Estimation of axial load in timber beams using resonance frequency analysis

Master’s Thesis in the Master’s Programme Structural Engineering and Building Performance Design

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Department of Civil and Environmental Engineering
Division of Structural Engineering
Steel and Timber Structures
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg, Sweden 2012
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Cover:
The picture shows frequency measurements on timber beams under different conditions. (Photo: Steve Laux, 2012)

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ABSTRACT

During the last years, the media reported about several timber roof collapses all across Europe, especially during rough winters. This is a serious problem that bears the risk of many fatal incidents and needs to be encountered in order to improve the safety of structures. One solution could be to estimate the axial loads in the structural members using resonance frequency analysis. The results could then be used to assess the safety of timber structures and to decide about their continued use or temporary closure. While this method had already been reported for steel structures, only little research had been carried out in the area of timber structures.

The aim of this project was therefore to investigate if it is possible to estimate the axial loads in timber beams using resonance frequency analysis and if yes, what are the precision requirements for the material properties and the measured frequencies. This was achieved by performing transversal frequency measurements on 32 timber specimens and an aluminium bar under tension. The latter hereby served as homogeneous reference for better interpretation of results. The two first frequencies, together with different values for the E-modulus were then used to estimate the axial load and the rotational stiffness at the boundaries. The numerical model behind the calculations was based on Timoshenko beam theory, allowing to include effects of shear deformations and rotary inertia. The material properties of the specimens were previously determined by static and dynamic tests. Finally, a sensitivity analysis was carried out to investigate the influence of errors in input parameters on the final results.

The best results were obtained using the E-modulus derived from transversal vibration tests and showed a mean error ranging from 7.6% to 46.6%, where the results generally improved for higher loads. When using the E-moduli from longitudinal vibration tests, the mean errors increased to 12.4% to 89.5%. It was also attempted to use the static E-modulus for the calculations, which led however to incorrect results. Dynamic values should be used for the parameter estimation with the presented resonance frequency method. The results of the sensitivity analysis showed that the sensitivity of the estimated axial load decreases for higher load levels, which could also be observed in the test results. The most influential parameters on the quality of the results were the measured frequencies and the clear beam length, followed by the density and the E-modulus.

Key words: resonance frequency analysis, timber beams, axial load, non-destructive testing, Timoshenko beam theory, dynamic E-modulus, modal analysis, parameter estimation
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Preface

This study investigated the possibility of using resonance frequency analysis for the estimation of axial loads in timber beams. The project was carried out as cooperation between the Divisions of Structural Engineering, Steel and Timber Structures of ETH Zurich, Switzerland and Chalmers University of Technology, Sweden. All tests were performed in the laboratory of the Division of Structural Engineering at Chalmers University of Technology.

The project was carried out under the supervision of PhD-student Thomas Kruglowa from Chalmers as well as Daniel Sandin from Midapro. Furthermore, much valuable input and support was also received from Michael Klippel, PhD-student at ETH Zurich. The examiners from the different universities were Professor Robert Kliger from Chalmers and Professor Andrea Frangi from ETH Zurich. I would like to thank all these participants for their help and support during this research.

Finally, it should be noted that the tests could never have been conducted without the sense of high quality and professionalism of the laboratory staff. Special thanks are directed to Lars Wahlström.

Gothenburg, June 2012

Steve Laux
### Notations

In the notation table, all variables occurring in the report are listed alphabetically.

#### Roman upper case letters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>Area of cross section</td>
</tr>
<tr>
<td>$A$</td>
<td>Equation coefficient</td>
</tr>
<tr>
<td>$B$</td>
<td>Equation coefficient</td>
</tr>
<tr>
<td>$C$</td>
<td>Equation coefficient</td>
</tr>
<tr>
<td>$C_i$</td>
<td>Equation coefficients</td>
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<tr>
<td>$C'_i$</td>
<td>Equation coefficients</td>
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<tr>
<td>$D$</td>
<td>Equation coefficient</td>
</tr>
<tr>
<td>$D_i$</td>
<td>Equation coefficients</td>
</tr>
<tr>
<td>$E$</td>
<td>E-modulus</td>
</tr>
<tr>
<td>$E_{0,\text{mean}}$</td>
<td>Mean E-modulus parallel to the grain for timber</td>
</tr>
<tr>
<td>$E_{i,b}$</td>
<td>Dynamic E-modulus from transversal vibration tests</td>
</tr>
<tr>
<td>$E_{i,L}$</td>
<td>Dynamic E-modulus from longitudinal vibration tests</td>
</tr>
<tr>
<td>$E_{\text{static}}$</td>
<td>Static E-modulus</td>
</tr>
<tr>
<td>$G$</td>
<td>G-modulus</td>
</tr>
<tr>
<td>$G_i$</td>
<td>Dynamic G-modulus from transversal vibration tests</td>
</tr>
<tr>
<td>$G_{\text{mean}}$</td>
<td>Mean G-modulus for timber</td>
</tr>
<tr>
<td>$H$</td>
<td>Factor in the continuous numerical model</td>
</tr>
<tr>
<td>$I$</td>
<td>Moment of inertia</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Polar moment of inertia</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Load levels for the static four-point bending test</td>
</tr>
<tr>
<td>$K_t$</td>
<td>Torsional constant</td>
</tr>
<tr>
<td>$L$</td>
<td>Beam length or clear span length</td>
</tr>
<tr>
<td>$L_{\text{max}}$</td>
<td>Maximum clear length for the beam in the tensile machine</td>
</tr>
<tr>
<td>$L_{\text{min}}$</td>
<td>Minimum clear length for the beam in the tensile machine</td>
</tr>
<tr>
<td>$M$</td>
<td>Bending moment</td>
</tr>
<tr>
<td>$M_t$</td>
<td>Torsional moment</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of terms in the Rayleigh-Ritz method</td>
</tr>
<tr>
<td>$S$</td>
<td>Axial load (positive in tension)</td>
</tr>
<tr>
<td>$S_E$</td>
<td>Euler buckling load for a simply supported beam</td>
</tr>
<tr>
<td>$S_{\text{est}}$</td>
<td>Estimated axial load</td>
</tr>
<tr>
<td>$S_{\text{max}}$</td>
<td>Maximum tensile load</td>
</tr>
<tr>
<td>$T^*$</td>
<td>Reference kinetic energy</td>
</tr>
<tr>
<td>$T_{\text{max}}$</td>
<td>Maximum kinetic energy</td>
</tr>
<tr>
<td>$V$</td>
<td>Shearing force</td>
</tr>
<tr>
<td>$V_{\text{max}}$</td>
<td>Maximum potential energy</td>
</tr>
<tr>
<td>$X$</td>
<td>Normal function or modal shape of a vibrating beam</td>
</tr>
<tr>
<td>$X_i$</td>
<td>Modal shapes of a vibrating beam</td>
</tr>
<tr>
<td>$Y$</td>
<td>Modal shape of a transversally vibrating beam</td>
</tr>
<tr>
<td>$Z$</td>
<td>Factor in the continuous numerical model</td>
</tr>
<tr>
<td>Greek letters</td>
<td>LaTeX Formulas</td>
</tr>
<tr>
<td>---------------</td>
<td>----------------</td>
</tr>
<tr>
<td>$a$</td>
<td>$\frac{E}{\rho}$</td>
</tr>
<tr>
<td>$a_i$</td>
<td>$\frac{G \cdot K_i}{(\rho \cdot I_p)}$</td>
</tr>
<tr>
<td>$b$</td>
<td>Beam height</td>
</tr>
<tr>
<td>$b_i$</td>
<td>Factor in the continuous numerical model</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Coefficients in Rayleigh-Ritz method</td>
</tr>
<tr>
<td>$f$</td>
<td>Resonance frequency</td>
</tr>
<tr>
<td>$f_i$</td>
<td>Resonance frequencies for different vibration modes</td>
</tr>
<tr>
<td>$f_{ib}$</td>
<td>Transversal resonance frequencies for different vibration modes</td>
</tr>
<tr>
<td>$f_{it}$</td>
<td>Torsional resonance frequencies for different vibration modes</td>
</tr>
<tr>
<td>$f_{li}$</td>
<td>Longitudinal resonance frequencies for different vibration modes</td>
</tr>
<tr>
<td>$f_{t,0,k}$</td>
<td>Characteristic tensile strength parallel to the grain</td>
</tr>
<tr>
<td>$f_{t,0,max}$</td>
<td>Maximum tensile strength parallel to the grain from tensile tests</td>
</tr>
<tr>
<td>$f_u$</td>
<td>Ultimate tensile strength</td>
</tr>
<tr>
<td>$h$</td>
<td>Beam height</td>
</tr>
<tr>
<td>$i$</td>
<td>Natural integer</td>
</tr>
<tr>
<td>$k$</td>
<td>Rotational stiffness at boundary</td>
</tr>
<tr>
<td>$k$</td>
<td>$\frac{\sqrt{\omega/\alpha}}}$</td>
</tr>
<tr>
<td>$k_i$</td>
<td>Translational and rotational stiffness at boundary</td>
</tr>
<tr>
<td>$k_i$</td>
<td>$\frac{\sqrt{\omega_i/\alpha}}}$</td>
</tr>
<tr>
<td>$k_{est}$</td>
<td>Estimated rotational stiffness at boundary</td>
</tr>
<tr>
<td>$k_s$</td>
<td>Shear factor, in general $k_s = 5/6$</td>
</tr>
<tr>
<td>$l$</td>
<td>Beam length, span or finite element length</td>
</tr>
<tr>
<td>$n$</td>
<td>Natural integer</td>
</tr>
<tr>
<td>$p$</td>
<td>Factor in the continuous numerical model</td>
</tr>
<tr>
<td>$s$</td>
<td>Factor in the continuous numerical model</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$u$</td>
<td>Longitudinal displacement</td>
</tr>
<tr>
<td>$v$</td>
<td>Transversal displacement</td>
</tr>
<tr>
<td>$w$</td>
<td>Moisture content or deflection</td>
</tr>
</tbody>
</table>
$w_i$  Moisture content or deflection in different points

$x, y$  Main directions in coordinate system

**Greek letters**

$\Delta$  Factor in the continuous numerical model

$\Theta$  $Y$ or $\Psi$

$\Psi$  Modal shape of a transversally vibrating beam

$\beta$  Angle of shear

$\varepsilon$  Strain

$\theta$  Rotation angle

$\phi$  $y$ or $\psi$

$\lambda_1$  Factor in the continuous numerical model

$\lambda_2$  Factor in the continuous numerical model

$\rho$  Density

$\rho_k$  Density

$\sigma$  Stress level

$\phi$  $k_s GA$ for the continuous numerical model

$\phi$  $12EI/(k_s GAL^2)$ for the discrete numerical model

$\psi$  Angle of rotation of cross section

$\omega$  Angular resonance frequency

$\omega_i$  Angular resonance frequencies for different vibration modes

**Signs and mathematical symbols**

$\%$  Percentage

$\int$  Integral

$[\ ]$  Matrix/vector parentheses

$\cos$  Cosinus function

$\sin$  Sinus function

$\tau$  Transposition of a matrix

$\cdot$  First derivate of a function

$\cdot\cdot$  Second derivate of a function

$\partial^i$  $i^{th}$ order derivate in multiple calculi

$d$  Infinite small increment

$\partial$  Derivate in multiple calculi

**Matrix notations (bold style)**

$C$  Coefficient vector

$\mathbf{C}$  Damping matrix

$CM$  Matrix for continuous numerical model

$K$  Stiffness matrix

$K^e$  Element stiffness matrix

$K^e_L$  Part of element stiffness matrix accounting for the axial load

$K^e_S$  Part of element stiffness matrix accounting for the strain

$M$  Mass matrix

$M^e$  Element mass matrix

$M^e_R$  Part of element mass matrix accounting for effects of rotary inertia
$M_T$  Part of element mass matrix accounting for effects of translatory inertia
$f(t)$  Time dependent force vector
$\nu$  Vector of displacement
$\phi$  Eigenvector to the eigenvalue problem

**Abbreviations**

°C  Degree Celsius
CEN  Comité Européen de Normalisation
E-modulus  Modulus of elasticity
FE(M)  Finite element (method)
FFT  Fast Fourier Transformation
FRF  Frequency response functions
G  Giga-
Glulam  Glued-laminated timber
G-modulus  Shear modulus
M  Mega-
N  Newton
OSB  Oriented Strand Boards
Pa  Pascal
TC  Technical Committee
k  kilo-
m  Meter
m  milli-
s  Second
1 Introduction

1.1 Background

Over the last years, the media reported about several roof collapses across Europe that occurred mostly during winter times after high snow precipitations. The fact that roof structures failed under the loads of snow and ice cost several human lives and caused many injuries. In January 2006, the Bad Reichenhall Ice Rink in Bavaria, Germany collapsed after continuing snowfalls. The failure of the 36-year-old roof claimed 15 lives and caused 34 injured. In February of the same year, the collapse of a market hall in Moscow claimed the lives of 50 Caucasian guest workers. These are only a few examples of collapses that occurred during this particularly strong winter. In February 2009, the roof of a three-year-old school sports hall in St. Gallen, Switzerland collapsed just two hours before start of classes. In January 2012 a one-year old roof structure of an ice-hockey hall in Slovakia collapsed during a match. All these examples show the severity of this safety issue that exists in many different countries and damages the public confidence in the construction sector.

The main reasons for the collapses consist in general of high snow loads, combined with moisture effects or wrong assumptions considering the structural behaviour. Branco (2010) stated that the “assessment of constructed timber trusses shows various differences in their structural model” and that “visual inspections are the basis of any analysis of timber structures” to ensure the detection of decay or mechanical damages. In the case that a snow removal does not occur, it is therefore important to assess the real behaviour of the structure that can considerably derive from the assumed model. Also a condition assessment of the structural members should be carried out to assure a realistic evaluation, especially for older structures. An example for this is further shown in Branco (2010), where two timber trusses from a construction situated near Caldonazzo Lake, Italy were analysed. The over 70 year old trusses were disassembled and reconstructed in the laboratory, where their real structural behaviour was assessed by visual grading as well as dynamic and static testing. Both trusses showed asymmetric behaviour even under symmetric loading, which is a clear deviation from the structural model. All this indicates that a lack of information about the structural behaviour of roofs can lead to incorrect or even unnecessary reinforcements or replacements.

The problem of uncertainties in structural integrity is furthermore enhanced by uncertainties in future snowfall development affected by climate change. Strasser (2008) indicates that climate change could have a harmful effect on snow loads, at least in colder regions. Even if higher temperatures will result in less frequent snowfall, extreme weather events and precipitations are likely to become more frequent. In regions that will remain cold enough these precipitations will occur in form of strong snowfalls, which most likely will exceed today’s design values. In other words, instead of many moderate snowfalls it is probable that there will be fewer snowfalls with higher peaks. It is therefore important to know if today’s structures can bear these increased loads.

The solution to these problems could lie in non-destructive testing (NDT) methods, namely frequency based identification methods. In contrast to strain gauges, these methods allow to estimate actual stress levels at any time and without the need of a reference state. This means that stresses inside the structural members caused by self-weight, creep or changes in temperature or moisture are included in the results, which
allows a realistic assessment of the structural behaviour and the detection of the most loaded, hence most vulnerable members. With this information, a better computer model can be created from which a more realistic assessment of the load bearing capacity can be drawn. This gives for example answers to the questions “What snow load levels can the roof bear?” and “When should a structure be temporary closed and reinforced for safety reasons?”.

Livingston (1994) was one of the first to introduce a method based on frequency measurements to estimate the axial load in members. His model furthermore allowed identifying the rotational restraint at the supports of prismatic beams. The principle is to transversally excite a beam under axial force and to determine the first frequencies with the help of an accelerometer linked to a computer software. Resonance frequency analysis then allows deriving the axial loads and boundary conditions from these frequencies. The method was tested on a square steel rod under tension to verify the results. Even though it could be observed that the axial loads were in general overestimated, the method showed promising results, especially for high-tension forces. The same technique was also used by Amabili (2010) to determine the in-situ tensile forces in steel rods being part of ancient masonry buildings in Italy. The results could then be used as a basis to decide whether the bars needed to be replaced or not. In Maille (2008) the method was used to assess the safety of an old breeding barn’s roof structure. The testing led to the axial loads in the steel members of the truss construction, which were then compared to the ones predicted by different computer models. The most accurate model could then be used to determine the bearing capacity of the structure. As these works show, the technique has been approved for steel, which is a homogeneous material with isotropic properties. However it seems that no experiments have yet been carried out to verify the method for inhomogeneous materials like timber, on which is the focus in this thesis.

Since one of the main application fields of timber products are roof structures and since the exact material behaviour is not entirely clear, a reliable non-destructive testing method would represent a major contribution to the safety assessment of existing structures.

1.2 Aim of the thesis

The aim of this thesis was to investigate the possibility of estimating axial loads in timber members by means of resonance frequency analysis. For promising results, this method could be used in the future to assess the safety of timber structures by determining the actual stress of state in the loaded members. The main difference to steel structures is that the material properties of timber are subject to a large spread given by its inhomogeneity and natural growth. It is therefore important to analyse what precision is required for the use of this method and what parameters have the highest influence on the estimation of the axial load.

1.3 Method

Based on a literature study carried out on resonance frequency testing for the determination of axial forces, the framework for the specimen dimensions and the test setup was elaborated. Static and dynamic testing was then performed on 32 timber specimens of Norway spruce to determine their material properties. Furthermore, the
same tests were carried out on an aluminium bar serving as reference for homogeneous material. Finally, transversal frequency measurements were performed on the specimen under different load levels. The collected data was then evaluated using Timoshenko beam theory to estimate boundary conditions and axial load. A sensitivity study was carried out to analyse the influence of the different input parameters on the estimation of the axial load and the boundary conditions.

1.4 Limitations
The main limitations in this research were a lack of time and available laboratory space and equipment. This led to small-scale experiments on 32 timber specimens of identical dimensions and an aluminium bar. No moisture variation or long-term effects could be considered. The available machine limited the research to tensile loads only.

1.5 Outlines
In the following, the contents of the different chapters and their chronological order are presented.

1. **Introduction**: This chapter gives the necessary background information as well as the aims of the thesis. Moreover, the methods to the respective aims are explained.

2. **Building material timber**: In this chapter, the material properties of timber are discussed. Furthermore, non-destructive methods for timber grading are explained.

3. **Determination of axial load and boundary conditions in slender beams**: A literature review is presented on resonance frequency analysis for the purpose of parameter estimation. Conclusions are drawn on how to chose the specimen size and test framework.

4. **Vibration theory**
The theory behind different types of beam vibrations is explained and the according assumptions are listed. The difference between Euler and Timoshenko theory is explained.

5. **Numerical modelling and parameter estimation**: In this chapter, different numerical models are presented and compared. The parameter estimation used in later data processing is explained. Finally, a sensitivity analysis is carried out to analyse the sensitivity of the axial load and the boundary conditions with regard to measured input parameters.

6. **Modal analysis**: This chapter explains the difference between theoretical and experimental modal analysis. The most important aspects of data acquisition and data processing are illustrated.
7. **Determination of material properties**: The specimens are presented along with information on the test equipment and the data processing. The material properties of each beam are determined using static and dynamic testing.

8. **Tension tests**: The tensile machine and setup of the final tests are explained. The results of the frequency measurements as well as the according estimation of axial loads and boundary conditions are presented. This is followed by a comparison and discussion of the results for timber and aluminium. Finally, tensile tests on some specimens are presented to determine their ultimate tensile strength.

9. **Conclusions and further research**: This chapter presents the final conclusions made from the data acquisition and the parameter analysis. In addition, suggestions for further research are made.

10. **References**: An alphabetical summary of the used literature is listed in this chapter.

11. **Appendices**: This chapter contains a summary of all test results, frequency response function plots and Matlab codes used in this paper.
2 Building material timber

The demand for construction timber has been rising considerably during the last decades. One main reason is the new awareness with regard to the environment caused by the climate change and resource scarcity. Timber meets this new requirement by being a naturally regrowing material with low energy consumption during the whole life cycle. Another reason for the raising use of timber materials is the high level of prefabrication. The different parts can be produced in indoor workshops prior to construction. This ensures high quality standards and weather independence. After the prefabrication, the different parts can then be transported on site and assembled in a quick and precise way.

A major disadvantage of timber is however its anisotropy. Unlike steel or other homogeneous materials, timber has different material properties in different directions, which requires much care and consideration during the design phase. This is given by its cell structure, which is illustrated in more detail in the next paragraph, together with the material properties of timber.

2.1 Characteristics of timber

When analysing the cell structure of wood, one has to distinguish softwood (coniferous wood) and hardwood (deciduous wood). From the cell structure illustrations in Figure 2-1, it can be seen that softwood has a more uniform composition parallel to the grain than hardwood, where the grain is penetrated by vessels variable in size and shape. This leads to a higher variation in material properties.

![Figure 2-1](image1.png)

It is however true for all types of clear wood, that the stiffness and strength properties generally increase with higher densities. Also the stiffness and strength properties are always considerably higher in grain direction than perpendicular to the grain. Figure 2-2 shows the variation of tensile and compression strength with the angle variation from the grain direction for strength class C24 according to SIA (2003). It is obvious that the strength perpendicular to the grain is very low, even negligible for tension, compared to the one in grain direction. This is also one of the most common failure modes observed in practice. The E-modulus for the same strength class is around 35 times higher in the grain direction than perpendicular to it SIA (2003).
The strength of a timber beam also depends on the loading mode, i.e. moment, tension, compression or shear. The according strengths are usually determined by direct testing methods under the assumption of elastic behaviour. Beams under tension generally show a brittle failure, while some plastic deformations are possible for compression forces, due to the buckling of the fibres.

The good properties of clear wood in grain direction are significantly reduced by random defects such as knots, oblique fibre orientation and resin pockets. The varying number and location of these defects cause a large spread in properties shown in Figure 2-3.

This spread let to the development of Engineered wood products (EWP), which in general aim to reduce the high variability of timber. For glue-laminated timber for instance, several lamellas of solid wood are glued together. This improves the material properties since local weak zones can redistribute stresses to adjacent stronger regions. Another approach is to shred the wood to smaller pieces and

Figure 2-2 Tensile and compression strength of C24 for different angles $\alpha$ to the grain direction SIA (2003)

Figure 2-3 Probability density functions for solid wood and glulam Thelandersson and Larsen (2003)
reassembling it with the use of adhesives as it is done for OSB panels. Figure 2-4 shows some examples of EWPs.

![Figure 2-4 Examples of Engineered wood products Bayledlumber.com (2012)](image)

According to Thelandersson and Larsen (2003), these are some of the main advantages of EWPs:

- Size is not limited by tree dimensions.
- Reduced effect of defects by distributing them over the whole beam.
- Reduced anisotropy
- Higher dimensional stability and tolerances

One should however also note that EWPs are generally more expensive than solid wood and that adhesives like glue are necessary for their production. The latter somewhat decrease the ecological character of the timber use.

Another consideration, that needs to be taken into account when dealing with timber, is that it interacts with the humidity in the environment until it reaches the equilibrium moisture content. This change in moisture can result in internal stresses causing deformations or even cracks, for example perpendicular to the grain. Also it has to be considered that the material properties vary with the moisture content. For high moisture contents, strength and stiffness values need to be reduced, while values around 12% at an ambient temperature of 20°C are optimal SIA (2003).

Another important aspect is that timber has a low shear modulus compared to homogeneous materials like steel. This means that shear deformations need to be taken into account for timber structures since they can be of the same order as deformations from bending. This effect becomes even more important for a high presence of knots since the reduction they cause in G-modulus is more important than the one in the E-modulus. The deformations of timber beams increase furthermore over time, since the strength decreases under permanent loading.

All these illustrations show that there are various uncertainties related to timber construction and that there is a need for non-destructive methods to estimate the material properties and make sure they are within desirable limits. The next paragraph gives an overview of methods used in practice for the grading of timber and the determination of some important properties of timber beams, namely the density, strength and stiffness.
2.2 Non-destructive methods for timber grading

According to Thelandersson and Larsen (2003), two types of strength grading can be distinguished:

- **Visual grading:** it is based on visual inspections to determine number and distribution of defects like knots, grain slope, compression wood, resin pockets etc. According to specific rules, the specimen can then be assigned to a strength class.

- **Machine grading:** there are different methods that can be used to measure several parameters including knots, ring width, density etc. The resulting information can then be combined to predict the strength and stiffness of the specimen.

The most numerous, yet also harmful defects are knots since they considerably reduce the stiffness and are in general also the cause for failure in ultimate bending or tensile tests.

One should keep in mind that even after timber grading there are still uncertainties concerning the material properties and that these values can only be considered as estimates based on more or less accurate correlations between different parameters. There is for example a good correlation between E-modulus and strength, even in presence of knots. The correlation between density and E-modulus on the other hand is only strong for clear wood specimens, which are not the general case Thelandersson and Larsen (2003).

**Determination of density**

The easiest way to determine the global density of a specimen is to calculate its weight-to-volume ratio. Since this is however not possible for existing structures without taking a probe, there is a need for other non-destructive methods that can be used on-site.

Accurate results can in this case be achieved by means of radiation techniques (x-ray or gamma rays). They allow to determine not only the global density, but even the density distribution of a beam. Furthermore it is possible to find the location as well as size of hidden knots since they have a higher density then the surrounding material (cf. Figure 2-5). An application of this technique can be found in Oja (2001) and (Schajer, 2001) where it was used to determine the density of logs and lumber from which it was then attempted to derive strength and stiffness.

![Illustration of the determination of wood density using x-rays Oja (2001)](image-url)
Determination of E-modulus

One of the most widely used methods to determine the static E-modulus is to perform continuous flatwise-bending tests. This is in fact a three-point loading test carried out along the beam length (cf. Figure 2-6). The result is the E-modulus distribution over the beam length, excluding the ends since it is not possible to carry out the measurements there.

Figure 2-6  Bending type machine for continuous timber grading, based on (Performancepanels.com, 2012)

A more recent approach to determine the E-modulus are dynamic testing methods. They have gained much importance during the last decades, since they provide a quick assessment of wood quality with relatively low means of equipment. In the following, some of these non-destructive dynamic grading methods are described in more detail.

Resonance frequency measurements are based on the fact that for free boundary conditions, the frequencies depend only on the density, the E-modulus and the geometry of the specimen. With known density it is then possible to estimate the E-modulus. The free boundary conditions can be simulated by suspending the specimen on rubber bands or by using soft foam pieces as support. The specimen is excited with a hammer and the response can be recorded using an accelerometer or a microphone. The frequencies can then be calculated by performing a Fast Fourier Transformation (FFT) on the raw time data, which can be done using a computer software Haines (1996). From these frequencies, longitudinal and transversal E-moduli can then be derived. The test setups for both kinds of frequency measurements are illustrated in Figure 2-7.

Figure 2-7  Illustration of the test-setup for longitudinal (top) and transversal (bottom) frequency measurement
The E-moduli calculated from dynamic testing are usually higher than the ones obtained from static testing. The reason for this is the difference in duration of the tests. Static tests are carried out during several minutes where the timber starts to creep, giving its viscoelastic material behaviour. This results in lower stiffness values. Dynamic tests only last seconds and therefore represent a short-time stiffness that is higher in value (Haines, 1996). The correlation between the different E-moduli is however quite strong, depending on the number of defects and the type of wood. In general it can be said that the correlation is higher for clear wood since defects like knots cause a local reduction of the stiffness and reduce the accuracy of the dynamic tests.

Ohlsson and Perstorper (1992) carried out both, flexure and longitudinal tests on a clear piece of Norway spruce to determine its E- and G-modulus using different numbers of frequencies together with Timoshenko theory. The test setup comprised several accelerometers to be able to determine the vibrations modes also for higher frequencies. The excitations were made using an instrumented hammer. The best results for the E-modulus could be achieved using only the two lowest longitudinal frequencies and the lowest flatwise bending frequency. The use of additional information from higher frequencies did not improve the results, but rather lowered the accuracy. For non-defect-free specimens it seems therefore reasonable to only use the lowest frequency.

Another possibility to determine the E-modulus from a known density is the ultrasonic test developed by Sandoz (1989), also known as Sylva-test. The test is based on the velocity of stress wave propagation in materials. In simple words, an impulse is induced at one end of a beam and the response is measured at the other end. The time delay of the impact over the length of the beam can then be used to compute the wave velocity from which the E-modulus can eventually be derived. The results presented in Sandoz (1989) show very good correlation between dynamic and static E-modulus. The general test-setup as well as the application in practice are illustrated in Figure 2-8.

![Schematic model of the Sylva-test](image)

**Figure 2-8** Schematic model of the Sylva-test (left) and application on a stress-laminated timber deck (right), Ross (1994)
Determination of G-modulus

The G-modulus can be determined in the same way as the E-modulus using torsional instead of bending vibrations Ohlsson and Perstorper (1992). To record the torsional modes, the accelerometer needs to be placed at the edge of the cross section at some distance from the neutral axis (cf. Figure 2-9). The beam then needs to be excited in an eccentric point to cause torsional vibrations. An additional accelerometer can be placed in the neutral axis to simplify the detection of the torsional frequencies. Comparison of the frequency plots for both accelerometers will then show that the torsional frequencies are only recorded by the eccentric accelerometer. As already mentioned, Ohlsson and Perstorper (1992) also presents good results for the measurement of the G-modulus, for which only the first two lowest torsional frequencies were used.

Figure 2-9 Test setup for transversal and torsional frequency measurement

Apart from the transversal excitation, all these methods can, at least in theory, be also applied to loaded structures. This means that material properties like density, E- and G-modulus can theoretically be determined on existing structures.
3 Determination of axial load and boundary conditions in slender beams

This chapter analysis research that was carried out in the past concerning the determination of axial loads and boundary conditions using of transversal frequency analysis. The different case studies are examined regarding the overall test setup, the testing equipment, the theoretical models and the accuracy of the results. Based on this analysis, conclusions are drawn on how to plan the laboratory tests and to process the data collected within this research.

3.1 Determination of axial loads

One of the first frequency based laboratory tests to determine the axial load in a prismatic Euler beam under tension was carried out by Livingston (1994). The setup consisted of a square steel rod subjected to loading and unloading cycles in a vertical Tinius Olsen tensile machine. A single accelerometer was attached at a distance of 4/5 of the specimen length, which was then excited with an impact hammer equipped with a soft tip. From the resulting frequency response function (FRF), the first three frequencies were then extracted for each loading and unloading cycle. Using a non-linear least square parameter technique described in Béliveau (1987) together with a continuous model, the axial force and the boundary conditions represented by rotational springs were then estimated, considering at first only the first two, then the first three frequencies. Even though the axial loads were in general overestimated, the results were promising, especially for higher forces. It can be said that the additional data provided by the third frequency did not improve the results of the experiment. The estimated boundary conditions varied over the loading and unloading of the rod and were not discussed in further detail.

A different approach was used by Tullini and Laudiero (2008). Instead of using several frequencies to determine the axial loads and boundary conditions, only the mode shape and frequency of the first mode were used. The necessary information was gathered by three accelerometers situated in each quarter point of the specimen. The displacements of the accelerometers were used to determine the shape of the first mode, whereas the frequency was again derived from the FRF. An impact hammer was used to measure the force function. This method was applied on both, a steel rod under tension and a slender box beam under compression. For both test setups, the results for the axial loads showed very good accordance with the real values. The estimation of the end constraint stiffness however was rather far away from the analytical values for the limit conditions pinned-pinned and fixed-fixed and was therefore not convincing.

Maille (2008) carried out in-situ experiments on an old breeding barn to assess its structural integrity for a full snow load. For this purpose, transversal frequency testing was carried out on the structural steel members to determine the axial tensile forces under different load combinations. The results of these measurements were then compared with different computer models to find out which one comes closest to the actual behaviour of the structure. Based on this model, the axial forces under a full snow load were then computed which could then be used for the structural assessment.
The tests were performed by striking every steel member in three different ways to excite as many frequencies as possible. These three methods included a steel hammer, a rubber mallet and plucking the members by hand. One single accelerometer was used to detect the frequencies. However, since in many cases not all of the frequencies could be detected, the author suggests for further research to use an additional accelerometer situated close to the end of the members. As the members were assumed to be pinned at both ends, the axial loads could be determined by a single parameter estimation using the first 6 measured frequencies and the formula for Euler beams Eq. (3.34) in Maille (2008). In addition, a dual parameter estimation was carried out by means of least square linear regression to also verify the E-modulus of each member. Even though most of them coincided well with the assumed value, there were still some outliers. For the testing of inhomogeneous materials with a high variation in material properties like timber, it is therefore reasonable to always verify the assumed parameters with additional measurements and estimations.

A somewhat different model was used in Italy to assess the safety of tie-rods in ancient masonry buildings, Amabili (2010). Using finite elements, the rods were modelled as Timoshenko beams, taking into account shear deformations and effects of rotary inertia. The equipment for the in-situ tests consisted of a single accelerometer situated at 2/5 of the span and an impact hammer to determine the first 6 resonance frequencies. The portion of the beam inside the wall at both sides was assumed to be elastically founded. The calculated values were found by applying Rayleigh-Ritz method (cf. Chapter 5). The unknown parameters were estimated by minimizing the weighted difference between the computed and measured frequencies. This was done in several ways, from first using only the two first modes from the estimation, then the first three and so on till finally all six modes. The advantage of this method is that measurement and modelling errors are minimized through the use of redundant data given by the several frequencies. It is however questionable if this could be done in the same way for timber since the accuracy requirements might not be fulfilled for higher frequencies.

As it can be seen, this literature study only yielded research that has been done on steel structures. It seems that not much research has been done on the possibility to determine the axial loads in timber members using resonance frequency analysis. There are however investigations on the boundary conditions of timber elements in frames or trusses.

### 3.2 Determination of boundary conditions

Crovella and Kyanka (2011) made use of vibration techniques to determine the rotational stiffness of timber joints. The main incentive for his research was the little amount of research available investigating the member joint properties, even though the failures usually occur in the joints. Transversal vibration tests were carried out on timber beams and frames of softwood and hardwood for different boundary conditions. The beams were unloaded, so that no axial tension needed to be taken into account. Furthermore the beams were modelled as Euler beams meaning that effects of shear deformations or rotary inertia were neglected. Shear deformations can however have a big influence on the vibration of timber beams, especially for the present span-to-depth ratios, that are 16 and 36. Chui and Smith (1990) suggests that shear deformations need to be taken into account for values below the region of 32 for simply supported beams. For stiffer boundary conditions, this limit value is even
lower. Here, the use of Timoshenko theory could have been further investigated to see if there is an improvement in results. A single accelerometer was used to record the beam vibrations that were then used to derive the first resonance frequency. With the above-mentioned assumptions a simple relation between the frequency and the joint stiffness could be used for the estimation. When looking at the continuum of joint stiffness plot in Figure 3-1, one can see that the ratio of the joint rotational stiffness and the beam flexural stiffness is of importance. If one of them is considerably higher than the other one, a change of the joint stiffness has a minimal effect on the frequency.

![Figure 3-1 Plot of continuum joint stiffness against the normalised minimum natural frequency, McGuire (1995)](image)

According to Leichti (2000), a 50% rigid connection is the maximum that can be achieved for timber joints. This theory seems to be supported by the results of Crovella and Kyanka (2011) that were all below that value, even for stiff connectors. To be able to assess the accuracy of the results, the rotational stiffness at the beam-ends was derived from deflection tests. The conclusion was that the results were good for semi-rigid joints, but rather poor for joints of low stiffness. This can possibly be explained by the small slope of the formula plot in low frequency regions, where small changes in frequency result in rather high changes for the estimated rotational stiffness. All these considerations can lead to the conclusion that it could be possible to determine also the rotational stiffness for beams under axial tension, at least if the boundary conditions are stiff enough and if Timoshenko theory is used.
3.3 Conclusion

For the determination of axial loads, most of the investigated members were steel bars with a high length compared to their sectional dimensions. The span-to-depth ratios ranged from around a 100 in Livingston (1994) to over 300 in Maille (2008). The reason for this is that for a given change in axial load, a change in frequency increases for a higher slenderness of the loaded member. This needs to be considered when choosing a sample size for the experiments in this research, since changes in axial load might not even be possible to detect for beams with a low slenderness. The tension machine used in this research is similar to the vertical one used in Livingston (1994). Since the friction grips at both ends of the beam are the same, it is reasonable to assume equal boundary conditions. The static system consequently is a simply supported beam with identical rotational springs at both ends. This leads to two unknown parameters, namely the axial force $S$ and the rotational stiffness $k$. This means that at least two bending frequencies are necessary to determine the parameters. It will however be investigated if additional information from higher frequencies can be used for error minimisation, even if this is doubtful given the inhomogeneous properties of timber and their influence on the precision of frequency measurement. In this case, a least linear regression is necessary to solve the over-determined system. It is necessary to keep the number of unknown parameters to a minimum to make sure not too much accuracy is lost, even if it would theoretically be possible to include more parameters, like for example material properties. For the actual tests, the use of one single accelerometer seems appropriate since the vibration modes are well known for the given system. The accelerometer needs to be placed away from modal nodes to make sure the desired frequencies can be recorded. A distance of 20% of the length is a good position to enable measurements of the first five frequencies. If necessary, an additional accelerometer placed closer to one end, can be used to record more frequencies. The beams should be excited on the weak axis, since the lower moment of inertia causes a higher variation of frequency for a change in axial force. Since the results seem to be sensible to the stress level, the specimen should be tested for a wide range of axial loads. As mentioned above, shear deformations have a big influence on the frequencies of timber beams, even if they are rather slender. It is therefore necessary to include them according to Timoshenko theory. This can be done in different ways, either by using an exact continuous model or using approximations through discrete modelling (finite elements) or Rayleigh-Ritz theory.
4 Vibration theory

The following paragraphs present derivations and applications of differential equations for several kinds of beam vibrations. All derivations were made according to Timoshenko (1974) and Howson and Williams (1973).

4.1 Derivations

4.1.1 Longitudinal vibrations

Figure 4-1 Illustration of a beam subjected to longitudinal vibration (a) and the according forces acting on an infinitesimal element (b)

Figure 4-1 (a) shows a prismatic beam of length \( l \) and the cross-section \( A \) subjected to longitudinal vibration. An according infinitesimal segment of length \( dx \), in a distance \( x \) from the left end of the beam, is illustrated in Figure 4-1 (b). Forming equilibrium of the acting forces yields

\[
S + \rho \cdot A \cdot dx \cdot \frac{\partial^2 u}{\partial t^2} - S - \frac{\partial S}{\partial x} \cdot dx = 0
\]  

(4.1)

where \( S \) is the internal force acting on the beam, \( \rho \) the density and \( u \) the longitudinal displacement of the segment, which makes \( \frac{\partial^2 u}{\partial t^2} \) its acceleration.

Applying Hooke’s law on the axial force \( S \), we obtain

\[
S = A \cdot \sigma = E \cdot A \cdot \varepsilon = E \cdot A \cdot \frac{\partial u}{\partial x}
\]  

(4.2)

where \( E \) is the modulus of elasticity, \( \sigma \) the axial stress and \( \varepsilon = \frac{\partial u}{\partial x} \) the axial strain. Substitution of Eq. (4.2) into Eq. (4.1) yields

\[
\frac{\partial^2 u}{\partial x^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial t^2}
\]  

(4.3)

If we introduce the wave velocity \( a = \sqrt{E/\rho} \), Eq. (4.3) can be rewritten as

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}
\]  

(4.4)
which is called the one-dimensional wave equation. For the natural vibration modes of the beam, the solution to Eq. (4.4) can be written as

$$u = X \cdot (A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t)) \quad (4.5)$$

where $X$ is called the normal function that defines the shape of the natural modes.

Substitution of Eq. (4.5) into Eq. (4.4) yields the time-independent differential equation for longitudinal vibration

$$\frac{d^2 X}{d x^2} + \frac{\omega^2}{a^2} \cdot X = 0 \quad (4.6)$$

which is the differential equation of motion for beams subjected to longitudinal vibrations according to Euler theory.

The solution to Eq. (4.6) is of the kind:

$$X = (C \cdot \cos\left(\frac{\omega x}{a}\right) + D \cdot \sin\left(\frac{\omega x}{a}\right)) \quad (4.7)$$

The constants $C$ and $D$ are determined by applying the boundary conditions of the beam. Examples are presented in Chapter 4.2.

### 4.1.2 Torsional vibrations

![Figure 4-2](https://via.placeholder.com/150)

Figure 4-2 Illustration of a beam subjected to torsional vibration (a) and the according moments acting on an infinitesimal element (b)

Considering the same beam subjected to torsion (Figure 4-2 (a)), the moment equilibrium can be derived according to Figure 4-2 (b):

$$M_t + \rho \cdot I_p \cdot dx \cdot \frac{\partial^2 \theta}{\partial t^2} - M_t - \frac{\partial M_t}{\partial x} \cdot dx = 0 \quad (4.8)$$

where $M_t$ is the torsional moment acting on the beam, $I_p$ the polar moment and $\theta$ the angular displacement of the segment, which makes $\partial^2 \theta / \partial t^2$ its acceleration.

Elementary torsion theory gives

$$M_t = G \cdot K_t \cdot \frac{\partial \theta}{\partial x} \quad (4.9)$$
where \( G \) is the modulus of shear, and \( K_t \) is the torsional constant that varies with the shape of the cross-section. For circular sections \( K_t \) is equal to the polar moment \( I_p \). For other cross-sections refer to Roymech.co.uk (2010).

Substitution of Eq. (4.9) into Eq. (4.8) yields

\[
\frac{\partial^2 \theta}{\partial x^2} = \frac{\rho I_p}{G K_t} \frac{\partial^2 \theta}{\partial t^2} \quad (4.10)
\]

Introducing \( b = \sqrt{G \cdot K_t / (\rho \cdot I_p)} \), Eq. (4.10) can be rewritten as

\[
\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{b^2} \frac{\partial^2 \theta}{\partial t^2} \quad (4.11)
\]

which has the same form than the one-dimensional wave equation from before, which is why the same solution as in Eq. (4.5) can again be used:

\[
\theta = X \cdot (A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t)) \quad (4.12)
\]

Substitution of Eq. (4.12) into Eq. (4.11) yields the time-independent differential equation for torsional vibration

\[
\frac{d^2 X}{dx^2} + \frac{\omega^2}{b^2} X = 0 \quad (4.13)
\]

which is the differential equation of motion for beams subjected to torsional vibrations according to Euler theory.

Since Eq. (4.13) is of the same kind as Eq. (4.6), its solution has also the same form as Eq. (4.7):

\[
X = (C \cdot \cos \left(\frac{\omega x}{b}\right) + D \cdot \sin \left(\frac{\omega x}{b}\right)) \quad (4.14)
\]

which can again be solved for different boundary conditions.

### 4.1.3 Transversal vibrations – Euler beam theory

The Euler beam theory is based on the assumptions that the cross-sections of a prismatic beam remain plane during vibration. This means that deformations caused by shear forces as well as rotary inertia effects are not taken into account when using this theory. The consequence is that it only gives accurate results if the deformations from shear are small compared to the ones from bending. This is in general the case for slender beams.
Figure 4-3 (a) shows a vibrating beam subjected to a tensile force $S$ and transversal vibration. Forming the equilibrium on an infinitesimal element (Figure 4-3 (b)) for both, vertical forces and moments, yields

\[ V - \rho \cdot A \cdot dx \cdot \frac{\partial^2 v}{\partial t^2} - V - \frac{\partial V}{\partial x} \cdot dx = 0 \tag{4.15} \]

and

\[ V \cdot dx - \frac{\partial M}{\partial x} \cdot dx = 0 \tag{4.16} \]

where $V$ is the shearing force acting on the beam, and $v$ the transversal displacement of the segment, which makes $\frac{\partial^2 v}{\partial t^2}$ its acceleration.

Solving Eq. (4.16) with respect to $V$ and substituting into Eq. (4.15) yields

\[ \frac{\partial^2 M}{\partial x^2} = -\rho \cdot A \cdot dx \cdot \frac{\partial^2 v}{\partial t^2} \tag{4.17} \]

The differential equation for the deflection of a beam subjected to a tensile force and a transverse load is

\[ M = EI \cdot \frac{\partial^4 v}{\partial x^4} - S \cdot v \tag{4.18} \]

Substitution of Eq. (4.18) into Eq. (4.17) leads to

\[ EI \cdot \frac{\partial^4 v}{\partial x^4} - S \cdot \frac{\partial^2 v}{\partial x^2} = -\rho \cdot A \cdot \frac{\partial^2 v}{\partial t^2} \tag{4.19} \]

If a beam vibrates in one of its natural modes, a solution to this equation is given by

\[ v = X \cdot (A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t)) \tag{4.20} \]

Substitution of Eq. (4.20) into Eq. (4.19), yields:

\[ EI \cdot \frac{\partial^4 X}{\partial x^4} - S \cdot \frac{\partial^2 X}{\partial x^2} = \rho \cdot A \cdot \omega^2 \cdot X \tag{4.21} \]

which is the differential equation of motion for beams under axial load subjected to transversal vibrations according to Euler theory.
4.1.4 Transversal vibrations – Timoshenko beam theory

This paragraph gives a derivation of the transversal vibrations of prismatic beams under axial load according to Timoshenko beam theory. In general, shear deformations can be neglected for flexural vibrations if a beam’s cross-sectional dimensions are small compared to its length. For timber, however, shear deformations can be of the same order as deformations from bending, even for slender beams. The reason is that the shear stiffness of the material is small compared to other materials. For this reason, it is important to include shear effects in the calculation, especially when studying the vibration modes of higher frequencies where they gain more and more of importance Timoshenko (1974), Howson and Williams (1973).

Figure 4-4 shows a deformed infinitesimal element to a transversally vibrating beam in the x-y plane. To include the shear deformations, it must be considered that the beam not only performs a translatory movement, but also rotates. The angle of rotation $\partial M / \partial x$ is the deflection slope and can be expressed as the sum of the angle of shear $\beta$ and a parameter $\psi$ that represents the angle of rotation of the cross-section:

$$\frac{\partial y}{\partial x} = \psi + \beta$$  \hspace{1cm} (4.22)

The rotation of the beam produces a moment $\rho \cdot l \cdot \frac{\partial^2 \psi}{\partial t^2} \cdot dx$, that needs to be taken into account when forming the dynamic moment equilibrium on an infinitesimal element of length $dx$:

$$V \cdot dx - \frac{\partial M}{\partial x} \cdot dx - \rho \cdot l \cdot \frac{\partial^2 \psi}{\partial t^2} \cdot dx - S \cdot \frac{\partial y}{\partial x} \cdot dx = 0$$  \hspace{1cm} (4.23)

The dynamic equilibrium for the forces in the vertical direction is

$$-V - \rho \cdot A \cdot dx \cdot \frac{\partial^2 y}{\partial t^2} + V + \frac{\partial v}{\partial x} \cdot dx = 0$$  \hspace{1cm} (4.24)

The differential equation for the deflection of a beam subjected to a tensile force and a transverse load is

$$M = -EI \cdot \frac{\partial^2 \psi}{\partial x^2}$$  \hspace{1cm} (4.25)
Furthermore, consideration of the shearing yields

\[ V = (k_s \cdot A \cdot G + S) \cdot \frac{\partial y}{\partial x} - k_s \cdot A \cdot G \cdot \psi \]  
(4.26)

Eliminating \( V, M, \) and either \( \psi \) or \( y \) by combining Eq. (4.22) - (4.26), leads to the complete differential equation for transverse vibrations of prismatic beams under axial load:

\[
EI \cdot \left(1 + \frac{S}{k_s \cdot A \cdot G}\right) \frac{\partial^4 \theta}{\partial x^4} - S \cdot \frac{\partial^2 \theta}{\partial x^2} + \rho \cdot A \cdot \frac{\partial^2 \theta}{\partial t^2}
\]

\[ - \rho \cdot I \cdot \left(1 + \frac{S}{k_s \cdot A \cdot G} + \frac{E}{k_s \cdot G}\right) \cdot \frac{\partial^4 \theta}{\partial x^2 \cdot \partial t^2} + \frac{\rho^2 \cdot I}{k_s \cdot G} \cdot \frac{\partial^4 \theta}{\partial t^4} = 0
\]  
(4.27)

where

\( \theta = y \) or \( \psi \)

If a beam vibrates in one of its natural modes, a solution to this equation is given by

\[ \theta = \Theta \cdot \left( A \cdot \cos(\omega \cdot t) + B \cdot \sin(\omega \cdot t) \right) \]  
(4.28)

Substitution of Eq. (4.28) into Eq. (4.27), leads to

\[
EI \cdot \left(1 + \frac{S}{k_s \cdot A \cdot G}\right) \frac{\partial^4 \Theta}{\partial x^4} - S \cdot \frac{\partial^2 \Theta}{\partial x^2} - \rho \cdot A \cdot \omega^2 \cdot \Theta
\]

\[ + \rho \cdot I \cdot \omega^2 \cdot \left(1 + \frac{S}{k_s \cdot A \cdot G} + \frac{E}{k_s \cdot G}\right) \cdot \frac{\partial^2 \Theta}{\partial x^2} + \frac{\rho^2 \cdot I}{k_s \cdot G} \cdot \omega^4 \cdot \Theta = 0
\]  
(4.29)

where

\( \Theta = Y \) or \( \Psi \)

and which is the differential equation of motion for beams under axial load subjected to transversal vibrations according to Timoshenko theory.

### 4.2 Applications

In the following, the equations of motion derived in Chapter 4.1 are used to determine the mode shapes and resonance frequencies for different boundary conditions.

#### 4.2.1 Longitudinal vibrations

**Beam with free ends**

For a free-free beam, the axial force, which is proportional to \( dX/dx \), has to be zero at both ends.

\[
\left( \frac{dX}{dx} \right)_{(x=0)} = 0 \quad \left( \frac{dX}{dx} \right)_{(x=l)} = 0
\]  
(4.30)

The first condition is only fulfilled if \( D = 0 \). To obtain a non-trivial solution, the second condition requests \( C \neq 0 \), which leads to

\[
\sin \left( \frac{\omega \cdot l}{a} \right) = 0
\]  
(4.31)

This is only fulfilled if
\[ \frac{\omega_i \cdot l}{a} = i \cdot \pi \]  

(4.32)

where \( i \) is an integer. Substituting \( \omega_i = 2\pi \cdot f_i \) and \( a = \sqrt{E/\rho} \) into Eq. (4.32) results in

\[ f_i = \frac{1}{2l} \cdot \sqrt{\frac{E}{\rho}} \cdot i \]  

(4.33)

which for \( i = 1, 2, 3 \ldots \) yields the natural frequencies of the longitudinal vibration modes of the beam. Together with Eq. (4.7), the first three frequencies with the corresponding modes can be determined to

\[ f_1 = \frac{1}{2l} \cdot \sqrt{\frac{E}{\rho}} \quad X_1 = C_1 \cdot \cos \left( \frac{\pi \cdot x}{l} \right) \]  

(4.34)

\[ f_2 = \frac{1}{l} \cdot \sqrt{\frac{E}{\rho}} \quad X_2 = C_2 \cdot \cos \left( \frac{2\pi \cdot x}{l} \right) \]  

(4.35)

\[ f_3 = \frac{3}{2l} \cdot \sqrt{\frac{E}{\rho}} \quad X_3 = C_3 \cdot \cos \left( \frac{3\pi \cdot x}{l} \right) \]  

(4.36)

The shape of the three first modes is illustrated in Figure 4-5.

![Figure 4-5](image)

Figure 4-5  Three first longitudinal vibration modes of a beam with free boundary conditions

**Beam with fixed ends**

For a beam with fixed ends, the displacement has to be zero at both ends.

\[ X_{(x=0)} = 0 \quad X_{(x=l)} = 0 \]  

(4.37)

The first condition is only fulfilled if \( C = 0 \). To obtain a non-trivial solution, the second condition requests \( D \neq 0 \), which brings us again to

\[ \sin \left( \frac{\omega \cdot l}{a} \right) = 0 \]  

(4.38)

and the according frequencies

\[ f_i = \frac{1}{2l} \cdot \sqrt{\frac{E}{\rho}} \cdot i \]  

(4.39)
which are the same as for the free-free condition. The modal shapes change however.

\[ f_1 = \frac{1}{2l} \sqrt{\frac{E}{\rho}} \quad X_1 = D_1 \cdot \sin \left( \frac{\pi x}{l} \right) \]  

\[ f_2 = \frac{1}{l} \sqrt{\frac{E}{\rho}} \quad X_2 = D_2 \cdot \sin \left( \frac{2\pi x}{l} \right) \]  

\[ f_3 = \frac{3}{2l} \sqrt{\frac{E}{\rho}} \quad X_3 = D_3 \cdot \sin \left( \frac{3\pi x}{l} \right) \]  

The shape of the three first modes is illustrated in Figure 4-6.

\[ \frac{dX}{dx} \bigg|_{x=0} = 0 \quad \frac{dX}{dx} \bigg|_{x=l} = 0 \]  

\[ \sin \left( \frac{\omega t}{b} \right) = 0 \]  

where \( i \) is an integer. Substituting \( \omega_i = 2\pi \cdot f_i \) and \( b = \sqrt{G \cdot K_i / (\rho \cdot I_p)} \) into Eq. (4.45) results in
\[ f_i = \frac{1}{2l} \cdot \sqrt{\frac{GK_t}{\rho l^2}} \cdot i \]  

which for \( i = 1, 2, 3 \ldots \) yields the natural frequencies of the torsional vibration modes of the beam. Together with Eq. (4.14), the first three frequencies with the corresponding modes can be determined to

\[ f_1 = \frac{1}{2l} \cdot \sqrt{\frac{GK_t}{\rho l^2}} \quad X_1 = C_1 \cdot \cos \left( \frac{\pi x}{l} \right) \]  

\[ f_2 = \frac{1}{l} \cdot \sqrt{\frac{GK_t}{\rho l^2}} \quad X_2 = C_2 \cdot \cos \left( \frac{2\pi x}{l} \right) \]  

\[ f_3 = \frac{3}{2l} \cdot \sqrt{\frac{GK_t}{\rho l^2}} \quad X_3 = C_3 \cdot \cos \left( \frac{3\pi x}{l} \right) \]  

The shape of the three first modes is illustrated in Figure 4-7.

![Figure 4-7 Three first torsional vibration modes of a beam with free boundary conditions](image)

**Beam with fixed ends**

For a beam with fixed ends, the rotation has to be zero at both ends.

\[ X_{(x=0)} = 0 \quad X_{(x=l)} = 0 \]  

(4.50)

The first condition is only fulfilled if \( C = 0 \). To obtain a non-trivial solution, the second condition requests \( D \neq 0 \), which leads again to

\[ \sin \left( \frac{\omega l}{b} \right) = 0 \]  

(4.51)

and the according frequencies

\[ f_i = \frac{1}{2l} \cdot \sqrt{\frac{GK_t}{\rho l^2}} \cdot i \]  

(4.52)

which are the same as for the free-free condition. The modal shapes however are different again.

\[ f_1 = \frac{1}{2l} \cdot \sqrt{\frac{GK_t}{\rho l^2}} \quad X_1 = D_1 \cdot \sin \left( \frac{\pi x}{l} \right) \]  

(4.53)
\[ f_2 = \frac{1}{l} \cdot \sqrt{\frac{G \cdot K_t}{\rho \cdot I_p}} \quad X_2 = D_2 \cdot \sin \left( \frac{2\pi \cdot x}{l} \right) \quad (4.54) \]

\[ f_3 = \frac{3}{2l} \cdot \sqrt{\frac{G \cdot K_t}{\rho \cdot I_p}} \quad X_3 = D_3 \cdot \sin \left( \frac{3\pi \cdot x}{l} \right) \quad (4.55) \]

The shape of the three first modes is illustrated in Figure 4-8.

![Figure 4-8 Three first torsional vibration modes of a beam with fixed boundary conditions](image)

As it can be seen, the modes and frequencies have the same shape for longitudinal and rotational vibrations for the same boundary conditions.

### 4.2.3 Transversal vibrations – Euler beam theory

**Beam with free ends, S=0**

The following boundary conditions apply for free-free beams:

\[
\left( \frac{d^2X}{dx^2} \right)_{(x=0)} = 0 \quad \left( \frac{d^3X}{dx^3} \right)_{(x=0)} = 0 \quad (4.56) \\
\left( \frac{d^2X}{dx^2} \right)_{(x=l)} = 0 \quad \left( \frac{d^3X}{dx^3} \right)_{(x=l)} = 0 \quad (4.57)
\]

The conditions in Eq. (4.56) are satisfied by the solution

\[ X = C \cdot (\cos(k \cdot x) + \cosh(k \cdot x)) + D \cdot (\sin(k \cdot x) + \sinh(k \cdot x)) \quad (4.58) \]

while the other two conditions in Eq. (4.57) yield the equations

\[ C \cdot (-\cos(k \cdot l) + \cosh(k \cdot l)) + D \cdot (-\sin(k \cdot l) + \sinh(k \cdot l)) = 0 \quad (4.59) \]

\[ C \cdot (\sin(k \cdot l) + \sinh(k \cdot l)) + D \cdot (-\cos(k \cdot l) + \cosh(k \cdot l)) = 0 \quad (4.60) \]

A solution for this system different from zero only exists if the determinant of Eq. (4.59) and Eq. (4.60) disappears:

\[ (-\cos(k \cdot l) + \cosh(k \cdot l))^2 - (\sin(k \cdot l))^2 - (\sinh(k \cdot l))^2 = 0 \quad (4.61) \]
With the equations
\[ \cosh(k \cdot l)^2 - \sinh(k \cdot l)^2 = 1 \]
\[ \cos(k \cdot l)^2 + \sin(k \cdot l)^2 = 1 \]
Eq. (4.61) can be rewritten as
\[ \cos(k \cdot l) \cdot \cosh(k \cdot l) = 1 \] (4.62)
for which the solution can be approximated to
\[ k_i \cdot l \approx (i + 1/2) \cdot \pi \] (4.63)
Substitution of \( k_i^4 = \frac{\omega_i^2}{a^2} \), \( a = \sqrt{\frac{E \cdot I}{\rho \cdot A}} \) and \( \omega_i = 2\pi \cdot f_i \), yields
\[ f_i = \frac{\pi}{8 \cdot l^2} \cdot \frac{E \cdot I}{\rho \cdot A} \cdot (2 \cdot i + 1)^2 \] (4.64)
The shape of the three first modes is illustrated in Figure 4-9.

![Figure 4-9 Three first transversal vibration modes of a beam with free boundary conditions](image)

**Beam with clamped ends, S=0**

For clamped ends, the boundary conditions are
\[ X_{(x=0)} = 0 \quad \frac{dX}{dx}_{(x=0)} = 0 \] (4.65)
\[ X_{(x=l)} = 0 \quad \frac{dX}{dx}_{(x=l)} = 0 \] (4.66)
The first two conditions in Eq. (4.65) are satisfied by the solution
\[ X = C \cdot (\cos(k \cdot x) - \cosh(k \cdot x)) + D \cdot (\sin(k \cdot x) - \sinh(k \cdot x)) \] (4.67)
while the other two conditions in Eq. (4.66) yield the equations
\[ C \cdot (\cos(k \cdot l) - \cosh(k \cdot l)) + D \cdot (\sin(k \cdot l) - \sinh(k \cdot l)) = 0 \] (4.68)
\[ C \cdot (\sin(k \cdot l) + \sinh(k \cdot l)) + D \cdot (-\cos(k \cdot l) + \cosh(k \cdot l)) = 0 \] (4.69)
A solution for this system different from zero only exists if the determinant of Eq. (4.68) and Eq. (4.69) disappears:

\[ (-\cos(k \cdot l) + \cosh(k \cdot l))^2 - (\sin(k \cdot l)^2 - \sinh(k \cdot l)^2) = 0 \]  
(4.70)

from which the same equation as in Eq. (4.62) can be deduced, which results in the same frequencies as for the free-free beam:

\[ f_i = \frac{\pi}{8 \cdot l^2} \cdot \frac{E \cdot l}{\rho \cdot A} \cdot (2 \cdot i + 1)^2 \]  
(4.71)

The shape of the three first modes is illustrated in Figure 4-10.

![Three first transversal vibration modes of a beam with fixed boundary conditions](image)

**Figure 4-10** Three first transversal vibration modes of a beam with fixed boundary conditions

**Beam with simple supports under axial load, S≠0**

For a simply supported beam, the boundary conditions are

\[ X(x=0) = 0 \quad X(x=l) = 0 \]  
(4.72)

which are satisfied for

\[ X_i = \sin\left(\frac{i \cdot \pi \cdot x}{l}\right) \]  
(4.73)

where \( i \) is an integer. Substituting Eq. (4.73) into Eq. (4.21) and replacing \( \omega_i = 2\pi \cdot f_i \) yields

\[ f_i = i^2 \cdot \frac{\pi}{2 \cdot l^2} \cdot \sqrt{\frac{E \cdot l}{\rho \cdot A}} \cdot \sqrt{1 + \frac{S^2 \cdot l^2}{i^2 \cdot E \cdot l \cdot \pi^2}} \]  
(4.74)
The three first mode shapes are illustrated in Figure 4-11.

![Mode Shapes](image)

**Figure 4-11** Three first transversal vibration modes of a beam under axial load with simply supported boundary conditions

For other boundary conditions there are no simple expressions, which states the need for numerical solutions. These can for example be found in Shaker (1975).

### 4.2.4 Transversal vibrations – Timoshenko beam theory

No simple expressions for an exact solution are available to solve Eq. (4.29). This makes it necessary to use numerical methods to determine the frequencies for beams with different boundary conditions, like for instance clamped-clamped or elastic supports, cf. Chapter 5.
5 Numerical modelling and parameter estimation

This chapter contains an overview of different means to calculate the resonance frequencies for a beam under axial load subjected to transversal vibrations for different boundary conditions using Timoshenko theory. The end conditions are modelled using translational and rotational springs (cf. Figure 5-1), based on the assumption that the supports show linearly elastic behaviour.

![Figure 5-1 Illustration of an axially loaded beam with translational and rotational spring supports, Livingston (1994)](image)

The according boundary conditions can be written as:

\[ V_x = 0 = k_1 \cdot (Y)_{x=0} \]  
\[ V_x = l = -k_3 \cdot (Y)_{x=l} \]  
\[ M_x = 0 = -k_2 \cdot (Y')_{x=0} \]  
\[ M_x = l = k_4 \cdot (Y')_{x=l} \]  

A dual parameter estimation method is furthermore presented that can be used to estimate axial load \( S \) and boundary conditions \( k \), assuming rigid transversal supports and identical rotational supports (cf. Figure 5-2).

![Figure 5-2 Illustration of a simply supported beam under axial load with rotational spring supports, Livingston (1994)](image)

At the end of the chapter, a sensitivity analysis is carried out to determine the sensitivity of the output parameters \( S \) and \( k \) with regard to errors in different input parameters.

5.1 Rayleigh-Ritz method

Since the exact determination of the first frequencies of a vibrating system is not always possible or even necessary, it can be reasonable to use approximation methods for this purpose. Such methods are for example Rayleigh’s method or its further development according to Ritz Harris (2002). While Rayleigh’s method gives only good results for the fundamental frequency, Ritz’s method can also be used to estimate some of the higher frequencies. Rayleigh’s method is based on the fact that the total energy in a vibrating system without damping is constant. When a beam
reaches its maximal deflection from its neutral state, the global potential energy is maximal and the kinetic energy is zero. When it passes through its equilibrium position, the opposite is true. For conservation of energy, these two energies therefore have to be equal, which is from where the fundamental frequency can be computed. Ritz’s method is based on the same principle with the difference that the deflection functions contain several undetermined parameters, which are adjusted in a way to minimize the frequency. It can be shown that frequencies found by using inexact shapes are always higher than the actual frequencies Den Hartog (1985). The accuracy of the result therefore depends largely on the choice of the deflection function. For the exact deflection function the computed frequency is the exact solution. Rayleigh–Ritz method is appropriate to determine the lower frequencies of simple systems.

In the case of the Timoshenko beam presented in Figure 5.1, the maximal potential and kinetic energies can be expressed as follows, according to Harris (2002):

\[
V_{\text{max}} = \frac{1}{2} \int_0^l \left[ EI \left( \frac{\partial^2 \theta}{\partial x^2} \right)^2 + k_5 GA \left( \frac{\partial v}{\partial x} - \theta \right)^2 \right] dx + \frac{1}{2} \int_0^l \left[ S \left( \frac{\partial v}{\partial x} \right)^2 \right] dx \\
+ \frac{1}{2} k_1 (v_{x=0})^2 + \frac{1}{2} k_3 (v_{x=l})^2 + \frac{1}{2} k_2 (\theta_{x=0})^2 + \frac{1}{2} k_4 (\theta_{x=l})^2 
\]

\[
T_{\text{max}} = \omega^2 \cdot T^* = \omega^2 \cdot \frac{1}{2} \int_0^l \left[ \rho A \dot{v}(x)^2 + \rho I \dot{\theta}(x)^2 \right] dx
\]

where \( T^* \) is the reference kinetic energy of the system.

For conservation of energy Eq. (5.5) and Eq. (5.6) have to be equal, which leads to the following equation, where \( V_{\text{max}} \) and \( T^* \) are functions of \( v \) and \( \theta \).

\[
V_{\text{max}}(v,\theta) - \omega^2 \cdot T^*(v,\theta) = 0
\]

If we are only interested in the first resonance frequency, we can assume functions for \( v(x) \) and \( \theta(x) \), substitute into Eq. (5.7) and solve for \( \omega \). The accuracy of the result will improve the more boundary conditions are fulfilled and the closer the functions are to the actual deflection shapes. However good results can already be obtained for rather poor input functions.

If we are however interested in resonance frequencies of higher order, Ritz’s improvement needs to be used to find accurate results. In this case the input functions are expressed as sums of functions multiplied by unknown parameters, as for example in:

\[
v(x) = \sum_{i=1}^N a_i \cdot \sin(i\pi x/l) \]

\[
\theta(x) = \sum_{i=1}^N b_i \cdot i \cdot \cos(i\pi x/l) \]

Substituting Eq. (5.8) and Eq. (5.9) into Eq. (5.7) and minimizing with regard to each unknown parameter yields a system of \( 2N \) equations with \( 2N \) unknowns. Written in matrix form, this system has a trivial solution only if the determinant equals to zero. The according equation can then be solved for the resonance frequencies. The accuracy will improve with a higher number of terms \( N \), so will however also the computational effort.
5.2 Discrete model

The discrete model is in fact a finite element approach, which means that the results are only approximations to the actual behaviour of a system. It is most suitable to determine the dynamic characteristics of complicated structures that are modelled using discrete mass, stiffness and damping matrices. These matrices have different shapes, depending on which element theory is used and how many degrees of freedom are included. In this paper, all the presented matrices were formed using Timoshenko beam elements.

5.2.1 Element mass matrix

The element mass matrix can be calculated using cubic polynomial shape functions and the principle of virtual work Friedman and Kosmatka (1993). The results are two element matrices, one accounting for effects of translatory, the other one for rotary inertia. The final element mass matrix is then obtained by forming the sum of these two matrices:

\[
M^e = M^e_T + M^e_R
\]

where

\[
M^e_T = \frac{\rho A l}{(1 + \phi)^3} \begin{bmatrix} 
\frac{13}{35} & \frac{7}{10} \phi + \frac{1}{3} \phi^2 \\
\frac{11}{210} + \frac{11}{120} \phi + \frac{1}{24} \phi^2 & \frac{11}{105} + \frac{1}{60} \phi + \frac{1}{120} \phi^2 \\
\frac{9}{70} + \frac{3}{10} \phi + \frac{1}{6} \phi^2 & \frac{9}{420} + \frac{3}{40} \phi + \frac{1}{24} \phi^2 \\
\frac{1}{105} + \frac{1}{60} \phi + \frac{1}{120} \phi^2 & \frac{1}{140} + \frac{1}{60} \phi + \frac{1}{120} \phi^2 \\
\end{bmatrix}
\]

\[
\text{symmetric}
\]

\[
M^e_R = \frac{\rho l}{(1 + \phi)^2} \begin{bmatrix} 
\frac{6}{5} & \frac{1}{10} - \frac{1}{2} \phi \\
\frac{2}{15} + \frac{1}{6} \phi + \frac{1}{3} \phi^2 & \frac{1}{10} + \frac{1}{2} \phi \\
\frac{6}{5} & \frac{1}{10} - \frac{1}{2} \phi \\
\frac{2}{15} + \frac{1}{6} \phi + \frac{1}{3} \phi^2 & \frac{1}{10} + \frac{1}{2} \phi \\
\end{bmatrix}
\]

\[
\text{symmetric}
\]

\[
\phi = 12EI/(k_s G A l^2)
\]

If shear effects and rotary effects can be neglected, these matrices can be reduced to Euler theory by setting \( \phi = 0 \) and \( M^e_R = 0 \). The reduced matrices are then the same as in Livingston (1994).

5.2.2 Element stiffness matrix

In a similar way, the element stiffness matrix can be formulated using again the principle of work. Two element matrices are obtained, one accounting for the strain of the beam, cf. Friedman and Kosmatka (1993), the other one for its axial load Paz (1997). The sum of these matrices yields the element stiffness matrix:

\[
K^e = K^e_S + K^e_L
\]
where

$$K_S^e = \frac{EI}{(1+\phi)l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & (4+\phi)l^2 & -6l & (2-\phi)l^2 \\ -12 & -6l & 12 & -6l \\ 6l & (2-\phi)l^2 & -6l & (4+\phi)l^2 \end{bmatrix}$$

$$K_L^e = \frac{S}{30l} \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}$$

$$\phi = \frac{12EI}{(k_s G A l^2)}$$

The according Euler element stiffness matrix can again be obtained by setting $$\phi = 0$$, Livingston (1994).

### 5.2.3 Element damping matrix

The damping matrix can be taken into account in different ways, depending on the damping properties of the system (for example viscous or hysteretic), cf. Harris (2002) for more information. In this thesis, damping is considered as a negligible parameter.

### 5.2.4 Global matrix formulation

According to Livingston (1994), there are three steps in the formulation of the global matrices:

1. Divide the system in a number of elements
2. Identify nodes between the elements and number consequently the degrees of freedom
3. Determine the mass and stiffness matrices for each element and add them into the global mass and stiffness matrices

This procedure is called the direct method.

As an example, consider the beam from Figure 5.1 with $$L = 3$$, $$E = 1$$, $$I = 1$$ and $$S = 0$$, modelled with three elements, each having a length of $$l = 1$$. Neglecting effects from shear and rotary inertia, the global mass and stiffness matrices are:
As it can be seen, the element matrices overlap and the first and last element stiffness matrices contain the boundary conditions of the beam.

The so obtained global matrices can be used to formulate the equation of motion for an undamped system under free vibration:

\[ M \ddot{v} + K v = 0 \]  

(5.12)

where \( v \) and \( \ddot{v} \) are vectors of transversal displacement and acceleration respectively, and \( M \) and \( K \) the global mass and stiffness matrices. The solution of the eigenvalue problem to Eq. (5.12) yields the natural frequencies of the examined system.

There are several FE softwares available that are based on the above or similar calculations. In this thesis, the Matlab based FE software SFVIBAT Akesson and O. (1980) is used in Chapter 5.4 for validation of the models.
5.3 Continuous model

The continuous model is most suitable for simple systems, like for example single bars or beams. For complex systems, the computational effort is too large, resulting in an inefficient use of memory. The advantage of the continuous model is however that it yields an exact solution since the differential Eq. (4.29) is solved directly. According to Howson and Williams (1973), this equation can be solved with the following functions:

\[ Y = C_1 \cosh \left( \frac{x}{l} \lambda_1 \right) + C_2 \sinh \left( \frac{x}{l} \lambda_1 \right) + C_3 \cos \left( \frac{x}{l} \lambda_2 \right) + C_4 \sin \left( \frac{x}{l} \lambda_2 \right) \]  
\[ \Psi = C_1' \sinh \left( \frac{x}{l} \lambda_1 \right) + C_2' \cosh \left( \frac{x}{l} \lambda_1 \right) + C_3' \sin \left( \frac{x}{l} \lambda_2 \right) + C_4' \cos \left( \frac{x}{l} \lambda_2 \right) \]

where

\[ \lambda_1 = \frac{b}{\sqrt{2(1 - s^2 p^2)}} \sqrt{\Delta + \sqrt{\Delta^2 + \frac{4}{b^2} (1 - s^2 p^2)(1 - b^2 r^2 s^2)}} \]

\[ \lambda_2 = \frac{b}{\sqrt{2(1 - s^2 p^2)}} \sqrt{\Delta + \sqrt{\Delta^2 + \frac{4}{b^2} (1 - s^2 p^2)(1 - b^2 r^2 s^2)}} \]

with

\[ b^2 = \rho Al^2 \omega^2 / EI \]
\[ r^2 = l / Al^2 \]
\[ \phi = k_s G A \]
\[ s^2 = EI / \phi l^2 \]
\[ p^2 = Pl^2 / EI \]
\[ \Delta = (p^2 / b^2) + r^2(1 - s^2 p^2) + s^2 \]

Furthermore it can be shown that Howson and Williams (1973):

\[ C_1' = HC_1 \]  
\[ C_2' = HC_2 \]  
\[ C_3' = -ZC_3 \]  
\[ C_4' = HC_4 \]

where

\[ H = \frac{(1 - s^2 p^2) \lambda_1^2 + b^2 s^2}{\lambda_1 l} \]
\[ Z = \frac{(1 - s^2 p^2) \lambda_2^2 - b^2 s^2}{\lambda_2 l} \]

Substituting Eq. (4.25) and Eq. (4.26) together with Eq. (5.13)-(5.18) into the boundary conditions (5.1)-(5.4) yields a system of four equations with four unknowns that can be written in the following matrix form:
\[ \mathbf{C}_M \cdot \mathbf{C} = 0 \]  
(5.19)

where

\[
\mathbf{C}_M = \begin{bmatrix}
<k_1> & <-\phi + \phi \frac{\lambda_1}{l} + S \frac{\lambda_1}{l}> \\
<k_3 c_2 - \phi H c_1 + \phi c_1 \frac{\lambda_1}{l} + Sc_1 \frac{\lambda_1}{l}> & <k_3 c_1 - \phi H c_2 + \phi c_2 \frac{\lambda_1}{l} + Sc_2 \frac{\lambda_1}{l}>
\end{bmatrix}
\]

\[
\begin{bmatrix}
<k_1> & <-\phi + \phi \frac{\lambda_2}{l} + S \frac{\lambda_2}{l}> \\
<k_3 c_4 + \phi Z c_3 - \phi c_3 \frac{\lambda_2}{l} - Sc_3 \frac{\lambda_2}{l}> & <k_3 c_3 - \phi Z c_4 + \phi c_4 \frac{\lambda_2}{l} + Sc_4 \frac{\lambda_2}{l}>
\end{bmatrix}
\]

\[
\mathbf{C} = \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}
\]

and

\[
c_1 = sinh(\lambda_1)
\]
\[
c_2 = cosh(\lambda_1)
\]
\[
c_3 = sin(\lambda_2)
\]
\[
c_4 = cos(\lambda_2)
\]

A non-trivial solution to this system can only be found if the determinant of \( \mathbf{C}_M \) vanishes, leading to

\[
|\mathbf{C}_M| = 0
\]  
(5.20)

This equation can be solved for the angular eigenfrequencies \( \omega_n \), from which the corresponding frequencies \( f_n \) can be computed. This can be done for different boundary conditions \( k_1 \) - \( k_4 \). In this thesis, the \( k_i \) are set to zero to model free support conditions, or they are set to values of \( 10^{15} \) to simulate rigid supports.

### 5.4 Model comparison

Table 5-1 and Table 5-2 shows the frequencies for different boundary conditions with and without axial load, calculated using the discrete and continuous model, cf. Appendix A, as well as the above-mentioned SFVIBAT software. Furthermore the frequencies according to Euler theory are presented to show the difference in the results compared to the Timoshenko theory. The chosen parameters are: \( E = 13000 \) MPa, \( G = 760 \) MPa, \( \rho = 400 \) kg/m\(^3\), \( L = 1.5m \), \( h = 75mm \) and \( b = 35mm \).
Table 5-1  Comparison of transversal frequencies of a beam with different boundary conditions calculated with different numerical models

<table>
<thead>
<tr>
<th>Beam</th>
<th>Continuous Model</th>
<th>Discrete Model (n=50)</th>
<th>SPVIBAT (n=50)</th>
<th>Discrete Model (n=50)</th>
</tr>
</thead>
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<td>90.64</td>
<td>90.19</td>
<td>91.16</td>
</tr>
<tr>
<td>pinned-pinned</td>
<td>245.49</td>
<td>246.06</td>
<td>243.51</td>
<td>251.28</td>
</tr>
<tr>
<td>f1</td>
<td>40.02</td>
<td>40.03</td>
<td>39.99</td>
<td>40.21</td>
</tr>
<tr>
<td>rotational springs</td>
<td>157.84</td>
<td>157.97</td>
<td>157.37</td>
<td>160.85</td>
</tr>
<tr>
<td>(k=10000 Nm)</td>
<td>59.49</td>
<td>59.50</td>
<td>59.44</td>
<td>60.11</td>
</tr>
<tr>
<td>f2</td>
<td>181.34</td>
<td>181.49</td>
<td>180.81</td>
<td>186.03</td>
</tr>
<tr>
<td>fixed-fixed</td>
<td>89.10</td>
<td>89.17</td>
<td>89.02</td>
<td>91.16</td>
</tr>
<tr>
<td>f2</td>
<td>239.19</td>
<td>239.49</td>
<td>238.45</td>
<td>251.28</td>
</tr>
</tbody>
</table>

Table 5-2  Comparison of transversal frequencies of an axially loaded beam with different boundary conditions calculated with different numerical models

<table>
<thead>
<tr>
<th>Beam</th>
<th>Continuous Model</th>
<th>Discrete Model (n=50)</th>
<th>SPVIBAT (n=50)</th>
<th>Discrete Model (n=50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>free-free</td>
<td>135.85</td>
<td>136.00</td>
<td>135.34</td>
<td>136.56</td>
</tr>
<tr>
<td>f2</td>
<td>288.11</td>
<td>288.78</td>
<td>285.83</td>
<td>293.71</td>
</tr>
<tr>
<td>pinned-pinned</td>
<td>60.97</td>
<td>60.98</td>
<td>60.92</td>
<td>61.10</td>
</tr>
<tr>
<td>f2</td>
<td>182.66</td>
<td>182.82</td>
<td>182.11</td>
<td>185.30</td>
</tr>
<tr>
<td>rotational springs</td>
<td>75.45</td>
<td>75.47</td>
<td>75.39</td>
<td>75.96</td>
</tr>
<tr>
<td>(k=10000 Nm)</td>
<td>203.46</td>
<td>203.65</td>
<td>202.87</td>
<td>207.70</td>
</tr>
<tr>
<td>fixed-fixed</td>
<td>102.52</td>
<td>102.54</td>
<td>102.43</td>
<td>104.54</td>
</tr>
<tr>
<td>f2</td>
<td>258.44</td>
<td>258.65</td>
<td>257.65</td>
<td>270.16</td>
</tr>
</tbody>
</table>

The discrete models calculate with \( n = 50 \) finite elements while the continuous models yield exact solutions for the respective model assumptions. Rigid supports are modelled setting \( k = 10^{15} \) Nm. The Euler model assumes a much stiffer behaviour of the system since shear deformations are neglected, leading to higher frequencies. They derive already considerably for the second bending mode. The comparison of the Timoshenko models shows good accordance for the first mode, while the differences are already higher for the second mode. The reason is the limited number of finite elements and leading to the fact that these models always yield approximate solutions. It can furthermore be observed that the continuous solution always lies in between the two discrete solutions, indicating a good validation of this model.

5.5 Parameter estimation

For the actual parameter estimation, the continuous model based on Timoshenko theory is used in the following. In most practical cases, as well as in the tests in Chapter 8, it is reasonable to assume that the translational supports can be modelled as rigid, while the rotational supports are modelled as springs. In some cases, it is even acceptable to assume identical support conditions at both sides. These considerations yield to the two systems illustrated in Figure 5-3 (a).

The first system in has two unknown parameters for which the solution requires therefore at least two resonance frequencies. In theory, also more frequencies could be used for error minimisation. The measurement of higher vibration modes might however be more likely to contain errors, which makes the error minimisation
unsuitable for some materials. This is for example the case for inhomogeneous timber, especially in presence of knots. Homogeneous materials like steel do not show these problems and allow also the accurate measurement of higher frequencies. If only two resonance frequencies are used, a plot similar to the one in Figure 5-3 (c) is the result. For a known rotational stiffness $k$, there exists only one axial load $S$ for which the beam vibrates in a given frequency. This way, a so-called contour curve can be depicted for each frequency by varying one of the parameters $k$ and $S$.

The second system in Figure 5-3 (b) has three unknown parameters for which the solution requires therefore at least three resonance frequencies. Instead of curves, each frequency is determined by a surface (cf. Figure 5-3 (d)). The intersections of these surfaces are the solutions to the system. Here, there are two solutions since $k_1$ and $k_2$ are exchangeable for symmetry reasons.

Figure 5-3 Simply supported beam under axial load with spring supports of identical stiffness (a), simply supported beam under axial load with spring supports of different stiffness (b), illustration of dual parameter estimation (c) and illustration of triple parameter estimation (d).

5.6 Sensitivity analysis for continuous model with $k_1 = k_2$

In the following, the sensitivities of the parameters $S$ and $k$ are investigated for different load levels and under the assumption of equal boundary conditions with regard to the following input parameters:

- Clear length $L$
- Density $\rho$
- E-modulus $E$
- G-modulus $G$
- Measured frequencies $f_i$

The errors of $S$ and $k$ are standardised to 25%, which allows a direct comparison of the values. A steep curve means a high sensitivity to the according parameter.
5.6.1 Sensitivity of $S$

Figure 5-4  Sensitivity of the axial load $S$ with regard to the clear length $L$ for different load levels

Figure 5-5  Sensitivity of the axial load $S$ with regard to the density $\rho$ for different load levels

Figure 5-6  Sensitivity of the axial load $S$ with regard to the E-modulus $E$ for different load levels
Figure 5-7  Sensitivity of the axial load $S$ with regard to the shear modulus $G$ for different load levels

Figure 5-8  Sensitivity of the axial load $S$ with regard to the first measured frequency for different load levels

Figure 5-9  Sensitivity of the axial load $S$ with regard to the second measured frequency for different load levels
5.6.2 Sensitivity of k

Figure 5-10  Sensitivity of the rotational stiffness $k$ with regard to the clear length $L$ for different load levels

Figure 5-11  Sensitivity of the rotational stiffness $k$ with regard to the density $\rho$ for different load levels

Figure 5-12  Sensitivity of the rotational stiffness $k$ with regard to the E-modulus $E$ for different load levels
Figure 5-13  Sensitivity of the rotational stiffness $k$ with regard to the shear modulus $G$ for different load levels

Figure 5-14  Sensitivity of the rotational stiffness $k$ with regard to the first measured frequency for different load levels

Figure 5-15  Sensitivity of the rotational stiffness $k$ with regard to the second measured frequency for different load levels
5.6.3 Discussion

In Figure 5-4 - Figure 5-9, it can be seen that the sensitivity of S with regard to other parameters decreases for high loads. The highest sensitivity can be observed for the measured frequencies, which shows the importance of a high measurement quality. The second highest sensitivity is associated with $L$. It is therefore important to choose a reasonable restraint length at the supports, which is not always easy, especially for complicated connections or when the load is transferred over more than one connection part. The next highest sensitivity is related to $\rho$ and $E$, followed by the shear modulus $G$. While the density can usually be determined with good accuracy, the estimation of $E$ and $G$ is not so straightforward.

The sensitivity of $k$ is less influenced by the size of the axial load. Figure 5-14 and Figure 5-15 show however that its sensitivity with regard to the measured frequencies increases for high axial loads, contrary to the sensitivity of $S$. Figure 5-10 - Figure 5-15 illustrate, that just as for the axial load, the sensitivity of $k$ is highest for the frequencies, followed by the length $L$ and finally the material properties $\rho$, $E$ and $G$. In general, one can say that the sensitivity of $k$ is higher than the one of $S$. 
6 Modal analysis

Modal analysis is used to describe a structure’s dynamic properties, like resonance frequencies, mode shapes and damping. Those are specific for every system and do not depend on the excitation, but only on the nature of the system and its components.

This chapter gives a short summary of different aspects from theoretical and experimental modal analysis with a main focus on data acquisition and processing.

6.1 Theoretical modal analysis (TMA)

In theoretical modal analysis, the modal parameters are determined by solving the differential equation of motion.

Every vibrating system can be described by using the equation of motion

\[ \mathbf{M} \cdot \ddot{\mathbf{v}} + \mathbf{C} \cdot \dot{\mathbf{v}} + \mathbf{K} \cdot \mathbf{v} = \mathbf{f}(t) \] (6.1)

where \( \mathbf{M} \) is the mass matrix, \( \mathbf{C} \) the damping matrix and \( \mathbf{K} \) the stiffness matrix. \( \mathbf{f}(t) \) is a vector containing a time-dependent excitation force. The vector \( \mathbf{u} \) describes the displacement of the system over time, \( \dot{\mathbf{v}} \) and \( \ddot{\mathbf{v}} \) the velocity and acceleration, respectively.

If we only consider the free vibration of Eq. (6.1) and neglect the damping

\[ \mathbf{M} \cdot \ddot{\mathbf{v}} + \mathbf{K} \cdot \mathbf{v} = 0 \] (6.2)

and chose a solution of the form

\[ \mathbf{v} = \mathbf{\phi} \cdot \cos(\omega \cdot t - \varphi) \] (6.3)

where \( \mathbf{\phi} \) is a vector, \( \omega \) the angular frequency and \( \varphi \) the phase angle, we obtain the following Eigenvalue problem:

\[ (-\omega^2 \cdot \mathbf{M} + \mathbf{K}) \cdot \mathbf{\phi} = 0 \] (6.4)

For a non-trivial solution, the determinant of Eq. (6.4) must vanish

\[ |-\omega^2 \cdot \mathbf{M} + \mathbf{K}| = 0 \] (6.5)

By solving Eq. (6.5) for \( \omega \), we obtain the angular resonance frequencies \( \omega_i \) of the system. Substitution of each eigenfrequency into Eq. (6.4) then yields the eigenvectors \( \mathbf{\phi}_i \) of the system. Each eigenvector \( \mathbf{\phi}_i \) represents a deflection over time at a frequency \( \omega_i \) and defines a specific modal shape of the free vibration. The deflection pattern of a structure subjected to any excitation force can in theory be expressed as the linear summation of its modal shapes.

While the solution of the eigenvalue problem is simple for harmonic force functions, it is not adaptable for pulses or other non-harmonic excitations. In this case, the resonance frequencies can be determined using a Fourier transformation.
6.2 Experimental modal analysis (EMA)

In practice, it is often not possible to use the analytical model because of different uncertainties. In this case the modal parameters can be determined experimentally. According to Dossing (1988) there are two different approaches in experimental modal analysis:

- Signal analysis (operational data)
- System analysis (modal data)

In signal analysis, only the response signal of the system is detected, while in system analysis also the force function is measured. As an example, consider the bridge in Figure 6-1. During service, it is subjected to wind forces that can cause undesirable vibrations. These vibrations can be measured and the obtained operational data can be used to study the bridges’ behaviour and to see what resonance frequencies are excited during operation and if there is need for adjustments. If one is however interested for example in specific resonance frequencies of the bridge, it is necessary to subject it to a force able to excite these frequencies. In this case the force can also be measured and considered in the modal analysis of the structure. The main difference is that the force has to be taken as given in the first case, while it is actively controlled in the second case.

![Figure 6-1 Illustration of a bridge with measurement of operational data (left) and modal data (right)](image)

For a given force function it is possible to detect which resonance frequencies are actually excited by having a look at the frequency response function (FRF). The FRF is defined as the ratio of response function (Output) to the force function (Input):

\[
FRF = \frac{Output}{Input} \tag{6.6}
\]

The force and response functions can be measured using accelerometers and force transducers. The collected data are time signals. Since however the frequencies are of interest, it is necessary to convert the signals from the time domain into the frequency domain (cf. Figure 6-2), which is done by performing a Fast Fourier Transformation (FFT).
6.3 Data acquisition

According to Avitabile (2001), there are two ways in system analysis to find the frequencies and mode shapes of a vibrating system (cf. Figure 6-3):

- Measure in one point and excite in several points
- Excite in one point and measure in several points

If only the frequencies are of interest and if the mode shapes can be predicted (or are actually known) since it is a simple structure like a beam with known boundary conditions, it is also possible to determine the according frequencies by only making one measurement and one excitation in the same point (drive point measurement). When doing this it is however very important that the point being measured is not a node for any of the desired mode shapes, since in this case the mode and its corresponding frequency are not being excited.

Accelerometers can be used for the measurement of the response signal. According to Dossing (1988), the main advantages are their low weight, wide frequency range and the simple mounting using either magnets, steel studs or bee-wax. The system can be excited either using a shaker or a hammer equipped with a piezoelectric transducer. While shakers are suitable to excite a system over longer time periods, hammers are used for short impulses. The hammers come in different sizes and tips, depending on what frequency range is of interest. The essential parameters are the weight of the hammer and the stiffness of the tips. The heavier a hammer is, the lower are the excited frequencies. The stiffer the tip, the higher are the excited frequencies. Soft tips (rubber, plastic) can therefore excite lower frequencies while hard tips (steel) are
better to excite higher frequencies. It is important to choose an adequate hammer and tips so that the energy of the hammer blow excites the desired frequency range and a good response is obtained. Figure 6-4 shows impulse shapes for different hammer tips.

![Figure 6-4 Impulse shape for different hammer tips, Kjaer (2012)](image)

When using accelerometers and additional equipment to acquire the modal data of a structure, it is also important that their weight is small compared to the one of the structure and its components. If this is not the case, they have to be considered as additional masses that change the mass properties of the system. The same applies for suspensions that can have a stiffening effect to the structure, which means a change in the stiffness properties. Furthermore it has to be investigated if the damping has to be taken into account to determine the frequencies or if it can be neglected.

The results acquired by accelerometers and force transducers represent timeline data that need to be further processed to obtain the eigenfrequencies. The advantage of measuring the input function is that errors originating for example from noise can be reduced.

### 6.4 Data processing

The time data acquired during measurements can be transformed to the frequency domain by performing a FFT.

Steps of the FFT according to Figure 6-5:

1. Analog signals must be filtered to remove high frequency signals that might be detected during testing without having any connection to the actual test (background noise)
2. The analog signals are approximated by conversion into digital signals
3. Weighting functions called Windows are used to reduce leakage (cf. below)
4. The actual FFT is performed to create linear spectra of the input and output data, transfer from time domain to frequency domain
5. The input, output and cross power spectra are computed.
6. These functions are then averaged and used to compute the FRF and the coherence function. While the FRF can be used to read out the resonance frequencies, the coherence function can be used for quality assessment of the data.
Figure 6-5  Illustration of the different steps of an FFT Analyser, Avitabile (2001)

Figure 6-6 shows an example of a FRF and the according coherence function. Peaks in the coherence function at a resonance frequency indicate a poor quality of the measurement. In this case, the quality of the results decreases for frequencies higher than approximately 4000 Hz. This is the maximum range of the hammer tip in use.

Figure 6-6  Example of a frequency response function and the according coherence function
Leakage is a signal processing error causing the distortion of data (cf. Figure 6-7). It is one of the most serious errors when processing data and has to be treated with special care. It occurs during the FFT transformation, which in theory requires a representation of data for all times or at least a periodic repetition. In practice, this is of course not possible since every signal has a finite observation period. There are however ways to reduce the leakage effect already during the data acquisition phase. On the one hand, the observation period should be chosen long enough so that the signal is not truncated before it has decayed to zero. One the other hand, it should not be longer than necessary since during the additional time only background noise can be recorded. In both cases, the result can be a blurry power spectrum with too low and unclear peaks.

![Figure 6-7 Illustration of the effect of truncation during data acquisition on result for the frequency response function, Dossing (1988)](image)

After data acquisition, it is possible to further increase the resolution by applying the already mentioned window functions. While there are different window functions, they all have more or less the same effect on the signal, by improving its periodicity and forcing it to zero. A popular example for a window is the Hanning function, which is good to improve the periodicity of the sample. It is bell-shaped and heavily reduces the beginning and end of the sample function to zero. Another example is the transient window that can for example be used to remove noise following an impact pulse. Exponential windows can in addition be used to process truncated data. It should be noted however that windows cause some data distortion themselves and should be avoided, if possible. They have a negative influence on the accuracy of peak amplitudes and the damping ratios. If only the frequencies are of interest, these effects can however be considered of secondary importance and are therefore acceptable.
7 Determination of material properties

7.1 Specimen dimensions and properties

The material for the specimen comes from previous experiments on glulam beams from Norway spruce (*Picea abies*). These beams were first sorted to make sure only intact timber was used to produce the new specimens. Further requirements were to minimize the amount of knots, finger joints and other natural defects like resin pockets. A total of 32 beams, each consisting of two lamellas with the dimensions $42 \times 45 \text{ mm}^2$, were then cut out to a length of 1.5 m from the original glulam beams. The thereby obtained pieces were then evenly planed to the following final measurements of the samples: $L \times H \times B = 1500 \times 75 \times 35 \text{ mm}^3$. Figure 7-1 shows one of the final specimens.

The choice of the sample size is based on the following conditions:

- The tensile machine limited the cross-sectional dimensions to a size smaller than $77 \times 45 \text{ mm}^2$, which is why a height of 75 mm was chosen.
- The height-to-width ratio had to be high enough to facilitate the measurement of torsional vibrations and to be able to distinguish the frequencies from the different vibration modes. The torsional frequencies are important to determine the shear modulus that is needed to take into account the effect of shear deformation on the bending modes. Also the proportions had to be realistic with regard to real-life constructions.
- The length-to-width ratio was chosen high enough to make sure that frequency changes for different stress levels are in a detectable range.

All these considerations yielded a bar-like timber beam of realistic proportions, such as they can be found for instance in roof structures, bridges or supports.

The original glulam beams were graded CE L40c according to European standards. This class is constituted of a combination of lamellas of different strength, the weaker ones situated on the inside, the stronger forming the outside layers (at least two strong lamellas per side). The specimens were cut out from the outer lamellas and can thus be considered as part of the higher quality timber. The mean values for the material properties of L40c are listed in Table 7-1.
Table 7-1 Material properties of the timber specimens according to SIA (2003)

<table>
<thead>
<tr>
<th>Material properties of L40c</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean modulus of elasticity parallel to the grain</td>
<td>$E_{0,\text{mean}}$</td>
<td>13000 MPa</td>
</tr>
<tr>
<td>Mean modulus of shear</td>
<td>$G_{\text{mean}}$</td>
<td>760 MPa</td>
</tr>
<tr>
<td>Tensile strength parallel to the grain</td>
<td>$f_{t,0,k}$</td>
<td>17.6 MPa</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho_k$</td>
<td>400 kg/m$^3$</td>
</tr>
</tbody>
</table>

The specimens were packed in plastic after being cut. Over the whole testing period, they were kept in a room with an average temperature of 20°C and a relative humidity of ranging from 30 to 40%. The moisture content of each specimen was measured in three places. The registered average moisture contents are shown in Appendix B. The mean moisture content of all the specimens was determined to 12.5%.

The actual densities for each sample were determined by weighing them. The results are also shown in Appendix B. The mean value of 488.6 kg/m$^3$ is well above the one given in Table 7-1. The reason could be that only higher quality lamellas were cut out of the initial beams, which in general have higher densities.

The tests carried out on the inhomogeneous timber specimen were also conducted on an aluminium bar, representing a homogeneous reference. The comparison of the final results should eventually allow identifying the material-related deviations. The aluminium bar had the dimensions 1495×50×10 mm$^3$. The general material properties of industrial aluminium are presented in Table 7-2 and the bar itself is illustrated in Figure 7-2.

![Illustration of reference aluminium specimen with body dimension](image)

Figure 7-2 Illustration of reference aluminium specimen with body dimension
Table 7-2  Material properties of the reference aluminium specimen according to Wikipedia.org (2012)

<table>
<thead>
<tr>
<th>Material property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of elasticity E</td>
<td>70000 MPa</td>
</tr>
<tr>
<td>Modulus of shear G</td>
<td>26000 MPa</td>
</tr>
<tr>
<td>Tensile strength $f_u$</td>
<td>45 MPa</td>
</tr>
<tr>
<td>Density $\rho_k$</td>
<td>2700 kg/m³</td>
</tr>
</tbody>
</table>

7.2 Test equipment and data processing

For all the dynamic tests carried out throughout this study, the same equipment was used. The beams were tapped with an instrumented hammer shown in Figure 7-3 (a). It was equipped with a piezoelectric force transducer PCB 208B05 that allowed measuring the impulse function. These hammers come with different exchangeable tips, depending on the desired frequency range. The response signal from the beams was recorded using uniaxial accelerometers PCB 303A02, cf. Figure 7-3 (b). Their weight of 2.8 g was small enough to exclude effects on the beam vibration. The accelerometers were attached to the beams with bee-wax. Hammer and accelerometers were connected over PCB 478A01 power suppliers to an HP 8-channel system linked to the computer.

![Impact hammer equipped with piezoelectric force transducer and steel tip (a) and uniaxial accelerometer (b)](image)

The software DAC Express from VTI Instruments was used for the acquisition of the raw signal data. The actual data processing was carried out using the software Matlab from the company MathWorks to transform the signal data from the time into the frequency domain by means of a Fast Fourier Transformation (FFT). The result of this transformation is the Frequency Response Function (FRF) from which the resonance frequencies of a beam can be extracted.
7.3 Determination of static E- and G-modulus

Four-point bending tests according to CEN/TC (2007) were carried out to determine the static E-modulus of each specimen. The beams were tested with two sets of weights for which the deflection at mid-point was measured. Furthermore, the deflections over the supports were measured and subtracted from the mid-deflection to take into account local compressions. The E-moduli were then calculated using (7.1):

\[
E = \frac{L^3(k_2-k_1)}{bh^3(w_2-w_1)} \left[ \frac{3a}{4L} \right] - \left( \frac{a}{L} \right)^3
\]

(7.1)

where \( h \) and \( b \) are the cross-sectional dimensions, \( L \) is the span, \( a \) is the distance between the loading and the nearest support, \( k \) are the total loads and \( w \) the according deflections (cf. Figure 7-4). The test setup is shown in Figure 7-5.

The test results are shown in Table 7-3.
Table 7-3  Measured deflections and computed E-moduli from the four-point bending test

<table>
<thead>
<tr>
<th>Beam No.</th>
<th>Deflection 1</th>
<th>Deflection 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Supp 1</td>
<td>Supp 2</td>
</tr>
<tr>
<td>1</td>
<td>0.166</td>
<td>0.205</td>
</tr>
<tr>
<td>2</td>
<td>0.235</td>
<td>0.372</td>
</tr>
<tr>
<td>3</td>
<td>0.141</td>
<td>0.467</td>
</tr>
<tr>
<td>4</td>
<td>0.138</td>
<td>0.393</td>
</tr>
<tr>
<td>5</td>
<td>0.181</td>
<td>0.359</td>
</tr>
<tr>
<td>6</td>
<td>0.290</td>
<td>0.364</td>
</tr>
<tr>
<td>7</td>
<td>0.127</td>
<td>0.405</td>
</tr>
<tr>
<td>8</td>
<td>0.127</td>
<td>0.448</td>
</tr>
<tr>
<td>9</td>
<td>0.376</td>
<td>0.401</td>
</tr>
<tr>
<td>10</td>
<td>0.163</td>
<td>0.323</td>
</tr>
<tr>
<td>11</td>
<td>0.130</td>
<td>0.261</td>
</tr>
<tr>
<td>12</td>
<td>0.192</td>
<td>0.342</td>
</tr>
<tr>
<td>13</td>
<td>0.142</td>
<td>0.398</td>
</tr>
<tr>
<td>14</td>
<td>0.226</td>
<td>0.400</td>
</tr>
<tr>
<td>15</td>
<td>0.206</td>
<td>0.385</td>
</tr>
<tr>
<td>16</td>
<td>0.148</td>
<td>0.301</td>
</tr>
<tr>
<td>17</td>
<td>0.398</td>
<td>0.687</td>
</tr>
<tr>
<td>18</td>
<td>0.149</td>
<td>0.375</td>
</tr>
<tr>
<td>19</td>
<td>0.155</td>
<td>0.214</td>
</tr>
<tr>
<td>20</td>
<td>0.303</td>
<td>0.563</td>
</tr>
<tr>
<td>21</td>
<td>0.193</td>
<td>0.477</td>
</tr>
<tr>
<td>22</td>
<td>0.126</td>
<td>0.408</td>
</tr>
<tr>
<td>23</td>
<td>0.175</td>
<td>0.433</td>
</tr>
<tr>
<td>24</td>
<td>0.128</td>
<td>0.400</td>
</tr>
<tr>
<td>25</td>
<td>0.188</td>
<td>0.321</td>
</tr>
<tr>
<td>26</td>
<td>0.226</td>
<td>0.401</td>
</tr>
<tr>
<td>27</td>
<td>0.167</td>
<td>0.689</td>
</tr>
<tr>
<td>28</td>
<td>0.127</td>
<td>0.412</td>
</tr>
<tr>
<td>29</td>
<td>0.179</td>
<td>0.370</td>
</tr>
<tr>
<td>30</td>
<td>0.191</td>
<td>0.499</td>
</tr>
<tr>
<td>31</td>
<td>0.217</td>
<td>0.528</td>
</tr>
<tr>
<td>32</td>
<td>0.145</td>
<td>0.548</td>
</tr>
</tbody>
</table>

Mean value: 12717  Std. dev.: 1127

The mean value of 12717 MPa shows good accordance with the mean E-modulus given in Table 7-1. The slightly lower values probably result from the fact that only two lamellas are left from the original beam, which increases the negative influence of local defects on material properties. Figure 7-6 shows a good correlation between E-moduli and densities of the samples. In general, it can be said that material properties improve with an increasing density. In this case, outliers from this rule result from extreme concentrations of knots and other defects.
Determination of longitudinal E-modulus

The longitudinal E-moduli were determined through dynamic tests on the beams under free boundary conditions. For this purpose, the beams were supported on foam pieces as shown in Figure 7-7. The beams were then tapped five times in one end with the instrumented hammer and the vibration was measured over a time period of 7 seconds using the uniaxial accelerometer situated on the other end. For this test, a steel tip was chosen that is suitable to excite higher frequencies.

The results for each experiment were the timelines for the accelerometer and the force transducer, upon which a Fast Fourier Transformation (FFT) was then carried out via the software Matlab. The results of this transformation were the input and output power spectra, the Coherence Function and finally the Frequency Response Function (FRF), cf. Figure 7-8.
Figure 7-8 Frequency plots and coherence function resulting from the data processing of the different signals recorded during the longitudinal tests.
It can be seen that the input power spectrum shows a break at around 4.2 kHz. The coherence function shows also high fluctuations from this point on. This is the maximum frequency range for the chosen hammer-tip combination. The first two longitudinal frequencies could be read out from the FRF and subsequently be used to calculate the longitudinal E-modulus with help of the following formula derived from (4.33):

$$E_L = \frac{4f^2l^2\rho}{t^2}$$

(7.2)

The E-moduli were calculated using either the first or the second frequencies. Furthermore the mean value was computed (cf. Table 7-4).

Table 7-4: Measured frequencies and computed E-moduli from the longitudinal tests

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Mean value: 14308 MPa, Std. dev.: 1128 MPa.

The best correlation with the static E-moduli from the bending tests could be obtained using only the first frequency. The mean value was found to be 14308 MPa, which means that the longitudinal E-moduli are on average 12.5% higher than the static ones. This is in accordance with previous studies Ilic (2001).

A summary of the correlation as well as the FRF plots can be found in Appendix B.
7.5 Determination of transversal E- and G-modulus

Transversal vibration tests were carried out to determine the E- and G-moduli of the beams. The determination of the G-moduli was necessary to include effects of shear deformations that become more important for higher modes. Just as for the longitudinal tests, the beams were tested under free boundary conditions. The setup is however different from the one in Chapter 7.4. The free-free conditions were achieved by hanging the beams vertically on soft rubber strings. These elastic strings are necessary to reduce the overall stiffness of the system to get as close as possible to a state of the free boundary conditions. The beams were tested in a vertical position to minimize the influence of the supports on the transversal waves. The support is illustrated in Figure 7-9 (a).

![Figure 7-9 Support conditions for the determination of the transversal E-modulus (a) and the according test setup (b)](image)

The equipment used for the tests consisted of two uniaxial accelerometers attached to the beam with bee-wax. For a free-free beam, the ends are antinodes that allow the measurement of frequencies for different vibration modes. This is why the accelerometers were placed on the end of the beam in two different spots according to Figure 7-9 (b). The specimens were excited five times with the instrumented hammer, equipped this time with a plastic tip, since the expected frequencies were in a lower range than for the longitudinal tests. The specimens had to be tapped on the edge to also excite torsional frequencies. The response signals were recorded for 15s and then transmitted to the computer where they could finally be processed to calculate the FRFs (cf. Figure 7-10).
Figure 7-10 Frequency plots and coherence function resulting from the data processing of the different signals recorded during the transversal tests.
Since A1 was placed in the neutral axis of the beam, it could only detect vibrations corresponding to bending modes. A2 however was placed on the edge and would therefore also record torsional vibrations. By comparing the FRFs of A1 and A2 it was then possible to distinguish bending and torsional frequencies (cf. Figure 7-11).

![Comparison between the frequency response function from the accelerometer situated in the neutral axis of the beam (left) and the one situated on the edge of the beam (right)](image)

The G-moduli were calculated using Eq (7.3) derived from Eq (4.46)

\[
G_F = \frac{4f^2l^2pl_p}{k_t^2} \quad \text{(7.3)}
\]

with the either the first or second frequency. Also the mean value of both G-moduli is presented (cf. Table 7-5).
Table 7-5 Measured torsional frequencies and computed G-moduli from the transversal tests

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Mean value: 791 | 811 | 801
Std. dev.: 65 | 59 | 61

Aluminium: 374.7 |

The transversal E-moduli were calculated using Timoshenko theory as well as Euler theory for comparison. Since no simple expression is available for a free-free beam, the self-developed Matlab code in Appendix A was used for the calculations. Again, the E-moduli were calculated for different frequencies, as shown in Table 7-6.
Table 7-6 Measured transversal frequencies and computed E-moduli from the transversal tests

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<td>12668</td>
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</table>

Mean value: 13795 MPa, 9.5% higher than the static one.

Appendix B presents a summary of the correlation and FRF plots.

7.6 Summary of results

In conclusion, one can say that the first frequencies are most suitable for the estimation of material properties since they are least influenced by material defects and hence can be measured with higher precision. The negative effect of defects increases with the number of modes, which excludes the use of higher frequencies for error minimisation.
Table 7-7 summarizes the results from the material properties estimation. As already mentioned, the dynamic E-moduli are higher than the static ones. Also, the mean of the estimated G-moduli is higher than the one in Table 7-1. This difference is mainly due to creeping that occurs during static testing. While the dynamic tests are carried out over a short time period of just a few seconds, the static tests last several minutes during which the timber starts to creep, resulting in the lower values for the static E-modulus.

Nevertheless, good correlations could be observed between the different E-moduli (cf. Figure 7-12 - Figure 7-14). The correlation between longitudinal and translational E-moduli is even nearly perfect.
Figure 7-12  Correlation between static and longitudinal E-moduli

Figure 7-13  Correlation between static and transversal E-moduli

Figure 7-14  Correlation between longitudinal and transversal E-moduli
Figure 7-15 gives a summary of the estimated E-moduli and their mean values.

Figure 7-15  Summary of the estimated E-moduli from the different tests and according mean values
8 Tension Tests

8.1 Setup and equipment

The tensile tests were carried out using the Aktiebolaget Alpha Sundbyberg machine shown in Figure 8-1. The beams were fixed at both ends with friction grips illustrated in Figure 8-2. The beams were chosen to stand out 10 mm on both sides to prevent them from sliding out of the grips for higher loads. The grips had a length of 120 mm which reduced the minimum possible clear length of the beams to $L_{\text{min}} = 1500 - 2 \cdot 10 - 2 \cdot 120 = 1240$ mm. Since the surface of these grips is not flat, but curved in both directions with a global maximum at approximately 60% of their length, the maximum possible clear length results in $L_{\text{max}} = L_{\text{min}} + 2 \cdot 50 = 1340$ mm. This caused difficulties in assessing the actual clear length of the beams, which is further discussed in Chapter 8.3.

![Figure 8-1](image-url) Tensile machine and setup of the accelerometer (left) and scale for the determination of the axial load (right)
The scale chosen for the tensile machine was 10000 kg, which allowed a precision of 10 kg. The accelerometer was attached with bee-wax at a distance of 250 mm from the lower grip, which equals approximately 20% of \( L_{\text{min}} \). This ensured the recording of the first three bending frequencies. The excitation was performed with 5 hammer blows right next to the accelerometer using an instrumented hammer equipped with a plastic tip. The technical equipment used for the modal testing is described in more detail in Chapter 7. The data acquisition settings are shown in Table 8-1. These settings allowed a resolution of the frequencies of approximately 0.1 Hz.

<table>
<thead>
<tr>
<th>Measurement function</th>
<th>Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trigger type</td>
<td>Accelerometer</td>
</tr>
<tr>
<td>Trigger level</td>
<td>0.1 V</td>
</tr>
<tr>
<td>Trigger delay</td>
<td>-0.01 V</td>
</tr>
<tr>
<td>Sampling rate</td>
<td>2560</td>
</tr>
<tr>
<td>Recording time</td>
<td>7 s</td>
</tr>
<tr>
<td>Blocksize</td>
<td>1024</td>
</tr>
<tr>
<td>Frequency range</td>
<td>1280 Hz</td>
</tr>
<tr>
<td>NFFT</td>
<td>( 2^{16} )</td>
</tr>
</tbody>
</table>

The frequency measurements were carried out for 6 different load levels between 5 kN and 30 kN with an interval of 5 kN. These levels were chosen on the assumption that the maximum tensile load was around 40 kN, corresponding to a tensile strength of 15 MPa. Later ultimate tensile tests described in Chapter 8.4 showed however that the mean tensile strength is considerably higher, namely 25.9 MPa for the tested beams.
8.2 Test results

The FRF plots and results from the frequency measurements are shown in Appendix C. Figure 8-3 shows the FRFs for the aluminium bar and one timber specimen for different load levels. It can be seen that the geometry of the aluminium bar was chosen in a way that the frequency increase due to a higher load is of a comparable magnitude as for a timber specimen. This is important for the comparison of parameter estimation results since the sensitivity study showed that the axial load is very sensitive to errors in frequency (cf. Chapter 5). Moreover one can see that the peaks are much more clear for the aluminium than for the timber specimen, which is mainly due to the differences in material homogeneity.

Table 8-2 lists the first three bending frequencies of the above specimen for the different load levels.

Table 8-2 Comparison of measured frequencies for aluminium (left) and timber (right)

<table>
<thead>
<tr>
<th>Applied load $S$</th>
<th>$f_{1a}$ [Hz]</th>
<th>$f_{2a}$ [Hz]</th>
<th>$f_{3a}$ [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1952</td>
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<td>93.1</td>
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</tr>
<tr>
<td>4022</td>
<td>40.0</td>
<td>98.9</td>
<td>184.0</td>
</tr>
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<td>44.5</td>
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</tr>
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<td>46.4</td>
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<td>197.6</td>
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<td>49.5</td>
<td>114.9</td>
<td>204.4</td>
</tr>
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<td>124.3</td>
<td>216.6</td>
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<tr>
<td>15696</td>
<td>57.8</td>
<td>129.4</td>
<td>223.3</td>
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</table>

<table>
<thead>
<tr>
<th>Applied load $S$</th>
<th>$f_{1b}$ [Hz]</th>
<th>$f_{2b}$ [Hz]</th>
<th>$f_{3b}$ [Hz]</th>
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</thead>
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<td></td>
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</tr>
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<td>123.6</td>
<td>319.0</td>
<td>584.7</td>
</tr>
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<td>19013</td>
<td>128.1</td>
<td>327.1</td>
<td>596.2</td>
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<td>331.9</td>
<td>600.8</td>
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<tr>
<td>27959</td>
<td>134.2</td>
<td>336.6</td>
<td>604.2</td>
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</table>

Figure 8-4 – Figure 8-6 show plots of the squares of the three first measured frequencies $f_k^2$ against the applied load $S$ under the assumption of equal boundary conditions $k$ at both ends of the beams. Since different friction grips were used for timber and aluminium, the respective restraint lengths are also different, namely 1260 mm and 1245 mm respectively. For the aluminium bar, the measured frequencies are all more or less arranged in a line, which shows that the boundary conditions remain nearly constant for different load levels. The assumption of equal restraints at both
ends is valid for the aluminium bar as the friction grips are identical and the aluminium is homogeneous. For the timber specimen only the frequencies of the three highest loads are situated on a line. The main reason for this is that the clear length changes over different load levels. The problem is that the pressure from the grips applied on the timber is not constant, but increases with the strain applied on the beam. Thereby, the grips penetrate deeper and deeper into the soft wood and through their curved surface, more and more reduce the clear length. In the next chapter, this will be taken into account by setting different lengths for the lower load levels. Another observation when comparing the figures is that the graph of the third frequency does not correspond well with the other two graphs. Already during data acquisition, it became clear that the peak of the third frequency is not clear enough to fulfil the precision requirements, at least not for the applied measurement method. The same conclusion could already be drawn in Chapter 7 during the estimation of the material properties. The higher the frequency modes, the lower are their accuracies and thus their use for parameter estimation. This is further discussed in the next paragraph.

Figure 8-4  Plots of axial load against square of calculated first frequency for different boundary conditions together with measured first frequencies for different load levels for aluminium (left) and timber (right)

Figure 8-5  Plots of axial load against square of calculated second frequency for different boundary conditions together with measured second frequencies for different load levels for aluminium (left) and timber (right)
Figure 8-6  Plots of axial load against square of calculated third frequency for different boundary conditions together with measured third frequencies for different load levels for aluminium (left) and timber (right)

8.3 Estimation of tensile force S

The bending frequencies resulting from the data processing in Chapter 8.2 are in the following used to estimate the axial load S and the boundary conditions at the restraints. In the first step, this is done using only the two first frequencies under the assumption of equal boundary conditions. In the second step, it is attempted to use also the third frequency either for error minimisation or to expand the model by a third parameter by dropping the assumption of equal boundary conditions. Since the stiffness of the timber specimen is much smaller compared to the steel grips, the rotational stiffness at the supports is mainly influenced by the E-modulus of the specimen. For the inhomogeneous timber, the E-modulus is however not constant over the whole specimen length, which can consequently lead to a difference in boundary conditions. This means that the use of an extra frequency should have a positive effect on the results, provided that it could be determined with an appropriate precision.

For the parameter estimation, Timoshenko theory was used to include effects of shear deformations and rotary inertia since they can have a rather high influence on the results (cf. Chapter 5). The Matlab code illustrated in Appendix B was used for this purpose.

While the material properties and the sectional dimensions had already previously been determined, the choice of the clear specimen length \( L \) was still unknown. As explained above, this length changes for different load levels due to characteristics of the tensile machine. This effect is illustrated in Figure 8-7 and Figure 8-8 for aluminium and timber, respectively.
The plots show the variation of the rotational stiffness $k$ for different lengths $L$ and load levels $S$. When comparing the two graphs, it is obvious that the effect of the grips is much less pronounced for aluminium than for timber. In Figure 8-8, the curves associated with the two lowest load levels show strong deviations from the pattern of the other levels. In Figure 8-7, the curves are much closer together and the deviation therefore less obvious. The conclusion was that the clear length could be assumed constant for the aluminium bar while it had to be varied for the timber beam, at least for the two lowest load levels, where the clear length turned out to be bigger compared to the other levels. For the timber specimen, the length for the lowest load was chosen to the maximum length $L_{\text{max}} = 1295 \text{ mm}$. Under the assumption of constant boundary conditions, the length for the second highest load was then determined to $L = 1275 \text{ mm}$, and the length for the remaining levels to $L = 1255 \text{ mm}$. For aluminium, $L$ was assumed to be $1245 \text{ mm}$ for all load levels. The difference in length for the two materials is related to the fact that different grips with different surface properties had to be used.

After all the input parameters were now determined, the parameter estimation could be initiated. Figure 8-9 shows the graphical estimation for aluminium and timber, previously explained in Chapter 5.
The use of two frequencies gave reasonable results of the axial load $S$ for both materials. The comparison of the graphs shows however that the third frequency is only of use for aluminium, while it is far off the crossing point of the two first frequencies for the timber specimen. As previously mentioned, the third frequency could not be determined with the required precision. This is why the parameter estimation was limited to the two first frequencies, which makes the assumption of equal boundary conditions indispensible and therefore excludes the expansion of the model to a third parameter. Table 8-3 and Table 8-4 show the according results for the aluminium bar and a timber specimen using the transversal as well as the longitudinal E-moduli to investigate the difference in results. It was also attempted to use the static E-modulus, but the results were far from accurate for both materials, showing that the estimation requires dynamic values for the material stiffness.

Table 8-3  
Results from the dual parameter estimation for the reference aluminium specimen using the transversal and longitudinal E-modulus

<table>
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<tr>
<th>Applied load $S$</th>
<th>% of yield</th>
<th>$S_0$</th>
<th>$f_0$</th>
<th>$f_0\prime$</th>
<th>$L$</th>
<th>$k_{x_0}$</th>
<th>$S_{x_0}$</th>
<th>Error on $S$</th>
<th>$k_{x_0\prime}$</th>
<th>$S_{x_0\prime}$</th>
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<td>12.5</td>
<td>53.10</td>
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Table 8-4  
Results from the dual parameter estimation for the timber specimen N°6 using the transversal and longitudinal E-modulus

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<th>Applied load $S$</th>
<th>% of yield</th>
<th>$S_0$</th>
<th>$f_0$</th>
<th>$f_0\prime$</th>
<th>$L$</th>
<th>$k_{x_0}$</th>
<th>$S_{x_0}$</th>
<th>Error on $S$</th>
<th>$k_{x_0\prime}$</th>
<th>$S_{x_0\prime}$</th>
<th>Error on $S$</th>
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<td>14.50</td>
<td>64.0</td>
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<td>14.0</td>
<td>26.8</td>
<td>14.0</td>
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<td>30.0</td>
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<td>26.8</td>
<td>14.0</td>
<td>26.8</td>
<td>14.0</td>
</tr>
</tbody>
</table>

The results for aluminium are very good, with absolute errors ranging from 15.2% to 3.2% for the transversal E-modulus. As expected, the results tend to improve for higher loads since the sensitivity of the system with regard to the input parameters...
decreases. There is however a small error spread that can be explained by the choice of a constant mean clear length and probably also some measurement errors. The results using the longitudinal E-modulus turned out to be less accurate than the ones using the transversal E-modulus. The same conclusions can be drawn for the timber specimens, where the results for this particular sample range from 68.1% to 10.3% when using the transversal E-modulus. The results for the other specimens are listed in Appendix C.

Table 8-5 shows the mean errors and deviations for the estimated axial load $S$ using different E-moduli for all 32 timber specimens. The variation of the mean error for different load levels is illustrated in Figure 8-10. A complete illustration of the individual errors and deviations for all specimens is listed in Appendix C.

Table 8-5  Mean errors of the estimated axial load $S_{est}$ for all timber specimens using the transversal and longitudinal E-moduli

<table>
<thead>
<tr>
<th>Applied load $S$ [N]</th>
<th>% of yield</th>
<th>$S/S_y$ [-]</th>
<th>Mean error on $S$ [%]</th>
<th>Std. dev. on error [%]</th>
<th>Error on $S$ [%]</th>
<th>Std. dev. on error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>7.4</td>
<td>0.25</td>
<td>46.6</td>
<td>89.5</td>
<td>68.4</td>
<td>102.5</td>
</tr>
<tr>
<td>10000</td>
<td>14.7</td>
<td>0.48</td>
<td>34.3</td>
<td>45.0</td>
<td>48.1</td>
<td>52.6</td>
</tr>
<tr>
<td>15000</td>
<td>22.1</td>
<td>0.70</td>
<td>27.8</td>
<td>39.9</td>
<td>41.5</td>
<td>48.3</td>
</tr>
<tr>
<td>20000</td>
<td>29.4</td>
<td>0.94</td>
<td>16.2</td>
<td>26.9</td>
<td>24.1</td>
<td>30.4</td>
</tr>
<tr>
<td>25000</td>
<td>36.8</td>
<td>1.17</td>
<td>9.4</td>
<td>23.7</td>
<td>14.4</td>
<td>26.2</td>
</tr>
<tr>
<td>30000</td>
<td>44.1</td>
<td>1.40</td>
<td>7.6</td>
<td>20.2</td>
<td>12.4</td>
<td>22.1</td>
</tr>
</tbody>
</table>

Figure 8-10  Graphical representation of the mean errors from Table 8-5 for the different load levels

The results show that the estimation of $S$ improves for higher load levels. Again, the best results could be obtained using the transversal E-modulus. It can be seen that the differences for the two E-moduli decrease for higher loads. The axial loads are in general overestimated, with mean errors ranging from 7.6% to 46.6%. The standard deviations are of the same order, which results in a very high spread for the results. The main reason is probably the clear lengths $L$ that was chosen the same for every specimen, even though there might be differences caused by varying material properties. This parameter was the most difficult to assess since the restraint length cannot be visually determined, but has a big influence on the results. Other errors
result from the assumption of equal boundary conditions and frequency measurement errors. Also, the applied model assumes that material properties are constant over the specimen length, which is clearly not the case for the inhomogeneous timber. Considering all these model uncertainties, the estimated axial loads appear quite reasonable and give an incentive for further research.

In addition to the estimated axial loads, the boundary conditions were analysed in more detail. Figure 8-11 shows the estimated rotational stiffness $k_{est}$ for different load levels of two timber specimens. The plots show rather random variations of the rotational stiffness. The only trend that can be recognized is that it seems to increase in the range of higher loads. This could however also be related to the choice of the varying clear length that has a high influence on the estimation for the rotational stiffness. Overestimated values of the clear length $L$ lead to higher values of $k$ (cf. Figure 5-10).

![Figure 8-11](image_url) Plots of the estimated rotational stiffness $k_{est}$ for different load levels and two different timber specimens using the transversal and longitudinal E-moduli

Figure 8-12 illustrates the correlation of the mean estimated rotational stiffness over different load levels for one specimen and its static E-modulus. The respective values are listed in Appendix C.
It was expected that $k$ would increase for a higher the specimen stiffness since the boundary conditions were believed to be mainly determined by the material properties. However no such trend could be verified from the data. Maybe this is due to the fact that the range of the E-moduli is rather small and that the errors to $k$ are rather high. The distribution of the boundary conditions seems rather constant with a mean value of 207695 Nm and a standard deviation of 169567 Nm when using the transversal E-moduli. For comparison, the mean value was 9358 Nm for the Aluminium bar. The difference is due to the smaller dimensions of the bar and the resulting much lower polar moment and bending stiffness.

### 8.4 Determination of ultimate tensile strength

Tensile tests according to CEN/TC (2007) were carried out on some of these specimens to determine an approximate tensile strength and to put it in relation to the E-moduli and the applied loads. The results are shown in Table 8.6 and Figure 8.13. Even though a small correlation between E-moduli and tensile strength is recognisable, the amount of data is too small to make final conclusions on this subject. The mean tensile strength was found to be 25.9 MPa with a standard deviation of 4.7 MPa for this amount of data.
Table 8-6  Results from the ultimate tensile tests

<table>
<thead>
<tr>
<th>Beam N°</th>
<th>E₀ [MPa]</th>
<th>S_{max} [N]</th>
<th>f_{l,0,max} [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>11311</td>
<td>66000</td>
<td>25.1</td>
</tr>
<tr>
<td>20</td>
<td>12842</td>
<td>61000</td>
<td>23.2</td>
</tr>
<tr>
<td>28</td>
<td>13376</td>
<td>55000</td>
<td>21.0</td>
</tr>
<tr>
<td>5</td>
<td>14615</td>
<td>86000</td>
<td>32.8</td>
</tr>
<tr>
<td>11</td>
<td>15165</td>
<td>60000</td>
<td>22.9</td>
</tr>
<tr>
<td>1</td>
<td>16805</td>
<td>80000</td>
<td>30.5</td>
</tr>
<tr>
<td>Mean value</td>
<td>14019</td>
<td>68000</td>
<td>25.9</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>1927</td>
<td>12280</td>
<td>4.7</td>
</tr>
</tbody>
</table>

Figure 8-13  Plot of the static E-modulus against the maximum tensile load for the tested specimens

Figure 8-14 illustrates the failure mode of the timber beams. The pressure caused by the friction grips induced local compression at the restraints, leading to a reduced section and hence stress peaks in this area. Resulting cracks at the restraint section of one lamella then relocated the tension on the remaining intact one, which in turn yielded at a knot or otherwise weakened section leading to ultimate failure.

Figure 8-14  Timber specimen after ultimate tensile test: local compression at the restraint (left) and final failure mode (right)
9 Conclusions and further research

9.1 Conclusions

The results of the dual parameter estimation showed that it is possible to estimate the axial loads in timber beams using resonance frequency analysis. The sensitivity analysis showed however that the quality of the results is very sensitive to errors in the input parameters, especially the measured frequencies, the clear length and the E-modulus.

It was therefore very important to ensure a high measurement quality and to only use a minimum of resonance frequencies since the precision decreased quickly for higher vibration modes. An error minimisation through additional frequency information was hence not possible in this research. The length of the measurement time and number of excitations was just too small to obtain clear peaks for the higher resonance frequencies of the timber specimens. For the reference aluminium specimen, these factors were however sufficient. This shows that the inhomogeneous character of timber, in association with its natural defects, requires more sophisticated measurement techniques than a homogeneous material. One possibility of improving the frequency resolution would be to use a modal vibration shaker instead of an instrumented hammer. This way more measurements can be made over a longer period of time, resulting in more clear frequency peaks.

The clear length of the beams was another factor that had a strong influence on the test results and caused a high standard deviation from the mean value. The problem was that the available tensile machine was not designed to apply a constant grip pressure on the beam ends. In fact, the grip pressure increased for higher load levels, leading to a varying clear length and also rotational stiffness at the restraints. In addition, these parameters varied for each specimen caused by the difference in material properties. It was therefore not possible to identify the exact clear length for each measurement. Hence, a mean clear length had to be chosen that caused the high spread in results for the parameter estimation. Note that the determination of the restraint length will also be of high importance when applying this method on real structures and therefore needs to be treated with special care.

The best results for the axial load were obtained using the E-modulus obtained from transversal vibration tests with a mean error ranging from 7.6% to 46.6%. When using the longitudinal E-moduli, the mean errors increased to 12.4% to 89.5%. Note, that in both cases the results consistently improved for higher loads. This was in accordance with the results from the sensitivity analysis. It was furthermore attempted to use the static E-modulus for the calculations, which led however to unusable results. This shows that dynamic values need to be used for the estimation of parameters when using the presented resonance frequency method. For the application of the method on real structures, it would therefore be possible to determine the E-modulus using longitudinal frequency measurements or maybe even time of flight measurements.

The model used for the parameter estimation was based on Timoshenko theory. This allowed to include effects of shear deformations and rotary inertia that had a major influence on results, especially for higher vibration modes.
9.2 Suggestions for further research

The results of the presented research were promising and hence raise the need for further research in this area. In the following, some suggestions are listed.

- **Influence of specimen geometry**

  One of the limitations of this research was the geometry of the specimen. It would be interesting to see how the sensitivity of the results changes for other specimen dimensions, especially bigger sizes. It is hereby important to use a machine that applies constant grip pressure to the specimen ends, so that the restraint length does not vary for different load levels and can be determined with more precision.

- **Testing of specimens under compression**

  Only specimens under tension were analysed in this thesis. The research could be expanded to specimens under compression to see if similar results can be obtained or if there are maybe additional effects that need to be taken into account in this case.

- **Tests on real structure**

  Finally, in-situ tests could be carried out on an existing structure to analyse the applicability of the method under real-life conditions. For this purpose, a modal vibration shaker might be necessary to improve the frequency resolution by increasing the measurement time. Also the frequency plots will probably be more complicated since other structural members might participate in vibration. One solution to this problem could be to attach additional accelerometers in different points of the structure to be able to allocate the frequencies to the respective members.
10 References


Roymech.co.uk. (2010), from http://www.roymech.co.uk/Useful Tables/Torsion/Torsion.html


11 Appendices

11.1 Appendix A – Matlab codes

11.1.1 Calculation of transversal frequencies

Discrete model – Euler beam theory

The program uses the input parameters to define element mass and stiffness matrices for each finite element, which are then put together to one global matrix. The solution of the according eigenvalue problem then yields the resonance frequencies.

```matlab
clear all
clc

%%Enter input data
E = 13000*10^6;     %%E-modulus in N/m2
G = 760*10^6;       %%G-modulus in N/m2
p = 400;            %%Specimen density in kg/m3
L = 1.5;            %%Clear length in m
H = 0.075;          %%Specimen height in m
B = 0.035;          %%Specimen width in m
S = 20000;          %%Axial load in N
k1 = 10^15;         %%Value for translational support in x=0 in N/m
k3 = 10^15;         %%Value for translational support in x=L in N/m
k2 = 10000;         %%Value for rotational support in x=0 in Nm
k4 = 10000;         %%Value for rotational support in x=L in Nm
n = 50;             %%Number of finite elements
nnf = 2;            %%Number of resonance frequencies

%%Calculation
A = B*H;
Iz = B^3*H/12;
el = L/n;

if (k1 == 0) && (k2 == 0) && (k3 == 0) && (k3 == 0);
nnf = nnf + 2;
end
```

%%Connection matrix
for ie = 1:n
    ktr = 2*ie;
    cn(ie,:) = [ktr-1:ktr+2];
end

%Consistent element mass matrix
em = zeros(4);
em = [ 156    22*el   54    -13*el;
      22*el  4*el^2  13*el   -3*el^2;
      54    13*el  156   -22*el;
     -13*el  -3*el^2 -22*el   4*el^2];
em = p*A*el/420 * em;

%Element stiffness matrix
ek = zeros(4);
ek = [ 6    3*el   -6     3*el;
      3*el  2*el^2  -3*el   el^2;
      -6   -3*el     6     -3*el;
      3*el  el^2   -3*el    2*el^2];
ek = (2*E*Iz/el^3)*ek;
gk = [ 36    3*el  -36     3*el;
      3*el  4*el^2  -3*el  -el^2;
     -36   -3*el     36    -3*el;
      3*el  -el^2   -3*el    4*el^2];
kg = (S/(30*el))*gk;
ek = ek + kg;

%Global matrix
k = zeros(ktr+2);
m = zeros(ktr+2);
for ie = 1:n
    index = cn(ie,:);
    m(index,index) = m(index,index) + em;
    k(index,index) = k(index,index) + ek;
end
k(1,1) = k(1,1) + k1;
k(2,2) = k(2,2) + k2;
k(ktr+1,ktr+1) = k(ktr+1,ktr+1) + k3;
k(ktr+2,ktr+2) = k(ktr+2,ktr+2) + k4;
%%Solve eigenvalue problem
[v,nu] = eig(k,m);
[w2,ie] = sort(diag(nu));
v = v(:,ie);

%%Print natural frequencies (Hz)
for ie = 1:nnf
    fcalc(ie) = sqrt(w2(ie))/(2*pi);
    fprintf('%f
', fcalc(ie))
end

Discrete model – Timoshenko beam theory
The program uses the same principles as for the Euler theory, except that the mass and stiffness matrices are composed of additional terms to take into account effects from shear deformations and rotary effects.

%%Remove all variables from the workspace and clear command window
clear all
clc

%%Enter input data
E = 13000*10^6;  %%E-modulus in N/m2
G = 760*10^6;    %%G-modulus in N/m2
p = 400;         %%Specimen density in kg/m3
L = 1.5;         %%Clear length in m
H = 0.075;       %%Specimen height in m
B = 0.035;       %%Specimen width in m
S = 20000;       %%Axial load in N
k1 = 10^15;      %%Value for translational support in x=0 in N/m
k3 = 10^15;      %%Value for translational support in x=L in N/m
k2 = 10000;      %%Value for rotational support in x=0 in Nm
k4 = 10000;      %%Value for rotational support in x=L in Nm
n = 50;          %%Number of finite elements
nnf = 2;         %%Number of resonance frequencies

%%Calculation
\[ A = B^*H; \]
\[ ks = 5/6; \]
\[ I_z = B^3*H/12; \]
\[ el = L/n; \]
\[ fi = 12*E*I_z/(ks*G*A*el^2); \]
\[
\text{if } (k1 == 0) && (k2 == 0) && (k3 == 0) && (k3 == 0);
\]
\[ nnf = nnf + 2; \]
\[
\text{Connection matrix}
\]
\[
\text{for } ie = 1:n
\]
\[ ktr = 2*ie; \]
\[ cn(ie,:) = [ktr-1:1:ktr+2]; \]
\[
\text{Consistent mass matrix}
\]
\[ em = zeros(4); \]
\[
\text{emt} = \begin{bmatrix}
13/35+7/10*fi+1/3*fi^2 & (11/210+11/120*fi+1/24*fi^2)*el \\
9/70+3/10*fi+1/6*fi^2 & -(13/420+3/40*fi+1/24*fi^2)*el \\
(1/105+1/60*fi+1/120*fi^2)*el^2 & (13/420+3/40*fi+1/24*fi^2)*el \\
9/70+3/10*fi+1/6*fi^2 & -(13/420+3/40*fi+1/24*fi^2)*el \\
\end{bmatrix};
\]
\[
\text{emr} = \begin{bmatrix}
6/5 & (1/10-1/2*fi)*el \\
(1/10-1/2*fi)*el & -6/5 \\
(1/10-1/2*fi)*el & (1/10-1/2*fi)*el \\
-6/5 & (1/10-1/2*fi)*el \\
\end{bmatrix};
\]
\[
\text{em} = p*A*el/(1+fi)^2*emt + p*A*I_z/((1+fi)^2*el)*emr;
\]
\[
\text{Stiffness matrix}
\]
\[ ek = zeros(4); \]
\[
\text{kb} = \begin{bmatrix}
12 & 6*el \\
6*el & (4*fi)*el^2 \\
-12 & -6*el \\
6*el & (2-2*fi)*el^2 \\
\end{bmatrix};
\]
\[
\text{kg} = \begin{bmatrix}
36 & 3*el \\
3*el & 4*el^2 \\
-36 & -3*el \\
3*el & -el^2 \\
\end{bmatrix};
\]
ek = (E*Iz/((1+fi)*el^3))*kb + (S/(30*el))*kg;

%%Global matrix
k = zeros(ktr+2);
m = zeros(ktr+2);
for ie = 1:n
    index = cn(ie,:);
    m(index,index) = m(index,index) + em;
    k(index,index) = k(index,index) + ek;
end
k(1,1) = k(1,1) + k1;
k(2,2) = k(2,2) + k2;
k(ktr+1,ktr+1) = k(ktr+1,ktr+1) + k3;
k(ktr+2,ktr+2) = k(ktr+2,ktr+2) + k4;

%%Solve eigenvalue problem
[v,nu] = eig(k,m);
[w2,ie] = sort(diag(nu));
v = v(:,ie);

%%Print natural frequencies (Hz)
for ie = 1:nnf
    fcalc(ie) = sqrt(w2(ie))/(2*pi);
end

Continuous model – Timoshenko beam theory
For the entered input parameters, the program determines the lowest transversal frequency for the chosen frequency range by finding the first root of equation (5.20). If no root exists for the entered frequency range, the output will be ‘change frequency range’. For higher frequencies, the range needs to be adjusted.

%%Remove all variables from the workspace and clear command window
clear all
cle
clc

%%Enter input data
E = 13000*10^6; %E-modulus in N/m²
G = 760*10^6; %G-modulus in N/m²
p = 400; %Specimen density in kg/m³
L = 1.5; %Clear length in m
H = 0.075; %Specimen height in m
B = 0.035; %Specimen width in m
S = 20000; %Axial load in N
k1 = 10^15; %Value for translational support in x=0 in N/m
k3 = 10^15; %Value for translational support in x=L in N/m
k2 = 10000; %Value for rotational support in x=0 in Nm
k4 = 10000; %Value for rotational support in x=L in Nm

%Define frequency range of interest
fmin = 20; %Minimum frequency
fmax = 77; %Maximum frequency

%Iteration
A = B*H;
ks = 5/6;
Iz = B^3*H/12;
step = 0.01;
i = 0;
for f0 = fmin:step:fmax
w = 2*pi*f0;
b2 = A*L^4*p*w^2/(E*Iz);
r2 = Iz/(A*L^2);
f1 = ks*A*G;
s2 = (E*Iz)/(f1*L^2);
p2 = -((L^2*S)/(E*Iz));
delta = p2/b2+r2*(1-p2*s2)+s2;
m1 = (sqrt(b2)*sqrt(sqrt((4*(1-p2*s2)*(1-b2*r2*s2))/b2+delta^2)-delta))/sqrt(2*(1-p2*s2));
m2 = (sqrt(b2)*sqrt(sqrt((4*(1-p2*s2)*(1-b2*r2*s2))/b2+delta^2)+delta))/sqrt(2*(1-p2*s2));
H = (b2*s2+m1^2*(1-p2*s2))/(L*m1);
Z = (m2^2*(1-p2*s2)-b2*s2)/(L*m2);
c1 = sinh(m1);
c2 = cosh(m1);
\[ c_3 = \sin(m_2); \]
\[ c_4 = \cos(m_2); \]
\[
M = \begin{bmatrix}
-k_1 & -\text{fi}*H+\text{fi}*m_1/L+m_1*S/L & -k_1 & -\text{fi}*Z+\text{fi}*m_2/L+m_2*S/L; \\
-k_3*c_2+\text{fi}*H*c_1+\text{fi}*m_1*c_1/L+\text{fi}*\text{si}c_1/L & -k_3*c_4+\text{fi}*Z*c_3+\text{fi}*m_2*c_3/L-m_2*S*c_3/L & -\text{fi}*Z*c_4+\text{fi}*m_2*c_4/L+m_2*S*c_4/L+k_3*c_3; \\
-k_3*c_2+\text{fi}*H*c_1+\text{fi}*m_1*c_1/L+\text{fi}*\text{si}c_1/L & -k_3*c_4+\text{fi}*Z*c_3+\text{fi}*m_2*c_3/L-m_2*S*c_3/L & -\text{fi}*Z*c_4+\text{fi}*m_2*c_4/L+m_2*S*c_4/L+k_3*c_3;
\end{bmatrix}
\]
\[
\text{Det} = \det(M);
\]
i=i+1;
\[
\text{result}(i,1) = f_0; \]
\[
\text{result}(i,2) = \text{Det}; \]
\[
\text{sig}(i) = \text{sign}(\text{Det}); \]
end
\[
\text{sig} = \text{sig}';
\]
for i=1:length(sig)-1
\[
\text{diff} = \text{sig}(i)-\text{sig}(i+1);
\]
if diff==0
\[
\text{continue}
\]
else break
end
end

%%Print output in [Hz]
if i == (fmax-fmin)/step
\[
\text{output} = '\text{change frequency range}'
\]
elseif abs(result(i,2)) < abs(result(i+1,2))
\[
f = \text{result}(i,1)
\]
else f = result(i+1,1)
end

### 11.1.2 Calculation of longitudinal E-moduli

After the data from the measurements is imported, the time signals are transformed into frequency data from which the first two resonance frequencies are extracted. These are then used to calculate the dynamic longitudinal E-moduli for the according specimen. The input and output power spectra are plotted along with the frequency response function and the coherence function. The output of the program is the two frequencies and the according E-moduli.
%%Remove all variables from the workspace, clear command window and close
%%%open windows

clear all

%%File import

[filename, pathname] = uigetfile('*.txt', 'Pick an Input File');
infile=[pathname filename];
importfile(infile)
clear('colheaders');
clear('textdata');

%%Variable declaration

s=data(:,1);                       %%time vector
V=data(:,2);                       %%input signal
V1=data(:,3);                      %%output signal
specnum = sscanf(filename,'T%f');  %%specimen number for density
SR = 10240;                        %%Sampling rate

% Densities of the 32 specimen in kg/m³

M = [527.7 511.7 485.6 496.5 515.0 497.0 483.6 466.5 511.5 505.4 532.8 522.2 511.7 464.8 500.8 493.0 447.5 494.0 462.0 460.7 488.6 484.1 460.4 507.4 501.3 465.0 479.0 517.1 447.5 459.7]
%%Density extraction
p = M(specnum);

%%Geometry
L=1.5; % Length in m

%%FFT into spectra
[x0,f] = pwelch(V,[],[],[],SR);
[x1,f] = pwelch(V1,[],[],[],SR);

%%Estimation of FRF
[x2,f] = tfestimate(V,V1,[],[],[],SR);

%%Estimation of Coherence function
[x3,f] = mscohere(V,V1,[],[],[],SR);

%%Logging for semi-logarithmic plot
P0=log(x0);
P1=log(x1);
P2=real(log(x2));
P3=real(log(x3));

%%Deleting of initial peaks
P0(1:20)=0;
P1(1:20)=0;
P2(1:20)=0;
P3(1:20)=0;

%%Plotting of input and output spectra, FRF and Coherence function
figure('Color', [1 1 1]);
plot(f,P0);
xlim([0 4200]);
title(['Specimen N=' num2str(specnum) ': Input Power Spectrum']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');
figure('Color', [1 1 1]);
plot(f,P1);
xlim([0 4200]);
title(['Specimen N∞',num2str(specnum),': Output Power Spectrum']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');
figure('Color',[1 1 1]);
plot(f,P3);
xlim([0 4200]);
title(['Specimen N∞',num2str(specnum),': Coherence Function']);
xlabel('Frequency [Hz]');
ylabel('Magnitude [-]');

%%Peak identification for first two frequencies
%%For some specimens, the Minpeakdistance needs to be adjusted

[PKS,LOCS] = findpeaks(P2,'SORTSTR','descend','MINPEAKDISTANCE',400);
for i=1:2
    peaks(i,1) = LOCS(i);
    peaks(i,2) = PKS(i);
end
peaks_sort = sortrows(peaks,1);

f1l=f(peaks_sort(1,1))
b1=P2(peaks_sort(1,1))
f2l=f(peaks_sort(2,1))
b2=P2(peaks_sort(2,1))

%%Printing frequencies
f1l=roundn(f1l,0)
f2l=roundn(f2l,0)

%%Marking of the identified frequencies
figure('Color',[1 1 1]);
plot(f,P2);
xlim([0 4200]);
title(['Specimen N∞',num2str(specnum),': Frequency Response Function']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');
hold on;
plot(f1l,b1+0.05,'k^','markerfacecolor','r');
plot(f2l,b2+0.05,'k^','markerfacecolor','r');
text(f1l-40,b1+0.3,num2str(f1l));
text(f2l-40,b2+0.3,num2str(f2l));
hold off;

%%Calculation of E-modulus using Euler theory
E1_Euler=round(4*f1l^2*L^2*p/(1^2)*10^(-6));
E2_Euler=round(4*f2l^2*L^2*p/(2^2)*10^(-6));

%%Print E
E1_Euler
E2_Euler

%%Create output file
fid = fopen('Output.out','w');
fprintf(fid, '%g
',specnum);
fprintf(fid, '%g
',f1l);
fprintf(fid, '%g
',f2l);
fprintf(fid, '%g
',E1_Euler);
fprintf(fid, '%g
',E2_Euler);

%%Plot FRF and Coherence
plotyy(f,P2,f,P3)
[AX,H1,H2] = plotyy(f,P2,f,P3,'plot');
set(get(AX(1),'Ylabel'),'String','Amplitude [dB]')
set(get(AX(2),'Ylabel'),'String','Magnitude [-]')
xlabel('Frequency [Hz]');
title(['Specimen N∞',num2str(specnum),': FRF and Coherence Function']);
set(AX(1),'XLim',[0 4200]);
set(AX(2),'XLim',[0 4200]);
hold on;
plot(f1l,b1+0.05,'k^','markerfacecolor','r');
plot(f2l,b2+0.05,'k^','markerfacecolor','r');
11.1.3 Calculation of transversal E-moduli

This script works very similar than the one for longitudinal frequencies. It is however distinguished between transversal and torsional frequencies. Also additional functions need to be used to calculate the different material properties. The output is the measured transversal and torsional frequencies as well as the calculated E- and G-moduli according to Euler and Timoshenko theory.

%%Remove all variables from the workspace, clear command window and close open windows
clc
clear all
close all

%%File import
[filename, pathname] = uigetfile('*.txt', 'Pick an Input File');
infile=[pathname filename];
importfile(infile)
clear('colheaders');
clear('textdata');

%%Variable declaration
s=data(:,1); % time vector
V=data(:,2); % input signal
V2=data(:,4); % output signal at the edge
specnum = sscanf(filename,'T%f') % specimen number for density
SR=2560; % Sampling rate
NFFT=2^16; % Number of FFT points

M = [527.7
     511.7
     485.6
     496.5
     515.0
     497.0
     483.6
     466.9];
%% Density extraction
p = M(specnum);

%% Geometry
L = 1.5;  \hspace{1cm} \text{% Length in m}
H = 0.075;  \hspace{1cm} \text{% Height in m}
B = 0.035;  \hspace{1cm} \text{% Width in m}
A = B * H;
ks = 5/6;
Iy = B * H^3 / 12;
Iz = B^3 * H / 12;
Ip = Iy + Iz;
Kt = H * B^3 / 3 * (1 - 0.630 * B / H + 0.052 * B^5 / H^5);

%% FFT into spectra
[x0, f] = pwelch(V, [], [], NFFT, SR);
[x2, f] = pwelch(V2, [], [], NFFT, SR);

%% Estimation of FRF
[x4, f] = tfestimate(V, V2, [], [], NFFT, SR);

%% Estimation of Coherence function
[x5, f] = mscohere(V, V2, [], [], NFFT, SR);

%% Logging for semi-logarithmic plot
P0 = \log(x0);
P2 = \log(x2);
P4 = \text{real}(\log(x4));
P5 = \log(x5);

%% Reducing data to the deserved range
P0(1:20) = 0;
P2(1:20) = 0;
P4(1:50) = 0;
P4(18049:32769) = 0;
P5(1:20) = 0;

%% Plotting of input and output spectra, FRF and Coherence function
figure('Color', [1 1 1]);
plot(f, P0);
xlim([0 700]);
title(['Specimen N∞', num2str(specnum), ': Input Power Spectrum']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');
figure('Color', [1 1 1]);
plot(f, P2);
xlim([0 700]);
title(['Specimen N∞', num2str(specnum), ': Output Power Spectrum']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');
figure('Color', [1 1 1]);
plot(f, P5);
xlim([0 700]);
title(['Specimen N∞', num2str(specnum), ': Coherence Function']);
xlabel('Frequency [Hz]');
ylabel('Magnitude [-]');

%% Peak identification for first four bending frequencies
%% For some specimens, the Minpeakdistance needs to be adjusted
[PKSb, LOCSb] = findpeaks(P4, 'SORTSTR', 'descend', 'MINPEAKDISTANCE', 300);
for i=1:6
    peaksb(i,1) = LOCSb(i);
    peaksb(i,2) = PKSb(i);
end
peaksb_sort = sortrows(peaksb,1);
f1b=f(peaksb_sort(1,1));
b1=P4(peaksb_sort(1,1));
f2b=f(peaksb_sort(2,1));
b2=P4(peaksb_sort(2,1));
f3b=f(peaksb_sort(4,1));
b3=P4(peaksb_sort(4,1));

%%Peak identification for first two torsional frequency
%%For some specimens, the Minpeakdistance needs to be adjusted
[PKSt,LOCSt] = findpeaks(P4,'SORTSTR','descend','MINPEAKDISTANCE',500);
for i=1:6
    peakst(i,1) = LOCSt(i);
    peakst(i,2) = PKSt(i);
end
peakst_sort = sortrows(peakst,1);
f1t=f(peakst_sort(3,1));
t1=P4(peakst_sort(3,1));
f2t=f(peakst_sort(5,1));
t2=P4(peakst_sort(5,1));

%%Printing frequencies
f1b=roundn(f1b,-1)
f2b=roundn(f2b,-1)
f3b=roundn(f3b,-1)
f1t=roundn(f1t,-1)
f2t=roundn(f2t,-1)

%%Calculation of G-modulus using Euler theory
G1_Euler=round(4*f1t^2*L^2*p*Ip/(Kt*1^2)*10^(-6));
G2_Euler=round(4*f2t^2*L^2*p*Ip/(Kt*2^2)*10^(-6));
Gm=(G1_Euler+G2_Euler)/2;

%%Calculation of flexural E-modulus using Euler theory
E1_Euler=Euler_b(f1b,p,A,Iz,L);
E2_Euler=Euler_b(f2b,p,A,Iz,L);
E3_Euler=Euler_b(f3b,p,A,Iz,L);

%%Calculation of flexural E-modulus using Timoshenko theory with Gm
E1_Timo=Timo_b(f1b,p,G1_Euler,A,ks,Iz,L);
E2_Timo=Timo_b(f2b,p,G1_Euler,A,ks,Iz,L);
E3_Timo=Timo_b(f3b,p,G1_Euler,A,ks,Iz,L);

%%Print G and E
G1_Euler
G2_Euler
E1_Euler
E2_Euler
E3_Euler
E1_Timo
E2_Timo
E3_Timo

%%Plot of FRF and Coherence
figure('Color',[1 1 1]);
plotyy(f,P4,f,P5)
[AX,H1,H2] = plotyy(f,P4,f,P5,'plot');
set(get(AX(1),'Ylabel'),'String','Amplitude [dB]')
set(get(AX(2),'Ylabel'),'String','Magnitude [-]')
xlabel('Frequency [Hz]');
title(['Specimen N∞',num2str(specnum),': FRF and Coherence Function']);
set(AX(1),'XLim',[0 700]);
set(AX(2),'XLim',[0 700]);
hold on;

%%Marking of the identified frequencies
plot(f1b,b1+0.05,'k^','markerfacecolor','r');
plot(f2b,b2+0.05,'k^','markerfacecolor','r');
plot(f3b,b3+0.05,'k^','markerfacecolor','r');
plot(f1t,t1+0.05,'k^','markerfacecolor','g');
plot(f2t,t2+0.05,'k^','markerfacecolor','g');
text(f1b-40,b1+0.3,num2str(f1b));
text(f2b-40,b2+0.3,num2str(f2b));
text(f3b-40,b3+0.3,num2str(f3b));
text(f1t-40,t1+0.3,num2str(f1t));
text(f2t-40,t2+0.3,num2str(f2t));
hold off

function E = Euler_b(f,p,A,Iz,L)

%%Definition of the range for E
Emin = 8000;
Emax = 20000;

%%Calculation
w = 2*pi*f;
i = 0;
for E0 = Emin:Emax
    E0 = E0*10^6;
    a = E0*Iz;
    b = 0;
    c = -A*p*w^2;
    l1 = sqrt(sqrt((-b/(2*a))^2-c/a)-b/(2*a));
    l2 = sqrt(sqrt((-b/(2*a))^2-c/a)+b/(2*a));
    c1 = sinh(L*l1);
    c2 = cosh(L*l1);
    c3 = sin(L*l2);
    c4 = cos(L*l2);
    M = [-(-a*l2^3-b*l2) 0 a*l1^3+b*l1 0; 0 -a*l2^2 0 a*l1^2; -c4*(a*l2^3-b*l2) c3*(a*l1^3+b*l1) c2*(a*l1^3+b*l1) c2*(a*l1^3+b*l1)]
\[ c_1 (a l_1^3 + b l_1) - a c_3 l_2^2 - a c_4 l_2^2 + a c_1 l_1^2 + a c_2 l_1^2 ] \]

\[ \text{Det} = \text{det}(M); \]

\[ i = i + 1; \]

\[ \text{result}(i,1) = E_0 \times 10^(-6); \]

\[ \text{result}(i,2) = \text{Det}; \]

\[ \text{sig}(i) = \text{sign}(\text{Det}); \]

\[ \text{end} \]

\[ \text{sig} = \text{sig}'; \]

\[ \text{for } i = 1: \text{length(sig)} - 1 \]

\[ \text{diff} = \text{sig}(i) - \text{sig}(i + 1); \]

\[ \text{if } \text{diff} == 0 \]

\[ \text{continue} \]

\[ \text{else break} \]

\[ \text{end} \]

\[ \text{end} \]

\[ \text{if abs(result}(i,2)) < \text{abs(result}(i+1,2)) \]

\[ E = \text{result}(i,1); \]

\[ \text{else } E = \text{result}(i+1,1); \]

\[ \text{end} \]

\[ \text{function } E = \text{Timo_b}(f,p,G,A,ks,Iz,L) \]

\[ \text{%%Definition of range for } E \]

\[ \text{Emin} = 8000; \]

\[ \text{Emax} = 20000; \]

\[ \text{%%Calculation} \]

\[ G = G \times 10^6; \]

\[ S = 0; \]

\[ k_0 = 0; \]

\[ k = 0; \]

\[ w = 2 \times \pi \times f; \]

\[ l = 0; \]

\[ \text{for } E_0 = \text{Emin}: \text{Emax} \]
\[ E_0 = E_0 \times 10^6; \]
\[ b_2 = A \times L^4 \times p \times w^2 / (E_0 \times I_z); \]
\[ r_2 = I_z / (A \times L^2); \]
\[ f_i = k_s \times A \times G; \]
\[ s_2 = (E_0 \times I_z) / (f_i \times L^2); \]
\[ p_2 = -((L^2 \times S) / (E_0 \times I_z)); \]
\[ \delta = p_2 / b_2 + r_2 \times (1 - p_2 \times s_2) + s_2; \]
\[ m_1 = \left( \sqrt{b_2} \times \sqrt{\sqrt{\left( \frac{4 \times (1 - p_2 \times s_2) \times (1 - b_2 \times r_2 \times s_2)}{b_2} + \delta^2} - \delta} \right) / \sqrt{2 \times (1 - p_2 \times s_2)}; \]
\[ m_2 = \left( \sqrt{b_2} \times \sqrt{\sqrt{\left( \frac{4 \times (1 - p_2 \times s_2) \times (1 - b_2 \times r_2 \times s_2)}{b_2} + \delta^2} + \delta} \right) / \sqrt{2 \times (1 - p_2 \times s_2)}; \]
\[ H = \left( b_2 \times s_2 + m_1^2 \times (1 - p_2 \times s_2) \right) / (L \times m_1); \]
\[ Z = (m_2^2 \times (1 - p_2 \times s_2) - b_2 \times s_2) / (L \times m_2); \]
\[ c_1 = \sinh(m_1); \]
\[ c_2 = \cosh(m_1); \]
\[ c_3 = \sin(m_2); \]
\[ c_4 = \cos(m_2); \]
\[ M = \begin{bmatrix} -k_a & -f_i \times H + f_i \times m_1 / L + m_1 \times S / L & -k_a & -f_i \times Z + f_i \times m_2 / L + m_2 \times S / L \\ \\ k_a \times c_2 - f_i \times H \times c_1 + f_i \times m_1 \times c_1 / L + m_1 \times S \times c_1 / L & -f_i \times H \times c_2 + f_i \times m_1 \times c_2 / L + m_1 \times S \times c_2 / L + k_a \times c_1 & k_a \times c_4 + f_i \times Z \times c_3 - f_i \times m_2 \times c_3 / L - m_2 \times S \times c_3 / L \end{bmatrix} \]
\[ \det = \det(M); \]
\[ i = i + 1; \]
\[ \text{result}(i, 1) = E_0 \times 10^6; \]
\[ \text{result}(i, 2) = \det; \]
\[ \text{sig}(i) = \text{sign} \left( \text{Det} \right); \]
\[ \text{end} \]
\[ \text{sig} = \text{sig}'; \]
\[ \text{for} \ i = 1 : \text{length}(\text{sig}) - 1 \]
\[ \text{diff} = \text{sig}(i) - \text{sig}(i + 1); \]
\[ \text{if} \ \text{diff} = 0 \]
\[ \text{continue} \]
else break
end

if abs(result(i,2)) < abs(result(i+1,2))
    E = result(i,1);
else E = result(i+1,1);
end

11.1.4 Dual parameter estimation

Extraction of frequencies

This program extracts the frequencies from the tensile tests. It works in the same way as the one for the free-free tests. Furthermore the input and output spectra as well as the frequency response and coherence function are depicted.

```matlab
%%Remove all variables from the workspace, clear command window and close open windows
clc
clear all
close all

%%File import
[filename, pathname] = uigetfile('*.txt', 'Pick an Input File');
infile=[pathname filename];
importfile(infile)
clear('colheaders');
clear('textdata');

%%Variable declaration
s=data(:,1);    %%time vector
V=data(:,2);    %%input signal
V1=data(:,3);   %%output signal at the edge
testnumber = sscanf(filename,'T%f_%f');   %%specimen number and load level
specnum = testnumber(1)
level = testnumber(2)
SR=2560; %%Sampling rate```
NFFT=2^16; \% Number of FFT points

\% FFT into spectra and Windowing using Welch method
[x0,f] = pwelch(V,[],[],NFFT,SR);
[x1,f] = pwelch(V1,[],[],NFFT,SR);

\% Estimation of FRF
[x2,f] = tfestimate(V,V1,[],[],NFFT,SR);

\% Estimation of Coherence function
[x3,f] = mscohere(V,V1,[],[],NFFT,SR);

\% Logging for semi-logarithmic plot
P0=log(x0);
P1=log(x1);
P2=real(log(x2));
P3=log(x3);
P2(1:20)=0;
P2(18049:32769)=0;
P3(18049:32769)=0;

\% Plotting of input and output spectra
figure('Color',[1 1 1]);
plot(f,P0);
xlim([0 700]);
title(['Specimen N\infty',num2str(specnum),': Input Power Spectrum']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');
figure('Color',[1 1 1]);
plot(f,P1);
xlim([0 700]);
title(['Specimen N\infty',num2str(specnum),': Output Power Spectrum']);
xlabel('Frequency [Hz]');
ylabel('Amplitude [dB]');

\% Peak identification for first three bending frequencies
\% For some specimens, the Minpeakdistance needs to be adjusted

[PKSb,LOCSb] = findpeaks(P2,'SORTSTR','descend','MINPEAKDISTANCE',1700);
for i=1:5
    peaksb(i,1) = LOCSb(i);
    peaksb(i,2) = PKSb(i);
end
peaksb_sort = sortrows(peaksb,1);
f1b=f(peaksb_sort(1,1));
b1=P2(peaksb_sort(1,1));
f2b=f(peaksb_sort(2,1));
b2=P2(peaksb_sort(2,1));
f3b=f(peaksb_sort(4,1));
b3=P2(peaksb_sort(4,1));

%%Printing frequencies
f1b=roundn(f1b,-1)
f2b=roundn(f2b,-1)
f3b=roundn(f3b,-1)

%%Plot of FRF and Coherence
figure('Color',[1 1 1]);
plotyy(f,P2,f,P3)
[AX,H1,H2] = plotyy(f,P2,f,P3,'plot');
set(get(AX(1),'Ylabel'),'String','Amplitude [dB]')
set(get(AX(2),'Ylabel'),'String','Magnitude [-]')
xlabel('Frequency [Hz]')
title(['Specimen N=',num2str(specnum),': FRF and Coherence Function S = ',num2str(level),' N'])
set(AX(1),'XLim',[0 700]);
set(AX(2),'XLim',[0 700]);
hold on;

%%Marking of the identified frequencies
plot(f1b,b1+0.05,'k^','markerfacecolor','r');
plot(f2b,b2+0.05,'k^','markerfacecolor','r');
plot(f3b,b3+0.05,'k^','markerfacecolor','r');
text(f1b-40,b1+0.3,num2str(f1b));
Parameter estimation

This script uses the entered frequencies to estimate the axial load $S$ and the rotational stiffness $k$ at the supports assuming equal boundary conditions. This is done by using each frequency to calculate $S$ for a range of different $k$. The results are two curves of which the crossing point is searched, which finally yields the result for the two unknown parameters. The two curves together with the solutions are depicted.

%%Remove all variables from the workspace, clear command window and clear open windows

```matlab
clear all
close all
clear

%%Enter input data
f1 = 134.3;         %%First measured frequency
f2 = 336.7;          %%Second measured frequency
E = 16805*10^6;      %%E-modulus in N/m2
G = 789*10^6;       %%G-modulus in N/m2
p = 527.7;          %%Specimen density in kg/m3
L = 1.255;          %%Clear length in m
H = 0.075;          %%Specimen height in m
B = 0.035;          %%Specimen width in m
ka = 10^15;         %%Value for translational support in x=0 in N/m

%%Definition of range for S and k
Smin = 0;
Smax = 1000000;
kmin = 0;
kmax = 750000;

%%Iteration
A = B*H;
kS = 5/6;
```
Iz = B^3*H/12;

iii=1;
n = 150;
diffk = kmax-kmin;
kount = 0;

while diffk > 0.1

%%Approximation of S1

dk = (kmax-kmin)/n;

for j = 1:(n+1)

  k0 = kmin+dk*(j-1);
  resultk(j) = k0;
  count = 0;
  dS = (Smax-Smin)/n;

  for i=1:(n+1)

    S0 = Smin+dS*(i-1);

    b2 = A*L^4*p*(2*pi*f1)^2/(E*Iz);
    r2 = Iz/(A*L^2);
    fi = ks*A*G;
    s2 = (E*Iz)/(fi*L^2);
    p2 = -((L^2*S0)/(E*Iz));

    delta = p2/b2+r2*(1-p2*s2)+s2;

    m1 = (sqrt(b2)*sqrt(sqrt((4*(1-p2*s2)*(1-b2*r2*s2))/b2+delta^2)-delta))/sqrt(2*(1-p2*s2));

    m2 = (sqrt(b2)*sqrt(sqrt((4*(1-p2*s2)*(1-b2*r2*s2))/b2+delta^2)+delta))/sqrt(2*(1-p2*s2));

    H = (b2*s2+m1^2*(1-p2*s2))/(L*m1);

    Z = (m2^2*(1-p2*s2)-b2*s2)/(L*m2);

    c1 = sinh(m1);
    c2 = cosh(m1);
    c3 = sin(m2);
    c4 = cos(m2);

    M = [-ka -fi*H+fi*m1/L+m1*S0/L -ka -fi*Z+fi*m2/L+m2*S0/L; ka*c2-fi*H*c1+fi*m1*c1/L+m1*S0*c1/L]
\[-\frac{\pi^2}{L^2} \cdot \frac{L^4 \cdot b_2}{2 \cdot \pi \cdot f_1^2} + \frac{L^4}{E \cdot I_z} \cdot r_2 + \frac{L^4}{E \cdot I_z} \cdot s_2 + \frac{p_2}{2 \cdot \pi \cdot f_1} \cdot \left(1 - p_2 \cdot s_2\right) + s_2;\]

\[
\text{Det} = \det(M); \\
\text{sig} = \text{sign}(	ext{Det}); \\
\text{if} \ \text{count} == 1 \\
\text{change} = \text{sig}/\text{prevsig}; \\
\text{if} \ \text{change} <= 0 \\
\text{force}(1,1) = \text{prevS}; \\
\text{force}(1,2) = S_0; \\
\text{break} \\
\text{end} \\
\text{end} \\
\text{prevS} = S_0; \\
\text{prevsig} = \text{sig}; \\
\text{count} = 1; \\
\text{end} \\
\text{error} = 0.1; \\
S_{\text{min1}} = \text{force}(1,1); \\
S_{\text{max1}} = \text{force}(1,2); \\
\text{diff} = S_{\text{max1}} - S_{\text{min1}}; \\
\text{while} \ \text{diff} > \text{error} \\
\text{count} = 0; \\
\text{dS}_1 = (S_{\text{max1}} - S_{\text{min1}})/n; \\
\text{for} \ i=1:(n+1) \\
S_0 = S_{\text{min1}} + \text{dS}_1 \cdot (i-1); \\
b_2 = A \cdot L^4 \cdot p \cdot (2 \cdot \pi \cdot f_1)^2 / (E \cdot I_z); \\
r_2 = I_z / (A \cdot L^2); \\
f_1 = k_s \cdot A \cdot G; \\
s_2 = (E \cdot I_z) / (f_1 \cdot L^2); \\
p_2 = -((L^2 \cdot S_0) / (E \cdot I_z)); \\
\text{delta} = p_2 / b_2 + r_2 \cdot (1 - p_2 \cdot s_2) + s_2;
\[ m_1 = \frac{\sqrt{b_2} \sqrt{\sqrt{\frac{4(1-p_2 s_2)(1-b_2 r_2 s_2)}{b_2+\delta^2}} - \delta}}{\sqrt{2(1-p_2 s_2)}}; \]
\[ m_2 = \frac{\sqrt{b_2} \sqrt{\sqrt{\frac{4(1-p_2 s_2)(1-b_2 r_2 s_2)}{b_2+\delta^2}} + \delta}}{\sqrt{2(1-p_2 s_2)}}; \]
\[ H = \frac{(b_2 s_2 + m_1^2)(1-p_2 s_2)}{(L m_1)}; \]
\[ Z = \frac{(m_2^2(1-p_2 s_2) - b_2 s_2)}{(L m_2)}; \]
\[ c_1 = \sinh(m_1); \]
\[ c_2 = \cosh(m_1); \]
\[ c_3 = \sin(m_2); \]
\[ c_4 = \cos(m_2); \]
\[ M = \begin{bmatrix} -k_a & -f_i H + f_i m_1/L + m_1 S_0/L & k_a \\ -f_i Z + f_i m_2/L + m_2 S_0/L & k_a c_2 - f_i H c_1 + f_i m_1 c_1/L + m_1 S_0 c_1/L \\ k_a c_4 + f_i Z c_3 - f_i m_2 c_3/L - m_2 S_0 c_3/L & -E I_z H m_1/L & H k_0 & E I_z Z m_2/L & Z k_0; \end{bmatrix} \]
\[ \text{Det} = \det(M); \]
\[ \text{sig} = \text{sign(} \text{Det}); \]
\[ \text{if} \ \text{count} == 1 \]
\[ \quad \text{change} = \text{sig}/\text{prevsig}; \]
\[ \quad \text{if} \ \text{change} <= 0 \]
\[ \quad \text{force}(1,1) = \text{prevS}; \]
\[ \quad \text{force}(1,2) = S_0; \]
\[ \quad \text{break} \]
\[ \text{end} \]
\[ \text{prevS} = S_0; \]
\[ \text{prevsig} = \text{sig}; \]
\[ \text{count} = 1; \]
\[ \text{end} \]
\[ \text{Smin1} = \text{force}(1,1); \]
\[ \text{Smax1} = \text{force}(1,2); \]
\[ \text{diff} = \text{Smax1} - \text{Smin1}; \]
\[ \text{resultS1}(j) = \text{Smin1}; \]
end
resultk = resultk';
resultS1 = resultS1';
figure(1);
scatter(resultk,resultS1,'o');
hold on;
%%End of Approximation of S1
%%Approximation of S2
dk = (kmax-kmin)/n;
for j = 1:(n+1)
    k0 = kmin+dk*(j-1);
    resultk(j) = k0;
    count = 0;
    dS = (Smax-Smin)/n;
    for i=1:(n+1)
        S0 = Smin+dS*(i-1);
        b2 = A*L^4*p*(2*pi*f2)^2/(E*Iz);
        r2 = Iz/(A*L^2);
        fi = ks*A*G;
        s2 = (E*Iz)/(fi*L^2);
        p2 = -(L^2*S0)/(E*Iz));
        delta = p2/b2+r2*(1-p2*s2)+s2;
        m1 = (sqrt(b2)*sqrt(sqrt((4*(1-p2*s2)*(1-b2*r2*s2))/(b2+delta^2)-delta))/sqrt(2*(1-p2*s2));
        m2 = (sqrt(b2)*sqrt(sqrt((4*(1-p2*s2)*(1-b2*r2*s2))/(b2+delta^2)+delta))/sqrt(2*(1-p2*s2));
        H = (b2*s2+m1^2*(1-p2*s2))/(L*m1);
        Z = (m2^2*(1-p2*s2)-b2*s2)/(L*m2);
        c1 = sinh(m1); c2 = cosh(m1);
        c3 = sin(m2); c4 = cos(m2);
\[ M = \begin{bmatrix} -k_a & -f_i H + f_i m_1/L + m_1 S_0/L & -k_a \\ -f_i Z + f_i m_2/L + m_2 S_0/L & k_a c_2 - f_i H c_1 + f_i m_1 c_1/L + m_1 S_0 c_1/L & -f_i H c_2 + f_i m_1 c_2/L + m_1 S_0 c_2/L + k_a c_1 \\ k_a c_4 + f_i Z c_3 - f_i m_2 c_3/L - m_2 S_0 c_3/L & -E I_z H m_1/L + H k_0 & E I_z Z m_2 c_4/L + Z k_0 c_3 \\ -E I_z H m_1 c_2/L + H k_0 c_1 & -E I_z H m_1 c_1/L + H k_0 c_2 & E I_z Z m_2 c_4/L + Z k_0 c_3 \\ -E I_z Z m_2 c_4/L + Z k_0 c_3 & E I_z Z m_2 c_3/L - Z k_0 c_4 \end{bmatrix}; \]

\[ \text{Det} = \det(M); \]
\[ \text{sig} = \text{sign(Det)}; \]
\[ \text{if count} == 1 \]
\[ \quad \text{change} = \text{sig}/\text{prevsig}; \]
\[ \quad \text{if change} <= 0 \]
\[ \quad \quad \text{force}(1,1) = \text{prevS}; \]
\[ \quad \quad \text{force}(1,2) = S_0; \]
\[ \quad \quad \text{break} \]
\[ \quad \end{end} \]
\[ \quad \text{prevS} = S_0; \]
\[ \quad \text{prevsig} = \text{sig}; \]
\[ \quad \text{count} = 1; \]
\[ \end{end} \]
\[ \quad \text{error} = 0.1; \]
\[ \quad \text{Smin1} = \text{force}(1,1); \]
\[ \quad \text{Smax1} = \text{force}(1,2); \]
\[ \quad \text{diff} = \text{Smax1} - \text{Smin1}; \]
\[ \quad \text{while diff} > \text{error} \]
\[ \quad \quad \text{count} = 0; \]
\[ \quad \quad \text{dS1} = (\text{Smax1} - \text{Smin1})/n; \]
\[ \quad \quad \text{for i}=1:(n+1) \]
\[ \quad \quad \quad \text{S0} = \text{Smin1} + \text{dS1}*(i-1); \]
\[ \quad \quad \quad \text{b2} = A L^4 p (2 pi f_2)^2/(E I_z); \]
\[ \quad \quad \quad \text{r2} = I_z/(A L^2); \]
\[ \quad \quad \quad \text{fi} = k_s A^2 G; \]
\[ \quad \quad \quad \text{s2} = (E I_z)/(f_i L^2); \]
\[ \quad \quad \quad \text{p2} = -(L^2 S_0)/(E I_z); \]
\[\delta = \frac{p_2}{b_2} + r_2(1-p_2s_2) + s_2;\]

\[m_1 = \frac{\sqrt{b_2} \cdot \sqrt{\sqrt{\frac{4(1-p_2s_2)(1-b_2r_2s_2)}{b_2} + \delta^2} - \delta}}{\sqrt{2(1-p_2s_2)}};\]

\[m_2 = \frac{\sqrt{b_2} \cdot \sqrt{\sqrt{\frac{4(1-p_2s_2)(1-b_2r_2s_2)}{b_2} + \delta^2} + \delta}}{\sqrt{2(1-p_2s_2)}};\]

\[H = \frac{(b_2s_2 + m_1^2(1-p_2s_2))}{L \cdot m_1};\]

\[Z = \frac{(m_2^2(1-p_2s_2) - b_2s_2)}{L \cdot m_2};\]

\[c_1 = \sinh(m_1);\]

\[c_2 = \cosh(m_1);\]

\[c_3 = \sin(m_2);\]

\[c_4 = \cos(m_2);\]

\[M = \begin{bmatrix} -k_a & -f_i + H \cdot c_1 + f_i \cdot m_1 / L + m_1 \cdot S_0 / L & -k_a \\ -f_i \cdot Z + f_i \cdot m_2 / L + m_2 \cdot S_0 / L & -f_i \cdot H + f_i \cdot c_2 + f_i \cdot m_1 / L + m_1 \cdot S_0 / L & -f_i \cdot Z + f_i \cdot m_2 / L + m_2 \cdot S_0 / L \\ -f_i \cdot Z \cdot c_3 - f_i \cdot m_2 / L - m_2 \cdot S_0 / c_3 \cdot L & -E \cdot I_z \cdot H \cdot m_1 / L \cdot H \cdot k_0 & E \cdot I_z \cdot Z \cdot m_2 / L \cdot Z \cdot k_0 \\ -E \cdot I_z \cdot H \cdot m_2 / c_3 / L \cdot H \cdot k_0 \cdot c_2 & -E \cdot I_z \cdot H \cdot m_1 / c_3 / L \cdot H \cdot k_0 \cdot c_2 & -E \cdot I_z \cdot Z \cdot m_2 / c_3 / L \cdot Z \cdot k_0 \cdot c_4 \end{bmatrix};\]

\[\text{Det} = \det(M);\]

\[\text{sig} = \text{sign(Det)};\]

\[\text{if} \ \text{count} == 1;\]

\[\text{change} = \text{sig}/\text{prevsig};\]

\[\text{if} \ \text{change} <= 0;\]

\[\text{force}(1,1) = \text{prevS};\]

\[\text{force}(1,2) = S_0;\]

\[\text{break};\]

\[\text{end};\]

\[\text{end};\]

\[\text{prevS} = S_0;\]

\[\text{prevsig} = \text{sig};\]

\[\text{count} = 1;\]

\[\text{end};\]

\[S_{\text{min}1} = \text{force}(1,1);\]

\[S_{\text{max}1} = \text{force}(1,2);\]

\[\text{diff} = S_{\text{max}1} - S_{\text{min}1};\]
```matlab
end
resultS2(j) = Smin1;
end
resultk = resultk';
resultS2 = resultS2';
scatter(resultk,resultS2,'x');

%% End of Approximation of S2
%% Determination of crossing point
diffS = resultS2-resultS1;
signdiff = sign(diffS);
[Y,I] = min(abs(diffS));
if diffS == 0
    kfin = resultk(I);
    Sfin = resultS1(I);
end
if signdiff(I) == signdiff(I+1)
    kmin = resultk(I-1);
    kmax = resultk(I);
    Smin = resultS2(I+1);
    Smax = resultS2(I-2);
else
    kmin = resultk(I);
    kmax = resultk(I+1);
    Smin = resultS2(I+2);
    Smax = resultS2(I-1);
end
diffk = kmax-kmin;
if kount == 0
    resultkplot = resultk;
    resultS1plot = resultS1;
    resultS2plot = resultS2;
```
end
kount = kount+1;
end

%%%Plot of iteration curves
figure(2)
plot(resultkplot.*0.001,resultS1plot.*0.001,'Color',[0 0 1])
hold on
plot(resultkplot.*0.001,resultS2plot.*0.001,'Color',[0 0.498039215803146 0])
scatter(kmin.*0.001,Smax.*0.001,'filled','red')
plot(0:0.1:kmin.*0.001,Smax.*0.001)
plot(kmin.*0.001,0:0.1:Smax.*0.001)
title(['Reference Aluminium Specimen S = 1245 N']);
xlabel('Rotational stiffness k [kNm]');
ylabel('Axial load S [kN]');
legend('f1','f2');

%%%Print results
kmin
Smax
11.2 Appendix B – Material properties

11.2.1 Density measurements

Data

Table 11-1 Weights and densities of the specimen

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<th>Beam No</th>
<th>Weight [kg]</th>
<th>Density [kg/m³]</th>
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Mean value 488.6
Std. dev. 24.2

Aluminium 1.976 2643.5
11.2.2 Moisture measurements

Data

Table 11-2  Moisture measurements in three different point and mean moisture contents of the specimen

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Mean value 12.5
Std. dev. 0.6
11.2.3 Four-point bending tests

Data

Table 11-3  Results of four-point bending tests

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<th>Sup2</th>
<th>Tot</th>
<th>Sup1</th>
<th>Mid</th>
<th>Sup2</th>
<th>Tot</th>
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11.2.4 Longitudinal vibration tests

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CHALMERS, Civil and Environmental Engineering, Master’s Thesis 2012:60
Correlations

**Figure 11-1** Correlation between longitudinal $E$-modulus found using $f_{1L}$ and static $E$-modulus

**Figure 11-2** Correlation between longitudinal $E$-modulus found using $f_{2L}$ and static $E$-modulus

**Figure 11-3** Correlation between longitudinal $E$-modulus found using both frequencies and static $E$-modulus
FRF plots
11.2.5 Transversal vibration tests

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<td>607.5</td>
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<td>315.4</td>
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<td>869</td>
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<td>427.9</td>
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<td>457.7</td>
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<td>642</td>
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<td>673</td>
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<td>14133</td>
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<td>219.2</td>
<td>422.5</td>
<td>312.9</td>
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<td>907</td>
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<td>233.2</td>
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<td>309.5</td>
<td>611.3</td>
<td>745</td>
<td>768</td>
<td>757</td>
<td>14014</td>
<td>13592</td>
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<tr>
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<td>226.8</td>
<td>437.2</td>
<td>307.9</td>
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<td>795</td>
<td>810</td>
<td>803</td>
<td>13246</td>
<td>13055</td>
</tr>
</tbody>
</table>

Mean value: 791.2, 811.0, 802.1, 13591.6, 12834.1, 12777.2, 13982.1

Aluminium

<table>
<thead>
<tr>
<th>Mean value</th>
<th>Std. dev.</th>
<th>Mean value</th>
<th>Std. dev.</th>
<th>Mean value</th>
<th>Std. dev.</th>
<th>Mean value</th>
<th>Std. dev.</th>
</tr>
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<tbody>
<tr>
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<td>67289</td>
</tr>
</tbody>
</table>

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Correlations

Figure 11-4  Correlation between transversal $E$-modulus found using $f_{1b}$ and static $E$-modulus

Figure 11-5  Correlation between transversal $E$-modulus found using $f_{2b}$ and static $E$-modulus

Figure 11-6  Correlation between transversal $E$-modulus found using $f_{3b}$ and static $E$-modulus
Figure 11-7  Correlation between transversal E-modulus found using all frequencies and static E-modulus
FRF plots
Specimen N15: FRF and Coherence Function

Specimen N14: FRF and Coherence Function

Specimen N15: FRF and Coherence Function
11.3 Appendix C - Tension tests

11.3.1 Results for each specimen

T1

Data

Table 11-6 Results of frequency measurements and dual parameter estimation for specimen N°1 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>T1</th>
<th>% of yield</th>
<th>S20</th>
<th>f02</th>
<th>f03</th>
<th>L</th>
<th>E</th>
<th>E2</th>
<th>Error on S</th>
<th>K</th>
<th>K2</th>
<th>Error on S</th>
</tr>
</thead>
<tbody>
<tr>
<td>6094</td>
<td>8.1</td>
<td>0.24</td>
<td>115.4</td>
<td>302.2</td>
<td>1.295</td>
<td>29300</td>
<td>6385</td>
<td>14.2</td>
<td>25600</td>
<td>6715</td>
<td>21.2</td>
</tr>
<tr>
<td>10301</td>
<td>15.1</td>
<td>0.44</td>
<td>120.6</td>
<td>312.1</td>
<td>1.275</td>
<td>20500</td>
<td>13860</td>
<td>27.0</td>
<td>18500</td>
<td>13500</td>
<td>31.0</td>
</tr>
<tr>
<td>14833</td>
<td>21.8</td>
<td>0.61</td>
<td>124.3</td>
<td>320.7</td>
<td>1.255</td>
<td>18200</td>
<td>12424</td>
<td>2.9</td>
<td>16400</td>
<td>11848</td>
<td>7.0</td>
</tr>
<tr>
<td>20311</td>
<td>35.6</td>
<td>0.82</td>
<td>128.6</td>
<td>335.9</td>
<td>1.235</td>
<td>17850</td>
<td>23938</td>
<td>18.9</td>
<td>15980</td>
<td>24627</td>
<td>22.6</td>
</tr>
<tr>
<td>24815</td>
<td>36.5</td>
<td>1.02</td>
<td>131.9</td>
<td>332.3</td>
<td>1.225</td>
<td>19530</td>
<td>20558</td>
<td>17.1</td>
<td>17730</td>
<td>29751</td>
<td>19.9</td>
</tr>
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<td>29334</td>
<td>41.0</td>
<td>1.20</td>
<td>134.3</td>
<td>336.7</td>
<td>1.215</td>
<td>243740</td>
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<td>7.9</td>
<td>213650</td>
<td>32588</td>
<td>10.1</td>
</tr>
</tbody>
</table>
FRF plots
Data

Table 11-7  Results of frequency measurements and dual parameter estimation for specimen N°2 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>S/E</th>
<th>f_0</th>
<th>f_0</th>
<th>L</th>
<th>using E_2</th>
<th>using E_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>N</td>
<td>[°]</td>
<td>[N]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5199</td>
<td>7.6</td>
<td>0.32</td>
<td>111.9</td>
<td>395.8</td>
<td>1.295</td>
<td>160910</td>
<td>11422</td>
</tr>
<tr>
<td>15500</td>
<td>15.1</td>
<td>0.46</td>
<td>137.5</td>
<td>502.4</td>
<td>1.275</td>
<td>113310</td>
<td>15880</td>
</tr>
<tr>
<td>19620</td>
<td>22.1</td>
<td>0.67</td>
<td>121.2</td>
<td>508.9</td>
<td>1.255</td>
<td>10070</td>
<td>17005</td>
</tr>
<tr>
<td>28338</td>
<td>28.9</td>
<td>0.88</td>
<td>124.5</td>
<td>517.1</td>
<td>1.255</td>
<td>133040</td>
<td>23340</td>
</tr>
<tr>
<td>29832</td>
<td>35.1</td>
<td>1.07</td>
<td>128.1</td>
<td>521.0</td>
<td>1.255</td>
<td>113290</td>
<td>34770</td>
</tr>
</tbody>
</table>
FRF plots

[Graphs showing FRF plots for different conditions, with peaks at specific frequencies labeled.]
### Data

Table 11-8 Results of frequency measurements and dual parameter estimation for specimen N°3 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>T3</th>
<th>Applied load (N)</th>
<th>% of yield</th>
<th>$e_0$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$k_2$</th>
<th>$k_1$</th>
<th>$k_0$</th>
<th>$e_{R2}$</th>
<th>$e_{R1}$</th>
<th>$e_{R0}$</th>
<th>Error on 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>6.2</td>
<td>0.30</td>
<td>110.7</td>
<td>289.3</td>
<td>1.275</td>
<td>2.205</td>
<td>216710</td>
<td>6740</td>
<td>56.4</td>
<td>20430</td>
<td>8866</td>
<td>58.5</td>
<td>10.1</td>
</tr>
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<td>1500</td>
<td>22.1</td>
<td>0.75</td>
<td>120.2</td>
<td>306.9</td>
<td>1.295</td>
<td>2.265</td>
<td>171170</td>
<td>13897</td>
<td>30.1</td>
<td>162740</td>
<td>13564</td>
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<td>128.7</td>
<td>313.8</td>
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<td>101150</td>
<td>29717</td>
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<td>2.255</td>
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<td>31053</td>
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<td>142010</td>
<td>31312</td>
<td>20.4</td>
<td>10.1</td>
</tr>
<tr>
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<td>130.0</td>
<td>324.3</td>
<td>1.225</td>
<td>2.205</td>
<td>127320</td>
<td>38547</td>
<td>25.0</td>
<td>121630</td>
<td>38661</td>
<td>20.1</td>
<td>10.1</td>
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</table>
FRF plots
### T4

#### Data

Table 11-9 Results of frequency measurements and dual parameter estimation for specimen N°4 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>$\text{Applied load} \ S$</th>
<th>$% \ \text{of yield}$</th>
<th>$S_{P0}$</th>
<th>$f_{a}$</th>
<th>$f_{m}$</th>
<th>$L$</th>
<th>$E_{aa}$</th>
<th>$E_{lm}$</th>
<th>Error on $E_{aa}$</th>
<th>Error on $E_{lm}$</th>
</tr>
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<tbody>
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<td>570.6</td>
<td>578.6</td>
<td>1.195</td>
<td>10.3120</td>
<td>7.6674</td>
<td>46.2</td>
<td>87360</td>
</tr>
<tr>
<td>9712</td>
<td>14.3</td>
<td>0.48</td>
<td>111.4</td>
<td>291.1</td>
<td>2.275</td>
<td>103.6520</td>
<td>135.4212</td>
<td>18.8</td>
<td>92573</td>
</tr>
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<td>14126</td>
<td>20.8</td>
<td>0.68</td>
<td>156.6</td>
<td>305.3</td>
<td>3.250</td>
<td>107.7874</td>
<td>177.3000</td>
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<td>0.96</td>
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<td>3.250</td>
<td>116.0230</td>
<td>227.9084</td>
<td>11.8</td>
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<tr>
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<td>1.14</td>
<td>124.7</td>
<td>314.0</td>
<td>2.230</td>
<td>103.8330</td>
<td>305.6000</td>
<td>27.9</td>
<td>83590</td>
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<td>128.0</td>
<td>320.3</td>
<td>2.230</td>
<td>107.0220</td>
<td>321.7500</td>
<td>11.6</td>
<td>114240</td>
</tr>
</tbody>
</table>

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FRF plots
### T5

#### Data

**Table 11-10** Results of frequency measurements and dual parameter estimation for specimen N°5 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load [N]</th>
<th>% of yield</th>
<th>$S_p$</th>
<th>$f_p$</th>
<th>$f_m$</th>
<th>$f_l$</th>
<th>$L$</th>
<th>$E_{tx}$</th>
<th>$E_{ty}$</th>
<th>Error on $S_p$</th>
<th>Error on $S_{ty}$</th>
<th>Error on $L$</th>
<th>Error on $E_{tx}$</th>
<th>Error on $E_{ty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5494</td>
<td>8.1</td>
<td>0.25</td>
<td>128.5</td>
<td>293.8</td>
<td>3.295</td>
<td>90134</td>
<td>13633</td>
<td>148.3</td>
<td>88430</td>
<td>14515</td>
<td>156.2</td>
<td>88640</td>
<td>14515</td>
</tr>
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<td>9810</td>
<td>16.4</td>
<td>0.43</td>
<td>114.3</td>
<td>294.0</td>
<td>3.275</td>
<td>85173</td>
<td>20955</td>
<td>113.1</td>
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</tr>
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<td>74189</td>
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<td>123.7</td>
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<td>31369</td>
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<td>23062</td>
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<td>127.4</td>
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<td>101280</td>
<td>30157</td>
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<td>31220</td>
<td>30889</td>
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<tr>
<td>27939</td>
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<td>1.29</td>
<td>130.4</td>
<td>525.2</td>
<td>3.255</td>
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<td>35208</td>
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<td>36063</td>
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<td>36063</td>
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</table>
FRF plots
T6

Data

Table 11-11  Results of frequency measurements and dual parameter estimation for specimen N°6 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>T6</th>
<th>Applied load [N]</th>
<th>% of yield</th>
<th>$E/\rho$</th>
<th>$f_0$</th>
<th>$f_0$</th>
<th>$l$</th>
<th>$E_{01}$</th>
<th>$E_{10}$</th>
<th>Error on $E_{01}$</th>
<th>$E_{\text{max}}$</th>
<th>$E_{\text{max}}$</th>
<th>Error on $E_{\text{max}}$</th>
</tr>
</thead>
<tbody>
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<td>5500</td>
<td>7.0</td>
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<td>110.2</td>
<td>390.1</td>
<td>1.294</td>
<td>140010</td>
<td>6861</td>
<td>28.8</td>
<td>443120</td>
<td>7064</td>
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</tr>
<tr>
<td>10000</td>
<td>14.7</td>
<td>0.44</td>
<td>116.8</td>
<td>300.6</td>
<td>1.275</td>
<td>103180</td>
<td>16885</td>
<td>68.1</td>
<td>90620</td>
<td>17188</td>
<td>71.9</td>
<td></td>
</tr>
<tr>
<td>16813</td>
<td>21.8</td>
<td>0.64</td>
<td>121.3</td>
<td>911.1</td>
<td>1.255</td>
<td>99768</td>
<td>21680</td>
<td>66.0</td>
<td>89338</td>
<td>22083</td>
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<td></td>
</tr>
<tr>
<td>18767</td>
<td>27.8</td>
<td>0.81</td>
<td>129.0</td>
<td>838.9</td>
<td>1.255</td>
<td>144940</td>
<td>21792</td>
<td>16.5</td>
<td>137220</td>
<td>22301</td>
<td>18.0</td>
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<tr>
<td>23512</td>
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<td>1.00</td>
<td>129.1</td>
<td>825.8</td>
<td>1.255</td>
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<td>136950</td>
<td>27904</td>
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<td>1.20</td>
<td>132.2</td>
<td>925.5</td>
<td>1.255</td>
<td>213950</td>
<td>30845</td>
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<td>1599480</td>
<td>31227</td>
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</table>
FRF plots

---

1. FRF plot for 5 = 525 N.
2. FRF plot for 5 = 1000 N.
3. FRF plot for 5 = 14013 N.

---

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T7

Data

Table 11-12  Results of frequency measurements and dual parameter estimation for specimen N°7 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>S/f,</th>
<th>f_m</th>
<th>f_n</th>
<th>L</th>
<th>E_m</th>
<th>E_n</th>
<th>Error on S</th>
<th>Error on L</th>
</tr>
</thead>
<tbody>
<tr>
<td>9908</td>
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<td>0.09</td>
<td>111.5</td>
<td>300.9</td>
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<td>257210</td>
<td>4209</td>
<td>-13.8</td>
<td>253180</td>
</tr>
<tr>
<td>16420</td>
<td>17.0</td>
<td>0.71</td>
<td>119.9</td>
<td>309.0</td>
<td>1.255</td>
<td>147370</td>
<td>15894</td>
<td>11.0</td>
<td>152220</td>
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<td>0.94</td>
<td>124.6</td>
<td>316.5</td>
<td>1.255</td>
<td>259710</td>
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<td>4.7</td>
<td>228240</td>
</tr>
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<td>23544</td>
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<td>138.5</td>
<td>324.8</td>
<td>1.255</td>
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<td>28743</td>
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<td>1.42</td>
<td>133.7</td>
<td>329.5</td>
<td>1.255</td>
<td>318670</td>
<td>29716</td>
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<td>259430</td>
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</table>
FRF plots
## T8

### Data

Table 11-13  Results of frequency measurements and dual parameter estimation for specimen N°8 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load L</th>
<th>% of yield</th>
<th>f E</th>
<th>f M</th>
<th>L</th>
<th>( E_{\text{long}} )</th>
<th>( E_{\text{trans}} )</th>
<th>Error on ( E_{\text{long}} )</th>
<th>Error on ( E_{\text{trans}} )</th>
</tr>
</thead>
<tbody>
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<td>292.3</td>
<td>1.295</td>
<td>79169</td>
<td>13469</td>
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<td>72350.5</td>
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<td>10104</td>
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<td>115.4</td>
<td>296.7</td>
<td>1.275</td>
<td>104500</td>
<td>17163</td>
<td>69.9</td>
<td>99079</td>
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<td>15009</td>
<td>22.1</td>
<td>120.7</td>
<td>307.3</td>
<td>1.255</td>
<td>89357</td>
<td>22872</td>
<td>52.4</td>
<td>81499</td>
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<td>19820</td>
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<td>128.3</td>
<td>313.0</td>
<td>1.255</td>
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<td>51359</td>
<td>59.0</td>
<td>70565</td>
</tr>
<tr>
<td>24329</td>
<td>35.8</td>
<td>128.7</td>
<td>322.0</td>
<td>1.255</td>
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<td>26659</td>
<td>21.8</td>
<td>137800</td>
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<td>29182</td>
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<td>122.9</td>
<td>329.0</td>
<td>1.255</td>
<td>180530</td>
<td>35532</td>
<td>21.1</td>
<td>158870</td>
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</table>
FRF plots
Data

Table 11-14  Results of frequency measurements and dual parameter estimation for specimen N°9 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>λ₁₀</th>
<th>λ₁₀</th>
<th>f₁₀</th>
<th>f₄₀</th>
<th>l</th>
<th>using λ₁₀</th>
<th>λ₁₀</th>
<th>Error on λ₁₀</th>
<th>using λ₄₀</th>
<th>λ₄₀</th>
<th>Error on λ₄₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N]</td>
<td>[%]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td></td>
<td></td>
<td>[m]</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>5396</td>
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<td>0.24</td>
<td>326.4</td>
<td>287.7</td>
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<td>151860</td>
<td>12787</td>
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<td>151930</td>
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</tr>
<tr>
<td>9810</td>
<td>14.4</td>
<td>0.43</td>
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<td>299.3</td>
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<td>8359</td>
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<td>9554</td>
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<td>0.62</td>
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<td>0.83</td>
<td>324.1</td>
<td>318.6</td>
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FRF plots
T10

Data

Table 11-15  Results of frequency measurements and dual parameter estimation for specimen N°10 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load S</th>
<th>% of yield</th>
<th>$\delta_s$</th>
<th>$f_s$</th>
<th>$f_l$</th>
<th>L</th>
<th>$x_{us}$</th>
<th>$x_{ls}$</th>
<th>Error on $S$</th>
<th>$x_{us}$</th>
<th>$x_{ls}$</th>
<th>Error on $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N]</td>
<td>[%]</td>
<td>[Å]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[Å]</td>
<td>[Nm]</td>
<td>[Nm]</td>
<td>[%]</td>
<td>[Nm]</td>
<td>[Nm]</td>
<td>[%]</td>
</tr>
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<td>4306</td>
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<td>25.8</td>
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<td>159050</td>
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<td>0.83</td>
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<td>1.255</td>
<td>203540</td>
<td>19853</td>
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<td>167280</td>
<td>20741</td>
<td>8.4</td>
</tr>
<tr>
<td>21838</td>
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<td>1.03</td>
<td>110.4</td>
<td>329.6</td>
<td>1.255</td>
<td>207510</td>
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<td>169700</td>
<td>27736</td>
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<td>153.8</td>
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<td>215870</td>
<td>31358</td>
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</table>
FRF plots
Data

Table 11-16  Results of frequency measurements and dual parameter estimation for specimen N°11 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load $F$ [N]</th>
<th>% of yield [%]</th>
<th>$\sigma_p$ [MPa]</th>
<th>$\mu_1$ [Hz]</th>
<th>$\mu_2$ [Hz]</th>
<th>$l$ [m]</th>
<th>( \lambda_a ) using $\mu_1$ [Nm]</th>
<th>$\sigma_{\lambda_a}$ [%]</th>
<th>Error on $\lambda_a$ [%]</th>
<th>( \lambda_a ) using $\mu_2$ [Nm]</th>
<th>$\sigma_{\lambda_a}$ [%]</th>
<th>Error on $\lambda_a$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
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<td>8.40</td>
<td>251.4</td>
<td>248.5</td>
<td>3.73</td>
<td>502.34</td>
<td>0.38</td>
<td>0.38</td>
<td>502.34</td>
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<td>0.38</td>
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<td>7.97</td>
<td>247.8</td>
<td>248.2</td>
<td>3.73</td>
<td>1002.50</td>
<td>0.38</td>
<td>0.38</td>
<td>1002.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
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<td>12.1</td>
<td>7.97</td>
<td>247.8</td>
<td>248.2</td>
<td>3.73</td>
<td>1502.50</td>
<td>0.38</td>
<td>0.38</td>
<td>1502.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
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<td>12.1</td>
<td>7.97</td>
<td>247.8</td>
<td>248.2</td>
<td>3.73</td>
<td>2002.50</td>
<td>0.38</td>
<td>0.38</td>
<td>2002.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>25000</td>
<td>12.3</td>
<td>8.38</td>
<td>248.2</td>
<td>248.2</td>
<td>3.73</td>
<td>2502.50</td>
<td>0.38</td>
<td>0.38</td>
<td>2502.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>30000</td>
<td>12.3</td>
<td>8.40</td>
<td>248.2</td>
<td>248.2</td>
<td>3.73</td>
<td>3002.50</td>
<td>0.38</td>
<td>0.38</td>
<td>3002.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>35000</td>
<td>12.3</td>
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<td>248.2</td>
<td>248.2</td>
<td>3.73</td>
<td>3502.50</td>
<td>0.38</td>
<td>0.38</td>
<td>3502.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>40000</td>
<td>12.3</td>
<td>8.40</td>
<td>248.2</td>
<td>248.2</td>
<td>3.73</td>
<td>4002.50</td>
<td>0.38</td>
<td>0.38</td>
<td>4002.50</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>45000</td>
<td>12.3</td>
<td>8.40</td>
<td>248.2</td>
<td>248.2</td>
<td>3.73</td>
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<td>0.38</td>
<td>4502.50</td>
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</table>
FRF plots

1. FRF plot for specimen N°11, load 5360 N.
2. FRF plot for specimen N°11, load 10660 N.
3. FRF plot for specimen N°11, load 14813 N.
T12

Data

Table 11-17  Results of frequency measurements and dual parameter estimation for specimen N°12 using the transversal and longitudinal E-modulus for different load levels

| Applied load [N] | % of yield | $S_1$ [%] | $T_0$ [Hz] | $T_\infty$ [Hz] | L [m] | $\rho_{\infty}$ [Nm] | $\rho_0$ [Nm] | Error on $\rho$ [%] | $\rho_{\infty}$ [Nm] | $\rho_0$ [Nm] | Error on $\rho$ [%] |
|-----------------|------------|----------|-------------|----------------|------|-----------------|------------|----------------|----------------|------------|----------------|------------|
| 5095            | 7.8        | 0.25     | 106.7       | 283.3          | 1.275 | 16010.2         | 32.7       | -40.6          | 153880         | 3973       | -26.4          |
| 10097           | 15.4       | 0.47     | 133.4       | 297.1          | 2.275 | 16010.2         | 90.7       | -13.5          | 153880         | 9920       | -5.5           |
| 15009           | 22.1       | 0.66     | 177.5       | 306.3          | 2.250 | 149120          | 117.9      | -21.6          | 152510         | 12772      | -14.9          |
| 20044           | 28.6       | 0.84     | 122.1       | 313.5          | 2.250 | 165120          | 188.4      | -3.0           | 157110         | 13984      | 2.4            |
| 24133           | 35.5       | 1.04     | 126.3       | 321.6          | 2.250 | 207050          | 21974      | -8.9           | 205070         | 22781      | -5.6           |
| 28057           | 41.3       | 1.21     | 128.8       | 327.0          | 2.250 | 469340          | 22735      | -18.9          | 455120         | 23370      | -16.7          |
FRF plots
The results for the second load levels were not calculated since the according coherence function showed strong irregularities.
FRF plots

![FRF plots](image-url)
### T14

**Data**

Table 11-19  Results of frequency measurements and dual parameter estimation for specimen N°14 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>$\Delta f$</th>
<th>$f_0$</th>
<th>$f_2$</th>
<th>L</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>Error on $E_1$</th>
<th>$E_3$</th>
<th>$E_4$</th>
<th>Error on $E_3$</th>
</tr>
</thead>
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<td>110.8</td>
<td>299.3</td>
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<td>152410</td>
<td>8049</td>
<td>-99.9</td>
</tr>
<tr>
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<td>0.10</td>
<td>116.5</td>
<td>325.6</td>
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<td>217110</td>
<td>1007</td>
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<td>333.0</td>
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<td>262640</td>
<td>18386</td>
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</tr>
<tr>
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<td>0.89</td>
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<td>323.3</td>
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<td>229310</td>
<td>15954</td>
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</table>
FRF plots
### Data

**Table 11-20**  Results of frequency measurements and dual parameter estimation for specimen $N^615$ using the transversal and longitudinal E-modulus for different load levels

<table>
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<th>Applied load $F$</th>
<th>% of yield</th>
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<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
<th>$L_4$</th>
<th>$L_5$</th>
<th>Error on $F_1$</th>
<th>$F_2$</th>
<th>Error on $F_2$</th>
<th>$F_3$</th>
<th>Error on $F_3$</th>
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<th>Error on $F_4$</th>
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<th>Error on $F_5$</th>
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</thead>
<tbody>
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<td>110.8</td>
<td>266.8</td>
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<td>20512</td>
<td>248.2</td>
<td>46658</td>
<td>21773</td>
<td>104.0</td>
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</table>
FRF plots
### Table 11-21 Results of frequency measurements and dual parameter estimation for specimen N°16 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>% of yield</th>
<th>%E (N)</th>
<th>f_n (Hz)</th>
<th>f_m (Hz)</th>
<th>L (m)</th>
<th>$\rho_{E}$</th>
<th>$\sigma_{E}$</th>
<th>Error on $\rho_{E}$ (%)</th>
<th>$\rho_{E}$</th>
<th>$\sigma_{E}$</th>
<th>Error on $\rho_{E}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1002</td>
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<td>121.6</td>
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<td>1.275</td>
<td>112470</td>
<td>32025</td>
<td>77.9</td>
<td>75486</td>
<td>15508</td>
<td>105.0</td>
</tr>
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<td>5.0</td>
<td>0.45</td>
<td>115.6</td>
<td>301.4</td>
<td>1.275</td>
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<td>45.3</td>
<td>75729</td>
<td>19035</td>
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<td>22.1</td>
<td>0.64</td>
<td>123.8</td>
<td>311.4</td>
<td>1.255</td>
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<td>31935</td>
<td>20.0</td>
<td>87748</td>
<td>24165</td>
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<td>0.82</td>
<td>128.7</td>
<td>320.4</td>
<td>1.255</td>
<td>167610</td>
<td>20447</td>
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<td>196100</td>
<td>21600</td>
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</tr>
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<td>0.90</td>
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<td>23191</td>
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<td>123300</td>
<td>26521</td>
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<td>132.2</td>
<td>321.0</td>
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<td>127110</td>
<td>31310</td>
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FRF plots
### T17

#### Data

**Table 11-22** Results of frequency measurements and dual parameter estimation for specimen N°17 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load $F$</th>
<th>% of yield</th>
<th>$f_p$</th>
<th>$f_p^*$</th>
<th>$f_p^*$</th>
<th>$L$</th>
<th>$E_{tr}$</th>
<th>$E_{tr}$</th>
<th>Error on $E_{tr}$</th>
<th>$E_{tr}$</th>
<th>$E_{tr}$</th>
<th>Error on $E_{tr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N]</td>
<td>[%]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[m]</td>
<td>[N/mm$^2$]</td>
<td>[N/mm$^2$]</td>
<td></td>
<td>[N/mm$^2$]</td>
<td>[N/mm$^2$]</td>
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</tr>
<tr>
<td>5129</td>
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<td>0.29</td>
<td>105.5</td>
<td>276.0</td>
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<td>124.6</td>
<td>60331</td>
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<td>0.57</td>
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<td>1.275</td>
<td>96618</td>
<td>15446</td>
<td>62483</td>
<td>18999</td>
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<td>15599</td>
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<td>0.79</td>
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<td>299.6</td>
<td>1.255</td>
<td>71751</td>
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<td>50576</td>
<td>23708</td>
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<td>121.6</td>
<td>307.0</td>
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<td>63127</td>
<td>29206</td>
<td>51.0</td>
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<td>125.1</td>
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<td>20780</td>
<td>69664</td>
<td>33071</td>
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<td>320.1</td>
<td>1.255</td>
<td>129840</td>
<td>22092</td>
<td>95113</td>
<td>34811</td>
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FRF plots
T18

Data

Table 11-23 Results of frequency measurements and dual parameter estimation for specimen N°18 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load [N]</th>
<th>% of yield</th>
<th>$S_i$ [N/mm²]</th>
<th>$f_a$ [Hz]</th>
<th>$f_l$ [Hz]</th>
<th>$l$ [m]</th>
<th>$E_{tr}$ [GPa]</th>
<th>$E_{l}$ [GPa]</th>
<th>Error on $S_i$ [%]</th>
<th>$E_{tr}$ [GPa]</th>
<th>$E_{l}$ [GPa]</th>
<th>Error on $S_i$ [%]</th>
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</thead>
<tbody>
<tr>
<td>5199</td>
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<td>0.28</td>
<td>127.5</td>
<td>295.6</td>
<td>1.295</td>
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<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
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<td>0.49</td>
<td>117.0</td>
<td>307.9</td>
<td>1.275</td>
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<td>/</td>
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<td>/</td>
<td>/</td>
<td>/</td>
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<tr>
<td>22500</td>
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<td>125.7</td>
<td>323.2</td>
<td>1.255</td>
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</table>

The results for this specimen were very erroneous for undeclared reasons, which is why they were not considered in following calculations.
FRF plots

![FRF plots](image_url)
Table 11-24  Results of frequency measurements and dual parameter estimation for specimen N°19 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load (N)</th>
<th>% of yield</th>
<th>ΔEγ</th>
<th>fα (Hz)</th>
<th>fβ (Hz)</th>
<th>L (m)</th>
<th>using fα</th>
<th>using fβ</th>
<th>Error on fα (%)</th>
<th>Error on fβ (%)</th>
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<td>488329</td>
<td>-22.1</td>
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<tr>
<td>16617</td>
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<td>0.65</td>
<td>113.7</td>
<td>292.7</td>
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<td>10741</td>
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<td>117.1</td>
<td>296.7</td>
<td>1.255</td>
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<td>128.8</td>
<td>320.5</td>
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T19

Data
FRF plots
Data

Table 11-25  Results of frequency measurements and dual parameter estimation for specimen N°20 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>$S/E$</th>
<th>$f_t$</th>
<th>$f_l$</th>
<th>L</th>
<th>$E_{t0}$</th>
<th>$E_{l0}$</th>
<th>Error on $f_t$</th>
<th>Error on $f_l$</th>
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<td>108.2</td>
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<td>1.299</td>
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<td>13941</td>
<td>15.0</td>
<td>157340</td>
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<tr>
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<td>23.1</td>
<td>0.80</td>
<td>205.9</td>
<td>308.5</td>
<td>1.255</td>
<td>1494500</td>
<td>17936</td>
<td>14.6</td>
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<td>19175 N</td>
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<td>124.8</td>
<td>313.5</td>
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<td>27019</td>
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T20
FRF plots
## Data

Table 11-26  Results of frequency measurements and dual parameter estimation for specimen N°21 using the transversal and longitudinal E-modulus for different load levels

<table>
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<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>f_{N1}</th>
<th>f_{N2}</th>
<th>f_{L}</th>
<th>l</th>
<th>using E_{N}</th>
<th>using E_{L}</th>
<th>using E_{N}</th>
<th>using E_{L}</th>
<th>Error on f_{N}</th>
<th>Error on f_{L}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[N]</td>
<td>[%]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[m]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[%]</td>
<td>[%]</td>
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</table>
FRF plots
Data

Table 11-27  Results of frequency measurements and dual parameter estimation for specimen N°22 using the transversal and longitudinal E-modulus for different load levels

<table>
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<tr>
<th>Σ Applied load</th>
<th>5% of yield</th>
<th>f₁₀</th>
<th>f₁₀₀</th>
<th>L</th>
<th>using E₁̂</th>
<th>using E₂̂</th>
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<tbody>
<tr>
<td>[N]</td>
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</table>
FRF plots

![FRF plot 1](image1)

![FRF plot 2](image2)

![FRF plot 3](image3)
### Data

**Table 11-28** Results of frequency measurements and dual parameter estimation for specimen N°23 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied Load [N]</th>
<th>% of Yield</th>
<th>$S_E$</th>
<th>$f_S$</th>
<th>$f_L$</th>
<th>$L$</th>
<th>$E_{TS}$</th>
<th>$E_{LT}$</th>
<th>Error on $E_{TS}$ (%)</th>
<th>$E_{LT}$</th>
<th>$E_{LS}$</th>
<th>Error on $E_{LT}$ (%)</th>
</tr>
</thead>
<tbody>
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<td>0.26</td>
<td>119.0</td>
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<td>180200</td>
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</table>
FRF plots
### T24

#### Data

Table 11-29  Results of frequency measurements and dual parameter estimation for specimen N°24 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>ū/σ</th>
<th>fₜ</th>
<th>fₜ</th>
<th>L</th>
<th>Lₜ</th>
<th>Lₜ</th>
<th>Lₜ</th>
<th>Lₜ</th>
<th>Error on ū</th>
<th>Error on ū</th>
<th>Error on ū</th>
<th>Error on ū</th>
</tr>
</thead>
<tbody>
<tr>
<td>5708</td>
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<td>112.4</td>
<td>287.8</td>
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<td>204.6</td>
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<td>20708</td>
<td>153.7</td>
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<td></td>
</tr>
<tr>
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<td>116.1</td>
<td>297.7</td>
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<td>99545</td>
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<td>118.3</td>
<td>95642</td>
<td>25379</td>
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<td>65510</td>
<td>24854</td>
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<td>3.255</td>
<td>121180</td>
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<td>1.36</td>
<td>132.3</td>
<td>323.2</td>
<td>3.255</td>
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<td>36.5</td>
<td>98170</td>
<td>42084</td>
<td>47.5</td>
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</tbody>
</table>
FRF plots
Table 11-30: Results of frequency measurements and dual parameter estimation for specimen NO.25 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Load (N)</th>
<th>% Yield</th>
<th>$f_1$ (Hz)</th>
<th>$f_2$ (Hz)</th>
<th>$f_3$ (Hz)</th>
<th>$f_4$ (Hz)</th>
<th>E (GPa)</th>
<th>Error (%)</th>
</tr>
</thead>
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<td>59.8</td>
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<td>1.295</td>
<td>113170</td>
</tr>
<tr>
<td>10200</td>
<td>15.0</td>
<td>0.46</td>
<td>114.1</td>
<td>298.2</td>
<td>1.275</td>
<td>115140</td>
<td>12032</td>
</tr>
<tr>
<td>15000</td>
<td>22.1</td>
<td>0.65</td>
<td>118.7</td>
<td>507.9</td>
<td>1.295</td>
<td>100960</td>
<td>16432</td>
</tr>
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<td>19820</td>
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<td>122.6</td>
<td>317.3</td>
<td>1.255</td>
<td>26040</td>
<td>14999</td>
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<td>523.5</td>
<td>1.295</td>
<td>170310</td>
<td>24850</td>
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<td>1.26</td>
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<td>210400</td>
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T25

Data
FRF plots
### Data

**Table 11-31** Results of frequency measurements and dual parameter estimation for specimen N°26 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied Load [N]</th>
<th>% of yield [%]</th>
<th>$f_0$ [Hz]</th>
<th>$f_1$ [Hz]</th>
<th>$L$ [m]</th>
<th>$E_{tx}$ [N/mm²]</th>
<th>$E_{tl}$ [N/mm²]</th>
<th>Error on $E_{tx}$ [%]</th>
<th>$E_{lx}$ [N/mm²]</th>
<th>$E_{ll}$ [N/mm²]</th>
<th>Error on $E_{ll}$ [%]</th>
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<tbody>
<tr>
<td>5096</td>
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<td>706.2</td>
<td>277.3</td>
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<td>50729</td>
<td>11075</td>
<td>65.3</td>
<td>58087</td>
<td>14204</td>
<td>170.3</td>
</tr>
<tr>
<td>10596</td>
<td>15.6</td>
<td>538.6</td>
<td>291.5</td>
<td>1.275</td>
<td>935042</td>
<td>173034</td>
<td>80.8</td>
<td>59988</td>
<td>20443</td>
<td>97.7</td>
</tr>
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<td>15009</td>
<td>22.1</td>
<td>317.9</td>
<td>500.2</td>
<td>1.255</td>
<td>699779</td>
<td>240277</td>
<td>86.4</td>
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<td>30181</td>
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<td>19914</td>
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<td>122.8</td>
<td>308.8</td>
<td>1.255</td>
<td>91681</td>
<td>27723</td>
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<td>94386</td>
<td>31305</td>
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<tr>
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<td>323.9</td>
<td>313.0</td>
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<td>130230</td>
<td>368279</td>
<td>23.3</td>
<td>89548</td>
<td>40317</td>
<td>34.1</td>
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FRF plots

![Graph 1](image1)

![Graph 2](image2)

![Graph 3](image3)
Data

Table 11-32  Results of frequency measurements and dual parameter estimation for specimen N°27 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load [N]</th>
<th>% of yield [%]</th>
<th>% of E</th>
<th>f₁ [Hz]</th>
<th>f₂ [Hz]</th>
<th>L [m]</th>
<th>( L_s ) [N/m]</th>
<th>( L_r ) [N/m]</th>
<th>Error on ( L_s ) [%]</th>
<th>Error on ( L_r ) [%]</th>
<th>( L_s ) [N/m]</th>
<th>( L_r ) [N/m]</th>
<th>Error on ( L_s ) [%]</th>
<th>Error on ( L_r ) [%]</th>
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</thead>
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<td>7.6</td>
<td>0.32</td>
<td>105.4</td>
<td>279.0</td>
<td>1.25</td>
<td>91245</td>
<td>14771</td>
<td>384.1</td>
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<td>50023</td>
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<td>65217</td>
<td>50023</td>
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<tr>
<td>1004</td>
<td>14.9</td>
<td>0.60</td>
<td>111.8</td>
<td>265.8</td>
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<td>91160</td>
<td>20447</td>
<td>382.4</td>
<td>65672</td>
<td>52908</td>
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<td>117.4</td>
<td>256.0</td>
<td>1.25</td>
<td>67947</td>
<td>28801</td>
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<td>48971</td>
<td>32573</td>
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<td>48971</td>
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<td>123.9</td>
<td>304.5</td>
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<td>23225</td>
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<td>65987</td>
<td>31309</td>
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<td>65987</td>
<td>31309</td>
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<td>24810</td>
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<td>125.5</td>
<td>310.9</td>
<td>1.25</td>
<td>111210</td>
<td>36139</td>
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<td>77413</td>
<td>38883</td>
</tr>
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<td>314.9</td>
<td>1.25</td>
<td>109879</td>
<td>41059</td>
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<td>73371</td>
<td>45128</td>
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<td>45128</td>
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</table>
FRF plots
## Data

Table 11-33 Results of frequency measurements and dual parameter estimation for specimen N°28 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>S/E</th>
<th>f_0</th>
<th>f</th>
<th>L</th>
<th>( \bar{K}_0 )</th>
<th>( \bar{K}_1 )</th>
<th>Error on S</th>
<th>Error on L</th>
</tr>
</thead>
<tbody>
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<td>82.6</td>
<td>277.8</td>
<td>3.25</td>
<td>247350</td>
<td>6656</td>
<td>-0.6</td>
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</tr>
<tr>
<td>9000</td>
<td>4.6</td>
<td>0.48</td>
<td>111.3</td>
<td>289.2</td>
<td>3.275</td>
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<td>0.70</td>
<td>151.2</td>
<td>298.3</td>
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<td>1.15</td>
<td>120.0</td>
<td>311.9</td>
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FRF plots
## Data

### Table 11-34  Results of frequency measurements and dual parameter estimation for specimen N°29 using the transversal and longitudinal E-modulus for different load levels

<table>
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<tr>
<th>Applied load $f$</th>
<th>% of yield</th>
<th>$%E$</th>
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<th>$f_{29}$</th>
<th>$L$</th>
<th>$k_{29}$</th>
<th>$k_{29}$</th>
<th>Error on $k$</th>
<th>$k_{29}$</th>
<th>$k_{29}$</th>
<th>Error on $k$</th>
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<td></td>
<td>[Hz]</td>
<td>[Hz]</td>
<td>[m]</td>
<td>[N/mm]</td>
<td>[N/mm]</td>
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<td>[N/mm]</td>
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<td>0.48</td>
<td>120.6</td>
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</table>
FRF plots
T30

Data

Table 11-35  Results of frequency measurements and dual parameter estimation for specimen N°30 using the transversal and longitudinal E-modulus for different load levels

<table>
<thead>
<tr>
<th>Applied load S</th>
<th>% of yield</th>
<th>%σt</th>
<th>Eα</th>
<th>Eγ</th>
<th>L</th>
<th>Error on S using Eα</th>
<th>Error on S using Eγ</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>(N)</td>
<td>(%)</td>
<td>(GPa)</td>
<td>(GPa)</td>
<td>(m)</td>
<td>(N)</td>
<td>(N)</td>
</tr>
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<td>/</td>
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</table>
FRF plots
T31

Data

Table 11-36  Results of frequency measurements and dual parameter estimation for specimen N°31 using the transversal and longitudinal E-modulus for different load levels

<table>
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<tr>
<th>Applied load</th>
<th>% of yield</th>
<th>%E</th>
<th>( f_0 )</th>
<th>( f_1 )</th>
<th>L</th>
<th>( \kappa_{E_1} )</th>
<th>( \mu_{E_1} )</th>
<th>Error on ( \kappa )</th>
<th>( \kappa_{E_2} )</th>
<th>( \mu_{E_2} )</th>
<th>Error on ( \mu )</th>
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<td>212770</td>
<td>9300</td>
<td>95.9</td>
<td>266480</td>
<td>3068</td>
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<td>123.6</td>
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<td>1.25</td>
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<td>33230</td>
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FRF plots
T32

Data

Table 11-37  Results of frequency measurements and dual parameter estimation for specimen N°32 using the transversal and longitudinal E-modulus for different load levels

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<th>% def</th>
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<th>$S_{T}$ [N]</th>
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FRF plots
Aluminium

Data

Table 11-38  Results of frequency measurements and dual parameter estimation for the reference aluminium specimen using the transversal and longitudinal E-modulus for different load levels

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<th>f&lt;sub&gt;y&lt;/sub&gt;</th>
<th>L&lt;sub&gt;1&lt;/sub&gt;</th>
<th>E&lt;sub&gt;x&lt;/sub&gt;</th>
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FRF plots
11.3.2 Summary of results

Table 11-39  Error on the estimated axial load S for each specimen and different load levels using the transversal E-modulus

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Mean value: 46.6, Std. dev.: 34.3

Table 11-40  Error on the estimated axial load S for each specimen and different load levels using the longitudinal E-modulus

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Mean value: 68.4, Std. dev.: 48.1
Table 11-41  Mean estimated rotational stiffness $k$ for each specimen using the transversal and longitudinal E-modulus and comparison with the static E-modulus

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Std. dev.  233373  64211  1217

Alu  16479  12045  70000