The giant component of the random bipartite graph

Master’s Thesis in Engineering Mathematics and Computational Science

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Abstract

The random bipartite graph $G(m,n;p)$ is the random graph obtained by taking the complete bipartite graph $K_{m,n}$ and deleting each edge independently with probability $1 - p$. Using a branching process argument, the threshold function for a giant connected component in $G(m,n;p)$ is found to be $1/\sqrt{mn}$, and the number of vertices in the giant component is proportional to $\sqrt{mn}$. An attempt is made to reduce the study of $G(m,n;p)$ to the well-studied random graph $G(m,p)$ using an original argument, and this is used to show that a.a.s. the giant of $G(m,n;p)$ is cyclic when $m = o(n)$ and $p\sqrt{mn}$ is large enough. Using a colouring argument, similar results are shown for Fortuin-Kasteleyn’s random-cluster model with parameter $1 \leq q \leq 2$ when $m = o(n)$. In particular, when $m = o(n)$ the critical inverse temperature for the Ising model on $K_{m,n}$ is found to be $-\frac{1}{2} \log \left(1 - \frac{2}{\sqrt{mn}}\right)$, and the number of vertices in the giant is proportional to $\sqrt{mn}$. 
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Introduction

Since its initiation in the late 1950s by Erdős and Rényi, the field of random graphs has become an important and largely independent part of combinatorics and probability theory. There have been several books published in the field, the most influential being Béla Bollobás’ 1985 monograph Random Graphs [2]. Inspired by that book, and Svante Janson, Tomasz Łuczak and Andrzej Ruciński’s 2001 book by the same name [10], this thesis extends classical results for the complete graph $K_m$ to the complete bipartite graph $K_{m,n}$. In particular, it focuses on results on the so-called giant component, which is historically important as one of the first remarkable results on random graphs.

The idea behind this thesis may be loosely described as starting with a certain set of vertices, randomly adding edges and studying the properties of the resulting graph. It is important to note that this may be done in any number of ways, and that random graphs may be obtained in other ways, e.g. by assigning numerical values to the vertices of the graph. For the most part, this thesis studies the independent Erdős-Rényi model, named after its originators, in which each edge is included with equal probability in the random graph, independently of all other edges. Other models include the closely related fixed-edge number Erdős-Rényi model, in which a fixed number $M$ of edges is included in the graph and any graph with $M$ edges is equiprobable.

This section sets the stage for the theorems of later sections, by introducing readers to the field of random graphs, and fixing the notation used throughout the thesis. Readers familiar with random graphs can skip Section 1 with no problem.

Section 2 presents proofs of the main theorems of the thesis, imitating the techniques of [10, Theorem 5.4] closely. The argument carries well through the transition from complete graphs to complete bipartite graph, with a slight alteration in which only one part of the bipartite graph $K_{m,n}$ is considered.

Only considering part of the vertices in the branching process argument suggests a method with which the study of random bipartite graph may be reduced to the already
established study of random complete graphs. Section 3 investigates this reduction method, and discusses its applications and limitations.

Following a colouring argument from a 1996 paper by Bollobás, Grimmett and Janson [3], Section 4 extends the results of previous sections to Fortuin-Kasteleyn’s random-cluster model. In particular, the section shows the existence and size of a giant component for the Ising model, important in statistical physics.

1.1 Basic graph theory

A graph is defined as a set \((V,E)\) of labelled vertices \(V\) and edges \(E \subseteq V \times V\). All graphs will be undirected, so that \((x,y) \in E\) if and only if \((y,x) \in E\). Throughout this thesis, \(V\) will be a finite set consisting of indexed letters; typically \(V = (a_1,a_2,...,a_m; b_1,b_2,...,b_n)\) or \(V = (a_1,...,a_m)\) for some positive integers \(m,n\). An edge \((x,y)\) between vertices \(x,y \in V\) will be denoted by the shorthand notation \(xy\).

For a given edge \(xy\), the vertices \(x\) and \(y\) are called the endpoints of \(xy\), and the vertices \(x\) and \(y\) are neighbours. Similarly, two edges \(e,f \in E\) are adjacent if they share an endpoint. A path from \(x\) to \(y\) is a set of edges \(e_0,...,e_k \in E\) such that \(e_{i-1}\) and \(e_i\) are adjacent for all \(1 \leq i \leq k\) and \(x\) and \(y\) are endpoints of \(e_0\) and \(e_k\), respectively. Equivalently, a path from \(x\) to \(y\) may be defined as a sequence of neighbours \(x = x_0,x_1,...,x_k = y\).

Let \(x \sim y\) if and only if there is a path from \(x\) to \(y\). This is an equivalence relation which partitions \(V\) into sets \(C_1,..,C_k\) called the (connected) components of \(G\). The connected component of \(x\) is the unique \(C_i = C(x)\) such that \(x \in C_i\).

![Figure 1.1: The complete graph \(K_6\), as defined in Section 1.1. While complete graphs will not be studied in this thesis, the results for complete bipartite graphs are inspired by and at some points derived from the corresponding results for complete graphs.](image)

A complete graph with \(m\) vertices, denoted \(K_m\), is a graph with vertex set \(V_m = (a_1,...,a_m)\) and edge set \(E_m = \{(a_i,a_j) : i \neq j\}\). A complete bipartite graph is a graph with \(V_{m,n} = (a_1,...,a_m; b_1,...,b_n)\) and \(E_{m,n} = A_m \times B_n\), where \(A_m = (a_1,...,a_m)\) and \(B_n = (b_1,...,b_n)\) make up a partition of \(V_{m,n}\). A vertex \(a_i \in A_m\) will be called even and
a vertex \( b_j \in B_n \) odd. Complete bipartite graphs are denoted \( K_{m,n} \). Throughout the thesis, we make the assumption that \( m = O(n) \), i.e. that there is a constant \( a \geq 1 \) such that \( m \leq an \) for all \( m,n \). We regard \( n \) as a function of \( m \), so that whenever \( n \) appears it should be read as \( n = n(m) \). Theorems will be shown to hold for \( m = an, a > 0 \) and/or \( m = o(n) \). We note that \( m \to \infty \) implies \( n \to \infty \).

A subgraph \( H = (V,F) \) of \( G = (V,E) \) is defined as a graph with the same vertex set as \( G \), and with edge set \( F \subseteq E \). In most cases it is convenient to call an edge \( e \in E \) open if \( e \in F \) and closed otherwise. Each subgraph corresponds to an edge-configuration \( \xi \in \{0,1\}^E \) where \( \xi(e) = 1 \) if \( e \) is open and \( \xi(e) = 0 \) otherwise. We define a partial ordering \( \preceq \) on the space \( \{0,1\}^E \) by \( \xi \preceq \eta \) if and only if \( \xi(x) \leq \eta(x) \) for all \( x \in E \).

A property of a graph \( (V,E) \) is defined as a family of subgraphs \( Q \subseteq \mathcal{P}(E) \). A subgraph \( (V,F) \) has property \( Q \) if \( F \in Q \). An increasing property is a property \( Q \) such that if \( F_1 \in Q \) and \( F_1 \subseteq F_2 \), then \( F_2 \in Q \). In other words, a property is increasing if adding edges to any graph with the property will produce a graph which also has that property.

### 1.2 Asymptotics

Asymptotic notation is frequent in random graphs, since results are typically shown “as \( m \to \infty \)”. We define for any function \( f \) and any positive function \( g \) the following:

\[
  f(x) = O(g(x)) \quad \text{if} \quad \limsup_{x \to \infty} \frac{|f(x)|}{g(x)} < \infty \tag{1.1}
\]

\[
  f(x) = o(g(x)) \quad \text{if} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \tag{1.2}
\]

Most results in this thesis are shown to hold asymptotically almost surely, or a.a.s. A sequence of random variables \( (X_k)_{k=1}^\infty \) has property \( Q \) a.a.s. if an only if \( \mathbb{P} \{ X_m \in Q \} \to 1 \). Furthermore, for a sequence \( X_m \) of random variables we define

\[
  X_m = o_p(a_m) \quad \text{if} \quad |X_m| = o(a_m) \text{ a.a.s.} \tag{1.3}
\]

The usage of \( o_p \) and a.a.s. is quite interchangeable. Typically, asymptotic bounds are stated a.a.s. and asymptotic values are stated using \( o_p \) terms.
1.3 Basic probability

Many results will rely heavily on inequalities related to the Markov inequality. For any non-negative random variable \(X\) with finite expectation, Markov’s inequality states

\[
P \{ X \geq a \} \leq \frac{\mathbb{E}[X]}{a}, \quad a > 0
\]  

Supposing further that \(X\) has finite second moment, Markov’s inequality applied on the random variable \(|X - \mathbb{E}[X]|^2\) implies Chebyshev’s inequality:

\[
P \{|X - \mathbb{E}[X]| \geq a\} \leq \frac{\text{Var}X}{a^2}, \quad a > 0
\]  

This will frequently be applied when a sequence of random variables \(X_m\) is shown to satisfy \(\sqrt{\text{Var}X} = o_p(\mathbb{E}[X_m])\). Chebyshev’s inequality implies

\[
P \{|X_m - \mathbb{E}[X_m]| \geq \mathbb{E}[X_m]\} \leq \frac{\text{Var}X_m}{\mathbb{E}[X_m]^2} = o(1)
\]  

so that \(X_m = \mathbb{E}[X_m] (1 + o_p(1))\).

The following bounds hold for all \(a \in \mathbb{R}\), and are commonly called Chernoff bounds.

\[
P \{X \geq a\} = P \{e^{tX} \geq e^{ta}\} \leq \frac{\mathbb{E}[e^{tX}]}{e^{ta}}, \quad t \geq 0
\]  

\[
P \{X \leq a\} = P \{e^{tX} \geq e^{ta}\} \leq \frac{\mathbb{E}[e^{tX}]}{e^{ta}}, \quad t < 0
\]  

For a random variable \(X\), the probability-generating function \(G_X\) is defined by \(G_X(s) = \mathbb{E}[s^X]\) when this expectation exists. The moment-generating function \(M_X\) is defined, when it exists, by \(M_X(t) = \mathbb{E}[e^{tX}] = G_X(e^t)\). The property of these functions that will be used most frequently is the following: Let \(N\) be a positive integer-valued random variable, and let \(Y = X_1 + \ldots + X_N\) where the \(X_i\) are i.i.d. and independent of \(N\). Then

\[
M_Y(t) = M_N(M_X(t)) \quad \text{and} \quad G_Y(s) = G_N(G_X(s))
\]  

whenever the functions exist. See e.g. [7].

A random variable \(X\) is Bernoulli distributed with parameter \(p \in [0,1]\), denoted \(X \in \text{Bern}(p)\), if \(P \{X = 1\} = p\) and \(P \{X = 0\} = 1 - p\). It is binomially distributed with parameters \(n \in \mathbb{N}\) and \(p \in [0,1]\), denoted \(X \in \text{Bi}(n,p)\), if for any integer \(0 \leq k \leq n\)

\[
P \{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}
\]  

The probability-generating function of \(X \in \text{Bi}(n,p)\) is given by \(G_X(s) = (1 - p + ps)^n\).
1.3.1 Branching processes

A Galton-Watson process or branching process is a stochastic process \((Z_k)_{k=0}^{\infty}\), defined as follows. We choose a non-negative integer-valued probability distribution \(A\) and call it the offspring distribution. Given a deterministic starting value \(Z_0\) (usually taken to be 1), each \(Z_k, k \geq 1\) is defined by \(Z_k = \sum_{i=1}^{Z_{k-1}} X_i\), where the \(X_i \in A\) are independent. The variables \(Z_k\) are interpreted as the number of individuals in the \(k\)-th generation of a population, and each \(X_k\) is the number of offsprings for an individual.

The extinction probability for a branching process is the probability of the event \(\{\exists k > 1 : Z_k = 0\}\). Let \(X \in A\). A classical result states that if \(E[X] \leq 1\), the extinction probability is 1, and if \(E[X] > 1\) the extinction probability is given by the unique solution in \((0, 1)\) to \(G_X(s) = s\). Here \(G_X\) denotes the probability-generating function of \(X\).

1.3.2 Stochastic domination

In Section 3 we shall need the measure-theoretic concept of stochastic domination. Given \(S \subseteq \mathbb{R}\), a partial ordering \(\preceq\) on a space \(\Omega\) and probability measures \(\mu, \nu\) on \(\Omega\), we say that \(\mu\) stochastically dominates \(\nu\), written \(\nu \preceq_D \mu\), if \(\nu(f) \leq \mu(f)\) for all increasing functions \(f: \Omega \to S\) (with respect to \(\preceq\)). Heuristically, this means that \(\mu\) prefers larger elements of \(\Omega\) than \(\nu\). In particular, for any increasing event \(B \subseteq S\) we have \(\nu(B) \leq \mu(B)\). See [6].

The following powerful result is used to characterize stochastic domination in relation to random graphs. It is a special case of a more general result called Holley’s Theorem, see e.g. [6]. Suppose \(x \in E\) and let \(X(x) = 1\) if \(x \in E\) is open and \(X(x) = 0\) otherwise. We define the event \(\{X = \xi \text{ off } x\}\) for \(\xi \in \{0,1\}^{E \setminus \{x\}}\) as the event \(\{X(y) = \xi(y)\}\) for all \(y \in E \setminus \{x\}\).

Holley’s Theorem. Let \(G = (V,E)\) be a graph and let \(\mu_1, \mu_2\) be probability measures on \(\{0,1\}^E\). If

\[
\mu_1(X(x) = 1 \mid X = \xi \text{ off } x) \leq \mu_2(X(x) = 1 \mid X = \xi' \text{ off } x)
\]

for all \(x \in E\) and all \(\xi, \xi' \in \{0,1\}^{E \setminus \{x\}}\) such that \(\xi \preceq \xi'\), then \(\mu_1 \preceq_D \mu_2\).

1.4 Random graph models

Let \(G = (V,E)\) be a graph. A random graph model, in all instances of this thesis, is a probability measure on either of the spaces \(S^V\) and \(S^E\) for some \(S \subseteq \mathbb{R}\). This section presents the different models that will be used or mentioned in the thesis.

1.4.1 The Erdős-Rényi model

The model that Erdős and Rényi introduced, which has been subsequently named the Erdős-Rényi model, comes in two closely linked forms. The one most suited for this
thesis is denoted $G(m,p)$, which generates a subgraph of the complete graph $K_m = (V_m, E_m)$ in which each edge is included independently of all others with probability $p$. Equivalently, it may be seen as starting with $K_{m,n}$ and removing each edge independently with probability $1 - p$. In other words, for any subgraph $H = (V_m, F)$ where $F \subseteq E_m$,

$$P\{F\} = \left( \binom{m}{2} \right)^{|F|} p^{|F|}(1-p)^{|E| - |F|}$$

(1.12)

For the complete bipartite graph $K_{m,n} = (A_m \cup B_n, A_m \times B_n)$ we denote the corresponding model $G(m,n;p)$. A subgraph $H = (A_m \cup B_n, F)$ is assigned probability

$$P\{F\} = \left( \frac{mn}{|F|} \right) p^{|F|}(1-p)^{mn - |F|}$$

(1.13)

### 1.4.2 Ising and Potts models

A physically important random graph model is the Ising model, invented by Wilhelm Lenz [11] and his student Ernst Ising [9] to model ferromagnetism. Mathematically, it is quite different from the Erdős-Rényi model in that it assigns probability to vertices and not to edges. Each vertex is assigned a spin, which here will be denoted by the values 1 or 2, and the model will prefer vertex configurations in which neighbours have equal spin. Formally, the Ising model on a finite graph $G = (V,E)$ consists of a probability measure on $\{1,2\}^V$ which to each $\sigma \in \{1,2\}^V$ assigns probability

$$\phi_\beta(\sigma) = \frac{1}{Z_\beta} \exp \left( -2\beta \sum_{x \sim y} I_{\{\sigma(x) \neq \sigma(y)\}} \right)$$

(1.14)

where $\beta \geq 0$, and $x \sim y$ if and only if $x, y \in V$ are neighbours. Here $Z_\beta$ is a normalizing constant. In ferromagnetic applications $\beta$ bears the interpretation of reciprocal temperature, i.e. $\beta = 1/T$ where $T$ is the temperature.

A natural generalization of the Ising model is the $q$-state Potts model [12], which assigns spins in the set $\{1,...,q\}$ for some integer $q \geq 2$. The Potts probability measure on $\{1,...,q\}^V$ is identical to (1.14) except for a new normalizing constant $Z_{\beta,q}$.

### 1.4.3 The random-cluster model

The random-cluster model, introduced by Cees Fortuin and Pieter Kasteleyn in a series of papers around 1970, see e.g. [8], is a random graph model that unifies the Erdős-Rényi, Ising and Potts models. For any $q > 0$ it assigns to any subgraph $(V,F)$ of a graph $(V,E)$ probability proportional to

$$\tilde{P}(F; E,p,q) = p^{|F|}(1-p)^{|E|-|F|} q^{c(V,F)}$$

(1.15)

where $c(V,F)$ is the number of components of $(V,F)$. Swendsen and Wang [13], followed by a simpler proof by Edwards and Sokal [4], showed that the random-cluster model at probability $p$ for any $q \in \mathbb{N}$ is equivalent to the $q$-state Potts model at inverse temperature $\beta = -\frac{1}{2} \log(1-p)$.
1.5 The giant component

Around 1960, Erdős and Renyi [5] proved the first result concerning the existence and non-existence of a so-called giant component in $G(m,p)$. Essentially, the result states that the graph $G(m,\lambda/m)$ a.a.s. contains one component which is much larger than all other components when $\lambda > 1$, while no such component exists for $\lambda < 1$. The following formulation of the result appears in [10]:

**Theorem.** Let $mp = \lambda$, where $\lambda > 0$ is a constant.

(i) If $\lambda < 1$, then a.a.s. the largest component of $G(m,p)$ has at most $\frac{3}{(1-\lambda)^2} \log m$ vertices.

(ii) Let $\lambda > 1$ and let $\theta = \theta(\lambda) \in (0,1)$ be the unique solution to $\theta + e^{-\lambda\theta} = 1$. Then $G(m,p)$ contains a giant component of $\theta m(1 + o_p(1))$ vertices. Furthermore, a.a.s. the size of the second largest component of $G(m,p)$ is at most $\frac{16\lambda}{(\lambda-1)^2} \log m$.

This thesis will mainly be concerned with proving a similar result for the bipartite graph $G(m,n; \lambda/\sqrt{mn})$, using proof techniques similar to those in [10].

In a 1996 paper [3], Bollobás et al. derived conditions for the existence and size of a giant component in the random-cluster model for any $q > 0$, by reducing to the $q = 1$ case and using known results. This thesis will partly emulate those methods on the bipartite graphs to prove similar results.
2

The branching process

Since the results contained in this thesis are concerned with component sizes, a method to measure the size of a component is needed. The following method is inspired by the one used in [10], and the process it describes will be called an even search process (or simply search process) and is closely related to a breadth-first search, common in computer graph algorithms. The algorithm finds the vertices in the connected component of an even vertex \(a\), and counts the number of even and odd vertices separately.

1. Let \(a_0 = a\) and \(S = \emptyset\), and mark \(a_0\) as saturated. All other vertices are initially unsaturated. Set \(k = 0\).

2. Find all unsaturated neighbours \(b_1,...,b_r\) of \(a_k\), and mark them as saturated. Let \(R_{k+1} = \{b_1,...,b_r\}\) and \(Y_{k+1} = |R_{k+1}|\).

3. For each \(b_i \in R_{k+1}\), find all unsaturated neighbours \(a_{i1},...,a_{is}\) of \(b_i\), \(s = s(i)\). Let \(S_{k+1} = \bigcup_{i=1}^{r} \{a_{i1},...,a_{is}\}\). Thus, \(S_{k+1}\) is the set of all unsaturated vertices at distance 2 from \(a_k\). Let \(X_{k+1} = |S_{k+1}|\).

4. Assign \(S := S \cup S_{k+1}\). If \(S\) is empty, stop the algorithm. If \(S\) is nonempty, let \(a_{k+1}\) be an arbitrary element of \(S\), remove \(a_{k+1}\) from \(S\) and mark \(a_{k+1}\) as saturated. Return to step 2 with \(k + 1\) in place of \(k\).

Here the search for odd vertices \(b_i\) should be considered an intermediate step: we are only interested in finding the even vertices of the component containing \(a\). So \(X_1\) is the number of “2-neighbours”, i.e. vertices at distance 2 from \(a\), and \(X_2, X_3,...\) are the number of vertices at distance 2 from vertices that have already been found in the process. The number of even vertices in the component is given by \(1 + \sum_{k=1}^{K} X_k\), where \(K\) is the total number of iterations of the algorithm.

This resembles a branching process, the difference being that a branching process as defined in Section 1.3.1 requires each \(X_i\) to be identically distributed. However, the
resemblance is in some sense close enough, in that we can find actual branching processes
bounding the search process from above and below.

It is clear that \(Y_1 \in \text{Bi}(n,p)\), since at the start all vertices are unsaturated, and \(a\)
has \(n\) neighbours. After this the distributions are more complicated, but we may bound
\(X_1\) from above by \(X_1^+ \in \text{Bi}(Y_1m,p)\), since it is the sum of \(Y_1\) random variables, each
bounded from above by \(\text{Bi}(m,p)\). Continuing in this fashion, we may bound
\(Y_k\) from above by \(Y_k^+ \in \text{Bi}(n,p)\) for all \(k\), and \(X_k\) from above by \(X_k^+ \in \text{Bi}(Y_k^+m,p)\). Using (1.9),
we have the following probability-generating function for \(X_k^+\):

\[
G_{X_k^+}(s) = (1 - p + p(1 - p + ps)^m)^n
\]  

(2.1)

Given information about the maximal size of a component, we may bound the process
from below by similar methods, as will be seen below.

2.1 Proportion of even and odd vertices

The even search process described above finds the number of even vertices of a component
in the graph \(G(m,n;p)\). To be able to state results about the total size of the component,
we show a result that relates the number of even vertices in a component to the number
of odd vertices in the same component. Heuristically, this section shows that the number
of odd vertices in a component is \(pn\) times the number of even vertices, if the number
of even vertices in the component is small enough. Note that this proportion equals the
expected degree of any even vertex. For this, we need a purely probabilistic lemma.

**Lemma 1.** If \(X \in \text{Bi}(n,p)\) where \(n\) and \(p = p(n)\) are such that \(E[X] = np \to \infty\) as
\(n \to \infty\), then \(X = np(1 + o_p(1))\)

**Proof.** We have \(E[X] = np\) and \(\text{Var}X = np(1 - p) = o(E[X]^2)\) so that by (1.6),
\(X = np(1 + o_p(1))\).

Lemma 1 is useful in proving the following.

**Lemma 2.** Let \(m_0 = m_0(m)\) be such that \(pm_0 \to \infty\) and \(pm_0 \to 0\) as \(m \to \infty\).
Let \(C\) be a component of \(G(m,n;p)\). Conditional on \(C\) having \(m_0\) even vertices, \(C\) has
\(pm_0n(1 + o_p(1))\) odd vertices.

**Proof.** In each step of the search process, a number \(Y_k\) of odd vertices is identified. These
random variables are each bounded from above by a random variable \(Y_k^+ \in \text{Bi}(n,p)\). Since
the number of odd vertices in the component, conditional on the component having
\(m_0\) even vertices, is a sum of \(m_0\) such variables, it must be bounded from above by
\(Y^+ \in \text{Bi}(m_0n,p)\). But by the assumption that \(pm_0n \to \infty\), we have \(E[Y^+] \to \infty\) and
by Lemma 1, \(Y^+ = pm_0n(1 + o_p(1))\).

Let \(n_+ = pm_0n(1 + o_p(1))\). Then the random variables \(Y_k\) must all be bounded
from below by \(Y_k^- \in \text{Bi}(n - n_+, p)\), since throughout the process there are at least
\(n - n_+\) odd vertices that are not yet saturated, and \(Y\) is bounded from below by \(Y^- \in\)
2.2. Subcritical case

This section is concerned with proving the non-existence of a giant component when \( \lambda < 1 \) using branching process arguments. It is assumed that \( m \leq n \), which is enough by symmetry. We shall need two lemmas.

**Lemma 4.** Let \( \lambda < 1 \), \( m_1 \leq m_2 \leq n \) and \( p_1 = \lambda / \sqrt{m_1 n} \), \( p_2 = \lambda / \sqrt{m_2 n} \). Then

\[
(1 - p_1 + p_1 \left(1 - p_1 + \frac{p_1}{\lambda}\right)^{m_1})^n \leq \left(1 - p_2 + p_2 \left(1 - p_2 + \frac{p_2}{\lambda}\right)^{m_2}\right)^n \tag{2.2}
\]

The proof of Lemma 4 is left to Appendix A.

**Lemma 5.** Let \( \lambda < 1 \). Given a random variable \( X \) with moment-generating function \( M_X(t) = (1 + p + p \left(1 - p + pe^t\right)^m)^n \), where \( m \leq n \) and \( p = \lambda / \sqrt{mn} \), we have

\[
\lambda \mathbb{E}[\lambda^{-X}] \leq \exp \left(-\frac{1}{6}(1 - \lambda)^3\right). \tag{2.3}
\]

**Proof.** First of all we note that

\[
\lambda \mathbb{E}[\lambda^{-X}] = \lambda \left(1 - p + p \left(1 - p + \frac{p}{\lambda}\right)^m\right)^n \tag{2.4}
\]

The proof is done in two steps. By Lemma 4 we can reduce this to showing

\[
\lambda \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} \left(1 - \frac{\lambda}{n} + \frac{1}{n}\right)^m\right)^n \leq \exp \left(-\frac{1}{6}(1 - \lambda)^3\right) \tag{2.5}
\]
This is done by noting that \((1 + \frac{1-\lambda}{n})^n \leq \exp(1-\lambda),\) and \(\left(1 + \frac{\lambda(\lambda - 1)}{n}ight)^n \leq \exp(\lambda e^{1-\lambda} - \lambda),\) so that
\[
\lambda \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} \left(1 - \frac{\lambda}{n} + \frac{1}{n}\right)^n\right)^n \leq \lambda \exp(\lambda e^{1-\lambda} - \lambda) \tag{2.6}
\]
The proof is finished by showing that this is no larger than \(\exp(- (1-\lambda)^3/6).\) This is equivalent to \(g(\lambda) \geq 0,\) where
\[
g'(\lambda) = \frac{\lambda^3}{6} - \frac{\lambda^2}{2} + \frac{3\lambda}{6} - \frac{1}{6} - \log \lambda - \lambda e^{1-\lambda} \tag{2.7}
\]
It is easy to show that \(g'(1) = g''(1) = 0,\) and \(g''(\lambda) > 0\) for all \(\lambda \in (0,1).\) Thus, \(g'(\lambda) < 0\) and \(g(\lambda) > 0\) for all \(\lambda \in (0,1),\) and the result follows.

\[\text{Theorem 6. If } \lambda < 1 \text{ and } m \leq n, \text{ then there is a.a.s. no component in } G(m,n; p) \text{ with more than } 7(1 - \lambda)^{-3} \log m \text{ even vertices.}\]

**Proof.** We bound the search process by a branching process with offspring distribution having moment-generating function \(M_X(t) = (1 - p + p(1 - p + pe^{t})^m)^n,\) as discussed at the beginning of Section 2. Consider a search process initiated at the even vertex \(a.\) The probability that the total number of even vertices found in the process is at least \(k\) is bounded as follows:
\[
P\left\{a \text{ belongs to a component of size at least } k\right\} \leq P\left\{\sum_{i=1}^{k} X_i \geq k - 1\right\} \tag{2.8}
\]

We can bound the quantity by the following Chernoff bound:
\[
P\left\{\sum_{i=1}^{k} X_i \geq k - 1\right\} \leq \frac{E\left[\exp(t(X_1 + \ldots + X_k))\right]}{\exp(t(k-1))} = \frac{E\left[e^{tX_1}\right]^k}{e^{t(k-1)}}, \quad t > 0 \tag{2.9}
\]
where we have used the fact that the \(X_i\) are \(i.i.d.\) to separate the expectation. The inequality holds for any \(t > 0,\) and in particular we may choose \(t = \log \left(\frac{1}{\lambda}\right)\) to obtain
\[
P\left\{\sum_{i=1}^{k} X_i \geq k - 1\right\} \leq \lambda^{k-1}E\left[\lambda^{-X_1}\right]^k = \frac{1}{\lambda} \left(\lambda E\left[\lambda^{-X_1}\right]\right)^k \tag{2.10}
\]
Using Lemma 5, and the \(m \leq n\) assumption, we have \(\lambda E\left[\lambda^{-X_1}\right] \leq \exp(-\frac{1}{6}(1-\lambda)^3).\)
2.3 Supercritical case

So letting \( k = 7(1 - \lambda)^{-3} \log m \), we have

\[
\mathbb{P} \{ \exists \text{ component with at least } k \text{ even vertices} \} \\
\leq \sum_{i=1}^{m} \mathbb{P} \{ a_i \text{ belongs to a component of size at least } k \} \\
\leq m \mathbb{P} \left\{ \sum_{i=1}^{k} X_i \geq k - 1 \right\} \\
\leq \frac{m}{\lambda} \exp \left( -\frac{k}{6} (1 - \lambda)^3 \right) \\
= \frac{m}{\lambda} m^{-7/6} \\
= o(1) \quad (2.11)
\]

\[\square\]

2.3 Supercritical case

This section is concerned with proving the existence, uniqueness and size of a giant component when \( \lambda > 1 \) using the even search process. Here we weaken the \( m \leq n \) assumption, since we will need to cover all cases in which \( m \sim n \). Instead we assume there is an \( a \geq 1 \) such that \( m \leq an \).

Lemma 7. Suppose there is an \( a \geq 1 \) such that \( m \leq an \). Let \( k_- = \frac{3 \log^2 m}{\lambda(\lambda - 1)^2} \) and \( k_+ = \frac{\sqrt{5m \log m}}{\lambda(\lambda - 1)} \). If \( \lambda > 1 \), then there is a.a.s. no component with \( k \) even vertices, where \( k_- \leq k \leq k_+ \).

Remark. Note, in particular, that \( k_+ > 2k_- \) for large \( m \). This means that two components with less than \( k_- \) even vertices can never be joined by a single edge to form a component with at least \( k_+ \) even vertices.

Proof. Starting at an even vertex \( a \), we show that the search process either terminates after fewer than \( k_- \) steps, or at the \( k \)-th step there are at least \( (\lambda - 1)k \) vertices in the component containing \( a \) that have been generated in the process but which are not yet saturated, for any \( k \) satisfying \( k_- \leq k \leq k_+ \). Indeed, if \( (\lambda - 1)k \) unsaturated vertices have been found, then the process continues since \( (\lambda - 1)k \geq (\lambda - 1)k_- \geq 1 \) for \( m \) large enough.

Since we consider only the part of the process where \( k \leq k_+ \), we need only identify at most \( k_+ + (\lambda - 1)k_+ = \lambda k_+ \) even vertices in the process, and we may bound \( X_i \) from below by \( X_i^{-} \in \text{Bi}(Y(m - \lambda k_+), p) \) where \( Y \in \text{Bi}(n, p) \). The probability that after \( k \) steps there are fewer than \( (\lambda - 1)k \) not yet saturated vertices, or that the process dies out after the first \( k \) steps, is smaller than

\[
\mathbb{P} \left\{ \sum_{i=1}^{k} X_i^{-} \leq k - 1 + (\lambda - 1)k \right\} = \mathbb{P} \left\{ \sum_{i=1}^{k} X_i^{-} \leq \lambda k - 1 \right\} \quad (2.12)
\]
As in the subcritical setting, we employ Chernoff bounds to bound this. First, we note that for any $X_i$ the moment-generating function is given by

$$M(t) = \left(1 - p + p \left(1 - p + pe^t\right)^{m - \lambda k_+}\right)^n$$

Putting $t = \frac{1}{\lambda k} \log \left(\frac{1}{\lambda}\right) < 0$, we have the following Chernoff bound:

$$P\left\{\sum_{i=1}^{k} X_i^+ \leq \lambda k - 1\right\} \leq \frac{M\left(\frac{1}{\lambda k} \log \left(\frac{1}{\lambda}\right)\right)^k}{\exp\left(\frac{1}{\lambda k} \log \left(\frac{1}{\lambda}\right) (\lambda k - 1)\right)} = \lambda^{1 - \frac{1}{\lambda k^2}} \left[M\left(\frac{1}{\lambda k} \log \left(\frac{1}{\lambda}\right)\right)\right]^k$$

Thus we wish to bound $M(t)$, which explicitly reads

$$\left(1 - p + p \left(1 - p + \frac{p}{\lambda k^2}\right)^{m - \lambda k_+}\right)^n$$

where $p = \lambda/\sqrt{mn}$. Using the inequality $(1 + x/n)^n \leq e^x$, which holds for all $x \in \mathbb{R}$ and all $n > 0$, we have

$$\left(1 - \frac{p}{\lambda k^2}\right)^{m - \lambda k_+} = \left(1 - \frac{\lambda - \lambda^1 - \frac{1}{\lambda k^2}}{\sqrt{mn}}\right)^{\sqrt{mn} (\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}})} \leq \exp\left\{-\left(\lambda - \lambda^1 - \frac{1}{\lambda k^2}\right) \left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right)\right\} \leq 1 - \frac{1}{\sqrt{a}} \left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right)$$

Where $a \geq 1$ is such that $\frac{m}{n} \leq a$ for all $m$. The second inequality holds since $e^{-x} \leq 1 - x/\lambda \sqrt{a}$ for $x \leq \lambda \sqrt{a} - 1$. Indeed, we have

$$\left(\lambda - \lambda^1 - \frac{1}{\lambda k^2}\right) \left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right) \leq \lambda \sqrt{a} - \lambda^1 - \frac{1}{\lambda k^2}\sqrt{a} \leq \lambda \sqrt{a} - 1$$

so the inequality is applicable. Putting (2.16) into (2.15) yields

$$\left(1 - p + p \left(1 - p + \frac{p}{\lambda k^2}\right)^{m - \lambda k_+}\right)^n \leq \left(1 + \frac{\lambda}{\sqrt{mn}} \left(1 - \frac{1}{\sqrt{a} \lambda} \left(\lambda - \lambda^1 - \frac{1}{\lambda k^2}\right) \left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right) - 1\right)\right)^n \leq \exp\left\{-\frac{1}{\sqrt{a}} \left(\lambda - \lambda^1 - \frac{1}{\lambda k^2}\right) \left(\sqrt{\frac{m}{n}} - \frac{\lambda k_+}{\sqrt{mn}}\right) \sqrt{\frac{n}{m}}\right\}$$

$$= \exp\left\{-\frac{1}{\sqrt{a}} \left(\lambda - \lambda^1 - \frac{1}{\lambda k^2}\right) \left(1 - \frac{\lambda k_+}{m}\right)\right\}$$
So, (2.14) becomes

\[
P \left\{ \sum_{i=1}^{k} X_i^- \leq \lambda k - 1 \right\} \leq \lambda^{1 - \frac{1}{\lambda}} \exp \left\{ -\frac{k}{\sqrt{a}} \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \left( 1 - \frac{\lambda k_+}{m} \right) \right\} \quad (2.19)
\]

\[
\leq \lambda \exp \left\{ \frac{1}{\sqrt{a}} \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \frac{\lambda k_+}{m} - \frac{k}{\sqrt{a}} \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \right\}
\]

Noting that \( k^- = k^2 \frac{\lambda \log m}{m} \), the probability that there is a component with \( k^- \leq k \leq k^+ \) even vertices is thus bounded as follows.

\[
m \sum_{k=k^-}^{k^+} P \left\{ \sum_{i=1}^{k} X_i^- \leq k - 1 + (\lambda - 1)k \right\}
\]

\[
\leq m k_+ \lambda \exp \left\{ \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \frac{\lambda k_+^2}{m \sqrt{a}} - \frac{k^-}{\sqrt{a}} \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \right\}
\]

\[
= m k_+ \lambda \exp \left\{ \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \frac{\lambda k_+^2}{m \sqrt{a}} - \log m \frac{\lambda k_+^2}{m \sqrt{a}} \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \right\}
\]

\[
= m k_+ \lambda \exp \left\{ \frac{\lambda k_+^2}{m \sqrt{a}} (1 - \log m) \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \right\}
\]

\[
= c_0 m^{3/2} \sqrt{\log(m^3)} \exp \left\{ -\frac{\log(m^3)}{\sqrt{a} \lambda (\lambda - 1)^2} (\log m - 1) \left( \lambda - \lambda^{1 - \frac{1}{\lambda}} \right) \right\}
\]

\[
= c_0 \sqrt{\log(m^3)} m^{3/2 - c_1 (\log m - 1)}
\]

\[
= o(1) \quad (2.20)
\]

where \( c_0, c_1 \) are some positive constants that depend on \( a, \lambda \) only.

\[\square\]

**Lemma 8.** If \( \lambda > 1 \), then a.a.s. there is at most one component with more than \( k_+ = \sqrt{m \log(m^3)} \) even vertices.

**Proof.** We argue again as in [10]. Suppose there exists at least one component of size at least \( k_+ \) (call such a component "large"), and let \( a_i \) and \( a_j \), \( i \neq j \) be two even vertices who each belong to large, possibly equal, components \( C_i \) and \( C_j \), respectively. We run two search processes starting at these two vertices. After \( k_+ \) steps, each process must have produced \( (\lambda - 1)k_+ \) not yet saturated even vertices. The probability that there does not exist an open path of length 2 between as yet unsaturated vertices of \( C_i \) and
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$C_j$ is bounded from above by

\[
P \{\forall l = 1, \ldots, n : \text{at least one of } a_ib_l, a_jb_l \text{ closed}\} \leq \left(1 - p^2\right)^{n[(\lambda - 1)k_+]^2} = \left(1 - \frac{\lambda^2}{mn}\right)^{n[(\lambda - 1)k_+]^2} \leq \exp\left\{-\frac{\lambda^2}{m}[(\lambda - 1)k_+]^2\right\} = \exp\left\{-\frac{\lambda^2}{m}[(\lambda - 1)k_+]^2\right\} \leq \exp\left\{-\frac{\lambda^2}{m}[(\lambda - 1)k_+]^2\right\} \leq \exp\left\{-\frac{\lambda^2}{m}[(\lambda - 1)k_+]^2\right\} = o(m^{-2})
\]

(2.21)

Since there are less than $m^2$ ways of choosing $a_i, a_j$, this implies that the probability that any pair of $a_i$ and $a_j$ belong to different large components tends to 0 as $m \to \infty$, and there cannot be more than one component of size larger than $k_+$. \qed

This means that we may partition the even vertices of $G(m,n;p)$ into "large" and "small" vertices, the former being the vertices that belong to the unique component of size at least $k_+$, if such a component exists. To see that the component indeed does exist and find its size, we count the number of small vertices.

**Theorem 9.** If $\lambda > 1$, then there is a component with $\theta_m m(1 + o(1))$ even vertices, where $\theta_m = \theta_m(\lambda) \in (0,1)$ is the unique solution to

\[
\theta_m + \exp\left\{\frac{\lambda^2}{\sqrt{a}} \left(\exp\left\{-\lambda \theta_m \sqrt{a}\right\} - 1\right)\right\} = 1 \quad \text{if} \quad m = an, a > 0
\]

\[
\theta_m + e^{-\lambda^2 \theta_m} = 1 \quad \text{if} \quad m = o(n)
\]

(2.22)

Furthermore, the second largest component a.a.s. has at most $\frac{3 \log^2 m}{\lambda(\lambda - 1)^2}$ even vertices.

**Proof.** The second statement follows directly from Lemmas 7 and 8.

For the first part, we calculate the probability that a given even vertex $a$ is small. Call this probability $\rho(m,n,p)$. As in [10], the probability is bounded from above by $\rho_+ = \rho_+(m,n,p)$, the extinction probability for the branching process in which the offspring distribution is given by $\Bi(Y(m - k_-),p) = \Bi(Y(m - o(m)),p)$, where $Y \in Bi(n,p)$ as usual. The probability $\rho$ is also bounded from below by $\rho_- + o(1)$, where $\rho_-$ is the extinction probability for the branching process with offspring distribution $\Bi(Ym,p)$. The $o(1)$ term bounds the probability that the process dies after more than $k_-$ steps.

The extinction probability $\rho_+$ is the unique positive solution to $G_X(\rho_+) = \rho_+$, where
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\[ G_X(x) = \left(1 - p + p(1 - p + px)^{m-o(m)}\right)^n. \] If \( m = o(n) \), we have

\[
(1 - p + px)^{m-o(m)} = \left(1 + \frac{\lambda(x-1)}{\sqrt{mn}} \right)^{m-o(m)}
\]

\[
= \sum_{k=0}^{m-o(m)} \binom{m-o(m)}{k} \left(\frac{\lambda(x-1)}{\sqrt{mn}} \right)^k
\]

\[
= 1 + \lambda(x-1)\sqrt{\frac{m-o(m)}{n}} + o\left(\frac{\sqrt{m}}{n}\right)
\] (2.23)

so that

\[
(1 - p + p(1 - p + px)^{m-o(m)})^n
\]

\[
= \left(1 + \frac{\lambda}{\sqrt{mn}} \right)^n
\]

\[
= \left(1 + \frac{\lambda^2(x-1)(1+o(1))}{n}\right)^n
\]

\[
= \exp\left\{\lambda^2(x-1)(1+o(1))\right\}
\] (2.24)

so as \( m \to \infty \), we have \( \rho_+ = 1 - \theta_m \), where \( e^{-\lambda^2\theta_m} = 1 - \theta_m \), and replacing \( m-o(m) \) by \( m \) we get \( \rho_- = 1 - \theta_m + o(1) \). Thus \( \rho = 1 - \theta_m + o(1) \), and the expected number of small even vertices is given by \( (1 - \theta_m)m(1 + o(1)) \). In other words, the expected number of even vertices in the largest component is \( \theta_m m(1 + o(1)) \).

Similarly, if \( m = an, a > 0 \), then

\[
\lim_{n \to \infty} \left(1 - \frac{\lambda}{\sqrt{an}} + \frac{\lambda}{\sqrt{an}} \left(1 - \frac{\lambda}{\sqrt{an}} + \frac{\lambda}{\sqrt{an}} x\right)^{an-o(n)}\right)^n
\]

\[
= \exp\left\{\frac{\lambda}{\sqrt{a}} \exp\left\{\lambda(x-1)\sqrt{a}(1+o(1))\right\} - \frac{\lambda}{\sqrt{a}}\right\}
\] (2.25)

and the expected number of even vertices in the giant component is again given by \( \theta_m m(1 + o(1)) \).

Lastly, we show that \( Y = \mathbb{E}[Y](1 + o_p(1)) \), where \( Y \) is the number of small vertices. From this follows that the giant component has size \( \theta_m m(1 + o_p(1)) \). Let \( \xi \) be 1 if vertex
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ai is small and 0 otherwise. Then

\[ E[\xi_i Y] = E[\xi_i Y | \xi_i = 1] P\{\xi_i = 1\} + E[\xi_i Y | \xi_i = 0] P\{\xi_i = 0\} \]

\[ = \sum_{k=1}^{k_-} E[Y | |C(ai)| = k] P\{|C(ai)| = k\} \rho(m,p) \]

\[ \leq \rho(m,p) (k_- + (m - \mathcal{O}(k_-))\rho(m - \mathcal{O}(k_-),p)) \]

so we have

\[ E[Y^2] = \sum_{i=1}^{m} E[\xi_i Y] \leq m\rho(m,p)(k_- + m\rho(m - \mathcal{O}(k_-),p)) \]

\[ = m^2\rho(m,p)(o(1) + \rho(m - \mathcal{O}(k_-),p)) \]

\[ = m^2\rho(m,p)^2(1 + o_p(1)) \]

\[ = E[Y^2](1 + o_p(1)) \] (2.26)

Thus, \( \text{Var} Y = E[Y^2] - E[Y]^2 = o_p(E[Y]^2) \) and by (1.6), \( Y = E[Y](1 + o_p(1)) \).

Figure 2.1 shows \( \theta_m \) as a function of \( \lambda \in [0,2] \) in the case \( m = o(n) \). □

2.4 At criticality

We conclude this section with a quick look into the case \( \lambda = 1 \). Here, the theory becomes much more involved and the tools we used above will not be enough to prove any detailed results. We confine ourselves to showing that the largest component has \( o_p(m) \) even vertices. This is done by showing that increasing properties are more likely when \( p \) increases; see Section 1.3.2.

Lemma 10. Let \( 0 \leq p_1 \leq p_2 \leq 1 \), and let \( \mu_1, \mu_2 \) be the measures on \( \{0,1\}^{E_{m,n}} \) in which each edge in \( K_{m,n} \) is open with probability \( p_1, p_2 \) respectively, independently of all other edges. Then \( \mu_1 \preceq \mu_2 \).

Proof. The proof is a simple application of Holley’s theorem. Let \( x \in E_{m,n} \) and let \( \xi_1, \xi_2 \in \{0,1\}^{E_{m,n}\setminus\{x\}} \) be such that \( \xi_1 \preceq \xi_2 \). By the independence of edges under \( \mu_1, \mu_2 \), we have

\[ \mu_i \{X(x) = 1 | X = \xi_i \text{ off } x\} = \mu_i \{X(x) = 1\} = p_i, \quad i = 1,2 \] (2.28)

so \( p_1 \leq p_2 \) implies

\[ \mu_1 \{X(x) = 1 | X = \xi_1 \text{ off } x\} \leq \mu_2 \{X(x) = 1 | X = \xi_2 \text{ off } x\} \] (2.29)
and by Holley’s theorem, $\mu_1 \preceq D \mu_2$.

**Theorem 11.** Let $\varepsilon > 0$. When $\lambda = 1$, the graph $G(m,n;\lambda/\sqrt{mn})$ a.a.s. has no component with more than $\varepsilon m$ even vertices. In other words, the largest component has $o_p(m)$ even vertices.

**Proof.** Let $\varepsilon > 0$, and let $\lambda > 1$ be such that $\theta_m = \theta_m(\lambda) < \varepsilon$ with $\theta_m(\lambda)$ as in Theorem 9. When $m = o(n)$, such a $\lambda$ exists by the continuity of $(\lambda, \theta_m) \mapsto e^{-\lambda^2 \theta_m + \theta_m - 1}$, and since $\theta_m = 0$ is the only solution of $e^{-\theta_m} + \theta_m - 1 = 0$. When $m \sim n$, a similar argument is used to ensure the existence of $\lambda$.

Let $Q_\varepsilon$ be the increasing property of $G(m,n;p)$ having a component with at least $\varepsilon m$ even vertices. With notation as in the proof of Lemma 10, let $p_1 = 1/\sqrt{m}$ and $p_2 = \lambda/\sqrt{mn}$. Then $\mu_1 \preceq D \mu_2$ and in particular $\mu_1(Q_\varepsilon) \leq \mu_2(Q_\varepsilon)$. But by Theorem 9, we have $\mu_2(Q_\varepsilon) = o(1)$. Thus,

$$\mathbb{P}\left\{ G(m,n;1/\sqrt{mn}) \in Q_\varepsilon \right\} = \mu_1(Q_\varepsilon) = o(1)$$

and the largest component of $G(m,n;1/\sqrt{mn})$ has $o_p(m)$ even vertices.

**2.5 The number of odd vertices**

Now that results have been shown for the even vertices of $G(m,n;p)$, Section 2.1 is used to deduce the total size of the largest component. The results can be summarized in the following theorem.
2.6. Concluding remarks

**Theorem 12.**

(a) (Subcriticality) If \( \lambda < 1 \), then there is a.a.s. no component with more than \( \frac{\tau \lambda \log m}{(1 - \lambda)^2} \sqrt{\frac{m}{n}} \) vertices.

(b) (Criticality) If \( \lambda = 1 \), then the largest component has \( o_p(m) \) even vertices and \( o_p(\sqrt{mn}) \) vertices in total.

(c) (Supercriticality) If \( \lambda > 1 \), then there is a component with \( \theta_m m \) even vertices, where \( \theta_m \) is given by (2.22). If \( m = o(n) \), the total number of vertices is \( \lambda \theta_m \sqrt{mn}(1 + o_p(1)) \). If \( m = an \), \( a > 0 \), the total number of vertices is \( (\theta_m m + \theta_n n)(1 + o_p(1)) \), where \( \theta_m \) is given by (2.22) and \( \theta_n \) is the unique positive solution to \( \theta_n + \exp\left\{ \frac{\lambda \sqrt{a}}{\theta_n} \left( \exp\left\{ -\frac{\lambda \sqrt{a}}{\theta_n} \right\} - 1 \right) \right\} = 1 \). In each case, the second largest component a.a.s. has at most \( \frac{3 \log m}{(\lambda - 1)^2} \sqrt{\frac{m}{n}} \) vertices in total.

**Proof.** The statements about even vertices are Theorems 6, 9 and 11. Statements about odd vertices in (a) and (b) follow from Corollary 3. The size of \( \theta_n \) in the case \( m = an \) is found by considering the even vertices for \( m = \frac{1}{a}n \).

The following lemma about the relative number of even and odd vertices in the giant shall be needed in Section 4. The proof is left to Appendix A.

**Lemma 13.** Suppose \( a > 0 \), \( \lambda > 1 \), and let \( \theta_m, \theta_n \) be the unique solutions in \((0, 1)\) to \( \theta_m + \exp\left\{ \frac{\lambda \sqrt{a}}{\theta_m} \left( \exp\left\{ -\frac{\lambda \sqrt{a}}{\theta_m} \right\} - 1 \right) \right\} = 1 \) and \( \theta_n + \exp\left\{ \frac{\lambda \sqrt{a}}{\theta_n} \left( \exp\left\{ -\frac{\lambda \sqrt{a}}{\theta_n} \right\} - 1 \right) \right\} = 1 \) respectively. Then \( \theta_n \geq \min(1, a) \theta_m \).

2.6 Concluding remarks

In the complete random graph setting, authors (e.g. [10]) frequently mention the following heuristic: *A giant component appears when the average vertex degree becomes greater than one.* Led by this heuristic, I initially looked for a threshold function of either of the forms \( \lambda / \min\{m, n\} \), \( \lambda / (m + n) \) or \( \lambda / \max\{m, n\} \), or similar. However, I could not make any of the formulas fit into the branching process argument, so instead started to work my way through the methods used in [2]. These combinatorial proofs quickly showed that \( m \) and \( n \) tend to appear as a product, which suggested the somewhat surprising use of a geometric mean. It is worth noting that the threshold function \( 1 / \sqrt{mn} \) is not consistent with the heuristic above as it stands, but instead we have seen in this section that the following would be the corresponding heuristic: *A giant component appears when the average number of paths of length 2 from a vertex becomes greater than one.*
3

The even projection

The branching process argument of the previous section was accomplished by considering only the even vertices of $K_{m,n}$. The usual branching process was replaced by a double-jump process which instead of searching for edges, considered paths of length 2 between even vertices. This suggests a method in which a subgraph $G \subseteq K_m$ is obtained from a subgraph $H \subseteq K_{m,n}$ by removing the odd vertices of $H$ and assigning an edge to $G$ if and only if the corresponding vertices in $H$ are connected by a path of length 2. See Figure 3.1. We will call $G$ the even projection of $H$. By using known results about random subgraphs on $K_m$, one may draw conclusions about properties of random subgraphs of $K_{m,n}$.

There are two major problems with this method. Firstly, the edges of the even projection are highly dependent, limiting the compatibility with the usual random graph theory which requires independent edges. Secondly, some properties of the even projection are difficult to translate back to the bipartite setting. We will see that this method is good enough to prove the existence, but not uniqueness, of a giant component when $\lambda > 1$, and with help of Section 2 it is shown that for $\lambda$ large enough, the giant is cyclic.

It should also be mentioned that there are no significant results in this section for the case $m \sim n$, since Holley’s theorem, which the proofs rely heavily upon, cannot be applied in this case to the author’s knowledge.

3.1 The even projection

Given a subgraph $G = (A_m \cup B_n, E)$ of $K_{m,n}$, we define the even projection of $G$ to be the graph with vertex set $A_m$ and in which any edge $a_i a_j$ is included if and only if there is a $1 \leq k \leq n$ such that $a_i b_k$ and $a_j b_k$ are both in $E$. Further, define the random even projection, denoted $G_\eta^e(m)$, as the random graph obtained by taking the even projection of $G(m,n;p)$.

Equivalently, given $\eta \in \{0,1\}^{E_{m,n}}$, we define the even projection of $\eta$ through a
3.1. The even projection

mapping \( \phi : \{0,1\}^{E_{m,n}} \to \{0,1\}^{E_m} \) by \( \phi(\eta) = \xi \), where for all \( i,j \)

\[
\phi(\eta)(a_i a_j) = \xi(a_i a_j) = \max_{1 \leq k \leq n} \eta(a_i b_k) \eta(a_j b_k). \tag{3.1}
\]

We define \( \mu^p_n \) to be the measure on \( \{0,1\}^{E_m} \) associated with even projections, i.e. the measure which to any \( \xi \in \{0,1\}^{E_m} \) assigns probability \( \mu^p_n(\xi) = \nu(\phi^{-1}(\xi)) \), where \( \nu \) is the i.i.d. measure on \( \{0,1\}^{E_{m,n}} \) which assigns probability \( p \) to each edge.

The \( p \) is moved out of its usual position in the notation to avoid misunderstanding: as we shall see, \( G^p_n(m) \) does not have edge-probability \( p \). The notation \( \nu \) will be kept for the i.i.d. measure on \( K_{m,n} \). We shall not need the following result, but it serves as a warmup for what is to come, and motivates the comparison of \( \mu^p_n \) to an i.i.d. measure with approximate edge-probability \( \lambda^2/m \approx 1 - (1 - p^2)^n \).

**Proposition 14.** The probability that an edge in \( G^p_n(m) \) is open is given by

\[
\mu^p_n \{a_i a_j \text{ open}\} = 1 - (1 - p^2)^n \tag{3.2}
\]

for all \( 1 \leq i,j \leq m \).

**Proof.** Here we introduce the event \( \{a_1 b_1 a_2 \text{ open}\} = \{a_1 b_1 \text{ open}\} \cap \{a_2 b_1 \text{ open}\} \) and let \( \{a_1 b_1 a_2 \text{ closed}\} \) be the complementary event \( \{a_1 b_1 \text{ closed}\} \cup \{a_2 b_1 \text{ closed}\} \).

This and similar results will be proved by passing to the random bipartite graph \( G(m,n;p) \). From the definition of \( G^p_n(m) \), we have

\[
\mu^p_n \{a_i a_j \text{ open}\} = \nu \{\exists k : a_i b_k a_j \text{ open}\} = 1 - \nu \{\forall k : a_i b_k a_j \text{ closed}\} = 1 - \nu \{a_i b_1 a_j \text{ closed}\}^n = 1 - (1 - \nu \{a_i b_1, a_j b_1 \text{ open}\})^n = 1 - (1 - p^2)^n \tag{3.3}
\]

We define \( \mu_\varepsilon \) as the measure on \( \{0,1\}^{E_m} \) for which each edge is open with probability \( (\lambda^2 - \varepsilon)/m \), independently of all other edges. The goal of the following two lemmas is to show that \( \mu_\varepsilon \preceq_D \mu^p_n \) for large \( m \) using Holley’s Theorem. See Section 1.3.2.

**Figure 3.1:** Left: Example subgraph \( H \) of \( K_{4,5} \). Right: The even projection of \( H \).
3.1. The even projection

![Diagram](image)

**Figure 3.2:** Example configuration $\xi$ on $K_8$. We set $x = a_1a_2$ so that $d_1 = 2$, $d_2 = 3$ and $d = 1$.

**Lemma 15.** Let $x \in E_n$. Denote the endpoints of $x$ by $a_1$, $a_2$ and let $\xi \in \{0,1\}^{E_m \setminus \{x\}}$ be an edge-configuration on the complete graph in which $a_1, a_2$ have $d_1, d_2$ neighbours respectively, and there are $d$ vertices which are neighbours to both $a_1$ and $a_2$. Then

$$
\mu^\mu_n (X(x) = 1 \mid X = \xi \text{ off } x) \geq 1 - (1 - p)^{2d} \left(1 - \frac{p^2}{(1 + p)^{2m - d_1 - d_2 - 4}}\right)^{n - d_1 - d_2} \tag{3.4}
$$

with equality if $d = 0$. In particular, we have

$$
\mu^\mu_n (X(x) = 1 \mid X = 0 \text{ off } x) \leq \mu^\mu_n (X(x) = 1 \mid X = \xi \text{ off } x) \tag{3.5}
$$

for all $\xi \in \{0,1\}^{E_m \setminus \{x\}}$ and all $x \in E_m$.

**Proof.** Let $x$ be the edge $a_1a_2$ and fix $\xi \in \{0,1\}^{E_m \setminus \{x\}}$. For simplicity, we will make the assumption that $\xi(a_1a_j) = 0$ whenever $a_1a_j$ is not adjacent to $a_1a_2$. This can be done without loss of generality, since $a_1a_j$ is independent of $a_1a_2$ when the edges are not adjacent.

Suppose $\eta \in \{0,1\}^{E_m \setminus \eta}$ is such that the even projection of $\eta$ is $\xi$. Then $\eta$ must contain $d_1 - d$ odd vertices $b_k$ such that $a_1b_k, a_i$ are open for $i = i_1, i_2, \ldots, i_{d_1 - d}$, and $d_2 - d$ odd vertices $b_k$ such that $a_2b_k, a_j$ is open for $j = j_1, j_2, \ldots, j_{d_2 - d}$, where $i_1, \ldots, i_{d_1 - d}, j_1, \ldots, j_{d_2 - d}$ are all distinct and greater than 2. The $b_k$ must all be distinct, since otherwise the projection would contain an open edge $a_1a_j$ which is not adjacent to $a_1a_2$.

Furthermore, for each vertex $a_i$ such that $a_1a_i$ and $a_2a_i$ are both open in $\xi$, there must be odd vertices $b_{k_1}, b_{k_2}$, possibly equal, such that $a_1b_{k_1}, a_i$ and $a_2b_{k_2}, a_i$ are both open. Note that if $k_1 = k_2$, then $a_1b_{k_1}a_2$ will be open, and $X(x) = 1$ follows immediately. Thus we assume that whenever $a_1a_i$ and $a_2a_i$ are both open in $\xi$, there are distinct $b_{k_1}, b_{k_2}$ defined as above. Since there are $d$ vertices $a_i$ satisfying this, we require $d$ pairs of vertices $b_{k_1}, b_{k_2}$ in $\eta$ to be such that $a_1b_{k_1}, a_i$ and $a_2b_{k_2}, a_i$ are both open.

By the argument above, there is in some sense a minimal configuration $\eta$ which has even projection $\xi$ and has a nontrivial probability of having $X(x) = 1$. Without loss of generality, we can suppose that $\eta$ has a canonical form obtained as follows.

1. Let $a_1b_{k_1}a_{k_2}$ be open for $k_1 = 1, \ldots, d_1$. 

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A

Denote the set of edges opened in this process by common neighbours. If \( R \) for all \( n, \xi \) closed. Let \( R \) since \( a \in R \)

The same will hold for \( a \) and \( b \).

Figure 3.3: Configuration \( \eta \) on \( K_{8,9} \) obtained from \( \xi \) in figure 3.2 by the algorithm described in the proof of Lemma 15. The set of edges in this figure is denoted \( A \). No edges can be added to \( b_1, b_3 \) or \( b_4 \) without changing the projection of \( \eta \), whence \( R = \{2, 5, 6, 7, 8, 9\} \). Opening either one of the 2d dashed edges will make \( a_1a_2 \) open in the projection without affecting other edges. This can otherwise only be achieved by, for some \( k > 5 \), opening the two edges \( a_1b_k \) and \( a_2b_k \).

2. Let \( a_2b_k,a_{k+2} \) be open for \( k = d_1 + 1, \ldots, d_1 + d_2 - d \).

3. Let \( a_2b_k,a_{k+2-d_k} \) be open for \( k = d_1 + d_2 - d + 1, \ldots, d_1 + d_2 \).

Denote the set of edges opened in this process by \( A \). Also, let \( B = \{a_ib_k : 1 \leq k \leq n, \xi (ij) = 0\} \). Then

\[
\mu^n \{X(x) = 1 \mid X = \xi \text{ off} \} \geq \nu \{\exists r : a_1b_r,a_2 \text{ open} \mid A \text{ open}, B \text{ closed}\} \quad (3.6)
\]

where "\( A \) open" should be read as "\( \pi \) open for all \( \pi \in A \)" and "\( B \) closed" means "\( \pi \) closed for all \( \pi \in B \)". Here the inequality comes from how we assigned odd vertices to the \( d \) common neighbours. If \( d = 0 \), there is no such freedom of choice and equality must hold.

Note that for \( r = 1, \ldots, d_1 - d \), we have \( a_2b_r \) closed since \( a_2b_r,a_{r+2} \in B \) but \( a_1b_r,a_{r+2} \in A \). Similarly, \( a_1b_r,a_{r+2} \in B \) while \( a_2b_r,a_{r+2} \in A \) for \( r = d_1 + 1, \ldots, d_1 + d_2 - d \) so \( a_1b_r \) is closed. Let \( R = \{1, \ldots, n\} \setminus \{\{1, \ldots, d_1 - d\} \cup \{d_1 + 1, \ldots, d_1 + d_2 - d\}\} \). Then

\[
\nu \{\exists r : a_1b_r,a_2 \text{ open} \mid A \text{ open}, B \text{ closed}\} = 1 - \nu \{\forall r \in R : a_1b_r,a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} \quad (3.7)
\]

Let \( R'_1 = \{d_1 - d + 1, \ldots, d_1\} \), \( R''_1 = \{d_2 - d + 1, \ldots, d_2\} \), and \( R_2 = \{d_2 + 1, \ldots, n\} \). Then \( R = R'_1 \cup R''_1 \cup R_2 \), and the sets are distinct. In the first case, say \( r = r_0 \in R'_1 \), we have, since \( a_1b_{r_0}a_2 \)

\[
\nu \{a_1b_{r_0}a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} = 1 - \nu \{a_1b_{r_0}a_2 \text{ closed} \mid a_1b_{r_0}a_{r_0+2} \text{ open}, a_1b_{r_0}a_j \text{ closed}, j > d_1 + 2\}
\]

\[
= \nu \{a_2b_{r_0} \text{ closed} \mid a_{r_0+2}b_{r_0} \text{ open}, a_jb_{r_0} \text{ closed}, j > d_1 + 2\}
\]

\[
= 1 - p \quad (3.8)
\]

The same will hold for \( r_0 \in R''_1 \). Now let \( r_0 \in R_2 \). We have

\[
\nu \{a_1b_{r_0}a_2 \text{ closed} \mid A \text{ open}, B \text{ closed}\} = 1 - \nu \{a_1b_{r_0}a_2 \text{ open} \mid A \text{ open}, B \text{ closed}\}
\]

\[
= 1 - \nu \{a_1b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \nu \{a_2b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed}\} \quad (3.9)
\]
3.1. The even projection

The two factors are similar. We have

\[
\nu \{ a_1 b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed} \} = \nu \{ a_1 b_{r_0} \text{ open}, a_j b_{r_0} \text{ closed} \mid j > d_1 + 2 \}
\]

\[
= \nu \{ a_1 b_{r_0} \text{ open}, a_j b_{r_0} \text{ closed} \mid j > d_1 + 2 \}
\]

\[
= \nu \{ a_1 b_{r_0} \text{ open}, b_{r_0} \text{ closed} \mid j > d_1 + 2 \}
\]

\[
= \frac{p(1 - p)^{m-d_1-2}}{(1 - p^2)^{m-d_1-2}}
\]

\[
= \frac{p}{(1 + p)^{m-d_1-2}}
\]

(3.10)

and the same formula with \(d_2\) holds for \(a_2 b_{r_0}\). Thus

\[
1 - \nu \{ a_1 b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed} \} \nu \{ a_2 b_{r_0} \text{ open} \mid A \text{ open}, B \text{ closed} \}
\]

\[
= 1 - \left( \frac{p}{(1 + p)^{m-d_1-2}} \right) \left( \frac{p}{(1 + p)^{m-d_2-2}} \right)
\]

\[
= 1 - \frac{p^2}{(1 + p)^{2m-d_1-d_2-4}}
\]

(3.11)

Returning to (3.7), we obtain

\[
\mu_n \{ X(x) = 1 \mid X = \xi \text{ off } x \}
\]

\[
\geq 1 - \nu \{ \forall r \in R : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed} \}
\]

\[
\times \nu \{ \forall r \in R'_1 : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed} \}
\]

\[
\times \nu \{ \forall r \in R'_2 : a_1 b_r a_2 \text{ closed} \mid A \text{ open}, B \text{ closed} \}
\]

\[
= 1 - \nu \{ a_1 b_{r_1} a_2 \text{ closed} \mid A \text{ open}, B \text{ closed} \} \big|_{R'_1}^{|R'_1|}
\]

\[
\times \nu \{ a_1 b_{r_2} a_2 \text{ closed} \mid A \text{ open}, B \text{ closed} \} \big|_{R'_2}^{|R'_2|}
\]

\[
\times \nu \{ a_1 b_{r_2} a_2 \text{ closed} \mid A \text{ open}, B \text{ closed} \} \big|_{R_2}^{|R_2|}
\]

\[
= 1 - (1 - p)^{2d} \left( 1 - \frac{p^2}{(1 + p)^{2m-d_1-d_2-4}} \right)^{n-d_1-d_2}
\]

(3.12)

which is the desired result. In particular,

\[
\mu_n \{ X(x) = 1 \mid X = 0 \text{ off } x \} = 1 - \left( 1 - \frac{p^2}{(1 + p)^{2m-4}} \right)^n
\]

(3.13)

and it is not hard to see that

\[
\mu_n \{ X(x) = 1 \mid X = 0 \text{ off } x \} \leq \mu_n \{ X(x) = 1 \mid X = \xi \text{ off } x \}
\]

(3.14)
3.1. The even projection

The following lemma is a consequence of Lemma 15.

**Lemma 16.** Let \( m = o(n) \) and \( \varepsilon > 0 \). Let \( \mu_\varepsilon \) be the measure on \( \{0,1\}^{E_m} \) which assigns probability \( \frac{\lambda^2 - \varepsilon}{m} \) to each edge independently. Then for \( m \) large enough,
\[
\mu_\varepsilon \left\{ X(x) = 1 \mid X = 0 \text{ off } x \right\} \leq \mu_n^p \left\{ X(x) = 1 \mid X = 0 \text{ off } x \right\}, \quad \forall x \in E_m \tag{3.15}
\]

**Proof.** By Lemma 15 and the independence of edges under \( \mu_\varepsilon \), this is equivalent to
\[
\frac{\lambda^2 - \varepsilon}{m} \leq 1 - \left( 1 - \left[ \frac{p}{(1+p)^{m-2}} \right]^2 \right)^n \tag{3.16}
\]
We have \( (1+p)^m = 1 + \lambda \sqrt{\frac{m}{n}}(1 + o(1)) \), so
\[
1 - \left( 1 - \left[ \frac{p}{(1+p)^{m-2}} \right]^2 \right)^n = 1 - \left( 1 - \left[ \frac{p(1+p)^2}{1 + \lambda \sqrt{\frac{m}{n}}(1 + o(1))} \right]^2 \right)^n
\]
\[
= 1 - \left( 1 - \left[ \frac{\lambda \sqrt{mn}(1 + 2 \lambda \sqrt{mn} + \lambda^2)}{1 + \lambda \sqrt{\frac{m}{n}}(1 + o(1))} \right]^2 \right)^n
\]
\[
= 1 - \left( 1 - \left[ \frac{\lambda^2}{mn} \right]^n \right)
\]
\[
= \frac{\lambda^2}{m} + o \left( \frac{1}{m} \right) \tag{3.17}
\]
and so for \( m \) large enough, the result follows. \( \square \)

**Remark.** The inequality (3.16) in fact does not hold for any sequence \( m,n \) such that \( m > n \). This is the reason that we study only the even vertices, just as in Section 2.

**Theorem 17.** Let \( m = o(n) \). For any \( \varepsilon > 0 \) we eventually have \( \mu_\varepsilon \leq_\mathcal{D} \mu_n^p \).

**Proof.** Using the edge-independence of \( \mu_\varepsilon \) followed by Lemmas 16 and 15, we have for large \( m \)
\[
\mu_\varepsilon \left\{ X(x) = 1 \mid X = \xi \text{ off } x \right\} = \mu_\varepsilon \left\{ X(x) = 1 \mid X = 0 \text{ off } x \right\}
\]
\[
\leq \mu_n^p \left\{ X(x) = 1 \mid X = 0 \text{ off } x \right\}
\]
\[
\leq \mu_n^p \left\{ X(x) = 1 \mid X = \xi' \text{ off } x \right\} \tag{3.18}
\]
for any \( \xi, \xi' \in \{0,1\}^{E_m \setminus \{x\}} \). Thus, \( \mu_\varepsilon \leq_\mathcal{D} \mu_n^p \) follows from Holley’s theorem. \( \square \)
3.2 Applications of even projection

Theorem 17 means that any increasing property of the i.i.d. graph with edge-probability \((\lambda^2 - \varepsilon)/m\) will eventually be satisfied by the even projection. Properties of \(G(m,n;p)\) may then be concluded from properties of its even projection. In this section we state corollaries of Theorem 17. First, we reprove the existence of a giant component without using any results from Section 2.

**Corollary 18.** Let \(m = o(n)\). If \(\lambda > 1\), then there is a.a.s. a component in \(G(m,n;p)\) with at least \(\theta m\) even vertices, where \(\theta = \theta_\lambda(\lambda) \in (0,1)\) is the unique solution to \(\theta + e^{-(\lambda^2 - \varepsilon)\theta} = 1\).

**Proof.** Let \(\varepsilon > 0\) be constant and such that \(\lambda^2 - \varepsilon > 1\). For \(m,n\) large enough the even projection measure satisfies \(\mu_\varepsilon \geq D \mu_\varepsilon\) and it follows that if we define \(\theta_\varepsilon \in (0,1)\) to be the unique solution to \(\theta + e^{-(\lambda^2 - \varepsilon)\theta} = 1\),

\[
\mu_\varepsilon \{\text{Component of size } \geq \theta_\varepsilon m(1 + o(1))\} \\
\geq \mu_\varepsilon \{\text{Component of size } \geq \theta_\varepsilon m(1 + o(1))\} \\
= 1 - o(1)
\]

by Theorem 9. By the continuity of \(\varepsilon \mapsto \theta_\varepsilon\), we must have

\[
\mu_\varepsilon \{\text{Component of size } \geq \theta_m m(1 + o(1))\} \rightarrow 1
\]

Thus the even projection has a component with at least \(\theta_m m(1 + o(1))\) even vertices, which means that the random bipartite graph must have at least \(\theta_m m(1 + o(1))\) even vertices which belong to the same component. 

Note that without using results from Section 2, this corollary only shows the existence of a large component, and it is possible that there is more than one component with \(\Theta(m)\) even vertices. However, results obtained from Theorem 17 typically show properties for any large component, and using results from Section 2 will then imply that the result holds for the unique giant component.

While reverting from the projected graph to \(G(m,n;p)\) without losing too much information was possible in Corollary 18, the statement of the following theorem is significantly weakened compared to the corresponding statement for complete graphs. The reason for this is, loosely speaking, that the property of having a giant component is a property relating to connectivity of vertices, while the property of containing a cycle relates to adjacency of edges.

**Theorem 19.** Let \(m = o(n)\). Then there exists a \(\lambda_0\) such that if \(\lambda > \lambda_0\), then a.a.s. the giant component in \(G(m,n;p)\) contains a cycle.

**Proof.** Let \(\delta, \varepsilon > 0\). According to [1], there is a \(\lambda_0 > 1\) such that the giant component of \(G(m,(\lambda^2 - \varepsilon)/m)\) contains a cycle of length at least \(\delta m\) for all \(\lambda > \lambda_0\). Since \(\mu_\varepsilon \leq D \mu_\varepsilon\)
for large $m$, there must be a large component with a cycle of length at least $\delta m$ in the even projection of $G(m,n; p)$.

Suppose there is no cycle in the giant component of $G(m,n; p)$. Then, since the even projection has a cycle of length $\delta m$, there must be an odd vertex $b_0$ such that $\deg b_0 \geq \delta m$, as this is the only way to obtain cycles in the even projection from an acyclic configuration. But we have $\deg b_0 \in Bi(m,p)$, so

$$
\nu \{ \exists k : \deg b_k \geq \delta m \} = 1 - \nu \{ \forall k : \deg b_k < \delta m \} = 1 - \nu \{ \deg b_1 < \delta m \}^n = 1 - \left( 1 - \sum_{j=\delta m}^{m} \binom{m}{j} p^j (1 - p)^{m-j} \right)^n = 1 - (1 - o(1/n))^n = o(1) \quad (3.21)
$$

so there must $a.a.s.$ be a cycle in the giant component of $G(m,n; p)$, if we can show that the sum is indeed $o(1/n)$. But

$$
n \sum_{j=\delta m}^{m} \binom{m}{j} p^j (1 - p)^{m-j} \leq \max_{j \geq \delta m} mn \binom{m}{j} p^j (1 - p)^{m-j} \leq \max_{j} \frac{m^j \lambda^j}{j! \sqrt{mn^j}} \leq \max_{j} \frac{m^{j+1} n \lambda^m}{(\delta m)! \sqrt{mn^j}} \leq \max_{j} \frac{m^{j+1} n \lambda^m}{(\delta m)^{\delta m} e^{-\delta m} \sqrt{mn^j}} \leq \frac{n^3 \lambda^m e^{\delta m}}{(\delta m)^{\delta m}} = o(1) \quad (3.22)
$$

and the result follows.
The random-cluster model

This section proves the existence of a giant component for the random-cluster model $G(m,n;p,q)$ with parameter $q \geq 1$ and $p = \lambda / \sqrt{mn}$. When $1 \leq q \leq 2$ and $m = o(n)$, a sharp threshold value of $\lambda$ is found for the existence, while for other cases partial results are found. Most importantly, a sharp threshold value is found for the Ising model when $m = o(n)$. The proof techniques imitate those of [3] closely, and this section shows that some of the reduction arguments used there apply directly to the bipartite setting.

We recall from Section 1.4.3 that the random-cluster model with parameter $q > 0$ assigns probability proportional to

$$\tilde{P}(F; E, p, q) = p^{|F|}(1 - p)^{|E| - |F|}q^{c(V,F)}$$

(4.1)

to any subgraph $(V,F)$ of $(V,E)$ which has $c(V,F)$ connected components.

4.1 Colouring argument

The proof techniques in [3] depend strongly on a colouring argument, which reduces the study of $G(m,n;p,q)$ to that of $G(M,N;p,1)$ for some random numbers $M,N$. Fix $0 \leq r \leq 1$. Given a random graph $G(m,n;p,q)$, each component is coloured red with probability $r$ and green with probability $1 - r$. The components are coloured independently of each other. Let the union of the red (green) components be the red (green) subgraph, and let $R$ denote the vertex set of the red subgraph. The following lemma relates $G(m,n;p,q)$ to a random-cluster graph with parameter $rq$.

Lemma 20. Let $V_1$ be a subset of $V_{m,n}$ with $m_1$ even and $n_1$ odd vertices. Conditional on $R = V_1$, the red subgraph of $G(m,n;p,q)$ is distributed as $G(m_1,n_1;p,rq)$ and the green subgraph as $G(m - m_1,n - n_1;p,(1-r)q)$; furthermore, the red subgraph is conditionally independent of the green subgraph.
4.2. Existence of a giant component

4. THE RANDOM-CLUSTER MODEL

Proof. Set \( V_2 = V \setminus V_1, m_2 = m - m_1, n_2 = n - n_1, \) and let \( E_1, E_2 \subseteq E_{m,n} \) be such that edges in \( E_i \) have endpoints in \( V_i \) only, \( i = 1, 2 \). Write \( c(V,E) \) for the number of components of a graph with vertex set \( V \) and edge set \( E \). Then \( c(V, E_1 \cup E_2) = c(V_1, E_1) + c(V_2, E_2) \) since \( E_1 \) and \( E_2 \) define disjoint components, and the probability that the red subgraph is \( (V_1, E_1) \) and the green subgraph is \( (V_2, E_2) \) satisfies

\[
P \{ G(m,n; p,q) = (V, E_1 \cup E_2) \text{ and } R = (V_1, E_1) \} = \left( \frac{p|E_1 \cup E_2| (1 - p)^{mn - |E_1 \cup E_2|} q^{c(V, E_1 \cup E_2)}}{Z(m,n; p,q)} \right) r^{c(V_1, E_1)} (1 - r)^{c(V_2, E_2)}
\]

Writing \( (V_1, E_1) \) satisfies

\[
C(m,n,p,q,m_1,n_1) p^{|E_1|} (1 - p)^{m_1 n_1 - |E_1|} (qr)^{c(V_1, E_1)}
\]

\[
\times p^{|E_2|} (1 - p)^{m_2 n_2 - |E_2|} (q(1 - r))^{c(V_2, E_2)}
\]

\[
= C(m,n,p,q,m_1,n_1) \mathbb{P} \{ E_1; V_1, p, q \} \mathbb{P} \{ E_2; V_2, p, (1 - r)q \}
\]

for some positive real \( C(m,n,p,q,m_1,n_1) \) depending only on \( m, n, p, q, m_1 \) and \( n_1 \). Hence, conditional on \( R = V_1 \) and the green subgraph being \( (V_2, E_2) \), the probability that the red subgraph is \( (V_1, E_1) \) is precisely \( \mathbb{P} \{ E_1; V_1, p, q \} \). \( \Box \)

Writing \( (M,N) = (m_1,n_1) \) for the number of red vertices, \( R \) is thus distributed as a random graph \( G(M,N; p, q) \) on a random number of vertices. In particular, it is distributed as \( G(M,N, p, 1) \) if \( q \geq 1 \) and \( r = q^{-1} \). By using distributional properties of \( M,N \) and using results from earlier sections, one may deduce properties of \( G(m,n; p,q) \).

4.2 Existence of a giant component

Imitating [3] further, we start the search for a giant component by proving the following lemma. We assume throughout this section that there is an \( a > 0 \) such that \( m \leq an \).

Lemma 21. Let \( q \geq 1 \). For any \( \lambda \neq 1 \), a.a.s. \( \mathbb{G}(m,n; \lambda/\sqrt{mn}, q) \) has at most one component such that at least one of the following happens:

- The component has at least \((mn)^{1/3}\) vertices
- The component has at least \(m^{3/4}\) even vertices
- The component has at least \(n^{3/4}\) odd vertices

Proof. Let \( L_{m,n,p,q} \) be the number of components of \( \mathbb{G}(m,n; p,q) \) having at least \((mn)^{1/3}\) vertices, or at least \(m^{3/4}\) even vertices. Suppose \( L_{m,n,p,q} \geq 2 \), and pick two of these in some arbitrary way. With probability \( r^2 \) both of these are coloured red. Setting \( r = q^{-1} \), we find that

\[
r^2 \mathbb{P} \{ L_{m,n,p,q} \geq 2 \} \leq \sum_{k,l} \mathbb{P} \{ L_{k,l,p,1} \geq 2 \} \mathbb{P} \{ |R| = (k,l) \}
\]

\[
\leq \max_{k,l} \mathbb{P} \{ L_{k,l,p,1} \geq 2 \} \to 0 \quad \text{as } m \to \infty \quad (4.3)
\]
4.2. Existence of a giant component

from which follows that a.a.s. \( L_{m,n,p,q} \leq 1 \), using known results about \( L_{k,l;p,1} \).

The last statement follows from \( (mn)^{1/3} \leq n^{3/4} \) for large \( m,n \), which follows from the assumption that \( m \leq an \) for some \( a > 0 \).

Let \( \Theta_m \in (0,1] \) be the maximal number such that a component has \( \Theta_m \) even vertices, and let \( \Theta_n \in (0,1] \) be the maximal number such that a component has \( \Theta_n \) odd vertices. Note that we need not (a priori) have a component with \( \Theta_m \Theta_n \). This issue is resolved in Theorem 28.

Lemma 22. Let \( q \geq 1 \) and \( r = q^{-1} \). With probability \( r \), \( G(m,n;p,q) \) has \( \Theta_m \) even red vertices, of which \( \Theta_m \) belong to the component with the most even vertices. With probability \( 1 - r \), there are \( r(1 - \Theta_m) \) even vertices and no component with more than \( m^{3/4} \) even vertices.

The same statements with \( m \) replaced by \( n \) hold for the odd case.

Proof. Suppose \( G(m,n;p,q) \) has \( \kappa \) components, and suppose that the components have \( \Theta_m, v_2, v_3, ..., v_\kappa \) even vertices. Call any even vertex not in the component with \( \Theta_m \) even vertices “small”. By the lemma above, we have \( \max v_i \leq m^{3/4} \). Conditional on \( \Theta_m \), the expected number of small even red vertices is

\[
\sum_{i=2}^{\kappa} v_ir = r(1 - \Theta_m)m
\]

and the variance is given by

\[
\sum_{i=2}^{\kappa} v_i^2r(1 - r) \leq \sum_{i=2}^{\kappa} v_i^2 \leq m \max v_i \leq m^{7/4}
\]

Hence there are \( r(1 - \Theta_m) \) small red even vertices by (1.6). The component which has \( \Theta_m \) even vertices is red with probability \( r \), and the result follows.

The same argument applies to the odd vertices, with \( \max v_i \leq n^{3/4} \).

Lemma 23. If \( m = o(n) \), then a.a.s. there is no component with at least \( \delta n \) odd vertices for any \( \delta > 0 \). In other words, we have \( \Theta_n \xrightarrow{P} 0 \) and \( N = rn(1 + o_p(1)) \).

Proof. Let \( \delta > 0 \) and let \( K_{m,n,p,q} \) be the number of components in \( G(m,n;p,q) \) which have at least \( \delta n \) odd vertices. Note that a.a.s. \( M = o(N) \) by Lemma 22, and we need only consider \( G(k,l;p,1) \) for \( k,l \) such that \( k = o(l) \). Thus,

\[
rP \{ K_{m,n,p,q} \geq 1 \} \leq \sum_{k=o(l)} P \{ K_{k,l,p,1} \geq 1 \} P \{|R| = (k,l)|
\leq \max_{k=o(l)} P \{ K_{k,l,p,1} \geq 1 \} \rightarrow 0
\]

which follows from earlier sections.
Lemma 23 cannot be applied. Let $\theta = \Theta_{m,n} \geq \theta_0$ for $G(m,n; \lambda/\sqrt{mn}, q)$.

(b) Let $m = an$, $a > 0$. If $\lambda > q \geq 1$, then there exists a $\theta_0 > 0$ such that a.a.s. $\Theta_m \geq \theta_0$ and $\Theta_n \geq \theta_0$ for $G(m,n; \lambda/\sqrt{mn}, q)$.

**Proof.** (a) For $q = 1$, the assertion was shown in Theorem 9. Hence assume $q > 1$ and thus $r = q^{-1} < 1$.

Let $\theta_0 = 1 - \left(\frac{\lambda + q}{2\lambda}\right)^2$, $\pi_m = P\{\Theta_m < \theta_0\}$, and $\varepsilon > 0$. By considering the event that the component with $\Theta_m m$ even vertices is coloured green we see that, with probability at least $(1-r)\pi_m + o(1)$, the number $M$ of red even vertices satisfies $M \geq r(1-\theta_0)m - \varepsilon m$, and there are no red components with at least $m^{3/4}$ even vertices. When this happens,

$$
\sqrt{MNp} \geq \sqrt{r(1-\theta_0)m - \varepsilon m} \sqrt{rn + o_p(n) \lambda \sqrt{mn}}
$$

$$
= \lambda \sqrt{r^2(1-\theta_0) - r\varepsilon + o_p(1)}
$$

$$
= \sqrt{\left(\frac{\lambda}{2q}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{\lambda}{2q} - r\varepsilon + o_p(1)}
$$

$$
> 1 \quad (4.7)
$$

for $m$ large enough with an appropriate choice of $\varepsilon > 0$. Thus the red subgraph is a supercritical Erdős-Rényi graph, and by Theorem 9 has a component with at least $\delta M \geq \delta(r(1-\theta_0) - \varepsilon)m$ even vertices for some $\delta > 0$. But this is eventually larger that $m^{3/4}$, so we must have $(1-r)\pi_m \to 0$ as $m \to 0$, and in particular $P\{\Theta_m < \theta_0\} \to 0$.

(b) The case $m = an$ is similarly handled, but more work needs to be done since Lemma 23 cannot be applied. Let $\theta_1 = 1 - \frac{\lambda + q}{2\lambda}$, $\pi_m = P\{\Theta_m < \theta_1\}$, $\pi_n = P\{\Theta_n < \theta_1\}$ and $\varepsilon > 0$. Arguing as in the previous proof, with probability $(1-r)^2\pi_m \pi_n + o(1)$ we have $M \geq r(1-\theta_1)m - \varepsilon m$ and $N \geq r(1-\theta_1)n - \varepsilon n$ and no red component with more than $m^{3/4}$ even vertices or more than $n^{3/4}$ odd vertices. When this happens,

$$
\sqrt{MNp} \geq \sqrt{r(1-\theta_1)m - \varepsilon m} \sqrt{r(1-\theta_1)n - \varepsilon n} \frac{\lambda}{\sqrt{mn}}
$$

$$
= \lambda (r(1-\theta_1) - \varepsilon)
$$

$$
= \frac{\lambda}{2q} + \frac{1}{2} - \varepsilon
$$

$$
> 1 \quad (4.8)
$$

for $m$ large enough. As above, it follows that $\pi_m \pi_n \to 0$, i.e. $\pi_m$, $\pi_n$ or both go to zero.

Suppose $\pi_m \to 0$. The other case is handled identically. We will show that $\Theta_n \geq \min(a, q^{-1})\Theta_m$. Define $\Theta_M$ as the proportion of red even vertices that belong to the red component with the most even vertices, and define $\Theta_N$ similarly. These numbers must satisfy $\Theta_M M = \Theta_m m$ and $\Theta_N N = \Theta_n n$. By Lemma 13 we a.a.s. have $\Theta_N \geq \min(A,1)\Theta_M$, where $A = M/N$. We have
4.3 Size of the giant component

\[ \Theta_n = \frac{\Theta N}{n} \geq \frac{N}{n} \min(A,1) \frac{\Theta_m}{M} \]  

(4.9)

but noting that \( N/M = 1/A \) and \( m/n = a \), we get

\[ \Theta_n \geq \min \left( a, \frac{a}{A} \right) \Theta_m. \]  

(4.10)

We a.a.s. have \( M \leq m \) and \( N \geq nq = n/q \), so \( A \leq qa \) and

\[ \Theta_n \geq \min(a,q^{-1}) \Theta_m. \]  

(4.11)

So letting \( \theta_0 = \min(a,q^{-1}) \theta_1 \), we a.a.s. have \( \Theta_m \geq \theta_0 \) and \( \Theta_n \geq \theta_0 \).

\[ \square \]

4.3 Size of the giant component

Given \( \lambda > 0 \) and \( a > 0 \), we define the functions \( \varphi_\lambda : \mathbb{R} \to \mathbb{R} \) and \( \psi_{\lambda,a} : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[ \varphi_\lambda(\theta) = e^{-\frac{\lambda^2}{q} \theta} - \frac{1 - \theta}{1 + (q-1)\theta} \]  

(4.12)

and

\[ \psi_{\lambda,a}(\theta_1,\theta_2) = \left( \exp \left\{ \frac{\lambda \sqrt{a}}{q} (1 + (q-1)\theta_2) \left( \exp \left\{ -\frac{\lambda}{\sqrt{a}} \theta_1 \right\} - 1 \right) \right\} - \frac{1-\theta_1}{1+\theta_1} \right) \]  

(4.13)

Lemma 25.

(a) Let \( m = o(n) \). If \( q \geq 1 \), then for any sequence \( \lambda = \lambda_m \) we have \( \varphi_\lambda(\Theta_m) \xrightarrow{P} 0 \).

(b) Let \( m = an \), \( a > 0 \). If \( q \geq 1 \), then for any sequence \( \lambda = \lambda_m \) we have \( \psi_{\lambda,a}(\Theta_m,\Theta_n) \xrightarrow{P} (0,0) \).

Proof. (a) For \( G(m,n;p,1) \), we have shown (Theorems 6, 9, 11) that for constant \( \lambda \in [0,\infty) \)

\[ e^{-\lambda^2 \Theta_m} + \Theta_m - 1 \xrightarrow{P} 0 \]  

(4.14)

Define \( \Theta_m = 1 \) if \( \lambda = \infty \), so that the convergence holds for all sequences in the compact set \([0,\infty]\). Let \( \lambda_m \) be one such sequence. By looking down convergent subsequences of \( \lambda_m \), we have

\[ e^{-\lambda_m^2 \Theta_m} + \Theta_m - 1 \xrightarrow{P} 0 \]  

(4.15)

If we apply this to the red subgraph, distributed as \( G(M,N;p,1) \), conditional on the component with \( \Theta_m \) even vertices being red we have for any sequence \( p_M \)

\[ e^{-p_M^2 \Theta_m MN} + \Theta_m - 1 \xrightarrow{P} 0 \]  

(4.16)
4.3. Size of the giant component

Since \( m = o(n) \), we have \( N = rn(1 + o_p(1)) \) by Lemma 23 and

\[
\exp\left\{ -\frac{\lambda^2_m}{q} \Theta_m \right\} = \frac{1 - \Theta_m}{1 + (q - 1)\Theta_m} - \frac{\Theta_m m}{1 + (q - 1)\Theta_m}
\]

\[
= \exp\left\{ -\frac{\lambda^2_m m}{n} \Theta_m \right\} + \frac{\Theta_m m}{1 + (q - 1)\Theta_m} - 1
\]

\[
= \exp\left\{ -\frac{\lambda^2_m, N(1 + o_p(1))}{n} \Theta_m \right\} + \frac{\Theta_m m}{(r + (1 - r)\Theta_m)m} - 1
\]  \hspace{1cm} (4.17)

Let \( p_m \) be such that \( \lambda_m = p_m \sqrt{mn} \). Then this equals

\[
\exp\left\{ -p_m^2 m N(1 + o_p(1)) \right\} + \frac{\Theta_m m}{M} - 1
\]

\[
\rightarrow 0
\]  \hspace{1cm} (4.18)

where \( p_M = p_m(1 + o_p(1)) \) is some sequence, and the result follows.

(b) Likewise, let \( m = an, a > 0 \). For \( q = 1 \), we have shown in Theorem 9 that for all sequences \( \lambda_M = p_M \sqrt{MN} \), the red subgraph \( G(M,N; \lambda_M/\sqrt{MN},1) \) must satisfy

\[
\Theta_M + \exp\left\{ \lambda_M \sqrt{\frac{N}{M}} \left( \exp\left\{ -\lambda_M \sqrt{\frac{M}{N}} \Theta_M \right\} - 1 \right) \right\} - 1 \rightarrow 0
\]  \hspace{1cm} (4.19)

Note that \( N = \frac{1}{q} (1 + (q - 1)\Theta_n) n \). For any sequence \( \lambda_m = p_m \sqrt{mn} = p_m m / \sqrt{a} = p_m n / \sqrt{a} \), we have

\[
\exp\left\{ \frac{\lambda_m}{q\sqrt{a}} (1 + (q - 1)\Theta_n) \left( \exp\left\{ -\lambda_m \sqrt{\Theta_m} \right\} - 1 \right) \right\} - \frac{1 - \Theta_m}{1 + (q - 1)\Theta_m}
\]

\[
= \exp\left\{ p_m \sqrt{a} \left( \exp\left\{ -p_m \Theta_m m \right\} - 1 \right) \right\} - \frac{1 - \Theta_m}{1 + (q - 1)\Theta_m}
\]

\[
= \frac{\Theta_m m}{(r + (1 - r)\Theta_m)m} + \exp\left\{ p_m \sqrt{a} \left( \exp\left\{ -p_m \Theta_m M \right\} - 1 \right) \right\} - 1
\]

\[
= \frac{\Theta_m m}{M} + \exp\left\{ p_m \sqrt{MN} \sqrt{\frac{N}{M}} \left( \exp\left\{ -p_m \sqrt{MN} \frac{M}{N} \Theta_M \right\} - 1 \right) \right\} - 1
\]

\[
\rightarrow 0
\]  \hspace{1cm} (4.20)

which follows from (4.19) by letting \( \lambda_M = p_m \sqrt{MN} \). The other part of the statement is proved similarly.

Because of the relative simplicity of \( \varphi_\lambda \), we may make more detailed claims about the giant component in the case \( m = o(n) \). For \( m = an \) however, the complexity of \( \psi_{\lambda,a} \) prevents us from proving more results at this moment.

To study \( \varphi_\lambda \), we define \( f(\theta) = \left( \frac{\lambda}{2} \log(1 + (q - 1)\theta) - \log(1 - \theta) \right) \) for \( 0 < \theta < 1 \), and note that \( \theta \) satisfies \( \varphi_\lambda(\theta) = 0 \) if and only if \( f(\theta) = \lambda \). Analytical properties of \( f \) are (indirectly) proved in [3], and we shall not reprove them here.
Theorem 26. Let $m = o(n)$.

(a) If $1 \leq q \leq 2$ and $\lambda < q$, or if $q > 2$ and $\lambda < \lambda_{\text{min}}$ where $\lambda_{\text{min}}$ is the unique minimum of $f(\theta)$, then $\Theta_m \xrightarrow{P} 0$ as $m \to \infty$.

(b) If $q \geq 1$ and $\lambda > q$, then $\Theta_m \xrightarrow{P} \theta(\lambda)$ where $\theta(\lambda)$ is the unique positive solution to $e^{-\lambda^2 \theta/q} = \frac{1-\theta}{1+(q-1)\theta}$.

Proof. The function $\varphi_\lambda$ is continuous on $[0,1]$. Let $Z_\lambda$ denote the set of zeros of $\varphi_\lambda$ for a fixed $\lambda$. By Lemma 25, we have $P \{ \Theta_m \in Z_\lambda + (-\varepsilon,\varepsilon) \} \to 1$ as $m \to \infty$. By [3], we have $Z_\lambda = \{0\}$ when $1 \leq q \leq 2$ and $\lambda < q$ or $q > 2$ and $\lambda < \lambda_{\text{min}}$, which implies $\Theta_m \xrightarrow{P} 0$. This is (a).

When $\lambda > q$ we have, again by [3], $Z_\lambda = \{0,\theta(\lambda)\}$. By Lemma 24, $P \{ \Theta_m > \delta \} \to 1$ for some $\delta > 0$. Thus, the only possibility is $\Theta_m \xrightarrow{P} \theta(\lambda)$. This is (b).  

When $m = o(n)$, Theorem 26 gives a sharp value of the threshold $\lambda_c(q)$ for $1 \leq q \leq 2$, while for $q > 2$ all we know is $\lambda_{\text{min}} \leq \lambda_c(q) \leq q$. Getting more detailed than this when $q > 2$ is likely to be much more involved than the proofs presented here, judging by the complexity of the proof in [3].

We have seen above that if $m = o(n)$, then $\Theta_n \xrightarrow{P} 0$. The following results looks closer at this convergence in the supercritical regime.

Theorem 27. Let $m = o(n)$, and suppose $q \geq 1$ and $\lambda > q$. Then $\Theta_n = \frac{\lambda^2(\theta(\lambda))}{q} \sqrt{n} (1 + o_p(1))$ where $\theta(\lambda)$ is the unique positive solution to $e^{-\lambda^2 \theta/q} = \frac{1-\theta}{1+(q-1)\theta}$.

Proof. We have seen in Theorem 26 that there a.a.s. is a component with $\Theta_m m$ even vertices, where $\Theta_m \xrightarrow{P} \theta$. With probability $r$, this component is coloured red, and by Lemma 22 the red subgraph is distributed as $G((r + (1-r)\theta)m, q, \lambda/\sqrt{m}, 1)$. To make this fit into Corollary 3 we make a change of variables.

Put $m' = (r + (1-r)\theta)m, n' = rn$. Then we have a component with $(r + (1-r)\theta)^{-1} \theta m'$ even vertices in the graph $G(m', n', \lambda \sqrt{r + (1-r)\theta} / \sqrt{m' n'}, 1)$. By Corollary 3, this component has

$$\frac{\lambda \sqrt{r + (1-r)\theta} / \sqrt{m' n'}}{r + (1-r)\theta} n' (1 + o_p(1))$$

odd vertices.

Using the proofs in this section, the following theorem resolves the issue described before Lemma 22. We show that the unique component containing a positive fraction of the even vertices must be the component with the most odd vertices.
4.4 Conclusion

We conclude this section by stating the collective results in a theorem.

**Theorem 29.** Suppose \( q \geq 1 \).

(i) Let \( m = o(n) \).

(a) (Subcritical) If \( 1 \leq q \leq 2 \) and \( \lambda \leq q \) or if \( q > 2 \) and \( \lambda < \lambda_{\min} \), then every component of \( G(m,n;\lambda/\sqrt{mn},q) \) has \( o_p(m) \) even vertices and \( o_p(n) \) odd vertices.

(b) (Supercritical) If \( q \geq 1 \) and \( \lambda > q \), then \( G(m,n;\lambda/\sqrt{mn},q) \) has a giant component with \( \lambda^\theta \sqrt{mn}(1 + o_p(1)) \) vertices, of which \( \theta m \) are even, where \( \theta \) is the unique positive solution to
\[
e^{-\lambda^2 \theta/q} = \frac{1-\theta}{1+q-1\theta}.
\]

(ii) Let \( m = an \), \( a > 0 \). Then a.a.s. the largest component of \( G(m,n;p,q) \) has \( \theta m m + \theta n n \) vertices, of which \( \theta m m \) are even and \( \theta n n \) odd, for some solution \( (\theta m, \theta n) \) to
\[
\psi_{\lambda,a}(\theta m, \theta n) = 0,
\]
with \( \psi_{\lambda,a} \) as in (4.13). If \( \lambda > q \), then \( \theta m > 0 \) and \( \theta n > 0 \).

In particular, we have the following important corollary concerning the Ising model. Note that no information is given in (b) for the case \( \lambda < 2 \).

**Corollary 30.** (a) If \( m = o(n) \), then under the Ising model at inverse temperature \( \beta = -\frac{1}{2} \log \left( 1 - \frac{\lambda}{\sqrt{mn}} \right) \) there is a.a.s. a giant component if and only if \( \lambda > 2 \). The giant component has \( \frac{\lambda^2}{2} \sqrt{mn}(1 + o_p(1)) \) vertices, of which \( \theta m (1 + o_p(1)) \) are even, where \( \theta \) is the unique positive solution to
\[
e^{-\lambda^2 \theta/q} = \frac{1-\theta}{1+q-1\theta}.
\]

(b) If \( m = an \), \( a > 0 \), the Ising model at inverse temperature \( \beta = -\frac{1}{2} \log \left( 1 - \frac{\lambda}{\sqrt{mn}} \right) \) a.a.s. has a giant component if \( \lambda > 2 \). The giant component has \( \theta m (1 + o_p(1)) \) even vertices and \( \theta n (1 + o_p(1)) \) odd vertices, where \( (\theta m, \theta n) \) is some pair of positive numbers satisfying
\[
\psi_{\lambda,a}(\theta m, \theta n) = 0.
\]

**Proof.** This follows immediately from Theorem 29 and the correspondence between the random-cluster model and the Ising model explained in Section 1.4.3. \( \square \)
Technical proofs

This appendix is intended to spell out the full proofs of some technical lemmas, which were left unproved so as to not clog up the main text. The lemmas will be restated and proved.

Lemma 4. Let $\lambda < 1$, $m_1 \leq m_2 \leq n$ and $p_1 = \frac{\lambda}{\sqrt{m_1 n}}$, $p_2 = \frac{\lambda}{\sqrt{m_2 n}}$. Then

$$
\left(1 - p_1 + p_1 \left(1 - p_1 + \frac{p_1}{\lambda}ight)^{m_1}\right)^n \leq \left(1 - p_2 + p_2 \left(1 - p_2 + \frac{p_2}{\lambda}ight)^{m_2}\right)^n \quad (2.2)
$$

Proof. Setting $b = \sqrt{n}$ and $x = m$, the result follows if we show that

$$
f(x) = -\frac{\lambda}{\sqrt{xb}} + \frac{\lambda}{\sqrt{xb}} \left(1 + \frac{1 - \lambda}{\sqrt{xb}}\right)^x
$$

is increasing for integers $x$ satisfying $1 \leq x \leq b^2$. Using the binomial expansion rule, this equals

$$
f(x) = \frac{\lambda}{\sqrt{xb}} \sum_{k=1}^{x} \left(1 - \frac{\lambda}{\sqrt{xb}}\right)^k \binom{x}{k} = \sum_{k=1}^{x} \frac{\lambda(1 - \lambda)^k (x)_k}{(\sqrt{xb})^{k+1} k!} \quad (A.2)
$$

Here $(x)_k = x(x-1)...(x-k+1)$. This is clearly increasing if each of the terms is increasing, i.e. if $(x)_k/((\sqrt{x})^{k+1})$ is increasing. Since $x \in \mathbb{N}$, this is equivalent to

$$
\frac{(x)_k}{\sqrt{x}^{k+1}} \leq \frac{(x+1)_k}{\sqrt{x+1}^{k+1}}, \quad \text{for all } x \geq k \geq 1 \quad (A.3)
$$

or in other words, using the definition of $(x)_k = x(x-1)...(x-k+1)$,

$$
\frac{x-k+1}{\sqrt{x}^{k+1}} \leq \frac{x+1}{\sqrt{x+1}^{k+1}}, \quad x \geq k \geq 1 \quad (A.4)
$$
The above is equivalent to
\[
\left(\frac{1-k}{2}\right) \log(x+1) + \left(\frac{k+1}{2}\right) \log(x) - \log(x-k+1) \geq 0, \quad x \geq k \geq 1 \quad (A.5)
\]
Denote the left-hand expression by \( g_k(x) \). The result will follow by showing that \( g_k(x) \) is decreasing and has limit 0 as \( x \to \infty \). We have
\[
g_k'(x) = \frac{1}{x+1} \left( \frac{1-k}{2} \right) + \frac{1}{x} \left( \frac{k+1}{2} \right) - \frac{1}{x-k+1}
\]
\[
= \frac{x(x-k+1)(1-k) + (x+1)(x-k+1)(k+1) - 2x(x+1)}{2x(x+1)(x-k+1)}
\]
\[
= \frac{x - kx - k^2 + 1}{2x(x+1)(x-k+1)}
\]
\[
= -\frac{x(k-1) + (k^2 - 1)}{2x(x+1)(x-k+1)}
\]
\[
\leq 0, \quad \forall x \geq k \geq 1 \quad (A.6)
\]
We also have
\[
g_k(x) = \log \left[ \frac{x^{k+1}}{(x+1)^{k+1}(x-k+1)} \right]
\]
\[
= \log \left[ \frac{1}{(1+x^{-1})^{k+1}(1-(k-1)x^{-1})} \right] \quad (A.7)
\]
so that \( \lim_{x \to \infty} g_k(x) = 0 \). The result follows. \(\square\)

**Lemma 13.** Suppose \( a > 0, \lambda > 1 \), and let \( \theta_m, \theta_n \) be the unique solutions in \((0,1)\) to \( \theta_m + \exp \left\{ \frac{\lambda}{\sqrt{a}} (\exp(-\lambda \sqrt{a} \theta_m) - 1) \right\} = 1 \) and \( \theta_n + \exp \left\{ \lambda \sqrt{a} \left( \exp \left\{ -\frac{\lambda}{\sqrt{a}} \theta_n \right\} - 1 \right) \right\} = 1 \) respectively. Then \( \theta_n \geq \min(1,a) \theta_m \).

**Proof.** Suppose \( a \geq 1 \), and let \( f(x) = x-1 + \exp \left\{ \frac{\lambda}{\sqrt{a}} (\exp(-\lambda \sqrt{a} x) - 1) \right\} \) for \( 0 < x < 1 \). This function is convex and satisfies \( f(0) = f(\theta_m) = 0 \). Thus, we may show that \( \theta_n \geq \theta_m \) by showing that \( f(\theta_n) \geq 0 \). We have, by the definition of \( \theta_n \),
\[
f(\theta_n) = \exp \left\{ \frac{\lambda}{\sqrt{a}} (\exp(-\lambda \sqrt{a} \theta_n) - 1) \right\} + \theta_n - 1 \quad (A.8)
\]
\[
= \exp \left\{ \frac{\lambda}{\sqrt{a}} (\exp(-\lambda \sqrt{a} \theta_n) - 1) \right\} - \exp \left\{ \lambda \sqrt{a} \left( \exp \left\{ -\frac{\lambda}{\sqrt{a}} \theta_n \right\} - 1 \right) \right\}
\]
Since \( e^x \) is a strictly increasing function, \( f(\theta_n) \geq 0 \) is equivalent to
\[ \frac{\lambda}{\sqrt{a}} \left( \exp \left\{ -\lambda \sqrt{a} \theta_n \right\} - 1 \right) - \lambda \sqrt{a} \left( \exp \left\{ -\frac{\lambda}{\sqrt{a}} \theta_n \right\} - 1 \right) \geq 0 \quad (A.9) \]

or

\[ \exp \left\{ -\lambda \sqrt{a} \theta_n \right\} - 1 - a \exp \left\{ -\frac{\lambda}{\sqrt{a}} \theta_n \right\} + a \geq 0 \quad (A.10) \]

To show that this indeed holds whenever \( a \geq 1 \), set \( y = \lambda \theta_n \) and \( z = \sqrt{a} \), and consider the function \( g(z) = e^{-yz} - 1 - z^2e^{-y/z} + z^2 \). This satisfies \( g(1) = 0 \) and

\[ g'(z) = y \left( e^{-y/z} - e^{-yz} \right) + 2z \left( 1 - e^{-y/z} \right) \geq 0. \quad (A.11) \]

Thus \( g(z) \geq 0 \) and \( \theta_n \geq \theta_m \) follows.

We argue similarly to show \( \theta_n \geq a \theta_m \) when \( a \leq 1 \). The convex function \( h(x) = \frac{x}{1-x} - 1 + \exp \left\{ \frac{-\lambda}{\sqrt{a}} x \right\} - 1 \) satisfies \( h(a \theta_m) = 0 \), so we show that \( h(\theta_n) \geq 0 \). By the definition of \( \theta_n \), we have \( \lambda \left( \exp \left\{ -\frac{\lambda}{\sqrt{a}} \theta_n \right\} - 1 \right) = \frac{1}{\sqrt{a}} \log(1 - \theta_n) \), so

\[
  h(\theta_n) = \frac{\theta_n}{a} - 1 + \exp \left\{ \frac{1}{a} \log(1 - \theta_n) \right\} \\
  = \frac{\theta_n}{a} - 1 + (1 - \theta_n)^{1/a} \quad (A.12)
\]

Letting \( k(x) = (1 - x)^{1/a} + x/a - 1 \) for \( 0 \leq x \leq 1 \), we have \( k(0) = 0 \) and \( k'(x) = a^{-1} \left( 1 - (1 - x)^{1/a-1} \right) \geq 0 \). So \( h(\theta_n) \geq 0 \) and \( \theta_n \geq a \theta_m \).

From this follows that \( \theta_n \geq \min(1,a) \theta_m \). \( \square \)
Bibliography


