The giant component of the random bipartite graph

Master's Thesis in Engineering Mathematics and Computational Science

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Abstract

The random bipartite graph $G(m,n;p)$ is the random graph obtained by taking the complete bipartite graph $K_{m,n}$ and deleting each edge independently with probability $1-p$. Using a branching process argument, the threshold function for a giant connected component in $\mathbb{G}(m,n;p)$ is found to be $1/\sqrt{mn}$, and the number of vertices in the giant component is proportional to $\sqrt{mn}$. An attempt is made to reduce the study of $\mathbb{G}(m,n;p)$ to the well-studied random graph $\mathbb{G}(m,p)$ using an original argument, and this is used to show that *a.a.s.* the giant of $\mathbb{G}(m,n;p)$ is cyclic when $m = o(n)$ and $p\sqrt{mn}$ is large enough. Using a colouring argument, similar results are shown for Fortuin-Kasteleyn's random-cluster model with parameter $1 \leq q \leq 2$ when $m = o(n)$. In particular, when $m = o(n)$ the critical inverse temperature for the Ising model on $K_{m,n}$ is found to be $-\frac{1}{2}\log\left(1 - \frac{2}{\sqrt{mn}}\right)$, and the number of vertices in the giant is proportional to $\sqrt{mn}$.

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1

Introduction

Since its initiation in the late 1950s by Erdős and Rényi, the field of random graphs has become an important and largely independent part of combinatorics and probability theory. There have been several books published in the field, the most influential being Béla Bollobás' 1985 monograph *Random Graphs* [2]. Inspired by that book, and Svante Janson, Tomasz Łuczak and Andrzej Ruciński's 2001 book by the same name [10], this thesis extends classical results for the complete graph K_m to the complete bipartite graph $K_{m,n}$. In particular, it focuses on results on the so-called giant component, which is historically important as one of the first remarkable results on random graphs.

The idea behind this thesis may be loosely described as starting with a certain set of vertices, randomly adding edges and studying the properties of the resulting graph. It is important to note that this may be done in any number of ways, and that random graphs may be obtained in other ways, e.g. by assigning numerical values to the vertices of the graph. For the most part, this thesis studies the independent Erdős-Rényi model, named after its originators, in which each edge is included with equal probability in the random graph, independently of all other edges. Other models include the closely related fixed-edge number Erdős-Rényi model, in which a fixed number M of edges is included in the graph and any graph with M edges is equiprobable.

This section sets the stage for the theorems of later sections, by introducing readers to the field of random graphs, and fixing the notation used throughout the thesis. Readers familiar with random graphs can skip Section 1 with no problem.

Section 2 presents proofs of the main theorems of the thesis, imitating the techniques of [10, Theorem 5.4] closely. The argument carries well through the transition from complete graphs to complete bipartite graph, with a slight alteration in which only one part of the bipartite graph $K_{m,n}$ is considered.

Only considering part of the vertices in the branching process argument suggests a method with which the study of random bipartite graph may be reduced to the already established study of random complete graphs. Section 3 investigates this reduction method, and discusses its applications and limitations.

Following a colouring argument from a 1996 paper by Bollobás, Grimmett and Janson [3], Section 4 extends the results of previous sections to Fortuin-Kasteleyn's random-cluster model. In particular, the section shows the existence and size of a giant component for the Ising model, important in statistical physics.

1.1 Basic graph theory

A *graph* is defined as a set (V,E) of labelled vertices V and edges $E \subseteq V \times V$. All graphs will be undirected, so that $(x,y) \in E$ if and only if $(y,x) \in E$. Throughout this thesis, V will be a finite set consisting of indexed letters; typically $V = (a_1, a_2, ..., a_m; b_1, b_2, ..., b_n)$ or $V = (a_1,...,a_m)$ for some positive integers m,n. An edge (x,y) between vertices $x, y \in V$ will be denoted by the shorthand notation xy.

For a given edge xy, the vertices x and y are called the *endpoints of xy*, and the vertices x and y are *neighbours*. Similarly, two edges $e,f \in E$ are *adjacent* if they share an endpoint. A *path* from x to y is a set of edges $e_0,...,e_k \in E$ such that e_{i-1} and e_i are adjacent for all $1 \leq i \leq k$ and x and y are endpoints of e_0 and e_k, respectively. Equivalently, a path from x to y may be defined as a sequence of neighbours $x = x_0, x_1,...,x_k = y$.

Let $x \sim y$ if and only if there is a path from x to y. This is an equivalence relation which partitions V into sets $C_1,...,C_k$ called the *(connected) components* of G. The *connected component of* x is the unique $C_i = C(x)$ such that $x \in C_i$.

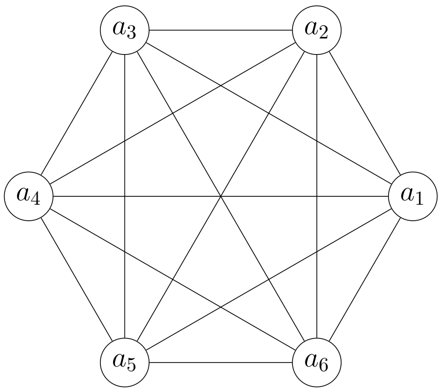

Figure 1.1: The complete graph K_6, as defined in Section 1.1. While complete graphs will not be studied in this thesis, the results for complete bipartite graphs are inspired by and at some points derived from the corresponding results for complete graphs.

A *complete* graph with m vertices, denoted K_m, is a graph with vertex set $V_m = (a_1,...,a_m)$ and edge set $E_m = \{(a_i,a_j) : i \neq j\}$. A *complete bipartite* graph is a graph with $V_{m,n} = (a_1,...,a_m; b_1,...,b_n)$ and $E_{m,n} = A_m \times B_n$, where $A_m = (a_1,...,a_m)$ and $B_n = (b_1,...,b_n)$ make up a partition of $V_{m,n}$. A vertex $a_i \in A_m$ will be called *even* and

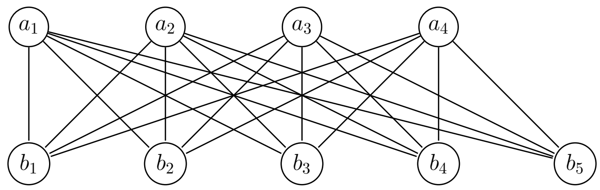

Figure 1.2: The complete bipartite graph $K_{4,5}$, as defined in Section 1.1. Graphs of this type are the object of study in the current thesis. Complete bipartite graphs will always be drawn with the *even* vertices $a_1,\ldots,a_m$ in the top row, and the *odd* vertices $b_1,\ldots,b_n$ in the bottom row.

a vertex $b_j \in B_n$ *odd*. Complete bipartite graphs are denoted $K_{m,n}$. Throughout the thesis, we make the assumption that $m = \mathcal{O}(n)$, i.e. that there is a constant $a \geq 1$ such that $m \leq an$ for all m,n. We regard n as a function of m, so that whenever n appears it should be read as $n = n(m)$. Theorems will be shown to hold for $m = an$, $a > 0$ and/or $m = o(n)$. We note that $m \to \infty$ implies $n \to \infty$.

A *subgraph* $H = (V,F)$ of $G = (V,E)$ is defined as a graph with the same vertex set as G, and with edge set $F \subseteq E$. In most cases it is convenient to call an edge $e \in E$ *open* if $e \in F$ and *closed* otherwise. Each subgraph corresponds to an *edge-configuration* $\xi \in \{0,1\}^E$ where $\xi(e) = 1$ if e is open and $\xi(e) = 0$ otherwise. We define a partial ordering $\preceq$ on the space $\{0,1\}^E$ by $\xi \preceq \eta$ if and only if $\xi(x) \leq \eta(x)$ for all $x \in E$.

A *property* of a graph (V,E) is defined as a family of subgraphs $Q \subseteq \mathcal{P}(E)$. A subgraph (V,F) *has property* Q if $F \in Q$. An *increasing property* is a property Q such that if $F_1 \in Q$ and $F_1 \subseteq F_2$, then $F_2 \in Q$. In other words, a property is increasing if adding edges to any graph with the property will produce a graph which also has that property.

1.2 Asymptotics

Asymptotic notation is frequent in random graphs, since results are typically shown "as $m \to \infty$". We define for any function f and any positive function g the following:

$$f(x) = \mathcal{O}(g(x)) \quad \text{if} \quad \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty \tag{1.1}$$

$$f(x) = o(g(x)) \quad \text{if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \tag{1.2}$$

Most results in this thesis are shown to hold *asymptotically almost surely*, or *a.a.s.* A sequence of random variables $(X_k)_{k=1}^{\infty}$ has property Q *a.a.s.* if an only if $\mathbf{P}\{X_m \in Q\} \to 1$. Furthermore, for a sequence X_m of random variables we define

$$X_m = o_p(a_m) \quad \text{if} \quad |X_m| = o(a_m) \ \textit{a.a.s.} \tag{1.3}$$

The usage of o_p and *a.a.s.* is quite interchangable. Typically, asymptotic bounds are stated *a.a.s.* and asymptotic values are stated using o_p terms.

1.3 Basic probability

Many results will rely heavily on inequalities related to the Markov inequality. For any non-negative random variable X with finite expectation, Markov's inequality states

$$\mathbf{P}\{X \geq a\} \leq \frac{\mathbf{E}[X]}{a}, \quad a > 0 \tag{1.4}$$

Supposing further that X has finite second moment, Markov's inequality applied on the random variable $|X - \mathbf{E}[X]|^2$ implies Chebyshev's inequality:

$$\mathbf{P}\{|X - \mathbf{E}[X]| \geq a\} = \mathbf{P}\left\{|X - \mathbf{E}[X]|^2 \geq a^2\right\} \leq \frac{\mathrm{Var}X}{a^2}, \quad a > 0 \tag{1.5}$$

This will frequently be applied when a sequence of random variables X_m is shown to satisfy $\sqrt{\mathrm{Var}\ X} = o_p(\mathbf{E}[X_m])$. Chebyshev's inequality implies

$$\mathbf{P}\{|X_m - \mathbf{E}[X_m]| \geq \mathbf{E}[X_m]\} \leq \frac{\mathrm{Var}\ X_m}{\mathbf{E}[X_m]^2} = o(1) \tag{1.6}$$

so that $X_m = \mathbf{E}[X_m](1 + o_p(1))$.

The following bounds hold for all $a \in \mathbb{R}$, and are commonly called Chernoff bounds.

$$\mathbf{P}\{X \geq a\} = \mathbf{P}\left\{e^{tX} \geq e^{ta}\right\} \leq \frac{\mathbf{E}\left[e^{tX}\right]}{e^{ta}}, \quad t \geq 0 \tag{1.7}$$

$$\mathbf{P}\{X \leq a\} = \mathbf{P}\left\{e^{tX} \geq e^{ta}\right\} \leq \frac{\mathbf{E}\left[e^{tX}\right]}{e^{ta}}, \quad t < 0 \tag{1.8}$$

For a random variable X, the *probability-generating function* G_X is defined by $G_X(s) = \mathbf{E}\left[s^X\right]$ when this expectation exists. The *moment-generating function* M_X is defined, when it exists, by $M_X(t) = \mathbf{E}\left[e^{tX}\right] = G_X(e^t)$. The property of these functions that will be used most frequently is the following: Let N be a positive integer-valued random variable, and let $Y = X_1 + ... + X_N$ where the X_i are *i.i.d.* and independent of N. Then

$$M_Y(t) = M_N(M_X(t)) \quad \text{and} \quad G_Y(s) = G_N(G_X(s)) \tag{1.9}$$

whenever the functions exist. See e.g. [7].

A random variable X is *Bernoulli distributed with parameter* $p \in [0,1]$, denoted $X \in \mathrm{Bern}(p)$, if $\mathbf{P}\{X = 1\} = p$ and $\mathbf{P}\{X = 0\} = 1 - p$. It is *binomially distributed with paramters* $n \in \mathbb{N}$ *and* $p \in [0,1]$, denoted $X \in \mathrm{Bi}(n,p)$, if for any integer $0 \leq k \leq n$

$$\mathbf{P}\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k} \tag{1.10}$$

The probability-generating function of $X \in \mathrm{Bi}(n,p)$ is given by $G_X(s) = (1 - p + ps)^n$.

1.3.1 Branching processes

A *Galton-Watson process* or *branching process* is a stochastic process $(Z_k)_{k=0}^{\infty}$, defined as follows. We choose a non-negative integer-valued probability distribution $\mathcal{A}$ and call it the *offspring distribution.* Given a deterministic starting value Z_0 (usually taken to be 1), each $Z_k, k \geq 1$ is defined by $Z_k = \sum_{i=1}^{Z_{k-1}} X_{k_i}$, where the $X_{k_i} \in \mathcal{A}$ are independent. The variables Z_k are interpreted as the number of individuals in the k-th generation of a population, and each X_{k_i} is the number of offsprings for an individual.

The *extinction probability* for a branching process is the probability of the event $\{\exists k > 1 : Z_k = 0\}$. Let $X \in \mathcal{A}$. A classical result states that if $\mathbf{E}[X] \leq 1$, the extinction probability is 1, and if $\mathbf{E}[X] > 1$ the extinction probability is given by the unique solution in (0,1) to $G_X(s) = s$. Here G_X denotes the probability-generating function of X.

1.3.2 Stochastic domination

In Section 3 we shall need the measure-theoretic concept of *stochastic domination.* Given $\mathcal{S} \subseteq \mathbb{R}$, a partial ordering $\preceq$ on a space Ω and probability measures μ, ν on Ω, we say that μ stochastically dominates ν, written $\nu \preceq_{\mathcal{D}} \mu$, if $\nu(f) \leq \mu(f)$ for all increasing functions $f : \Omega \to \mathcal{S}$ (with respect to $\preceq$). Heuristically, this means that μ prefers larger elements of Ω than ν. In particular, for any increasing event $B \subseteq \mathcal{S}$ we have $\nu(B) \leq \mu(B)$. See [6].

The following powerful result is used to characterize stochastic domination in relation to random graphs. It is a special case of a more general result called Holley's Theorem, see e.g. [6]. Suppose $x \in E$ and let $X(x) = 1$ if $x \in E$ is open and $X(x) = 0$ otherwise. We define the event $\{X = \xi \text{ off } x\}$ for $\xi \in \{0{,}1\}^{E \setminus \{x\}}$ as the event $\{X(y) = \xi(y) \text{ for all } y \in E \setminus \{x\}\}$.

Holley's Theorem. *Let $G = (V,E)$ be a graph and let μ_1, μ_2 be probability measures on $\{0{,}1\}^E$. If*

$$\mu_1(X(x) = 1 \mid X = \xi \textit{ off } x) \leq \mu_2(X(x) = 1 \mid X = \xi' \textit{ off } x) \tag{1.11}$$

for all $x \in E$ and all $\xi, \xi' \in \{0{,}1\}^{E \setminus \{x\}}$ such that $\xi \preceq \xi'$, then $\mu_1 \preceq_{\mathcal{D}} \mu_2$.

1.4 Random graph models

Let $G = (V,E)$ be a graph. A random graph model, in all instances of this thesis, is a probability measure on either of the spaces $\mathcal{S}^V$ and $\mathcal{S}^E$ for some $\mathcal{S} \subseteq \mathbb{R}$. This section presents the different models that will be used or mentioned in the thesis.

1.4.1 The Erdős-Rényi model

The model that Erdős and Rényi introduced, which has been subsequently named the *Erdős-Rényi model*, comes in two closely linked forms. The one most suited for this thesis is denoted $\mathbb{G}(m,p)$, which generates a subgraph of the complete graph $K_m = (V_m, E_m)$ in which each edge is included independently of all others with probability p. Equivalently, it may be seen as starting with $K_{m,n}$ and removing each edge independently with probability $1-p$. In other words, for any subgraph $H = (V_m, F)$ where $F \subseteq E_m$,

$$\mathbf{P}\{F\} = \binom{\binom{m}{2}}{|F|} p^{|F|}(1-p)^{\binom{m}{2}-|F|} \tag{1.12}$$

For the complete bipartite graph $K_{m,n} = (A_m \cup B_n, A_m \times B_n)$ we denote the corresponding model $\mathbb{G}(m,n;p)$. A subgraph $H = (A_m \cup B_n, F)$ is assigned probability

$$\mathbf{P}\{F\} = \binom{mn}{|F|} p^{|F|}(1-p)^{mn-|F|} \tag{1.13}$$

1.4.2 Ising and Potts models

A physically important random graph model is the Ising model, invented by Wilhelm Lenz [11] and his student Ernst Ising [9] to model ferromagnetisism. Mathematically, it is quite different from the Erdős-Rényi model in that it assigns probability to vertices and not to edges. Each vertex is assigned a *spin*, which here will be denoted by the values 1 or 2, and the model will prefer vertex configurations in which neighbours have equal spin. Formally, the Ising model on a finite graph $G = (V,E)$ consists of a probability measure on $\{1,2\}^V$ which to each $\sigma \in \{1,2\}^V$ assigns probability

$$\phi_\beta(\sigma) = \frac{1}{Z_\beta} \exp\left(-2\beta \sum_{x \sim y} I_{\{\sigma(x) \neq \sigma(y)\}}\right) \tag{1.14}$$

where $\beta \geq 0$, and $x \sim y$ if and only if $x, y \in V$ are neighbours. Here Z_β is a normalizing constant. In ferromagnetic applications β bears the interpretation of reciprocal temperature, i.e. $\beta = 1/T$ where T is the temperature.

A natural generalization of the Ising model is the q-state Potts model [12], which assigns spins in the set $\{1,...,q\}$ for some integer $q \geq 2$. The Potts probability measure on $\{1,...,q\}^V$ is identical to (1.14) except for a new normalizing constant $Z_{\beta,q}$.

1.4.3 The random-cluster model

The random-cluster model, introduced by Cees Fortuin and Pieter Kasteleyn in a series of papers around 1970, see e.g. [8], is a random graph model that unifies the Erdős-Rényi, Ising and Potts models. For any $q > 0$ it assigns to any subgraph (V,F) of a graph (V,E) probability proportional to

$$\widetilde{P}(F; E,p,q) = p^{|F|}(1-p)^{|E|-|F|} q^{c(V,F)} \tag{1.15}$$

where $c(V,F)$ is the number of components of (V,F). Swendsen and Wang [13], followed by a simpler proof by Edwards and Sokal [4], showed that the random-cluster model at probability p for any $q \in \mathbb{N}$ is equivalent to the q-state Potts model at inverse temperature $\beta = -\frac{1}{2}\log(1-p)$.

1.5 The giant component

Around 1960, Erdős and Renyi [5] proved the first result concerning the existence and non-existence of a so-called giant component in $\mathbb{G}(m,p)$. Essentially, the result states that the graph $\mathbb{G}(m,\lambda/m)$ *a.a.s.* contains one component which is much larger than all other components when $\lambda > 1$, while no such component exists for $\lambda < 1$. The following formulation of the result appears in [10]:

Theorem. *Let $mp = \lambda$, where $\lambda > 0$ is a constant.*

(i) If $\lambda < 1$, then a.a.s. the largest component of $\mathbb{G}(m,p)$ has at most $\frac{3}{(1-\lambda)^2}\log m$ vertices.

(ii) Let $\lambda > 1$ and let $\theta = \theta(\lambda) \in (0,1)$ be the unique solution to $\theta + e^{-\lambda\theta} = 1$. Then $\mathbb{G}(m,p)$ contains a giant component of $\theta m(1 + o_p(1))$ vertices. Furthermore, a.a.s. the size of the second largest component of $\mathbb{G}(m,p)$ is at most $\frac{16\lambda}{(\lambda-1)^2}\log m$.

This thesis will mainly be concerned with proving a similar result for the bipartite graph $\mathbb{G}(m,n;\lambda/\sqrt{mn})$, using proof techniques similar to those in [10].

In a 1996 paper [3], Bollobás et al. derived conditions for the existence and size of a giant component in the random-cluster model for any $q > 0$, by reducing to the $q = 1$ case and using known results. This thesis will partly emulate those methods on the bipartite graphs to prove similar results.

2

The branching process

SINCE the results contained in this thesis are concerned with component sizes, a method to measure the size of a component is needed. The following method is inspired by the one used in [10], and the process it describes will be called a *even search process* (or simply *search process*) and is closely related to a breadth-first search, common in computer graph algorithms. The algorithm finds the vertices in the connected component of an even vertex a, and counts the number of even and odd vertices separately.

1. Let $a_0 = a$ and $S = \emptyset$, and mark a_0 as *saturated*. All other vertices are initially *unsaturated*. Set $k = 0$.

2. Find all unsaturated neighbours b_1,...,b_r of a_k, and mark them as saturated. Let $R_{k+1} = \{b_1,...,b_r\}$ and $Y_{k+1} = |R_{k+1}|$.

3. For each $b_i \in R_{k+1}$, find all unsaturated neighbours $a_{i1}, ..., a_{is}$ of b_i, $s = s(i)$. Let $S_{k+1} = \cup_{i=1}^{r}\{a_{i1},...,a_{is}\}$. Thus, S_{k+1} is the set of all unsaturated vertices at distance 2 from a_k. Let $X_{k+1} = |S_{k+1}|$.

4. Assign $S := S \cup S_{k+1}$. If S is empty, stop the algorithm. If S is nonempty, let a_{k+1} be an arbitrary element of S, remove a_{k+1} from S and mark a_{k+1} as saturated. Return to step 2 with $k + 1$ in place of k.

Here the search for odd vertices b_i should be considered an intermediate step: we are only interested in finding the even vertices of the component containing a. So X_1 is the number of "2-neighbours", i.e. vertices at distance 2 from a, and X_2, X_3,... are the number of vertices at distance 2 from vertices that have already been found in the process. The number of even vertices in the component is given by $1 + \sum_{k=1}^{K} X_k$, where K is the total number of iterations of the algorithm.

This resembles a branching process, the difference being that a branching process as defined in Section 1.3.1 requires each X_i to be identically distributed. However, the resemblance is in some sense close enough, in that we can find actual branching processes bounding the search process from above and below.

It is clear that $Y_1 \in \mathrm{Bi}(n,p)$, since at the start all vertices are unsaturated, and a has n neighbours. After this the distributions are more complicated, but we may bound X_1 from above by $X_1^+ \in \mathrm{Bi}(Y_1 m,p)$, since it is the sum of Y_1 random variables, each bounded from above by $\mathrm{Bi}(m,p)$. Continuing in this fashion, we may bound Y_k from above by $Y_k^+ \in \mathrm{Bi}(n,p)$ for all k, and X_k from above by $X_k^+ \in \mathrm{Bi}(Y_k^+ m,p)$. Using (1.9), we have the following probability-generating function for X_k^+:

$$G_{X_k^+}(s) = (1 - p + p(1 - p + ps)^m)^n \tag{2.1}$$

Given information about the maximal size of a component, we may bound the process from below by similar methods, as will be seen below.

2.1 Proportion of even and odd vertices

The even search process described above finds the number of even vertices of a component in the graph $\mathbb{G}(m,n;p)$. To be able to state results about the total size of the component, we show a result that relates the number of even vertices in a component to the number of odd vertices in the same component. Heuristically, this section shows that the number of odd vertices in a component is pn times the number of even vertices, if the number of even vertices in the component is small enough. Note that this proportion equals the expected degree of any even vertex. For this, we need a purely probabilistic lemma.

Lemma 1. *If $X \in Bi(n,p)$ where n and $p = p(n)$ are such that $\boldsymbol{E}[X] = np \to \infty$ as $n \to \infty$, then $X = np(1 + o_p(1))$*

Proof. We have $\mathbf{E}[X] = np$ and $\mathrm{Var}X = np(1-p) = o(\mathbf{E}[X]^2)$ so that by (1.6), $X = np(1 + o_p(1))$. □

Lemma 1 is useful in proving the following.

Lemma 2. *Let $m_0 = m_0(m)$ be such that $pm_0n \to \infty$ and $pm_0 \to 0$ as $m \to \infty$. Let C be a component of $\mathbb{G}(m,n;p)$. Conditional on C having m_0 even vertices, C has $pm_0n(1 + o_p(1))$ odd vertices.*

Proof. In each step of the search process, a number Y_k of odd vertices is identified. These random variables are each bounded from above by a random variable $Y_k^+ \in \mathrm{Bi}(n,p)$. Since the number of odd vertices in the component, conditional on the component having m_0 even vertices, is a sum of m_0 such variables, it must be bounded from above by $Y^+ \in \mathrm{Bi}(m_0n,p)$. But by the assumption that $pm_0n \to \infty$, we have $\mathbf{E}[Y^+] \to \infty$ and by Lemma 1, $Y^+ = pm_0n(1 + o_p(1))$.

Let $n_+ = pm_0n(1 + o_p(1))$. Then the random variables Y_k must all be bounded from below by $Y_k^- \in \mathrm{Bi}(n - n_+,p)$, since throughout the process there are at least $n - n_+$ odd vertices that are not yet saturated, and Y is bounded from below by $Y^- \in$

Bi($m_0(n-n_+)$,p). But $pm_0(n-n_+) = pm_0n(1+o(1))$ since $n_+ = o_p(n)$, so $\mathbf{E}[Y^-] \to \infty$ and we have $Y^- = pm_0n(1+o_p(1))$.

Thus, conditional on the component having m_0 even vertices, we must have $Y = pm_0n(1+o_p(1))$. □

Corollary 3. *Let $\lambda > 0$ and $p = \frac{\lambda}{\sqrt{mn}}$. Suppose (a) $m = o(n)$ or (b) $m = an$, $a > 0$, and $m_0 = o(m)$. Let C be a component of $\mathbb{G}(m,n;p)$, and m_0 be such that $m_0 \to \infty$ as $m \to \infty$. Conditional on C having m_0 even vertices, it has $\lambda m_0\sqrt{\frac{n}{m}}(1+o_p(1))$ odd vertices.*

Proof. (a) We have $pm_0n = \lambda m_0\sqrt{\frac{n}{m}} \to \infty$ and $pm_0 = \lambda\frac{m_0}{\sqrt{mn}} \le \lambda\sqrt{\frac{m}{n}} \to 0$ as $m \to \infty$, and the result follows from Lemma 2.

(b) We have $pm_0n = \frac{\lambda}{\sqrt{a}}m_0 \to \infty$ and $pm_0 \le \lambda m_0/m = o(1)$ so the result follows from Lemma 2. □

Corollary 3 leaves only the case $m_0 = \theta m$, $m = an$ for some $\theta \in (0,1]$ and $a > 0$. In this case, the number of even and odd vertices will both need to be calculated explicitly.

2.2 Subcritical case

This section is concerned with proving the non-existence of a giant component when $\lambda < 1$ using branching process arguments. It is assumed that $m \le n$, which is enough by symmetry. We shall need two lemmas.

Lemma 4. *Let $\lambda < 1$, $m_1 \le m_2 \le n$ and $p_1 = \lambda/\sqrt{m_1n}$, $p_2 = \lambda/\sqrt{m_2n}$. Then*

$$\left(1 - p_1 + p_1\left(1 - p_1 + \frac{p_1}{\lambda}\right)^{m_1}\right)^n \le \left(1 - p_2 + p_2\left(1 - p_2 + \frac{p_2}{\lambda}\right)^{m_2}\right)^n \tag{2.2}$$

The proof of Lemma 4 is left to Appendix A.

Lemma 5. *Let $\lambda < 1$. Given a random variable X with moment-generating function $M_X(t) = \left(1 - p + p\left(1 - p + pe^t\right)^m\right)^n$, where $m \le n$ and $p = \lambda/\sqrt{mn}$, we have*

$$\lambda\boldsymbol{E}\left[\lambda^{-X}\right] \le \exp\left(-\frac{1}{6}(1-\lambda)^3\right). \tag{2.3}$$

Proof. First of all we note that

$$\lambda\mathbf{E}\left[\lambda^{-X}\right] = \lambda\left(1 - p + p\left(1 - p + \frac{p}{\lambda}\right)^m\right)^n \tag{2.4}$$

The proof is done in two steps. By Lemma 4 we can reduce this to showing

$$\lambda\left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}\left(1 - \frac{\lambda}{n} + \frac{1}{n}\right)^n\right)^n \le \exp\left(-\frac{1}{6}(1-\lambda)^3\right) \tag{2.5}$$

This is done by noting that $\left(1+\frac{1-\lambda}{n}\right)^n \leq \exp(1-\lambda)$, and $\left(1+\frac{\lambda(e^{1-\lambda}-1)}{n}\right)^n \leq \exp(\lambda e^{1-\lambda}-\lambda)$, so that

$$\lambda\left(1-\frac{\lambda}{n}+\frac{\lambda}{n}\left(1-\frac{\lambda}{n}+\frac{1}{n}\right)^n\right)^n \leq \lambda\exp\left(\lambda e^{1-\lambda}-\lambda\right) \tag{2.6}$$

The proof is finished by showing that this is no larger than $\exp(-(1-\lambda)^3/6)$. This is equivalent to $g(\lambda) \geq 0$, where

$$g(\lambda) = \frac{\lambda^3}{6} - \frac{\lambda^2}{2} + \frac{3\lambda}{2} - \frac{1}{6} - \log\lambda - \lambda e^{1-\lambda} \tag{2.7}$$

It is easy to show that $g(1) = g'(1) = 0$, and $g''(\lambda) > 0$ for all $\lambda \in (0,1)$. Thus, $g'(\lambda) < 0$ and $g(\lambda) > 0$ for all $\lambda \in (0,1)$, and the result follows.

□

Theorem 6. *If $\lambda < 1$ and $m \leq n$, then there is a.a.s. no component in $\mathbb{G}(m,n;p)$ with more than $7(1-\lambda)^{-3}\log m$ even vertices.*

Proof. We bound the search process by a branching process with offspring distribution having moment-generating function $M_X(t) = \left(1-p+p\left(1-p+pe^t\right)^m\right)^n$, as discussed at the beginning of Section 2. Consider a search process initiated at the even vertex a. The probability that the total number of even vertices found in the process is at least k is bounded as follows:

$$\mathbf{P}\left\{a \text{ belongs to a component of size at least } k\right\} \leq \mathbf{P}\left\{\sum_{i=1}^{k} X_i \geq k-1\right\} \tag{2.8}$$

We can bound the quantity by the following Chernoff bound:

$$\mathbf{P}\left\{\sum_{i=1}^{k} X_i \geq k-1\right\} \leq \frac{\mathbf{E}\left[\exp(t(X_1+...+X_k))\right]}{\exp(t(k-1))} = \frac{\mathbf{E}\left[e^{tX_1}\right]^k}{e^{t(k-1)}}, \quad t>0 \tag{2.9}$$

where we have used the fact that the X_i are *i.i.d.* to separate the expectation. The inequality holds for any $t > 0$, and in particular we may choose $t = \log\left(\frac{1}{\lambda}\right)$ to obtain

$$\mathbf{P}\left\{\sum_{i=1}^{k} X_i \geq k-1\right\} \leq \lambda^{k-1}\mathbf{E}\left[\lambda^{-X_1}\right]^k = \frac{1}{\lambda}\left(\lambda\mathbf{E}\left[\lambda^{-X_1}\right]\right)^k \tag{2.10}$$

Using Lemma 5, and the $m \leq n$ assumption, we have $\lambda\mathbf{E}\left[\lambda^{-X_1}\right] \leq \exp\left(-\frac{1}{6}(1-\lambda)^3\right)$.

So letting $k = 7(1-\lambda)^{-3}\log m$, we have

$$\begin{aligned}
& \mathbf{P}\{\exists \text{ component with at least } k \text{ even vertices}\} \\
\leq\ & \sum_{i=1}^{m} \mathbf{P}\{a_i \text{ belongs to a component of size at least } k\} \\
\leq\ & m\mathbf{P}\left\{\sum_{i=1}^{k} X_i \geq k-1\right\} \\
\leq\ & \frac{m}{\lambda}\exp\left(-\frac{k}{6}(1-\lambda)^3\right) \\
=\ & \frac{m}{\lambda} m^{-7/6} \\
=\ & o(1) \qquad (2.11)
\end{aligned}$$

□

2.3 Supercritical case

This section is concerned with proving the existence, uniqueness and size of a giant component when $\lambda > 1$ using the even search process. Here we weaken the $m \leq n$ assumption, since we will need to cover all cases in which $m \sim n$. Instead we assume there is an $a \geq 1$ such that $m \leq an$.

Lemma 7. *Suppose there is an $a \geq 1$ such that $m \leq an$. Let $k_- = \frac{3\log^2 m}{\lambda(\lambda-1)^2}$ and $k_+ = \frac{\sqrt{3m\log m}}{\lambda(\lambda-1)}$. If $\lambda > 1$, then there is a.a.s. no component with k even vertices, where $k_- \leq k \leq k_+$.*

Remark. Note, in particular, that $k_+ > 2k_-$ for large m. This means that two components with less than k_- even vertices can never be joined by a single edge to form a component with at least k_+ even vertices.

Proof. Starting at an even vertex a, we show that the search process **either** terminates after fewer than k_- steps, **or** at the k-th step there are at least $(\lambda-1)k$ vertices in the component containing a that have been generated in the process but which are not yet saturated, for any k satisfying $k_- \leq k \leq k_+$. Indeed, if $(\lambda-1)k$ unsaturated vertices have been found, then the process continues since $(\lambda-1)k \geq (\lambda-1)k_- \geq 1$ for m large enough.

Since we consider only the part of the process where $k \leq k_+$, we need only identify at most $k_+ + (\lambda-1)k_+ = \lambda k_+$ even vertices in the process, and we may bound X_i from below by $X_i^- \in \mathrm{Bi}(Y(m-\lambda k_+), p)$ where $Y \in \mathrm{Bi}(n,p)$. The probability that after k steps there are fewer than $(\lambda-1)k$ not yet saturated vertices, or that the process dies out after the first k steps, is smaller than

$$\mathbf{P}\left\{\sum_{i=1}^{k} X_i^- \leq k-1+(\lambda-1)k\right\} = \mathbf{P}\left\{\sum_{i=1}^{k} X_i^- \leq \lambda k - 1\right\} \qquad (2.12)$$

As in the subcritical setting, we employ Chernoff bounds to bound this. First, we note that for any X_i the moment-generating function is given by

$$M(t) = \left(1 - p + p\left(1 - p + pe^t\right)^{m-\lambda k_+}\right)^n \tag{2.13}$$

Putting $t = \frac{1}{\lambda k}\log\left(\frac{1}{\lambda}\right) < 0$, we have the following Chernoff bound:

$$\begin{aligned}\mathbf{P}\left\{\sum_{i=1}^{k} X_i^- \leq \lambda k - 1\right\} &\leq \frac{M\left(\frac{1}{\lambda k}\log\left(\frac{1}{\lambda}\right)\right)^k}{\exp\left(\frac{1}{\lambda k}\log\left(\frac{1}{\lambda}\right)(\lambda k - 1)\right)} \\ &= \lambda^{1-\frac{1}{\lambda k}}\left[M\left(\frac{1}{\lambda k}\log\left(\frac{1}{\lambda}\right)\right)\right]^k \end{aligned} \tag{2.14}$$

Thus we wish to bound $M(t)$, which explicitly reads

$$\left(1 - p + p\left(1 - p + \frac{p}{\lambda^{\frac{1}{\lambda k}}}\right)^{m-\lambda k_+}\right)^n \tag{2.15}$$

where $p = \lambda/\sqrt{mn}$. Using the inequality $(1 + x/n)^n \leq e^x$, which holds for all $x \in \mathbb{R}$ and all $n > 0$, we have

$$\begin{aligned}\left(1 - p + \frac{p}{\lambda^{\frac{1}{\lambda k}}}\right)^{m-\lambda k_+} &= \left(1 - \frac{\lambda - \lambda^{1-\frac{1}{\lambda k}}}{\sqrt{mn}}\right)^{\sqrt{mn}\left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right)} \\ &\leq \exp\left\{-\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right)\right\} \\ &\leq 1 - \frac{1}{\sqrt{a}\lambda}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right) \end{aligned} \tag{2.16}$$

Where $a \geq 1$ is such that $\frac{m}{n} \leq a$ for all m. The second inequality holds since $e^{-x} \leq 1 - x/\lambda\sqrt{a}$ for $x \leq \lambda\sqrt{a} - 1$. Indeed, we have

$$\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right) \leq \lambda\sqrt{a} - \lambda^{1-\frac{1}{\lambda k}}\sqrt{a} \leq \lambda\sqrt{a} - 1 \tag{2.17}$$

so the inequality is applicable. Putting (2.16) into (2.15) yields

$$\begin{aligned}&\left(1 - p + p\left(1 - p + \frac{p}{\lambda^{\frac{1}{\lambda k}}}\right)^{m-\lambda k_+}\right)^n \\ \leq\ &\left(1 + \frac{\lambda}{\sqrt{mn}}\left(1 - \frac{1}{\sqrt{a}\lambda}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right) - 1\right)\right)^n \\ \leq\ &\exp\left\{-\frac{1}{\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right)\sqrt{\frac{n}{m}}\right\} \\ =\ &\exp\left\{-\frac{1}{\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(1 - \frac{\lambda k_+}{m}\right)\right\} \end{aligned} \tag{2.18}$$

So, (2.14) becomes

$$\mathbf{P}\left\{\sum_{i=1}^{k} X_i^- \le \lambda k - 1\right\} \le \lambda^{1-\frac{1}{\lambda k}} \exp\left\{-\frac{k}{\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\left(1 - \frac{\lambda k_+}{m}\right)\right\} \tag{2.19}$$
$$\le \lambda \exp\left\{\frac{1}{\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\frac{\lambda k k_+}{m} - \frac{k}{\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\right\}$$

Noting that $k_- = k_+^2 \frac{\lambda \log m}{m}$, the probability that there is a component with $k_- \le k \le k_+$ even vertices is thus bounded as follows.

$$\begin{aligned}
& m \sum_{k=k_-}^{k_+} \mathbf{P}\left\{\sum_{i=1}^{k} X_i^- \le k - 1 + (\lambda - 1)k\right\} \\
\le\ & m k_+ \lambda \exp\left\{\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\frac{\lambda k_+^2}{m\sqrt{a}} - \frac{k_-}{\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\right\} \\
=\ & m k_+ \lambda \exp\left\{\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\frac{\lambda k_+^2}{m\sqrt{a}} - \log m \frac{\lambda k_+^2}{m\sqrt{a}}\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\right\} \\
=\ & m k_+ \lambda \exp\left\{\frac{\lambda k_+^2}{m\sqrt{a}}(1 - \log m)\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\right\} \\
=\ & c_0 m^{3/2}\sqrt{\log(m^3)} \exp\left\{-\frac{\log(m^3)}{\sqrt{a}\lambda(\lambda-1)^2}(\log m - 1)\left(\lambda - \lambda^{1-\frac{1}{\lambda k}}\right)\right\} \\
=\ & c_0 \sqrt{\log(m^3)} m^{\frac{3}{2} - c_1(\log m - 1)} \\
=\ & o(1)
\end{aligned} \tag{2.20}$$

where c_0, c_1 are some positive constants that depend on a, λ only. □

Lemma 8. *If $\lambda > 1$, then a.a.s. there is at most one component with more than $k_+ = \frac{\sqrt{m \log(m^3)}}{\lambda(\lambda-1)}$ even vertices.*

Proof. We argue again as in [10]. Suppose there exists at least one component of size at least k_+ (call such a component "large"), and let a_i and a_j, $i \ne j$ be two even vertices who each belong to large, possibly equal, components C_i and C_j, respectively. We run two search processes starting at these two vertices. After k_+ steps, each process must have produced $(\lambda - 1)k_+$ not yet saturated even vertices. The probability that there does not exist an open path of length 2 between as yet unsaturated vertices of C_i and

C_j is bounded from above by

$$\begin{aligned}
& \mathbf{P}\left\{\forall l = 1,...,n : \text{at least one of } a_i b_l, a_j b_l \text{ closed}\right\}^{[(\lambda-1)k_+]^2} \\
= & \left(1-p^2\right)^{n[(\lambda-1)k_+]^2} \\
= & \left(1-\frac{\lambda^2}{mn}\right)^{n[(\lambda-1)k_+]^2} \\
\leq & \exp\left\{-\frac{\lambda^2}{m}\left[(\lambda-1)k_+\right]^2\right\} \\
= & \exp\left\{-\frac{\lambda^2}{m}(\lambda-1)^2\left(\frac{\sqrt{m\log(m^3)}}{\lambda(\lambda-1)}\right)^2\right\} \\
= & \exp\left\{-\log(m^3)\right\} \\
= & o(m^{-2})
\end{aligned} \tag{2.21}$$

Since there are less than m^2 ways of choosing a_i, a_j, this implies that the probability that any pair of a_i and a_j belong to different large components tends to 0 as $m \to \infty$, and there cannot be more than one component of size larger than k_+. □

This means that we may partition the even vertices of $\mathbb{G}(m,n;p)$ into "large" and "small" vertices, the former being the vertices that belong to the unique component of size at least k_+, if such a component exists. To see that the component indeed does exist and find its size, we count the number of small vertices.

Theorem 9. *If $\lambda > 1$, then there is a component with $\theta_m m(1+o_p(1))$ even vertices, where $\theta_m = \theta_m(\lambda) \in (0,1)$ is the unique solution to*

$$\begin{aligned}
\theta_m + \exp\left\{\tfrac{\lambda}{\sqrt{a}}\left(\exp\left\{-\lambda\theta_m\sqrt{a}\right\}-1\right)\right\} &= 1 \quad \textit{if} \quad m = an, a > 0 \\
\theta_m + e^{-\lambda^2\theta_m} &= 1 \quad \textit{if} \quad m = o(n)
\end{aligned} \tag{2.22}$$

Furthermore, the second largest component a.a.s. has at most $\frac{3\log^2 m}{\lambda(\lambda-1)^2}$ even vertices.

Proof. The second statement follows directly from Lemmas 7 and 8.

For the first part, we calculate the probability that a given even vertex a is small. Call this probability $\rho(m,n,p)$. As in [10], the probability is bounded from above by $\rho_+ = \rho_+(m,n,p)$, the extinction probability for the branching process in which the offspring distribution is given by $\mathrm{Bi}(Y(m-k_-),p) = \mathrm{Bi}(Y(m-o(m)),p)$, where $Y \in \mathrm{Bi}(n,p)$ as usual. The probability ρ is also bounded from below by $\rho_- + o(1)$, where ρ_- is the extinction probability for the branching process with offspring distribution $\mathrm{Bi}(Ym,p)$. The $o(1)$ term bounds the probability that the process dies after more than k_- steps.

The extinction probability ρ_+ is the unique positive solution to $G_X(\rho_+) = \rho_+$, where

$G_X(x) = \left(1 - p + p\left(1 - p + px\right)^{m-o(m)}\right)^n$. If $m = o(n)$, we have

$$\begin{aligned}
(1-p+px)^{m-o(m)} &= \left(1 + \frac{\lambda(x-1)}{\sqrt{mn}}\right)^{m-o(m)} \\
&= \sum_{k=0}^{m-o(m)} \binom{m-o(m)}{k} \left(\frac{\lambda(x-1)}{\sqrt{mn}}\right)^k \\
&= 1 + \lambda(x-1)\sqrt{\frac{m-o(m)}{n}} + o\left(\sqrt{\frac{m}{n}}\right) \qquad (2.23)
\end{aligned}$$

so that

$$\begin{aligned}
& (1-p+p(1-p+px)^{m-o(m)})^n \\
= & \left(1 + \frac{\lambda}{\sqrt{mn}}\left(-1 + 1 + \lambda(x-1)\sqrt{\frac{m-o(m)}{n}} + o\left(\sqrt{\frac{m}{n}}\right)\right)\right)^n \\
= & \left(1 + \frac{\lambda^2(x-1)(1+o(1))}{n}\right)^n \\
= & \exp\left\{\lambda^2(x-1)(1+o(1))\right\} \qquad (2.24)
\end{aligned}$$

so as $m \to \infty$, we have $\rho_+ = 1 - \theta_m$, where $e^{-\lambda^2\theta_m} = 1 - \theta_m$, and replacing $m - o(m)$ by m we get $\rho_- = 1 - \theta_m + o(1)$. Thus $\rho = 1 - \theta_m + o(1)$, and the expected number of small even vertices is given by $(1-\theta_m)m(1+o(1))$. In other words, the expected number of even vertices in the largest component is $\theta_m m(1+o(1))$.

Similarly, if $m = an$, $a > 0$, then

$$\begin{aligned}
& \lim_{n\to\infty} \left(1 - \frac{\lambda}{\sqrt{an}} + \frac{\lambda}{\sqrt{an}}\left(1 - \frac{\lambda}{\sqrt{an}} + \frac{\lambda}{\sqrt{an}}x\right)^{an-o(n)}\right)^n \\
= & \exp\left\{\frac{\lambda}{\sqrt{a}}\exp\left\{\lambda(x-1)\sqrt{a}(1+o(1))\right\} - \frac{\lambda}{\sqrt{a}}\right\} \qquad (2.25)
\end{aligned}$$

and the expected number of even vertices in the giant component is again given by $\theta_m m(1+o(1))$.

Lastly, we show that $Y = \mathbf{E}[Y](1+o_p(1))$, where Y is the number of small vertices. From this follows that the giant component has size $\theta_m m(1+o_p(1))$. Let ξ_i be 1 if vertex

a_i is small and 0 otherwise. Then

$$\begin{aligned}\mathbf{E}[\xi_i Y] &= \mathbf{E}[\xi_i Y \mid \xi_i = 1]\,\mathbf{P}\{\xi_i = 1\} + \mathbf{E}[\xi_i Y \mid \xi_i = 0]\,\mathbf{P}\{\xi_i = 0\} \\ &= \mathbf{E}[Y \mid \xi_i = 1]\,\rho(m,p) \\ &= \sum_{k=1}^{k_-} \mathbf{E}[Y \mid |C(a_i)| = k]\,\mathbf{P}\{|C(a_i)| = k\}\,\rho(m,p) \\ &= \rho(m,p) \sum_{k=1}^{k_-} (k + \mathbf{E}[Y - k \mid Y \geq k])\,\mathbf{P}\{|C(a_i)| = k\} \\ &\leq \rho(m,p)\,(k_- + (m - \mathcal{O}(k_-))\rho(m - \mathcal{O}(k_-),p)) \\ &\leq \rho(m,p)(k_- + m\rho(m - \mathcal{O}(k_-),p)) \qquad (2.26)\end{aligned}$$

so we have

$$\begin{aligned}\mathbf{E}\left[Y^2\right] &= \sum_{i=1}^{m} \mathbf{E}[\xi_i Y] \\ &\leq m\rho(m,p)(k_- + m\rho(m - \mathcal{O}(k_-),p)) \\ &= m^2\rho(m,p)(o(1) + \rho(m - \mathcal{O}(k_-),p)) \\ &= m^2\rho(m,p)^2(1 + o_p(1)) \\ &= \mathbf{E}[Y]^2\,(1 + o_p(1)) \qquad (2.27)\end{aligned}$$

Thus, $\text{Var}\, Y = \mathbf{E}\left[Y^2\right] - \mathbf{E}[Y]^2 = o_p(\mathbf{E}[Y]^2)$ and by (1.6), $Y = \mathbf{E}[Y]\,(1 + o_p(1))$. □

Figure 2.1 shows θ_m as a function of $\lambda \in [0,2]$ in the case $m = o(n)$.

2.4 At criticality

We conclude this section with a quick look into the case $\lambda = 1$. Here, the theory becomes much more involved and the tools we used above will not be enough to prove any detailed results. We confine ourselves to showing that the largest component has $o_p(m)$ even vertices. This is done by showing that increasing properties are more likely when p increases; see Section 1.3.2.

Lemma 10. *Let $0 \leq p_1 \leq p_2 \leq 1$, and let μ_1, μ_2 be the measures on $\{0,1\}^{E_{m,n}}$ in which each edge in $K_{m,n}$ is open with probability p_1,p_2 respectively, independently of all other edges. Then $\mu_1 \preceq_{\mathcal{D}} \mu_2$.*

Proof. The proof is a simple application of Holley's theorem. Let $x \in E_{m,n}$ and let $\xi_1, \xi_2 \in \{0,1\}^{E_{m,n}\setminus\{x\}}$ be such that $\xi_1 \preceq \xi_2$. By the independence of edges under μ_1, μ_2, we have

$$\mu_i\{X(x) = 1 \mid X = \xi_i \text{ off } x\} = \mu_i\{X(x) = 1\} = p_i, \quad i = 1,2 \qquad (2.28)$$

so $p_1 \leq p_2$ implies

$$\mu_1\{X(x) = 1 \mid X = \xi_1 \text{ off } x\} \leq \mu_2\{X(x) = 1 \mid X = \xi_2 \text{ off } x\} \qquad (2.29)$$

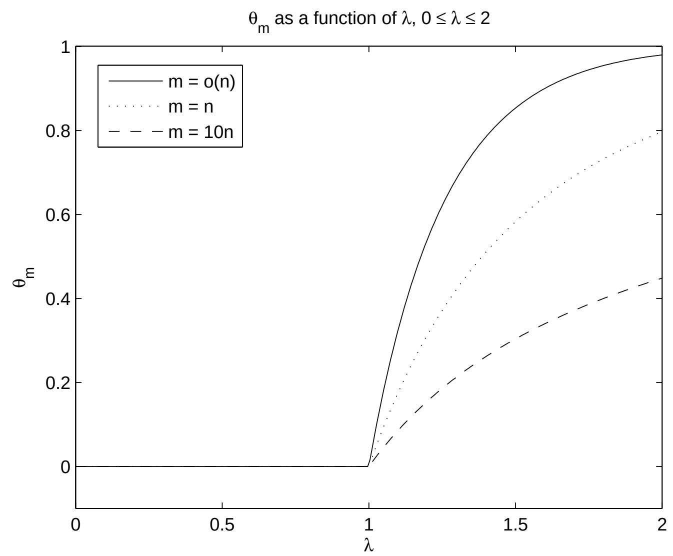

Figure 2.1: The proportion of even vertices belonging to the giant component for $\lambda \in [0,2]$ when $m = o(n)$, n or $10n$ as $m \to \infty$. The values of θ_m are given by Theorem 9.

and by Holley's theorem, $\mu_1 \preceq_{\mathcal{D}} \mu_2$. □

Theorem 11. *Let $\varepsilon > 0$. When $\lambda = 1$, the graph $\mathbb{G}(m,n;\lambda/\sqrt{mn})$ a.a.s. has no component with more than εm even vertices. In other words, the largest component has $o_p(m)$ even vertices.*

Proof. Let $\varepsilon > 0$, and let $\lambda > 1$ be such that $\theta_m = \theta_m(\lambda) < \varepsilon$ with $\theta_m(\lambda)$ as in Theorem 9. When $m = o(n)$, such a λ exists by the continuity of $(\lambda, \theta_m) \longmapsto e^{-\lambda^2\theta_m} + \theta_m - 1$, and since $\theta_m = 0$ is the only solution of $e^{-\theta_m} + \theta_m - 1 = 0$. When $m \sim n$, a similar argument is used to ensure the existence of λ.

Let Q_ε be the increasing property of $\mathbb{G}(m,n;p)$ having a component with at least εm even vertices. With notation as in the proof of Lemma 10, let $p_1 = 1/\sqrt{mn}$ and $p_2 = \lambda/\sqrt{mn}$. Then $\mu_1 \preceq_{\mathcal{D}} \mu_2$ and in particular $\mu_1(Q_\varepsilon) \le \mu_2(Q_\varepsilon)$. But by Theorem 9, we have $\mu_2(Q_\varepsilon) = o(1)$. Thus,

$$\mathbf{P}\left\{\mathbb{G}(m,n;1/\sqrt{mn}) \in Q_\varepsilon\right\} = \mu_1(Q_\varepsilon) = o(1) \tag{2.30}$$

and the largest component of $\mathbb{G}(m,n;1/\sqrt{mn})$ has $o_p(m)$ even vertices. □

2.5 The number of odd vertices

Now that results have been shown for the even vertices of $\mathbb{G}(m,n;p)$, Section 2.1 is used to deduce the total size of the largest component. The results can be summarized in the following theorem.

Theorem 12.

(a) (Subcriticality) If $\lambda < 1$*, then there is a.a.s. no component with more than* $\frac{7\lambda \log m}{(1-\lambda)^3}\sqrt{\frac{n}{m}}$ *vertices.*

(b) (Criticality) If $\lambda = 1$*, then the largest component has* $o_p(m)$ *even vertices and* $o_p(\sqrt{mn})$ *vertices in total.*

(c) (Supercriticality) If $\lambda > 1$*, then there is a component with* $\theta_m m(1 + o_p(1))$ *even vertices, where* θ_m *is given by (2.22). If* $m = o(n)$*, the total number of vertices is* $\lambda\theta_m\sqrt{mn}(1 + o_p(1))$*. If* $m = an$*,* $a > 0$*, the total number of vertices is* $(\theta_m m+\theta_n n)(1+o_p(1))$*, where* θ_m *is given by (2.22) and* θ_n *is the unique positive solution to* $\theta_n + \exp\left\{\lambda\sqrt{a}\left(\exp\left\{-\frac{\lambda\theta_n}{\sqrt{a}}\right\} - 1\right)\right\} = 1$*. In each case, the second largest component a.a.s. has at most* $\frac{3\log^2 m}{(\lambda-1)^2}\sqrt{\frac{n}{m}}$ *vertices in total.*

Proof. The statements about even vertices are Theorems 6, 9 and 11. Statements about odd vertices in (a) and (b) follow from Corollary 3. The size of θ_n in the case $m = an$ is found by considering the even vertices for $m = \frac{1}{a}n$. □

The following lemma about the relative number of even and odd vertices in the giant shall be needed in Section 4. The proof is left to Appendix A.

Lemma 13. *Suppose* $a > 0$*,* $\lambda > 1$*, and let* θ_m, θ_n *be the unique solutions in* (0,1) *to* $\theta_m + \exp\left\{\frac{\lambda}{\sqrt{a}}(\exp\{-\lambda\sqrt{a}\theta_m\} - 1)\right\} = 1$ *and* $\theta_n + \exp\left\{\lambda\sqrt{a}\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_n\right\} - 1\right)\right\} = 1$ *respectively. Then* $\theta_n \geq \min(1,a)\theta_m$*.*

2.6 Concluding remarks

In the complete random graph setting, authors (e.g. [10]) frequently mention the following heuristic: *A giant component appears when the average vertex degree becomes greater than one.* Led by this heuristic, I initially looked for a threshold function of either of the forms $\lambda/\min\{m,n\}$, $\lambda/(m+n)$ or $\lambda/\max\{m,n\}$, or similar. However, I could not make any of the formulas fit into the branching process argument, so instead started to work my way through the methods used in [2]. These combinatorial proofs quickly showed that m and n tend to appear as a product, which suggested the somewhat surprising use of a geometric mean. It is worth noting that the threshold function $1/\sqrt{mn}$ is not consistent with the heuristic above as it stands, but instead we have seen in this section that the following would be the corresponding heuristic: *A giant component appears when the average number of paths of length* 2 *from a vertex becomes greater than one.*

3

The even projection

The branching process argument of the previous section was accomplished by considering only the even vertices of $K_{m,n}$. The usual branching process was replaced by a double-jump process which instead of searching for edges, considered paths of length 2 between even vertices. This suggests a method in which a subgraph $G \subseteq K_m$ is obtained from a subgraph $H \subseteq K_{m,n}$ by removing the odd vertices of H and assigning an edge to G if and only if the corresponding vertices in H are connected by a path of length 2. See Figure 3.1. We will call G the even projection of H. By using known results about random subgraphs on K_m, one may draw conclusions about properties of random subgraphs of $K_{m,n}$.

There are two major problems with this method. Firstly, the edges of the even projection are highly dependent, limiting the compatibility with the usual random graph theory which requires independent edges. Secondly, some properties of the even projection are difficult to translate back to the bipartite setting. We will see that this method is good enough to prove the existence, but not uniqueness, of a giant component when $\lambda > 1$, and with help of Section 2 it is shown that for λ large enough, the giant is cyclic.

It should also be mentioned that there are no significant results in this section for the case $m \sim n$, since Holley's theorem, which the proofs rely heavily upon, cannot be applied in this case to the author's knowledge.

3.1 The even projection

Given a subgraph $G = (A_m \cup B_n, E)$ of $K_{m,n}$, we define the *even projection of* G to be the graph with vertex set A_m and in which any edge $a_i a_j$ is included if and only if there is a $1 \le k \le n$ such that $a_i b_k$ and $a_j b_k$ are both in E. Further, define the *random even projection*, denoted $\mathbb{G}_n^p(m)$, as the random graph obtained by taking the even projection of $\mathbb{G}(m,n;p)$.

Equivalently, given $\eta \in \{0,1\}^{E_{m,n}}$, we define the even projection of η through a mapping $\phi : \{0,1\}^{E_{m,n}} \longrightarrow \{0,1\}^{E_m}$ by $\phi(\eta) = \xi$, where for all i,j

$$\phi(\eta)(a_i a_j) = \xi(a_i a_j) = \max_{1 \le k \le n} \eta(a_i b_k)\eta(a_j b_k). \tag{3.1}$$

We define μ_n^p to be the measure on $\{0,1\}^{E_m}$ associated with even projections, i.e. the measure which to any $\xi \in \{0,1\}^{E_m}$ assigns probability $\mu_n^p(\xi) = \nu(\phi^{-1}(\xi))$, where ν is the *i.i.d.* measure on $\{0,1\}^{E_{m,n}}$ which assigns probability p to each edge.

The p is moved out of its usual position in the notation to avoid misunderstanding: as we shall see, $\mathbb{G}_n^p(m)$ does not have edge-probability p. The notation ν will be kept for the *i.i.d.* measure on $K_{m,n}$. We shall not need the following result, but it serves as a warmup for what is to come, and motivates the comparison of μ_n^p to an *i.i.d.* measure with approximate edge-probability $\lambda^2/m \approx 1 - (1-p^2)^n$.

Proposition 14. *The probability that an edge in $\mathbb{G}_n^p(m)$ is open is given by*

$$\mu_n^p \{a_i a_j \text{ open}\} = 1 - (1-p^2)^n \tag{3.2}$$

for all $1 \le i,j \le m$.

Proof. Here we introduce the event $\{a_1 b_1 a_2 \text{ open}\} = \{a_1 b_1 \text{ open}\} \cap \{a_2 b_1 \text{ open}\}$ and let $\{a_1 b_1 a_2 \text{ closed}\}$ be the complementary event $\{a_1 b_1 \text{ closed}\} \cup \{a_2 b_1 \text{ closed}\}$

This and similar results will be proved by passing to the random bipartite graph $\mathbb{G}(m,n;p)$. From the definition of $\mathbb{G}_n^p(m)$, we have

$$\begin{aligned} \mu_n^p \{a_i a_j \text{ open}\} &= \nu \{\exists k : a_i b_k a_j \text{ open}\} \\ &= 1 - \nu \{\forall k : a_i b_k a_j \text{ closed}\} \\ &= 1 - \nu \{a_i b_1 a_j \text{ closed}\}^n \\ &= 1 - (1 - \nu \{a_i b_1, a_j b_1 \text{ open}\})^n \\ &= 1 - (1-p^2)^n \end{aligned} \tag{3.3}$$

□

We define μ_ε as the measure on $\{0,1\}^{E_m}$ for which each edge is open with probability $(\lambda^2 - \varepsilon)/m$, independently of all other edges. The goal of the following two lemmas is to show that $\mu_\varepsilon \preceq_{\mathcal{D}} \mu_n^p$ for large m using Holley's Theorem. See Section 1.3.2.

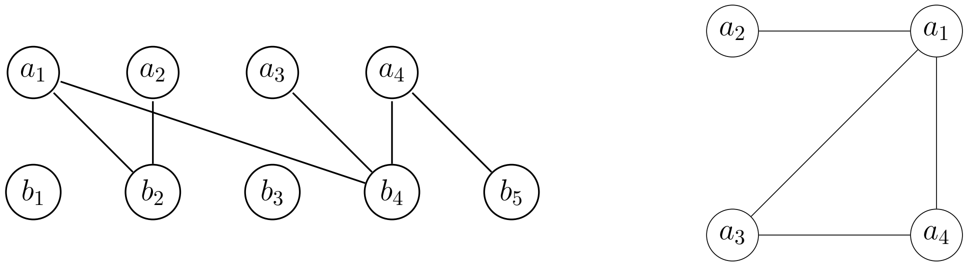

Figure 3.1: Left: Example subgraph H of $K_{4,5}$. Right: The even projection of H.

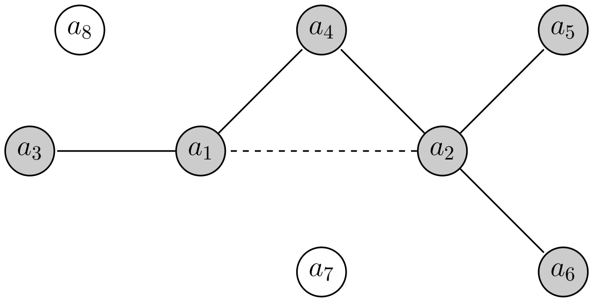

Figure 3.2: Example configuration ξ on K_8. We set $x = a_1a_2$ so that $d_1 = 2$, $d_2 = 3$ and $d = 1$.

Lemma 15. *Let $x \in E_m$. Denote the endpoints of x by a_1, a_2 and let $\xi \in \{0,1\}^{E_m \setminus \{x\}}$ be an edge-configuration on the complete graph in which a_1, a_2 have d_1,d_2 neighbours respectively, and there are d vertices which are neighbours to both a_1 and a_2. Then*

$$\mu_n^p(X(x) = 1 \mid X = \xi \text{ off } x) \geq 1 - (1-p)^{2d}\left(1 - \frac{p^2}{(1+p)^{2m-d_1-d_2-4}}\right)^{n-d_1-d_2} \tag{3.4}$$

with equality if $d = 0$. In particular, we have

$$\mu_n^p(X(x) = 1 \mid X = 0 \text{ off } x) \leq \mu_n^p(X(x) = 1 \mid X = \xi \text{ off } x) \tag{3.5}$$

for all $\xi \in \{0,1\}^{E_m \setminus \{x\}}$ and all $x \in E_m$.

Proof. Let x be the edge a_1a_2 and fix $\xi \in \{0,1\}^{E_m \setminus \{x\}}$. For simplicity, we will make the assumption that $\xi(a_ia_j) = 0$ whenever a_ia_j is not adjacent to a_1a_2. This can be done without loss of generality, since a_ia_j is independent of a_1a_2 when the edges are not adjacent.

Suppose $\eta \in \{0,1\}^{E_{m,n}}$ is such that the even projection of η is ξ. Then η must contain $d_1 - d$ odd vertices b_k such that $a_1b_ka_i$ is open for $i = i_1, i_2, ..., i_{d_1-d}$, and $d_2 - d$ odd vertices b_k such that $a_2b_ka_j$ is open for $j = j_1,j_2,...,j_{d_2-d}$, where $i_1,...,i_{d_1-d},j_1,...,j_{d_2-d}$ are all distinct and greater than 2. The b_k must all be distinct, since otherwise the projection would contain an open edge a_ia_j which is not adjacent to a_1a_2.

Furthermore, for each vertex a_i such that a_1a_i and a_2a_i are both open in ξ, there must be odd vertices b_{k_1},b_{k_2}, possibly equal, such that $a_1b_{k_1}a_i$ and $a_2b_{k_2}a_i$ are both open. Note that if $k_1 = k_2$, then $a_1b_{k_1}a_2$ will be open, and $X(x) = 1$ follows immediately. Thus we assume that whenever a_1a_i and a_2a_i are both open in ξ, there are distinct b_{k_1},b_{k_2} defined as above. Since there are d vertices a_i satisfying this, we require d pairs of vertices b_{k_1}, b_{k_2} in η to be such that $a_1b_{k_1}a_i$ and $a_2b_{k_2}a_i$ are both open.

By the argument above, there is in some sense a minimal configuration η which has even projection ξ and has a nontrivial probability of having $X(x) = 1$. Without loss of generality, we can suppose that η has a canonical form obtained as follows.

1. Let $a_1b_ka_{k+2}$ be open for $k = 1,...,d_1$.

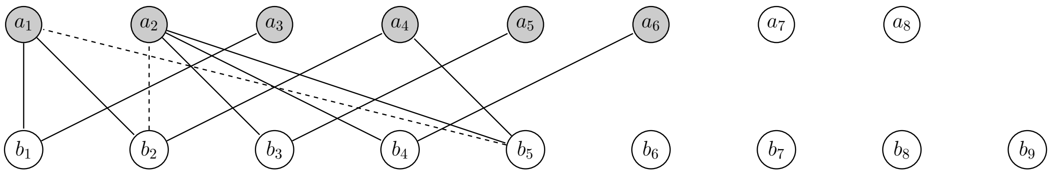

Figure 3.3: Configuration η on $K_{8,9}$ obtained from ξ in figure 3.2 by the algorithm described in the proof of Lemma 15. The set of edges in this figure is denoted A. No edges can be added to b_1, b_3 or b_4 without changing the projection of η, whence $R = \{2,5,6,7,8,9\}$. Opening either one of the $2d$ dashed edges will make a_1a_2 open in the projection without affecting other edges. This can otherwise only be achieved by, for some $k > 5$, opening the two edges a_1b_k and a_2b_k.

2. Let $a_2b_ka_{k+2}$ be open for $k = d_1 + 1,...,d_1 + d_2 - d$.

3. Let $a_2b_ka_{k+2-d_2}$ be open for $k = d_1 + d_2 - d + 1,...,d_1 + d_2$.

Denote the set of edges opened in this process by A. Also, let $B = \{a_ib_ka_j : 1 \leq k \leq n, \xi(ij) = 0\}$. Then

$$\mu_n^p \{X(x) = 1 \mid X = \xi \text{ off } x\} \geq \nu \{\exists r : a_1b_ra_2 \text{ open} \mid A \text{ open}, B \text{ closed}\} \tag{3.6}$$

where "A open" should be read as "π open for all $\pi \in A$", and "B closed" means "π closed for all $\pi \in B$". Here the inequality comes from how we assigned odd vertices to the d common neighbours. If $d = 0$, there is no such freedom of choice and equality must hold.

Note that for $r = 1,...,d_1 - d$, we have a_2b_r closed since $a_2b_ra_{r+2} \in B$ but $a_1b_ra_{r+2} \in A$. Similarly, $a_1b_ra_{r+2} \in B$ while $a_2b_ra_{r+2} \in A$ for $r = d_1 + 1,...,d_1 + d_2 - d$ so a_1b_r is closed. Let $R = \{1,...,n\} \setminus (\{1,...,d_1 - d\} \cup \{d_1 + 1,...,d_1 + d_2 - d\})$. Then

$$\begin{aligned} & \nu \{\exists r : a_1b_ra_2 \text{ open} \mid A \text{ open}, B \text{ closed}\} \\ = \; & \nu \{\exists r \in R : a_1b_ra_2 \text{ open} \mid A \text{ open}, B \text{ closed}\} \\ = \; & 1 - \nu \{\forall r \in R : a_1b_ra_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} \end{aligned} \tag{3.7}$$

Let $R_1' = \{d_1 - d + 1,...,d_1\}$, $R_1'' = \{d_2 - d + 1,...,d_2\}$, and $R_2 = \{d_2 + 1,...,n\}$. Then $R = R_1' \cup R_1'' \cup R_2$, and the sets are distinct. In the first case, say $r = r_0 \in R_1'$, we have, since $a_1b_{r_0}a_2$ depends only on events in A and B containing $a_1b_{r_0}$ or $a_2b_{r_0}$,

$$\begin{aligned} & \nu \{a_1b_{r_0}a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} \\ = \; & \nu \{a_1b_{r_0}a_2 \text{ closed} \mid a_1b_{r_0}a_{r_0+2} \text{ open}, a_1b_{r_0}a_j \text{ closed}, j > d_1 + 2\} \\ = \; & \nu \{a_2b_{r_0} \text{ closed} \mid a_{r_0+2}b_{r_0} \text{ open}, a_jb_{r_0} \text{ closed}, j > d_1 + 2\} \\ = \; & 1 - p \end{aligned} \tag{3.8}$$

The same will hold for $r_0 \in R_1''$. Now let $r_0 \in R_2$. We have

$$\begin{aligned} & \nu \{a_1b_{r_0}a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} \\ = \; & 1 - \nu \{a_1b_{r_0}a_2 \text{ open} \mid A \text{ open}, B \text{ closed}\} \\ = \; & 1 - \nu \{a_1b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \, \nu \{a_2b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \end{aligned} \tag{3.9}$$

The two factors are similar. We have

$$
\begin{aligned}
& \nu\{a_1 b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \\
= \; & \nu\{a_1 b_{r_0} \text{ open} \mid a_1 b_{r_0} a_j \text{ closed}, j > d_1 + 2\} \\
= \; & \frac{\nu\{a_1 b_{r_0} \text{ open}, a_1 b_{r_0} a_j \text{ closed}, j > d_1 + 2\}}{\nu\{a_1 b_{r_0} a_j \text{ closed}, j > d_1 + 2\}} \\
= \; & \frac{\nu\{a_1 b_{r_0} \text{ open}, a_j b_{r_0} \text{ closed}, j > d_1 + 2\}}{\nu\{a_1 b_{r_0} a_j \text{ closed}, j > d_1 + 2\}} \\
= \; & \frac{p(1-p)^{m-d_1-2}}{(1-p^2)^{m-d_1-2}} \\
= \; & \frac{p}{(1+p)^{m-d_1-2}} \qquad (3.10)
\end{aligned}
$$

and the same formula with d_2 holds for $a_2 b_{r_0}$. Thus

$$
\begin{aligned}
& 1 - \nu\{a_1 b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \, \nu\{a_2 b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \\
= \; & 1 - \left(\frac{p}{(1+p)^{m-d_1-2}}\right)\left(\frac{p}{(1+p)^{m-d_2-2}}\right) \\
= \; & 1 - \frac{p^2}{(1+p)^{2m-d_1-d_2-4}} \qquad (3.11)
\end{aligned}
$$

Returning to (3.7), we obtain

$$
\begin{aligned}
& \mu_n^p\{X(x) = 1 \mid X = \xi \text{ off } x\} \\
\geq \; & 1 - \nu\{\forall r \in R : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} \\
= \; & 1 - \nu\left\{\forall r \in R_1' : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\right\} \\
& \times \nu\left\{\forall r \in R_1'' : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\right\} \\
& \times \nu\{\forall r \in R_2 : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} \\
= \; & 1 - \nu\left\{a_1 b_{r_1'} a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\right\}^{|R_1'|} \\
& \times \nu\left\{a_1 b_{r_1''} a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\right\}^{|R_1''|} \\
& \times \nu\{a_1 b_{r_2} a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\}^{|R_2|} \\
= \; & 1 - (1-p)^{2d}\left(1 - \frac{p^2}{(1+p)^{2m-d_1-d_2-4}}\right)^{n-d_1-d_2} \qquad (3.12)
\end{aligned}
$$

which is the desired result. In particular,

$$
\mu_n^p\{X(x) = 1 \mid X = 0 \text{ off } x\} = 1 - \left(1 - \frac{p^2}{(1+p)^{2m-4}}\right)^n \qquad (3.13)
$$

and it is not hard to see that

$$
\mu_n^p\{X(x) = 1 \mid X = 0 \text{ off } x\} \leq \mu_n^p\{X(x) = 1 \mid X = \xi \text{ off } x\} \qquad (3.14)
$$

□

The following lemma is a consequence of Lemma 15.

Lemma 16. *Let $m = o(n)$ and $\varepsilon > 0$. Let μ_ε be the measure on $\{0,1\}^{E_m}$ which assigns probability $\frac{\lambda^2-\varepsilon}{m}$ to each edge independently. Then for m large enough,*

$$\mu_\varepsilon \{X(x) = 1 \mid X = 0 \text{ off } x\} \leq \mu_n^p \{X(x) = 1 \mid X = 0 \text{ off } x\}, \quad \forall x \in E_m \tag{3.15}$$

Proof. By Lemma 15 and the independence of edges under μ_ε, this is equivalent to

$$\frac{\lambda^2 - \varepsilon}{m} \leq 1 - \left(1 - \left[\frac{p}{(1+p)^{m-2}}\right]^2\right)^n \tag{3.16}$$

We have $(1+p)^m = 1 + \lambda\sqrt{\frac{m}{n}}(1+o(1))$, so

$$\begin{aligned}
1 - \left(1 - \left[\frac{p}{(1+p)^{m-2}}\right]^2\right)^n &= 1 - \left(1 - \left[\frac{p(1+p)^2}{1+\lambda\sqrt{\frac{m}{n}}(1+o(1))}\right]^2\right)^n \\
&= 1 - \left(1 - \left[\frac{\frac{\lambda}{\sqrt{mn}}\left(1 + 2\frac{\lambda}{\sqrt{mn}} + \frac{\lambda^2}{mn}\right)}{1+\lambda\sqrt{\frac{m}{n}}(1+o(1))}\right]^2\right)^n \\
&= 1 - \left(1 - \left[\frac{\lambda(mn + 2\lambda\sqrt{mn} + \lambda^2)}{mn\sqrt{mn} + \lambda m^2 n(1+o(1))}\right]^2\right)^n \\
&= 1 - \left(1 - \left[\frac{\lambda}{\sqrt{mn}}(1+o(1))\right]^2\right)^n \\
&= 1 - \left(1 - \frac{\lambda^2}{mn}(1+o(1))\right)^n \\
&= \frac{\lambda^2}{m} + o\left(\frac{1}{m}\right)
\end{aligned} \tag{3.17}$$

and so for m large enough, the result follows. □

Remark. The inequality (3.16) in fact does not hold for any sequence m,n such that $m > n$. This is the reason that we study only the even vertices, just as in Section 2.

Theorem 17. *Let $m = o(n)$. For any $\varepsilon > 0$ we eventually have $\mu_\varepsilon \preceq_{\mathcal{D}} \mu_n^p$.*

Proof. Using the edge-independence of μ_ε followed by Lemmas 16 and 15, we have for large m

$$\begin{aligned}
\mu_\varepsilon \{X(x) = 1 \mid X = \xi \text{ off } x\} &= \mu_\varepsilon \{X(x) = 1 \mid X = 0 \text{ off } x\} \\
&\leq \mu_n^p \{X(x) = 1 \mid X = 0 \text{ off } x\} \\
&\leq \mu_n^p \{X(x) = 1 \mid X = \xi' \text{ off } x\}
\end{aligned} \tag{3.18}$$

for any $\xi, \xi' \in \{0,1\}^{E_m \setminus \{x\}}$. Thus, $\mu_\varepsilon \preceq_{\mathcal{D}} \mu_n^p$ follows from Holley's theorem.

□

3.2 Applications of even projection

Theorem 17 means that any increasing property of the *i.i.d.* graph with edge-probability $(\lambda^2 - \varepsilon)/m$ will eventually be satisfied by the even projection. Properties of $\mathbb{G}(m,n;p)$ may then be concluded from properties of its even projection. In this section we state corollaries of Theorem 17. First, we reprove the existence of a giant component without using any results from Section 2.

Corollary 18. *Let $m = o(n)$. If $\lambda > 1$, then there is a.a.s. a component in $\mathbb{G}(m,n;p)$ with at least θm even vertices, where $\theta_m = \theta_m(\lambda) \in (0,1)$ is the unique solution to $\theta_m + e^{-\lambda^2\theta_m} = 1$.*

Proof. Let $\varepsilon > 0$ be constant and such that $\lambda^2 - \varepsilon > 1$. For m,n large enough the even projection measure satisfies $\mu_n^p \succeq_{\mathcal{D}} \mu_\varepsilon$ and it follows that if we define $\theta_\varepsilon \in (0,1)$ to be the unique solution to $\theta + e^{-(\lambda^2-\varepsilon)\theta} = 1$,

$$\begin{aligned} & \mu_n^p \{\text{Component of size } \geq \theta_\varepsilon m(1 + o(1))\} \\ \geq \; & \mu_\varepsilon \{\text{Component of size } \geq \theta_\varepsilon m(1 + o(1))\} \\ = \; & 1 - o(1) \end{aligned} \tag{3.19}$$

by Theorem 9. By the continuity of $\varepsilon \mapsto \theta_\varepsilon$, we must have

$$\mu_n^p \{\text{Component of size } \geq \theta_m m(1 + o(1))\} \to 1 \tag{3.20}$$

Thus the even projection has a component with at least $\theta_m m(1+o_p(1))$ even vertices, which means that the random bipartite graph must have at least $\theta_m m(1 + o_p(1))$ even vertices which belong to the same component. $\square$

Note that without using results from Section 2, this corollary only shows the existence of a large component, and it is possible that there is more than one component with $\Theta(m)$ even vertices. However, results obtained from Theorem 17 typically show properties for any large component, and using results from Section 2 will then imply that the result holds for the unique giant component.

While reverting from the projected graph to $\mathbb{G}(m,n;p)$ without losing too much information was possible in Corollary 18, the statement of the following theorem is significantly weakened compared to the corresponding statement for complete graphs. The reason for this is, loosely speaking, that the property of having a giant component is a property relating to connectivity of vertices, while the property of containing a cycle relates to adjacency of edges.

Theorem 19. *Let $m = o(n)$. Then there exists a λ_0 such that if $\lambda > \lambda_0$, then a.a.s. the giant component in $\mathbb{G}(m,n;p)$ contains a cycle.*

Proof. Let $\delta, \varepsilon > 0$. According to [1], there is a $\lambda_0 > 1$ such that the giant component of $\mathbb{G}(m, (\lambda^2 - \varepsilon)/m)$ contains a cycle of length at least δm for all $\lambda > \lambda_0$. Since $\mu_\varepsilon \preceq_{\mathcal{D}} \mu_n^p$

for large m, there must be a large component with a cycle of length at least δm in the even projection of $\mathbb{G}(m,n;p)$.

Suppose there is no cycle in the giant component of $\mathbb{G}(m,n;p)$. Then, since the even projection has a cycle of length δm, there must be an odd vertex b_0 such that $\deg b_0 \geq \delta m$, as this is the only way to obtain cycles in the even projection from an acyclic coniguraftion. But we have $\deg b_0 \in \mathrm{Bi}(m,p)$, so

$$\begin{aligned}
\nu\{\exists k : \deg b_k \geq \delta m\} &= 1 - \nu\{\forall k : \deg b_k < \delta m\} \\
&= 1 - \nu\{\deg b_1 < \delta m\}^n \\
&= 1 - \left(1 - \sum_{j=\delta m}^{m} \binom{m}{j} p^j (1-p)^{m-j}\right)^n \\
&= 1 - (1 - o(1/n))^n \\
&= o(1) \qquad (3.21)
\end{aligned}$$

so there must *a.a.s.* be a cycle in the giant component of $\mathbb{G}(m,n;p)$, if we can show that the sum is indeed $o(1/n)$. But

$$\begin{aligned}
n \sum_{j=\delta m}^{m} \binom{m}{j} p^j (1-p)^{m-j} &\leq \max_{j \geq \delta m} mn \binom{m}{j} p^j (1-p)^{m-j} \\
&\leq \max_j mn \frac{m^j \lambda^j}{j! \sqrt{mn}^j} \\
&\leq \max_j \frac{m^{j+1} n \lambda^m}{(\delta m)! \sqrt{mn}^j} \\
&\leq \max_j \frac{m^{j+1} n \lambda^m}{(\delta m)^{\delta m} e^{-\delta m} \sqrt{mn}^j} \\
&\leq \frac{m^3 \lambda^m e^{\delta m}}{(\delta m)^{\delta m}} \\
&= o(1) \qquad (3.22)
\end{aligned}$$

and the result follows. □

4

The random-cluster model

This section proves the existence of a giant component for the random-cluster model $\mathbb{G}(m,n;p,q)$ with parameter $q \geq 1$ and $p = \lambda/\sqrt{mn}$. When $1 \leq q \leq 2$ and $m = o(n)$, a sharp threshold value of λ is found for the existence, while for other cases partial results are found. Most importantly, a sharp threshold value is found for the Ising model when $m = o(n)$. The proof techniques imitate those of [3] closely, and this section shows that some of the reduction arguments used there apply directly to the bipartite setting.

We recall from Section 1.4.3 that the random-cluster model with parameter $q > 0$ assigns probability proportional to

$$\widetilde{P}(F; E,p,q) = p^{|F|}(1-p)^{|E|-|F|}q^{c(V,F)} \tag{4.1}$$

to any subgraph (V,F) of (V,E) which has $c(V,F)$ connected components.

4.1 Colouring argument

The proof techniques in [3] depend strongly on a colouring argument, which reduces the study of $\mathbb{G}(m,n;p,q)$ to that of $\mathbb{G}(M,N;p,1)$ for some random numbers M,N. Fix $0 \leq r \leq 1$. Given a random graph $\mathbb{G}(m,n;p,q)$, each component is coloured *red* with probability r and *green* with probability $1-r$. The components are coloured independently of each other. Let the union of the red (green) components be the *red (green) subgraph*, and let R denote the vertex set of the red subgraph. The following lemma relates $\mathbb{G}(m,n;p,q)$ to a random-cluster graph with parameter rq.

Lemma 20. *Let V_1 be a subset of $V_{m,n}$ with m_1 even and n_1 odd vertices. Conditional on $R = V_1$, the red subraph of $\mathbb{G}(m,n;p,q)$ is distributed as $\mathbb{G}(m_1,n_1;p,rq)$ and the green subgraph as $\mathbb{G}(m-m_1,n-n_1;p,(1-r)q)$; furthermore, the red subgraph is conditionally independent of the green subgraph.*

Proof. Set $V_2 = V \setminus V_1$, $m_2 = m - m_1$, $n_2 = n - n_1$, and let $E_1, E_2 \subseteq E_{m,n}$ be such that edges in E_i have endpoints in V_i only, $i = 1,2$. Write $c(V,E)$ for the number of components of a graph with vertex set V and edge set E. Then $c(V, E_1 \cup E_2) = c(V_1,E_1) + c(V_2,E_2)$ since E_1 and E_2 define disjoint components, and the probability that the red subgraph is (V_1,E_1) and the green subgraph is (V_2,E_2) satisfies

$$\begin{aligned}
& \mathbf{P}\left\{\mathbb{G}(m,n;p,q) = (V,E_1 \cup E_2) \text{ and } R = (V_1,E_1)\right\} \\
= & \left(\frac{p^{|E_1\cup E_2|}(1-p)^{mn-|E_1\cup E_2|}q^{c(V,E_1\cup E_2)}}{Z(m,n;p,q)}\right) r^{c(V_1,E_1)}(1-r)^{c(V_2,E_2)} \\
= & C(m,n,p,q,m_1,n_1)p^{|E_1|}(1-p)^{m_1n_1-|E_1|}(qr)^{c(V_1,E_1)} \\
& \times p^{|E_2|}(1-p)^{m_2n_2-|E_2|}(q(1-r))^{c(V_2,E_2)} \\
= & C(m,n,p,q,m_1,n_1)\mathbf{P}\left\{E_1; V_1,p,rq\right\}\mathbf{P}\left\{E_2; V_2,p,(1-r)q\right\} \qquad (4.2)
\end{aligned}$$

for some positive real $C(m,n,p,q,m_1,n_1)$ depending only on m,n,p,q,m_1 and n_1. Hence, conditional on $R = V_1$ and the green subgraph being (V_2,E_2), the probability that the red subgraph is (V_1,E_1) is precisely $\mathbf{P}\{E_1; V_1,p,rq\}$. □

Writing $(M,N) = (m_1,n_1)$ for the number of red vertices, R is thus distributed as a random graph $\mathbb{G}(M,N;p,rq)$ on a random number of vertices. In particular, it is distributed as $\mathbb{G}(M,N,p,1)$ if $q \geq 1$ and $r = q^{-1}$. By using distributional properties of M,N and using results from earlier sections, one may deduce properties of $\mathbb{G}(m,n;p,q)$.

4.2 Existence of a giant component

Imitating [3] further, we start the search for a giant component by proving the following lemma. We assume throughout this section that there is an $a > 0$ such that $m \leq an$.

Lemma 21. *Let $q \geq 1$. For any $\lambda \neq 1$, a.a.s. $\mathbb{G}(m,n;\lambda/\sqrt{mn},q)$ has at most one component such that at least one of the following happens:*

- *The component has at least $(mn)^{1/3}$ vertices*
- *The component has at least $m^{3/4}$ even vertices*
- *The component has at least $n^{3/4}$ odd vertices*

Proof. Let $L_{m,n,p,q}$ be the number of components of $\mathbb{G}(m,n;p,q)$ having at least $(mn)^{1/3}$ vertices, or at least $m^{3/4}$ even vertices. Suppose $L_{m,n,p,q} \geq 2$, and pick two of these in some arbitrary way. With probability r^2 both of these are coloured red. Setting $r = q^{-1}$, we find that

$$\begin{aligned}
r^2\mathbf{P}\left\{L_{m,n,p,q} \geq 2\right\} &\leq \sum_{\substack{k,l \\ kl\geq(mn)^{1/3} \text{ or } k\geq m^{3/4}}} \mathbf{P}\left\{L_{k,l,p,1} \geq 2\right\}\mathbf{P}\left\{|R| = (k,l)\right\} \\
&\leq \max_{\substack{k,l \\ kl\geq(mn)^{1/3} \text{ or } k\geq m^{3/4}}} \mathbf{P}\left\{L_{k,l,p,1} \geq 2\right\} \to 0 \quad \text{as } m \to \infty \qquad (4.3)
\end{aligned}$$

from which follows that *a.a.s.* $L_{m,n,p,q} \leq 1$, using known results about $L_{k,l,p,1}$.

The last statement follows from $(mn)^{1/3} \leq n^{3/4}$ for large m,n, which follows from the assumption that $m \leq an$ for some $a > 0$. □

Let $\Theta_m \in (0,1]$ be the maximal number such that a component has $\Theta_m m$ even vertices, and let $\Theta_n \in (0,1]$ be the maximal number such that a component has $\Theta_n n$ odd vertices. Note that we need not (a priori) have a component with $\Theta_m m + \Theta_n n$ vertices, because two different components might define Θ_m,Θ_n. This issue is resolved in Theorem 28.

Lemma 22. *Let $q \geq 1$ and $r = q^{-1}$. With probability r, $\mathbb{G}(m,n;p,q)$ has $\Theta_m m + r(1 - \Theta_m)m + o_p(m)$ even red vertices, of which $\Theta_m m$ belong to the component with the most even vertices. With probability $1 - r$, there are $r(1 - \Theta_m)m + o_p(m)$ even vertices and no component with more than $m^{3/4}$ even vertices.*

The same statements with m replaced by n hold for the odd case.

Proof. Suppose $\mathbb{G}(m,n;p,q)$ has κ components, and suppose that the components have $\Theta_m m, v_2, v_3,\ldots,v_\kappa$ even vertices. Call any even vertex not in the component with $\Theta_m m$ even vertices "small". By the lemma above, we have $\max v_i \leq m^{3/4}$. Conditional on Θ_m, the expected number of small even red vertices is

$$\sum_{i=2}^{\kappa} v_i r \quad = \quad r(1 - \Theta_m)m \tag{4.4}$$

and the variance is given by

$$\sum_{i=2}^{\kappa} v_i^2 r(1-r) \leq \sum_{i=2}^{\kappa} v_i^2 \leq m \max_{i \geq 2} v_i \leq m^{7/4} \tag{4.5}$$

Hence there are $r(1 - \Theta_m)m + o_p(m)$ small red even vertices by (1.6). The component which has $\Theta_m m$ even vertices is red with probability r, and the result follows.

The same argument applies to the odd vertices, with $\max v_i \leq n^{3/4}$. □

Lemma 23. *If $m = o(n)$, then a.a.s. there is no component with at least δn odd vertices for any $\delta > 0$. In other words, we have $\Theta_n \xrightarrow{P} 0$ and $N = rn(1 + o_p(1))$.*

Proof. Let $\delta > 0$ and let $K_{m,n,p,q}$ be the number of components in $\mathbb{G}(m,n;p,q)$ which have at least δn odd vertices. Note that *a.a.s.* $M = o(N)$ by Lemma 22, and we need only consider $\mathbb{G}(k,l;p,1)$ for k,l such that $k = o(l)$. Thus,

$$\begin{aligned} r\mathbf{P}\left\{K_{m,n,p,q} \geq 1\right\} &\leq \sum_{k=o(l)} \mathbf{P}\left\{K_{k,l,p,1} \geq 1\right\} \mathbf{P}\left\{|R| = (k,l)\right\} \\ &\leq \max_{k=o(l)} \mathbf{P}\left\{K_{k,l,p,1} \geq 1\right\} \to 0 \end{aligned} \tag{4.6}$$

which follows from earlier sections. □

Lemma 24.

(a) Let $m = o(n)$. If $\lambda > q \geq 1$, then there exists a $\theta_0 > 0$ such that a.a.s. $\Theta_m \geq \theta_0$ for $\mathbb{G}(m,n;\lambda/\sqrt{mn},q)$.

(b) Let $m = an$, $a > 0$. If $\lambda > q \geq 1$, then there exists a $\theta_0 > 0$ such that a.a.s. $\Theta_m \geq \theta_0$ and $\Theta_n \geq \theta_0$ for $\mathbb{G}(m,n;\lambda/\sqrt{mn},q)$.

Proof. (a) For $q = 1$, the assertion was shown in Theorem 9. Hence assume $q > 1$ and thus $r = q^{-1} < 1$.

Let $\theta_0 = 1 - \left(\frac{\lambda+q}{2\lambda}\right)^2$, $\pi_m = \mathbf{P}\{\Theta_m < \theta_0\}$, and $\varepsilon > 0$. By considering the event that the component with $\Theta_m m$ even vertices is coloured green we see that, with probability at least $(1-r)\pi_m + o(1)$, the number M of red even vertices satisfies $M \geq r(1-\theta_0)m - \varepsilon m$, and there are no red components with at least $m^{3/4}$ even vertices. When this happens,

$$\begin{aligned}
\sqrt{MNp} &\geq \sqrt{r(1-\theta_0)m - \varepsilon m}\sqrt{rn + o_p(n)}\frac{\lambda}{\sqrt{mn}} \\
&= \lambda\sqrt{r^2(1-\theta_0) - r\varepsilon + o_p(1)} \\
&= \sqrt{\left(\frac{\lambda}{2q}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{\lambda}{2q} - r\varepsilon + o_p(1)} \\
&> 1 \qquad (4.7)
\end{aligned}$$

for m large enough with an appropriate choice of $\varepsilon > 0$. Thus the red subgraph is a supercritical Erdős-Rényi graph, and by Theorem 9 has a component with at least $\delta M \geq \delta(r(1-\theta_0) - \varepsilon)m$ even vertices for some $\delta > 0$. But this is eventually larger that $m^{3/4}$, so we must have $(1-r)\pi_m \to 0$ as $m \to 0$, and in particular $\mathbf{P}\{\Theta_m < \theta_0\} \to 0$.

(b) The case $m = an$ is similarly handled, but more work needs to be done since Lemma 23 cannot be applied. Let $\theta_1 = 1 - \frac{\lambda+q}{2\lambda}$, $\pi_m = \mathbf{P}\{\Theta_m < \theta_1\}$, $\pi_n = \mathbf{P}\{\Theta_n < \theta_1\}$ and $\varepsilon > 0$. Arguing as in the previous proof, with probability $(1-r)^2\pi_m\pi_n + o(1)$ we have $M \geq r(1-\theta_1)m - \varepsilon m$ and $N \geq r(1-\theta_1)n - \varepsilon n$ and no red component with more than $m^{3/4}$ even vertices or more than $n^{3/4}$ odd vertices. When this happens,

$$\begin{aligned}
\sqrt{MNp} &\geq \sqrt{r(1-\theta_1)m - \varepsilon m}\sqrt{r(1-\theta_1)n - \varepsilon n}\frac{\lambda}{\sqrt{mn}} \\
&= \lambda\left(r(1-\theta_1) - \varepsilon\right) \\
&= \frac{\lambda}{2q} + \frac{1}{2} - \varepsilon \\
&> 1 \qquad (4.8)
\end{aligned}$$

for m large enough. As above, it follows that $\pi_m\pi_n \to 0$, i.e. π_m, π_n or both go to zero.

Suppose $\pi_m \to 0$. The other case is handled identically. We will show that $\Theta_n \geq \min(a, q^{-1})\Theta_m$. Define Θ_M as the proportion of red even vertices that belong to the red component with the most even vertices, and define Θ_N similarly. These numbers must satisfy $\Theta_M M = \Theta_m m$ and $\Theta_N N = \Theta_n n$. By Lemma 13 we *a.a.s.* have $\Theta_N \geq \min(A,1)\Theta_M$, where $A = M/N$. We have

$$\Theta_n = \frac{\Theta_N N}{n} \geq \frac{N}{n}\min(A,1)\Theta_m \frac{m}{M} \tag{4.9}$$

but noting that $N/M = 1/A$ and $m/n = a$, we get

$$\Theta_n \geq \min\left(a, \frac{a}{A}\right)\Theta_m. \tag{4.10}$$

We *a.a.s.* have $M \leq m$ and $N \geq rn = n/q$, so $A \leq qa$ and

$$\Theta_n \geq \min(a, q^{-1})\Theta_m \tag{4.11}$$

So letting $\theta_0 = \min(a,q^{-1})\theta_1$, we *a.a.s.* have $\Theta_m \geq \theta_0$ and $\Theta_n \geq \theta_0$.

□

4.3 Size of the giant component

Given $\lambda > 0$ and $a > 0$, we define the functions $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ and $\psi_{\lambda,a} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\varphi_\lambda(\theta) = e^{-\frac{\lambda^2}{q}\theta} - \frac{1-\theta}{1+(q-1)\theta} \tag{4.12}$$

and

$$\psi_{\lambda,a}(\theta_1,\theta_2) = \begin{pmatrix} \exp\left\{\frac{\lambda\sqrt{a}}{q}(1+(q-1)\theta_2)\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_1\right\}-1\right)\right\} - \frac{1-\theta_1}{1+(q-1)\theta_1} \\ \exp\left\{\frac{\lambda}{q\sqrt{a}}(1+(q-1)\theta_1)\left(\exp\left\{-\lambda\sqrt{a}\theta_2\right\}-1\right)\right\} - \frac{1-\theta_2}{1+(q-1)\theta_2} \end{pmatrix} \tag{4.13}$$

Lemma 25.
(a) Let $m = o(n)$. If $q \geq 1$, then for any sequence $\lambda = \lambda_m$ we have $\varphi_\lambda(\Theta_m) \xrightarrow{P} 0$.
(b) Let $m = an$, $a > 0$. If $q \geq 1$, then for any sequence $\lambda = \lambda_m$ we have $\psi_{\lambda,a}(\Theta_m,\Theta_n) \xrightarrow{P} (0,0)$.

Proof. (a) For $\mathbb{G}(m,n;p,1)$, we have shown (Theorems 6, 9, 11) that for constant $\lambda \in [0,\infty)$

$$e^{-\lambda^2\Theta_m} + \Theta_m - 1 \xrightarrow{P} 0 \tag{4.14}$$

Define $\Theta_m = 1$ if $\lambda = \infty$, so that the convergence holds for all sequences in the compact set $[0,\infty]$. Let λ_m be one such sequence. By looking down convergent subsequences of λ_m, we have

$$e^{-\lambda_m^2\Theta_m} + \Theta_m - 1 \xrightarrow{P} 0 \tag{4.15}$$

If we apply this to the red subgraph, distributed as $\mathbb{G}(M,N;p,1)$, conditional on the component with $\Theta_m m$ even vertices being red we have for any sequence p_M

$$e^{-p_M^2\Theta_M MN} + \Theta_M - 1 \xrightarrow{P} 0 \tag{4.16}$$

Since $m = o(n)$, we have $N = rn(1 + o_p(1))$ by Lemma 23 and

$$\begin{aligned} & \exp\left\{-\frac{\lambda_m^2}{q}\Theta_m\right\} - \frac{1-\Theta_m}{1+(q-1)\Theta_m} \\ = & \exp\left\{-\lambda_m^2 \frac{rn}{n}\Theta_m\right\} + \frac{q\Theta_m}{1+(q-1)\Theta_m} - 1 \\ = & \exp\left\{-\frac{\lambda_m^2 N(1+o_p(1))}{n}\Theta_m\right\} + \frac{\Theta_m m}{(r+(1-r)\Theta_m)m} - 1 \end{aligned} \tag{4.17}$$

Let p_m be such that $\lambda_m = p_m\sqrt{mn}$. Then this equals

$$\begin{aligned} & \exp\left\{-p_m^2\Theta_m mN(1+o_p(1))\right\} + \frac{\Theta_m m}{M} - 1 \\ = & \exp\left\{-p_M^2\Theta_M MN\right\} + \Theta_M - 1 \\ \xrightarrow{P} & \; 0 \end{aligned} \tag{4.18}$$

where $p_M = p_m(1 + o_p(1))$ is some sequence, and the result follows.

(b) Likewise, let $m = an$, $a > 0$. For $q = 1$, we have shown in Theorem 9 that for all sequences $\lambda_M = p_M\sqrt{MN}$, the red subgraph $\mathbb{G}(M,N;\lambda_M/\sqrt{MN},1)$ must satisfy

$$\Theta_M + \exp\left\{\lambda_M\sqrt{\frac{N}{M}}\left(\exp\left\{-\lambda_M\sqrt{\frac{M}{N}}\Theta_M\right\} - 1\right)\right\} - 1 \xrightarrow{P} 0 \tag{4.19}$$

Note that $N = \frac{1}{q}(1+(q-1)\Theta_n)n$. For any sequence $\lambda_m = p_m\sqrt{mn} = p_m m/\sqrt{a} = p_m n\sqrt{a}$, we have

$$\begin{aligned} & \exp\left\{\frac{\lambda_m}{q\sqrt{a}}(1+(q-1)\Theta_n)\left(\exp\left\{-\lambda_m\sqrt{a}\Theta_m\right\} - 1\right)\right\} - \frac{1-\Theta_m}{1+(q-1)\Theta_m} \\ = & \exp\left\{p_m N\left(\exp\left\{-p_m\Theta_m m\right\} - 1\right)\right\} - \frac{1-\Theta_m}{1+(q-1)\Theta_m} \\ = & \frac{\Theta_m m}{(r+(1-r)\Theta_m)m} + \exp\left\{p_m N\left(\exp\left\{-p_m\Theta_M M\right\} - 1\right)\right\} - 1 \\ = & \frac{\Theta_m m}{M} + \exp\left\{p_m\sqrt{MN}\sqrt{\frac{N}{M}}\left(\exp\left\{-p_m\sqrt{MN}\sqrt{\frac{M}{N}}\Theta_M\right\} - 1\right)\right\} - 1 \\ \xrightarrow{P} & \; 0 \end{aligned} \tag{4.20}$$

which follows from (4.19) by letting $\lambda_M = p_m\sqrt{MN}$. The other part of the statement is proved similarly. □

Because of the relative simplicity of φ_λ, we may make more detailed claims about the giant component in the case $m = o(n)$. For $m = an$ however, the complexity of $\psi_{\lambda,a}$ prevents us from proving more results at this moment.

To study φ_λ, we define $f(\theta) = \left(\frac{q}{\theta}(\log(1+(q-1)\theta) - \log(1-\theta))\right)^{1/2}$ for $0 < \theta < 1$, and note that θ satisfies $\varphi_\lambda(\theta) = 0$ if and only if $f(\theta) = \lambda$. Analytical properties of f are (indirectly) proved in [3], and we shall not reprove them here.

Theorem 26. *Let $m = o(n)$.*

(a) If $1 \le q \le 2$ and $\lambda < q$, or if $q > 2$ and $\lambda < \lambda_{\min}$ where $\lambda_{\min}$ is the unique minimum of $f(\theta)$, then $\Theta_m \xrightarrow{P} 0$ as $m \to \infty$.

(b) If $q \ge 1$ and $\lambda > q$, then $\Theta_m \xrightarrow{P} \theta(\lambda)$ where $\theta(\lambda)$ is the unique positive solution to $e^{-\lambda^2\theta/q} = \frac{1-\theta}{1+(q-1)\theta}$.

Proof. The function φ_λ is continuous on [0,1]. Let Z_λ denote the set of zeros of φ_λ for a fixed λ. By Lemma 25, we have $\mathbf{P}\{\Theta_m \in Z_\lambda + (-\varepsilon,\varepsilon)\} \to 1$ as $m \to \infty$. By [3], we have $Z_\lambda = \{0\}$ when $1 \le q \le 2$ and $\lambda < q$ or $q > 2$ and $\lambda < \lambda_{\min}$, which implies $\Theta_m \xrightarrow{P} 0$. This is (a).

When $\lambda > q$ we have, again by [3], $Z_\lambda = \{0,\theta(\lambda)\}$. By Lemma 24, $\mathbf{P}\{\Theta_m > \delta\} \to 1$ for some $\delta > 0$. Thus, the only possibility is $\Theta_m \xrightarrow{P} \theta(\lambda)$. This is (b).

□

When $m = o(n)$, Theorem 26 gives a sharp value of the threshold $\lambda_c(q)$ for $1 \le q \le 2$, while for $q > 2$ all we know is $\lambda_{\min} \le \lambda_c(q) \le q$. Getting more detailed than this when $q > 2$ is likely to be much more involved than the proofs presented here, judging by the complexity of the proof in [3].

We have seen above that if $m = o(n)$, then $\Theta_n \xrightarrow{P} 0$. The following results looks closer at this convergence in the supercritical regime.

Theorem 27. *Let $m = o(n)$, and suppose $q \ge 1$ and $\lambda > q$. Then $\Theta_n = \frac{\lambda\theta(\lambda)}{q}\sqrt{\frac{m}{n}}(1 + o_p(1))$ where $\theta(\lambda)$ is the unique positive solution to $e^{-\lambda^2\theta/q} = \frac{1-\theta}{1+(q-1)\theta}$.*

Proof. We have seen in Theorem 26 that there *a.a.s.* is a component with $\Theta_m m$ even vertices, where $\Theta_m \xrightarrow{P} \theta$. With probability r, this component is coloured red, and by Lemma 22 the red subgraph is distributed as $\mathbb{G}((r + (1-r)\theta)m, rn, \lambda/\sqrt{mn}, 1)$. To make this fit into Corollary 3 we make a change of variables.

Put $m' = (r+(1-r)\theta)m$, $n' = rn$. Then we have a component with $(r+(1-r)\theta)^{-1}\theta m'$ even verties in the graph $\mathbb{G}(m', n', \lambda\sqrt{r + (1-r)\theta}\sqrt{r}/\sqrt{m'n'}, 1)$. By Corollary 3, this component has

$$\begin{aligned} & \frac{\lambda\sqrt{r + (1-r)\theta}\sqrt{r}}{\sqrt{m'n'}} \frac{\theta m'}{r + (1-r)\theta} n'(1 + o_p(1)) \\ = \; & \lambda r\theta \sqrt{\frac{m'}{r + (1-r)\theta}} \sqrt{\frac{n'}{r}} (1 + o_p(1)) \\ = \; & \frac{\lambda\theta}{q}\sqrt{mn}(1 + o_p(1)) \end{aligned} \tag{4.21}$$

odd vertices. □

Using the proofs in this section, the following theorem resolves the issue described before Lemma 22. We show that the unique component containing a positive fraction of the even vertices must be the component with the most odd vertices.

Theorem 28. *Suppose $\Theta_m > 0$. Then there is a.a.s. one unique component containing $\Theta_m m$ even vertices and $\Theta_n n$ odd vertices.*

Proof. Suppose $m = o(n)$, and let C be any component but the one containing $\Theta_m m$ even vertices. By Lemma 21, this component has $o_p(m)$ even vertices. Going through the proof of Theorem 27 for this component, it is seen that it has $o_p(\sqrt{mn})$ odd vertices, which is clearly less than $\Theta_n n$.

Suppose $m = an$, $a > 0$. By Lemma 25, there is a $\delta > 0$ such that *a.a.s.*, $\Theta_m > \delta$ and $\Theta_n > \delta$. Suppose C_1 is a component with $\Theta_m m$ even vertices and C_2 is a component with $\Theta_n n$ odd vertices. Since $m = an$, both of these components *a.a.s.* have at least $(mn)^{1/3}$ vertices, and by Lemma 21 we must have $C_1 = C_2$. □

4.4 Conclusion

We conclude this section by stating the collective results in a theorem.

Theorem 29. *Suppose $q \geq 1$.*
(i) Let $m = o(n)$.

> *(a) (Subcritical) If $1 \leq q \leq 2$ and $\lambda < q$ or if $q > 2$ and $\lambda < \lambda_{\min}$, then every component of $\mathbb{G}(m,n;\lambda/\sqrt{mn},q)$ has $o_p(m)$ even vertices and $o_p(n)$ odd vertices.*
>
> *(b) (Supercritical) If $q \geq 1$ and $\lambda > q$, then $\mathbb{G}(m,n;\lambda/\sqrt{mn},q)$ has a giant component with $\frac{\lambda\theta}{q}\sqrt{mn}(1 + o_p(1))$ vertices, of which θm are even, where θ is the unique positive solution to $e^{-\lambda^2\theta/q} = \frac{1-\theta}{1+(q-1)\theta}$.*

(ii) Let $m = an$, $a > 0$. Then a.a.s. the largest component of $\mathbb{G}(m,n;p,q)$ has $\theta_m m + \theta_n n$ vertices, of which $\theta_m m$ are even and $\theta_n n$ odd, for some solution (θ_m, θ_n) to $\psi_{\lambda,a}(\theta_m, \theta_n) = 0$, with $\psi_{\lambda,a}$ as in (4.13). If $\lambda > q$, then $\theta_m > 0$ and $\theta_n > 0$.

In particular, we have the following important corollary concerning the Ising model. Note that no information is given in (b) for the case $\lambda < 2$.

Corollary 30. *(a) If $m = o(n)$, then under the Ising model at inverse temperature $\beta = -\frac{1}{2}\log\left(1 - \frac{\lambda}{\sqrt{mn}}\right)$ there is a.a.s. a giant component if and only if $\lambda > 2$. The giant component has $\frac{\lambda\theta}{2}\sqrt{mn}(1 + o_p(1))$ vertices, of which $\theta m(1 + o_p(1))$ are even, where θ is the unique positive solution to $e^{-\lambda^2\theta/q} = \frac{1-\theta}{1+\theta}$.*

(b) If $m = an$, $a > 0$, the Ising model at inverse temperature $\beta = -\frac{1}{2}\log\left(1 - \frac{\lambda}{\sqrt{mn}}\right)$ a.a.s. has a giant component if $\lambda > 2$. The giant component has $\theta m(1 + o_p(1))$ even vertices and $\theta_n n(1 + o_p(1))$ odd vertices, where (θ_m,θ_n) is some pair of positive numbers satisfying $\psi_{\lambda,a}(\theta_m, \theta_n) = 0$.

Proof. This follows immediately from Theorem 29 and the correspondence between the random-cluster model and the Ising model explained in Section 1.4.3. □

A

Technical proofs

THIS appendix is intended to spell out the full proofs of some technical lemmas, which were left unproved so as to not clog up the main text. The lemmas will be restated and proved.

Lemma 4. *Let* $\lambda < 1$*,* $m_1 \leq m_2 \leq n$ *and* $p_1 = \lambda/\sqrt{m_1 n}$*,* $p_2 = \lambda/\sqrt{m_2 n}$*. Then*

$$\left(1 - p_1 + p_1\left(1 - p_1 + \frac{p_1}{\lambda}\right)^{m_1}\right)^n \leq \left(1 - p_2 + p_2\left(1 - p_2 + \frac{p_2}{\lambda}\right)^{m_2}\right)^n \tag{2.2}$$

Proof. Setting $b = \sqrt{n}$ and $x = m$, the result follows if we show that

$$f(x) = -\frac{\lambda}{\sqrt{x}b} + \frac{\lambda}{\sqrt{x}b}\left(1 + \frac{1-\lambda}{\sqrt{x}b}\right)^x \tag{A.1}$$

is increasing for integers x satisfying $1 \leq x \leq b^2$. Using the binomial expansion rule, this equals

$$f(x) = \frac{\lambda}{\sqrt{x}b}\left[\sum_{k=1}^{x}\left(\frac{1-\lambda}{\sqrt{x}b}\right)^k\binom{x}{k}\right] = \sum_{k=1}^{x}\frac{\lambda(1-\lambda)^k}{(\sqrt{x}b)^{k+1}}\frac{(x)_k}{k!} \tag{A.2}$$

Here $(x)_k = x(x-1)...(x-k+1)$. This is clearly increasing if each of the terms is increasing, i.e. if $(x)_k/(\sqrt{x}^{k+1})$ is increasing. Since $x \in \mathbb{N}$, this is equivalent to

$$\frac{(x)_k}{\sqrt{x}^{k+1}} \leq \frac{(x+1)_k}{\sqrt{x+1}^{k+1}}, \quad \text{for all } x \geq k \geq 1 \tag{A.3}$$

or in other words, using the definition of $(x)_k = x(x-1)...(x-k+1)$,

$$\frac{x-k+1}{\sqrt{x}^{k+1}} \leq \frac{x+1}{\sqrt{x+1}^{k+1}}, \quad x \geq k \geq 1 \tag{A.4}$$

The above is equivalent to

$$\left(\frac{1-k}{2}\right)\log(x+1)+\left(\frac{k+1}{2}\right)\log(x)-\log(x-k+1)\geq 0, \quad x\geq k\geq 1 \tag{A.5}$$

Denote the left-hand expression by $g_k(x)$. The result will follow by showing that $g_k(x)$ is decreasing and has limit 0 as $x\to\infty$. We have

$$\begin{aligned} g_k'(x) &= \frac{1}{x+1}\left(\frac{1-k}{2}\right)+\frac{1}{x}\left(\frac{k+1}{2}\right)-\frac{1}{x-k+1} \\ &= \frac{x(x-k+1)(1-k)+(x+1)(x-k+1)(k+1)-2x(x+1)}{2x(x+1)(x-k+1)} \\ &= \frac{x-kx-k^2+1}{2x(x+1)(x-k+1)} \\ &= -\frac{x(k-1)+(k^2-1)}{2x(x+1)(x-k+1)} \\ &\leq 0, \qquad \forall x\geq k\geq 1 \end{aligned} \tag{A.6}$$

We also have

$$\begin{aligned} g_k(x) &= \log\left[\frac{x^{\frac{k+1}{2}}}{(x+1)^{\frac{k-1}{2}}(x-k+1)}\right] \\ &= \log\left[\frac{1}{(1+x^{-1})^{\frac{1-k}{2}}(1-(k-1)x^{-1})}\right] \end{aligned} \tag{A.7}$$

so that $\lim_{x\to\infty} g_k(x)=0$. The result follows. □

Lemma 13. *Suppose $a>0$, $\lambda>1$, and let θ_m,θ_n be the unique solutions in (0,1) to $\theta_m+\exp\left\{\frac{\lambda}{\sqrt{a}}(\exp\{-\lambda\sqrt{a}\theta_m\}-1)\right\}=1$ and $\theta_n+\exp\left\{\lambda\sqrt{a}\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_n\right\}-1\right)\right\}=1$ respectively. Then $\theta_n\geq\min(1,a)\theta_m$.*

Proof. Suppose $a\geq 1$, and let $f(x)=x-1+\exp\left\{\frac{\lambda}{\sqrt{a}}(\exp\{-\lambda\sqrt{a}x\}-1)\right\}$ for $0<x<1$. This function is convex and satisfies $f(0)=f(\theta_m)=0$. Thus, we may show that $\theta_n\geq\theta_m$ by showing that $f(\theta_n)\geq 0$. We have, by the definition of θ_n,

$$\begin{aligned} f(\theta_n) &= \exp\left\{\frac{\lambda}{\sqrt{a}}\left(\exp\{-\lambda\sqrt{a}\theta_n\}-1\right)\right\}+\theta_n-1 \qquad \text{(A.8)} \\ &= \exp\left\{\frac{\lambda}{\sqrt{a}}\left(\exp\{-\lambda\sqrt{a}\theta_n\}-1\right)\right\}-\exp\left\{\lambda\sqrt{a}\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_n\right\}-1\right)\right\} \end{aligned}$$

Since e^x is a strictly increasing function, $f(\theta_n)\geq 0$ is equivalent to

$$\frac{\lambda}{\sqrt{a}}\left(\exp\left\{-\lambda\sqrt{a}\theta_n\right\}-1\right)-\lambda\sqrt{a}\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_n\right\}-1\right)\geq 0 \qquad (A.9)$$

or

$$\exp\left\{-\lambda\sqrt{a}\theta_n\right\}-1-a\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_n\right\}+a\geq 0 \qquad (A.10)$$

To show that this indeed holds whenever $a \geq 1$, set $y = \lambda\theta_n$ and $z = \sqrt{a}$, and consider the function $g(z) = e^{-yz} - 1 - z^2 e^{-y/z} + z^2$. This satisfies $g(1) = 0$ and

$$g'(z) = y\left(e^{-\frac{y}{z}} - e^{-yz}\right) + 2z\left(1 - e^{-\frac{y}{z}}\right) \geq 0. \qquad (A.11)$$

Thus $g(z) \geq 0$ and $\theta_n \geq \theta_m$ follows.

We argue similarly to show $\theta_n \geq a\theta_m$ when $a \leq 1$. The convex function $h(x) = \frac{x}{a} - 1 + \exp\left\{\frac{\lambda}{\sqrt{a}}\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}x\right\}-1\right)\right\}$ satisfies $h(a\theta_m) = 0$, so we show that $h(\theta_n) \geq 0$. By the definition of θ_n, we have $\lambda\left(\exp\left\{-\frac{\lambda}{\sqrt{a}}\theta_n\right\}-1\right) = \frac{1}{\sqrt{a}}\log(1-\theta_n)$, so

$$\begin{aligned} h(\theta_n) &= \frac{\theta_n}{a} - 1 + \exp\left\{\frac{1}{a}\log(1-\theta_n)\right\} \\ &= \frac{\theta_n}{a} - 1 + (1-\theta_n)^{1/a} \end{aligned} \qquad (A.12)$$

Letting $k(x) = (1-x)^{1/a} + x/a - 1$ for $0 \leq x \leq 1$, we have $k(0) = 0$ and $k'(x) = a^{-1}\left(1-(1-x)^{1/a-1}\right) \geq 0$. So $h(\theta_n) \geq 0$ and $\theta_n \geq a\theta_m$.

From this follows that $\theta_n \geq \min(1,a)\theta_m$. □

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