrm{d}a - \zeta \mathrm{d}\bar{a}\big) \\ \mathrm{d}\xi_e &= i\mathrm{d}\theta_e + \pi R\big(\zeta^{-1}\mathrm{d}a + \zeta \mathrm{d}\bar{a}\big) \ . \end{aligned} \tag{5.30}$$

The holomorphic 2-form ϑ is an element of $\Omega^{(2,0)}(\mathcal{M})$ for each $\zeta \in \mathbb{P}$ in complex structure J^ζ. This implies that both Darboux coordinate differentials are of type (1,0) i.e holomorphic one forms in J^ζ.

Taking $\xi_e = \log \mathcal{X}_e$ and using that the electric unit charge is $\gamma = (0,1)$ gives Z_γ as the coordinate $a \in \mathcal{B}$ and $\theta_\gamma = \theta_e$. Written out ξ_e is

$$\xi_e = \pi R \zeta^{-1} a + i\theta_e + \pi R \zeta \bar{a} \tag{5.31}$$

which is the same expression as in (5.10). The conclusion is that the electric coordinate contribution to the symplectic form is the classical one and the instanton corrections are all in the magnetic coordinate ξ_m. The problem of finding the instanton corrected Darboux functions over the twistor space is thus restricted to finding the magnetic solution. The straightforward way would be to make an ansatz for $\mathcal{X}_m$ and then solve the corresponding Cauchy-Riemann equations on $\mathcal{M}$.

In [13] a different approach is presented for the solution, and instead of demanding holomorphicity given the geometry they look for a solution that is validated by giving the right symplectic form ϑ. By linearity the splitting in semiflat and instanton corrected contributions are transferred to the symplectic form as

$$\begin{aligned} \vartheta(\zeta) &= \vartheta^{sf}(\zeta) + \vartheta^{inst}(\zeta) \\ \vartheta^{sf}(\zeta) &= -\frac{1}{4\pi^2 R} \mathrm{d}\xi_e \wedge \big[i\mathrm{d}\theta_m + 2\pi i A^{sf} + \pi i V^{sf}(\zeta^{-1}\mathrm{d}a - \zeta\mathrm{d}\bar{a})\big] \\ \vartheta^{inst}(\zeta) &= -\frac{1}{4\pi^2 R} \mathrm{d}\xi_e \wedge \big[2\pi i A^{inst} + \pi i V^{inst}(\zeta^{-1}\mathrm{d}a - \zeta\mathrm{d}\bar{a})\big] \end{aligned} \tag{5.32}$$

identifying $\mathrm{d}\xi_m^{sf}$ as the right factor of the second line. In the semiflat part above the term $A^{sf} \sim \mathrm{d}\theta_e$ carries a dependence on the electric torus coordinate θ_e. In constructing the Darboux functions with the origin in the magnetic-electric duality frame we want to separate the dependence on θ_m and θ_e. The solution to this is to simply subtract a multiple of the electric coordinate $\mathrm{d}\xi_e$ such that the θ_e dependence cancels. This maneuver preserves the symplectic form

$$\vartheta^{sf} \sim \mathrm{d}\xi_e \wedge (\mathrm{d}\xi_m^{sf} - k\mathrm{d}\xi_e) = \mathrm{d}\xi_e \wedge \mathrm{d}\xi_m^{sf} \tag{5.33}$$

and the explicit semiflat magnetic coordinate is

$$\begin{aligned} \mathrm{d}\xi_m^{sf} &= i\mathrm{d}\theta_m + 2\pi i A^{sf} + \pi i V^{sf}(\zeta^{-1}\mathrm{d}a - \zeta \mathrm{d}\bar{a}) - \frac{iq^2}{4\pi}\big(\log\frac{a}{\Lambda} - \log\frac{\bar{a}}{\bar{\Lambda}}\big)\mathrm{d}\xi_e \\ &= \frac{Rq^2}{2i}\zeta^{-1}\mathrm{d}(a\log\frac{a}{\Lambda} - a) + i\mathrm{d}\theta_m - \frac{Rq^2}{2i}\zeta\mathrm{d}(\bar{a}\log\frac{\bar{a}}{\bar{\Lambda}} - \bar{a})\ . \end{aligned} \tag{5.34}$$

where we recognise the multiple of $\mathrm{d}\xi_e$ as the rightmost term in the first line. This expression coincides with the ansatz (5.12) for a unit magnetic charge $\gamma = (1{,}0)$, and for the central charge function $Z_m = \frac{q^2}{2\pi i}(a\log\frac{a}{\Lambda} - a)$.

The form of the instanton-corrected magnetic solution worked out in [13] is the ingenious ansatz

$$\begin{aligned} \mathcal{X}_m = \mathcal{X}_m^{sf}\exp\Big[&\frac{iq}{4\pi}\int_{l_+}\frac{\mathrm{d}\eta}{\eta}\frac{\eta+\zeta}{\eta-\zeta}\log[1-\mathcal{X}_e(\eta)^q] \\ &- \frac{iq}{4\pi}\int_{l_-}\frac{\mathrm{d}\eta}{\eta}\frac{\eta+\zeta}{\eta-\zeta}\log[1-\mathcal{X}_e(\eta)^{-q}]\Big] \end{aligned} \tag{5.35}$$

where the paths $l_\pm$ are any paths from 0 to ∞ on $\mathbb{P}$ belonging respectively to the half-planes

$$H_\pm = \{\zeta : \pm\Re\frac{a}{\zeta} < 0\} \tag{5.36}$$

which are viewed as two hemispheres of $\mathbb{P}$. The two contours are to be read as the contributions from positively and negatively winded state excitations around S^1. The verification of the ansatz (5.35) reduce to checking that

$$\vartheta(\zeta) = -\frac{1}{4\pi^2 R}\mathrm{d}\log\mathcal{X}_e \wedge \mathrm{d}\log\mathcal{X}_m \tag{5.37}$$

reproduces the symplectic form (5.29) corresponding to the Ooguri-Vafa metric. The differential of the magnetic Darboux coordinate is

$$\mathrm{d}\log\mathcal{X}_m = \mathrm{d}\log\mathcal{X}_m^{sf} + I_+ + I_- \tag{5.38}$$

for the integrals

$$
\begin{aligned}
I_{\pm} &= \pm\frac{iq}{4\pi}\mathrm{d}\Big(\int_{l_\pm}\frac{\mathrm{d}\eta}{\eta}\frac{\eta+\zeta}{\eta-\zeta}\log[1-\mathcal{X}_e(\eta)^{\pm q}]\Big) \\
&= \pm\frac{iq}{4\pi}\cdot\mp q\int_{l_\pm}\frac{\mathrm{d}\eta}{\eta}\frac{\eta+\zeta}{\eta-\zeta}\frac{\mathcal{X}_e^{\pm q}(\eta)}{1-\mathcal{X}_e^{\pm q}(\eta)}\frac{\mathrm{d}\mathcal{X}_e(\eta)}{\mathcal{X}_e(\eta)} \\
&= -\frac{iq^2}{4\pi}\int_{l_\pm}\frac{\mathrm{d}\eta}{\eta}\frac{\eta+\zeta}{\eta-\zeta}\frac{\mathcal{X}_e^{\pm q}(\eta)}{1-\mathcal{X}_e^{\pm q}(\eta)}\mathrm{d}\log\mathcal{X}_e(\eta)\,,
\end{aligned}
\tag{5.39}
$$

remembering that the $\mathbb{P}$ parameter ζ is constant with respect to the exterior derivative. The expression (5.38) and the splitting (5.32) implies that the correction part must obey

$$
\mathrm{d}\log\mathcal{X}_e\wedge(I_+ + I_-) = \mathrm{d}\log\mathcal{X}_e\wedge\left[2\pi i A^{inst}+\pi i V^{inst}(\zeta^{-1}\mathrm{d}a-\zeta\mathrm{d}\bar{a})\right]. \tag{5.40}
$$

This is the equation that we verify by evaluating the left hand side. To perform the integrals we deform the contours $l_\pm$ to the specific case

$$
l_\pm = \{\eta : \pm\frac{a}{\eta}\in\mathbb{R}_-\} \tag{5.41}
$$

which are the two rays centered on the half planes $H_\pm$. First we note that the modulus of the electric Darboux function

$$
|\mathcal{X}_e(\eta)|<1 \quad \forall \quad \eta\in l_+ \tag{5.42}
$$

since the real part of ξ_e is strictly negative on this path. The analogous statement is that $|\mathcal{X}_e|>1$ on l_-. The wedge products on the left in (5.40) may be rewritten by moving the one form under the integral yielding the expression

$$
\begin{aligned}
\frac{\eta+\zeta}{\eta-\zeta}\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge\mathrm{d}\log\mathcal{X}_e(\eta) &= \frac{\eta+\zeta}{\eta-\zeta}\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge[\mathrm{d}\log\mathcal{X}_e(\eta)-\mathrm{d}\log\mathcal{X}_e(\zeta)] \\
&= \pi R\,\frac{\eta+\zeta}{\eta-\zeta}\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge\left[(\eta^{-1}-\zeta^{-1})\mathrm{d}a+(\eta-\zeta)\mathrm{d}\bar{a}\right] \\
&= -\pi R\,\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge\left[(\eta^{-1}\mathrm{d}a-\eta\mathrm{d}\bar{a})+(\zeta^{-1}\mathrm{d}a-\zeta\mathrm{d}\bar{a})\right]
\end{aligned}
\tag{5.43}
$$

which contains terms of order -1, 0 and 1 in η. Inserting this expression in (5.40) gives the integrals

$$
\begin{aligned}
\frac{iq^2R}{4}\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge\Big\{&\int_{l_+}\frac{\mathrm{d}\eta}{\eta}\left[(\eta^{-1}\mathrm{d}a-\eta\mathrm{d}\bar{a})+(\zeta^{-1}\mathrm{d}a-\zeta\mathrm{d}\bar{a})\right]\frac{\mathcal{X}_e^{q}(\eta)}{1-\mathcal{X}_e^{q}(\eta)} \\
&+\int_{l_-}\frac{\mathrm{d}\eta}{\eta}\left[(\eta^{-1}\mathrm{d}a-\eta\mathrm{d}\bar{a})+(\zeta^{-1}\mathrm{d}a-\zeta\mathrm{d}\bar{a})\right]\frac{\mathcal{X}_e^{-q}(\eta)}{1-\mathcal{X}_e^{-q}(\eta)}\Big\}
\end{aligned}
\tag{5.44}
$$

which may be evaluated in closed form. We consider the different orders of η separately. The l_+-integral over the ζ- and ζ^{-1}-terms is

$$(\zeta^{-1}\mathrm{d}a - \zeta\mathrm{d}\bar{a})\int_{l_+}\frac{\mathrm{d}\eta}{\eta}\frac{\mathcal{X}_e^q(\eta)}{1-\mathcal{X}_e^q(\eta)} \tag{5.45}$$

and since the modulus of the Darboux function is less than one we may rewrite the integral part above as a geometric series. This allows for moving out the oscillating θ_e-factor leaving a Bessel integral

$$\begin{aligned}\int_{l_+}\frac{\mathrm{d}\eta}{\eta}\frac{\mathcal{X}_e^q(\eta)}{1-\mathcal{X}_e^q(\eta)} &= \int_{l_+}\frac{\mathrm{d}\eta}{\eta}\sum_{n>0}\mathcal{X}_e^{qn}(\eta)\\ &= \sum_{n>0}\exp[iqn\theta_e]\int_{l_+}\frac{\mathrm{d}\eta}{\eta}\exp[\pi Rqn(\frac{a}{\eta}+\eta\bar{a})]\\ &= \sum_{n>0}\exp[iqn\theta_e]\int_{\mathbb{R}+}\frac{\mathrm{d}x}{x}\exp[-\pi Rqn(\frac{1}{x}+x|a|^2)]\\ &= 2\sum_{n>0}\exp[iqn\theta_e]K_0(2\pi R|qan|)\ .\end{aligned} \tag{5.46}$$

Over the ray l_+ from 0 to ∞ the integral is evaluated as a real integral, changing variable to $x=\eta/a<0$ giving $\mathrm{d}\eta/\eta=\mathrm{d}x/x$. This ensures that the exponent is always real and negative and hence a convergent Bessel integral. The corresponding integral over l_- gives by the same construction $x>0$ and the negative sign is due to the negative charge q over this ray. The geometric sum is thus effectively over the negative integers and together the integrals contribute by

$$\begin{aligned}&\frac{iq^2R}{2}\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge(\zeta^{-1}\mathrm{d}a-\zeta\mathrm{d}\bar{a})\sum_{n\neq 0}\exp[iqn\theta_e]K_0(2\pi R|qan|) = \\ &\quad \mathrm{d}\log\mathcal{X}_e(\zeta)\wedge i\pi V^{inst}(\zeta^{-1}\mathrm{d}a-\zeta\mathrm{d}\bar{a})\end{aligned} \tag{5.47}$$

to (5.40). To evaluate the integrals multiplying the $(\eta^{-1}\mathrm{d}a-\eta\mathrm{d}\bar{a})$ factor in (5.44) we treat first the l_+-case. The first integral is

$$\begin{aligned}&\mathrm{d}a\int_{l_+}\frac{\mathrm{d}\eta}{\eta^2}\frac{\mathcal{X}_e^q(\eta)}{1-\mathcal{X}_e^q(\eta)} = \mathrm{d}a\int_{l_+}\frac{\mathrm{d}\eta}{\eta^2}\sum_{n>0}\mathcal{X}_e^{qn}(\eta) = \\ &\mathrm{d}a\sum_{n>0}\exp[iqn\theta_e]\int_{l_+}\frac{\mathrm{d}\eta}{\eta^2}\exp[\pi Rqn(\frac{a}{\eta}+\eta\bar{a})]\\ &= -2\,\mathrm{d}a\sum_{n>0}\frac{|a|}{a}\exp[iqn\theta_e]K_1(2\pi R|qan|)\end{aligned} \tag{5.48}$$

where we picked up a total minus sign from the $\mathrm{d}\eta/\eta^2$-term which is negative for the l_+-ray and the factor of $|a|$ is due to the Bessel integral. In the same spirit the second integral (of order η in (5.44)) is evaluated to

$$\begin{aligned}
&\mathrm{d}\bar{a}\int_{l_+}\frac{\mathrm{d}\eta}{\eta}\eta\frac{\mathcal{X}_e^q(\eta)}{1-\mathcal{X}_e^q(\eta)} = \\
&\mathrm{d}\bar{a}\sum_{n>0}\exp[iqn\theta_e]\int_{l_+}\mathrm{d}\eta\exp[\pi Rqn(\frac{a}{\eta}+\eta\bar{a})] \\
&= -2\,\mathrm{d}\bar{a}\sum_{n>0}\frac{|a|}{\bar{a}}\exp[iqn\theta_e]K_1(2\pi R|qan|)\ .
\end{aligned} \tag{5.49}$$

Adding the results of the ray l_-, with sums over negative integers, and an overall sign due to the positiveness of the integral parameter the result is

$$\begin{aligned}
&-\frac{iq^2R}{2}\mathrm{d}\log\mathcal{X}_e(\zeta)\wedge\sum_{n\neq 0}\mathrm{sgn}(n)|a|\exp[iqn\theta_e]K_1(2\pi R|qan|) \\
&= \mathrm{d}\log\mathcal{X}_e(\zeta)\wedge 2\pi i A^{sf}
\end{aligned} \tag{5.50}$$

which completes the check that the ansatz actually gives the magnetic function that reproduces the metric. Comparing (5.40) to the integral contributions (5.47) and (5.50) we see that all instanton contributions agree.

We note here that the generalisation to higher rank, or manifolds of higher dimension, is carried out following the same line of reasoning. In coordinates $x^I = (a^I, \theta_e^I)$ and $\theta_{m,I}$ chosen such that the $U(1)^r$ isometry Killing vectors are $\partial_{\theta_{m,I}}$ the metric is

$$g = V(x)^{-1,IJ}(\frac{\mathrm{d}\theta_{m,I}}{2\pi}+A_I(x))(\frac{\mathrm{d}\theta_{m,J}}{2\pi}+A_J(x)) + V(x)_{IJ}\mathrm{d}x^I\cdot\mathrm{d}x^J \tag{5.51}$$

and to keep the hyperkähler structure the relation between A_I and V_{IJ} are chosen such that the Kähler forms

$$\omega^a = \mathrm{d}x^{a,I}\wedge(\mathrm{d}\theta_{m,I}+A_I) + \frac{1}{2}V_{IJ}\varepsilon^{abc}\mathrm{d}x^{b,I}\wedge\mathrm{d}x^{c,J} \tag{5.52}$$

all are closed, in accordance with the existence of the Kähler forms (5.19). Analogous expressions for the potentials V and A are obtained and it is found that the electric Darboux functions are the semiflat ones. The instanton corrections are accounted for in the magnetic functions. Details of these calculations are found in [13].

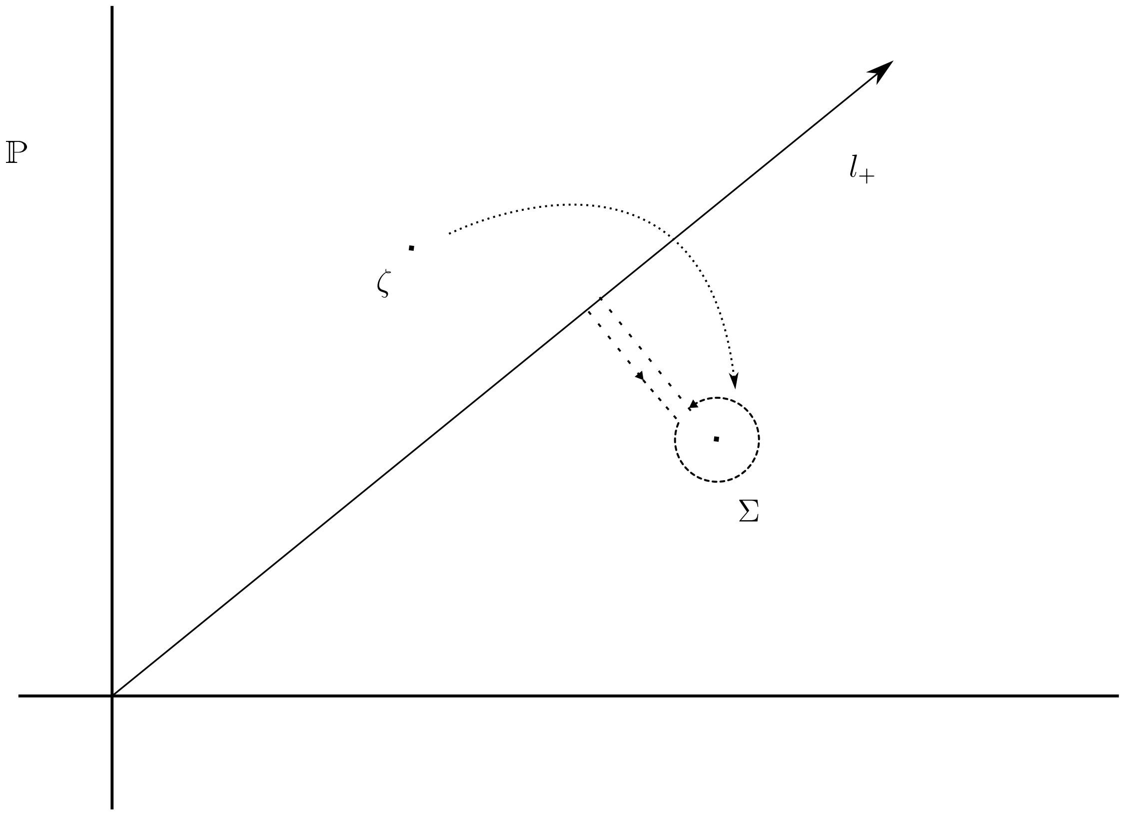

Figure 3: *When ζ move along a path across the integration contour l_+ the contour encircles the pole by the curve Σ.*

6 Wall-Crossing in Twistor Space

This section is devoted to the relation between the Kontsevich-Soibelman wall-crossing formula introduced in section 4.2, and the general problem of finding Darboux functions $\mathcal{X}_\gamma$ that reproduces the symplectic form of the hyperkähler moduli space $\mathcal{M}$. The main point is to accentuate that a continuous metric on the moduli space is in one-to-one correspondence with the validity of the wall-crossing formula. This problem contains lot of subtleties and details, many of them which we leave out of this part. This section should be viewed as an introduction to the problem and gives an intuition for its solution and main features.

We want to investigate how the solutions $\mathcal{X}_{m,e}$ behave close to a ray l. The electrical and the semiflat magnetic Darboux functions are regular in ζ, except for the the singularities at zero and the infinity point. The full solution for $\mathcal{X}_m$ is holomorphic in ζ only on the complement of the rays $l_\pm$. Consider the first integral of (5.35) and let ζ follow a path crossing the ray l_+ clockwise. When crossing the contour of integration the pole of the integrand crosses the ray as well and hence the integral may be splitted

$$\oint_{l_+} \to \oint_{l_+} + \oint_{\Sigma} \tag{6.1}$$

where Σ is a closed positively oriented curve enclosing the pole on the other side of the curve, as in figure 3. The limits when $\zeta \to l_+$ clockwise and counterclockwise respectively are denoted $(\mathcal{X}_m)^{\pm}_{l_+}$ and correspondingly for the path l_-. Due to the exponent in the solution (5.35) the contribution from the Σ-integral is multiplicative. By the residue theorem the closed contour integral is

$$\begin{aligned} &\frac{iq}{4\pi}\oint_{\Sigma}\frac{\mathrm{d}\eta}{\eta}\frac{\eta+\zeta}{\eta-\zeta}\log[1-\mathcal{X}_e(\eta)^q] = 2\pi i\frac{iq}{4\pi}\cdot\frac{\zeta+\zeta}{\zeta}\log[1-\mathcal{X}_e(\zeta)^q] \\ &= \log(1-\mathcal{X}_e(\zeta)^q)^{-q} \end{aligned} \tag{6.2}$$

and the contribution to $\mathcal{X}_m$ is thus $(1-\mathcal{X}_e(\zeta)^q)^{-q}$. Equating the expressions on both sides of the rays yields

$$(\mathcal{X}_m)^+_{l_+} = (\mathcal{X}_m)^-_{l_+}(1-\mathcal{X}_e(\zeta)^q)^{-q} \tag{6.3}$$

and the corresponding expression

$$(\mathcal{X}_m)^+_{l_-} = (\mathcal{X}_m)^-_{l_-}(1-\mathcal{X}_e(\zeta)^{-q})^{q} \tag{6.4}$$

for the path l_-. We see that these expressions resembles the behaviour of the torus coordinates $X_{m,e}$ under the KS transformation introduced in section 4.2. In the following we make this relation precise.

The functions $\mathcal{X}_m$ and $\mathcal{X}_e$ constitutes, for fixed values of a, R and ζ, a map

$$\mathcal{X} : \mathcal{M}_a \to T_a \tag{6.5}$$

where $\mathcal{M}_a$ is the torus fiber of the hyperkähler fibration $\mathcal{M} \to \mathcal{B}$ with coordinates (θ_m, θ_e). One may restate the relation, saying that $\mathcal{X}_\gamma$ is the pullback $\mathcal{X}_\gamma(\theta) = X_\gamma(\mathcal{X}(\theta))$ to the torus fiber $\mathcal{M}_a$ of the coordinates $X_{m,e}$. T_a is a complexified torus with coordinates (X_m, X_e) introduced in section 4.2. As functions on the fiber $\mathcal{X}_{m,e} = \exp[i(\theta_{m,e} + f(\theta))]$ where f is due to possible monodromy transformations around $a = 0$. They both hence take S^1-values and this is the reason for the map (6.5) to take values on a torus T_a.

The solutions $\mathcal{X}$ are valued on the torus T_a and hence the KS transformations (4.22) apply to the Darboux functions. Actually the discontinuities of $\mathcal{X}_m$ in (6.3)-(6.4) may be stated as

$$\mathcal{X}^+_m = \mathcal{K}_{(\pm q,0)}\mathcal{X}^-_m \tag{6.6}$$

at the rays $l_\pm$. This is the link between the work of Kontsevich and Soibelman and the description of the moduli space in the simplest case. The description of the Darboux coordinates over $\mathcal{M}$ has necessarily discontinuities, which are accounted for as KS transformations. So far we have considered mutually local particles and we now turn to the general case.

We now turn to the case where *not* all particles are mutually local i.e there is no duality frame where all particles may be considered electrically charged. As in the previous case the full hyperkähler metric g receives corrections from instantons originating from massive BPS states in four dimensions. The contribution from each BPS state of charge γ are expected to be weighted by the BPS index $\Omega(\gamma)$. From section 4.1 we know that these degeneracies are only piecewise constant over the base $\mathcal{B}$. Hence the instanton corrections have discontinuous jumps at some loci on the Coulomb branch and so should the full metric.

From the Lagrangian of the compactified theory we expect the metric g to be everywhere smooth, since it is specified by the holomorphic period matrix τ. The tower of instanton contributions and their discontinuities must therefore be carefully balanced so to make g smooth. The authors of [13] found that this is achieved if the BPS index obey the Kontsevich-Soibelman wall-crossing formula. This statement is at the heart of the physical interpretation of the wall-crossing formula. In the following we motivate why this has to be true.

A Riemann-Hilbert problem is the problem of finding holomorphic functions $f_+(z)$ and $f_-(z)$ on the interior and exterior respectively of a closed curve Σ. For all $z \in \Sigma$ they obey a boundary condition $\alpha(z)f_+(z) + \beta(z)f_-(z) = \gamma(z)$ for some given holomorphic functions α, β and γ.

This is the precise setup needed to describe the Darboux functions as piecewise analytic over the full manifold $\mathcal{M}$. The KS symplectomorphisms are maps

$$\mathcal{K}_\gamma : T_a \to T_a \tag{6.7}$$

which for charges that support BPS states gives contribution to the wall-crossing formula. To make contact with the contours l of the solution (5.35) for the magnetic Darboux function the subset

$$\Gamma_{a,l} = \{\gamma \in \Gamma_a \,|\, Z_\gamma(a)/\zeta \in \mathbb{R}_-\}\,, \quad \zeta \in l \tag{6.8}$$

of the local charge lattice Γ_a is defined. The rays where $\Gamma_{a,l} \neq \emptyset$ are named BPS rays. The corresponding transformation

$$S_l = \prod_{\gamma \in \Gamma a,l} \mathcal{K}_\gamma^{\Omega(\gamma)} \tag{6.9}$$

over BPS states is thus over all possible charges whose central charges line up along the ray l. If the set $\Gamma_{a,l}$ is empty then S_l is 1. Note that this is very similar to the construction of $\overset{\frown}{A}$ in section 4.2. However for $\gamma \in \Gamma_{a,l}$ all KS transformations commute, since the charges in this subset are all real multiples of each other. This gives $\langle \gamma, \gamma' \rangle = 0$ and thus the Lie algebra generators all commute from which it follows that no particular ordering of the operator product need to be stated.

The boundary condition of the Riemann-Hilbert problem over the ray l is set to

$$\mathcal{X}^{+} = S_l \mathcal{X}^{-} \tag{6.10}$$

where $\mathcal{X}^{\pm}$ are the limits of $\mathcal{X}$ when ζ approaches l clockwise or counterclockwise respectively. This is the generalisation of the discontinuity over l for the unit charge functions $\mathcal{X}_{m,e}$ in (6.6). For each ray where the central charges share their complex argument a Riemann-Hilbert problem is formulated as above. The solutions are piecewise holomorphic functions of ζ away from the rays.

So far the discussion of the solution has been about the ζ-dependence of the Darboux functions. The metric is not dependent on the twistor parameter and we want to say something about the solutions as functions on $\mathcal{M}$. Away from the marginal stability walls in $\mathcal{B}$ the solutions are continuous since the BPS spectrum is constant at these loci. Let a_0 be any point on the wall and consider the case when $a \in \mathcal{B}$ approaches this wall from one side. The BPS rays corresponding to charges $\gamma = m\gamma_1 + n\gamma_2$ will rotate towards each other eventually approaching a single ray while keeping their mutual ordering. The Riemann-Hilbert discontinuity of $\mathcal{X}$ at the wall is thus

$$\overset{\frown}{A} = \overset{\frown}{\prod_{\gamma = m\gamma_1 + n\gamma_2}} \mathcal{K}_{\gamma}^{\Omega(\gamma, a_+)} \qquad m,\, n > 0 \tag{6.11}$$

and if the limit $\lim_{a \to a_0} \mathcal{X}$ exists from this side of the wall it is a solution to the Riemann-Hilbert problem at the ray l. If one makes the same argument, but consider a approaching the wall from the other side gives a analogous discontinuity, but with the product in the reverse order. $\mathcal{X}$ is continuous at a_0 if the two limits agree at this point and is thus the solution to the same Riemann-Hilbert problem. Recall that the wall-crossing formula (4.19) is precisely this statement if $\mathcal{X}$ is continuous over the wall of marginal stability.

The full instanton corrected Darboux functions are solutions to the integral equation

$$\mathcal{X}_{\gamma}(\zeta) = \mathcal{X}_{\gamma}^{sf} \exp\Big[-\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma', a) \langle \gamma, \gamma' \rangle \int_{l_{\gamma'}} \frac{\mathrm{d}\eta}{\eta} \frac{\eta + \zeta}{\eta - \zeta} \log\{1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\eta)\} \Big],$$

which is a crucial part of the main result of Gaiotto, Moore and Neitzke in [13]. If the Kontsevich-Soibelman wall-crossing formula is satisfied by the BPS index there is an piecewise analytic solution in a and ζ for the Darboux functions over $\mathcal{M}$. It follows that the metric constructed out of the symplectic form on the hyperkähler manifold is continuous. The transformations $\mathcal{K}_\gamma$ over the wall are all symplectomorphisms and hence they are compatible with differentiable transition functions gluing the charts of $\mathcal{M}$ together to a manifold.

Appendices

A Hyperkähler Geometry

In appendix A.1 a brief background on hyperkähler geometry is presented. The basic ingredients and relations are introduced on a working level for application in section 5. In A.2 the construction of a hyperkähler geometry via Darboux coordinates are introduced. Making use of the full Riemann sphere of complex structures and the symplectic form on the manifold allows for Darboux coordinate functions $\mathcal{X}(u,\zeta)$ with certain properties, all used in section 5, and presented in part two of this appendix. See e.g [17].

A.1 Foundations

A hyperkähler manifold $\mathcal{M}$ is a Riemannian manifold of $4n$ real dimensions. It carries three covariantly constant complex structures J_1, J_2 and J_3 giving three different Kähler forms ω_1, ω_2 and ω_3 respectively. The complex structures are (1,1) tensor fields obeying the quaternion algebra

$$J_a J_b = -\delta_{ab} + \varepsilon_{abc} J_c \,. \tag{A.1}$$

The space of Kähler metrics is infinite dimensional since a Kähler metric may be transformed to another Kähler metric by adding any function $f(z,\bar{z}) \in C^\infty$ to the Kähler potential. The set of hyperkähler geometries are much more restricted, since a Kähler transformation has to preserve the compatibility with all complex structures and it is known that the set of isometry classes is finite [16]

For any point $(a^1, a^2, a^3) \in S^2$ we may form the linear combination $a^i J_i$ which is also a covariantly constant complex structure on $\mathcal{M}$ since

$$(a^i J_i)^2 = a^i a^j J_i J_j = -a^i a^i = -1 \tag{A.2}$$

and where the cross terms vanish due to antisymmetry. The map

$$a^1 = i\frac{\bar{\zeta}-\zeta}{1+\bar{\zeta}\zeta} \quad a^2 = -\frac{\bar{\zeta}+\zeta}{1+\bar{\zeta}\zeta} \qquad a^3 = \frac{1-\bar{\zeta}\zeta}{1+\bar{\zeta}\zeta} \tag{A.3}$$

realises the isomorphism $S^2 \approx \mathbb{CP}^1 = \mathbb{P}$. Hence we have a family of complex structures and corresponding Kähler forms, parametrised by a complex parameter

$\zeta \in \mathbb{P}$ and

$$J^{\zeta} = -\frac{1}{1+\bar{\zeta}\zeta}\big(i(\bar{\zeta}-\zeta)J_1 - (\bar{\zeta}+\zeta)J_2 + (1-\bar{\zeta}\zeta)J_3\big) \qquad \text{(A.4)}$$
$$= -\frac{1}{1+\bar{\zeta}\zeta}\big(i\bar{\zeta}J_+ - i\zeta J_- + (1-\bar{\zeta}\zeta)J_3\big)$$

for $J_\pm = J_1 \pm iJ_2$. We will consider the form

$$\vartheta(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i\zeta}{2}\omega_- \qquad \text{(A.5)}$$

where correspondingly $\omega_\pm = \omega_1 \pm i\omega_2$. At the points 0 and ∞ of the Riemann sphere this form must be multiplied by ζ and ζ^{-1} respectively to make sense. Hence ϑ is a section of the line bundle $\mathcal{O}_{\mathbb{P}}(2)$. This (1,1)-form is symplectic by construction and since the complex structures are covariantly constant it is closed. It is holomorphic with respect to the complex structure J^ζ i.e for each fixed ζ we have

$$J^{\zeta}\vartheta(\zeta) = i\vartheta(\zeta) \qquad \text{(A.6)}$$

as seen by taking

$$\begin{aligned} J^{\zeta}\vartheta(\zeta) &= -\frac{1}{1+\bar{\zeta}\zeta}\big(i\bar{\zeta}J_+ - i\zeta J_- + (1-\bar{\zeta}\zeta)J_3\big)\big(-\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i\zeta}{2}\omega_-\big) \qquad \text{(A.7)}\\ &= -g\frac{1}{1+\bar{\zeta}\zeta}\big[i\bar{\zeta}J_+J_3 + \frac{1}{2}\bar{\zeta}\zeta J_+J_- - \frac{1}{2}J_-J_+ - i\zeta J_-J_3 \\ &\qquad - \frac{i}{2}\zeta^{-1}(1-\bar{\zeta}\zeta)J_3J_+ - \frac{i}{2}\zeta(1-\bar{\zeta}\zeta)J_3J_- - (1-\bar{\zeta}\zeta)\big] \\ &= -\frac{1}{1+\bar{\zeta}\zeta}\big[(-\bar{\zeta} - \frac{1}{2}\zeta^{-1}(1-\bar{\zeta}\zeta))\omega_+ + (-\zeta + \frac{1}{2}\zeta(1-\bar{\zeta}\zeta))\omega_- \\ &\qquad - i(1+\bar{\zeta}\zeta)\omega_3\big] \\ &= i\vartheta(\zeta) \end{aligned}$$

where we note that the algebra gets modified a bit in the $J_\pm$ structures. The existence of a holomorphic symplectic form may be restated as the existence of an isomorphism between the holomorphic tangent and cotangent space. As this must be non-degenerate we have for $m = \dim T^{(1,0)}\mathcal{M}$

$$0 \neq \det\vartheta_{ij} = \det(-\vartheta_{ji}) = (-1)^m \det\vartheta_{ji} \;\Rightarrow\; m \in 2\mathbb{Z} \qquad \text{(A.8)}$$

and the even dimension of both the holomorphic and the anti-holomorphic tangent spaces gives that $\dim_{\mathbb{R}}\mathcal{M}$ is necessarily a multiple of four.

A hyperkähler manifold is a holomorphic symplectic manifold as a complex manifold in the complex structure J^ζ. This motivates an example of a hyperkähler manifold - the K3 surface. It is a Calabi-Yau 2-fold and hence it carries a non-vanishing holomorphic top-form, which in this low-dimensional case is the holomorphic symplectic form of the hyperkähler manifold. Note that the K3 surface is a compact manifold in opposition to the hyperkähler manifold considered in this thesis. Compactness of the K3 follows from the description of the manifold e.g as a the vanishing locus of a degree four polynomial in $\mathbb{P}^3$, which is compact.

A.2 Twistor Space Construction of a Hyperkähler Metric

The symplectic form (A.5) suggests to consider the fibration of $\mathcal{M}$ over $\mathbb{P}$ with the simple projection $\pi(u, \zeta) = \zeta$ for $u \in \mathcal{M}$. This is the twistor space $\mathcal{Z}$ of the hyperkähler manifold which topologically is the product $\mathcal{M} \times \mathbb{P}$. The symplectic form is a holomorphic 2-form section valued in $\Omega^2_{\mathcal{M}_\zeta} \otimes \mathcal{O}_{\mathbb{P}}(2)$, giving a holomorphic symplectic form for each fiber $\mathcal{M}_\zeta = \pi^{-1}(\zeta)$. The line bundle factor is due to the polynomial transition functions of order 2 over $\mathbb{P}$.

The twistorial construction of a hyperkähler metric uses a set of holomorphic functions $\mathcal{X}_{\gamma^i}(u, \zeta)$ on the twistor space. These are related to the choice of Darboux coordinates by exponentiation and may be called Darboux functions defined such that the holomorphic symplectic form is

$$\vartheta(\zeta) = K \langle \mathrm{d} \log(\mathcal{X}), \mathrm{d} \log(\mathcal{X}) \rangle \tag{A.9}$$

for some constant K and the wedge product implicitly assumed in the symplectic pairing. In the physical case we consider the functions $\mathcal{X}_{\gamma^i}$ enumerated by the magnetic-electric charge γ^i and in this case the constant K is supported by a antisymmetric inverse form $\alpha^{ij} = \langle \gamma^i, \gamma^j \rangle^{-1}$ as

$$\vartheta(\zeta) = K \alpha^{ij} \, \mathrm{d}(\log \mathcal{X}_{\gamma^i}) \wedge \mathrm{d}(\log \mathcal{X}_{\gamma^j}) \tag{A.10}$$

At the end of this section we describe how the metric is obtained from this form.

The functions $\mathcal{X}_\gamma$ are assumed to have some basic properties compatible with the structure of a twistor space [16]. The enumerating charges add up under multiplication $\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$. This multiplicativity allow for unit magnetic and electric charge solutions to be combined into a solution of any charge.

There is a reality condition on how $\mathcal{X}$ transforms under the antipodal map $\zeta \mapsto -1/\bar{\zeta}$ on $\mathbb{P}$. The Darboux functions obey $\bar{\mathcal{X}}_{-\gamma}(\zeta) = \mathcal{X}_\gamma(-1/\bar{\zeta})$ under this transformation. In the solution of the wall crossing problem as in [13] one has to assume that the Darboux functions are at least piecewise holomorphic in u, which is enough to make $\vartheta(\zeta)$ globally defined. This is equivalent to the statement that $\mathcal{X}$ is solution to the Cauchy-Riemann equations on $(\mathcal{M},g)$ which take the form for $u \in \mathcal{M}$ and for any fixed $\zeta \in \mathbb{P}$

$$\partial_u \mathcal{X} = \mathcal{A}^k_u \mathcal{X} \tag{A.11}$$

where $\mathcal{A}^k$ are first order differential operators acting on the torus fiber $\mathcal{M}_u \to \mathcal{M} \to \mathcal{B}$. We do not get into the details of the Cauchy-Riemann equations on $\mathcal{M}$ here but refer to [13].

Once computed, the symplectic form $\vartheta(\zeta)$ may be used to recover the metric by reading of the coefficients of order 1, -1 and 0 in ζ. That is, we pick out the three Kähler forms ω_i and then, since the inverse of a complex structure is the negative of it self

$$\omega_{2\,ab}(\omega_1^{-1})^{bc} = g_{ad}J^d_{2\,b}(gJ_1)^{-1\,bc} = -g_{ad}J^d_{2\,b}g^{ec}J^b_{1\,e} = g_{ad}g^{ce}J^d_{3\,e} = J^c_{3\,a} \tag{A.12}$$

The metric is then obtained from ω_3 which is obtained by lowering the indicies of J_3 by the metric. By acting with the inverse complex structure from the right one gets $g_{ab} = -\omega_{3\,ad}J^d_{3\,b}$. Note that since ϑ is a (2,0)-form in complex structure J^ζ so is the Kähler form ω_3. However ω_3 is of type (1,1) in it's corresponding structure J_3 which gives g as the Kähler metric on $\mathcal{M}$.

B Special Geometry and Riemann Surfaces

The rigid special geometry structure also occur in the analysis of Riemann surfaces and we briefly introduce this framework here [9] [23] . A readable introduction is found in [28]. Considering families of Riemann surfaces there is a moduli space of these manifolds analogous to the field theory moduli space. The moduli space is parametrised by a set of variables u^i. The period matrix, central in the physical picture, have a natural interpretation in terms of the surface homology. For a Riemann surface Σ of genus g there is a homology basis of 1-cycles α^A and β_B obeying

$$\alpha^A \cdot \beta_B = -\beta_B \cdot \alpha^A = \delta^A_B \qquad \alpha^2 = \beta^2 = 0 \qquad A, B = 1, \ldots, g \tag{B.1}$$

which repeats the pattern of the duality frame introduced above. By Poincaré duality this basis corresponds to g independent, holomorphic 1-forms η_i on Σ. We may therefore introduce the periods of these 1-forms with respect to the homology basis as

$$\Big(\int_{\alpha^A}\eta_i, \int_{\beta^B}\eta_i\Big) \tag{B.2}$$

which for each i is a symplectic vector with $2g$ components. Since only half of them is linearly independent they are related by a transformation τ as

$$\int_{\beta^B}\eta_i = \tau_{BA}\int_{\alpha^A}\eta_i \ . \tag{B.3}$$

Riemann's first and second relation [14] implies that $\tau_{AB} = \tau_{BA}$ and that $\Im\tau > 0$. If there exists a meromorphic one-form $\lambda(u)$ (the Seiberg-Witten differential) such that it is holomorphic in u^i over the moduli space and

$$\int_{\alpha^A}\eta_i = \frac{\partial}{\partial u^i}\int_{\alpha^A}\lambda(u) \quad , \qquad \int_{\beta^B}\eta_i = \frac{\partial}{\partial u^i}\int_{\beta^B}\lambda(u) \tag{B.4}$$

then locally there is a solution $\mathcal{F}(a)$ such that a basis (dual basis) $\{a_A\}$ ($\{\tilde{a}_B\}$) for the moduli space of Σ is

$$a^A = \int_{\alpha^A}\lambda(u) \qquad \tilde{a}_B = \frac{\partial\mathcal{F}(a)}{\partial a^B} = \int_{\beta^B}\lambda(u) \tag{B.5}$$

Inserting the basis in (B.4) and then using (B.3) implies that

$$\frac{\partial}{\partial u^i}\frac{\partial\mathcal{F}(a)}{\partial a^B} = \tau_{BA}\frac{\partial a_A}{\partial u^i} \qquad \Rightarrow \qquad \frac{\partial^2\mathcal{F}(a)}{\partial a^A\partial a^B} = \tau_{AB} \tag{B.6}$$

and the transformation τ is thus analogous to the period matrix we already know from the prepotential.

One may also, if choosing the one-form basis $\eta_A = \mathrm{PD}(\alpha^A)$ (PD denotes Poincaré dual) rewrite (B.3) as

$$\int_{\beta^B}\mathrm{PD}(\alpha^A)(u) = \tau_{BC}(u)\int_{\alpha^C}\mathrm{PD}(\alpha^A)(u) = \tau_{BC}(u)\delta^A_C = \tau_{BA}(u) \tag{B.7}$$

which represents the period matrix as the homology intersection form, holomorphically varying over the moduli space of Σ. The Poincaré dual carries the dependence of the moduli parameters, and this viewpoint somewhat justifies the name period matrix, representing it as the periods of the homology basis.

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