Thesis for the Degree of Doctor of Philosophy

Residue currents on singular varieties

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Abstract

This thesis concerns various aspects of the theory of residue currents. Particularly, we study residue currents on singular varieties and duality theorems for such currents.

On a singular variety, there are various notions of holomorphic functions. In Paper I, we study how to extend the definition of Coleff-Herrera products and Bochner-Martinelli type residue currents from the case of strongly holomorphic functions to weakly holomorphic functions, and investigate how various properties known in the strongly holomorphic case transform into the weakly holomorphic case.

The duality theorem for Coleff-Herrera products on a complex manifold is one of the key properties of the Coleff-Herrera product. On a singular variety, the duality theorem for Coleff-Herrera products is in general false. In Paper II, we discuss necessary and sufficient conditions for when the duality theorem holds, and in particular we show that on any singular variety, one can find examples where the duality principle fails.

Another important property of the Coleff-Herrera product is the transformation law. In Paper III, we describe a comparison formula for Andersson-Wulcan currents, generalizing the transformation law. Applications of this formula include giving a proof by means of residue currents of a theorem of Hickel related to the Jacobian of a holomorphic mapping, and constructing a current on a singular variety satisfying the duality principle.

The failure of the duality theorem for Coleff-Herrera products leads to the search for an alternative. In Paper IV, we elaborate on the construction in Paper III, of a current satisfying the duality principle for an arbitrary ideal. In particular, using the comparison formula, we explain how we can view this construction as an intrinsic construction on the variety, generalizing the construction of Andersson and Wulcan.

Keywords: weakly holomorphic functions, duality theorem, Coleff-Herrera products, Bochner-Martinelli currents, residue currents, singular varieties, analytic spaces, local analytic geometry

Preface

This thesis consists of the following papers.

- ▹ Richard Lärkäng,
  "*Residue currents associated with weakly holomorphic functions*",
  in *Ark. Mat.* **50** (2012), no. 1, 135–164.
- ▹ Richard Lärkäng,
  "*On the duality theorem on an analytic variety*",
  accepted for publication in *Mat. Ann.*
- ▹ Richard Lärkäng,
  "*A comparison formula for residue currents*",
  preprint.
- ▹ Richard Lärkäng,
  "*Residue currents with prescribed annihilator ideals on singular varieties*",
  preprint.

In order to treat a more coherent theme, the following papers are not included.

- ▹ Richard Lärkäng & Håkan Samuelsson Kalm,
  "*Various approaches to products of residue currents*",
  in *J. Funct. Anal.* **264** (2013), no. 1, 118–138.
- ▹ Richard Lärkäng & Elizabeth Wulcan,
  "*Computing residue currents of monomial ideals using comparison formulas*",
  preprint.

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Richard Lärkäng
Göteborg, December 2012

Contents

Part I

INTRODUCTION

Introduction

1. Introduction

In algebraic geometry one studies solutions to polynomial equations. For example, in the plane, the set of real solutions to the equation $x^2+y^2 = 1$ is the unit circle, see Figure 1 (left), and the solutions to the equation $x^5 = y^2$ are shown in Figure 1 (right). This second geometric figure is called a cusp. One could consider instead the solutions to these equations as the solutions to $x^2 + y^2 - 1 = 0$ and $x^5 - y^2 = 0$, i.e., as the zero sets of the polynomials on the left-hand sides. Such objects are called *varieties*. Notice the difference between these two curves; the circle is smooth everywhere, while the cusp is smooth outside the origin, and "crumpled up" at the origin, it has a *singularity* there. Much of the focus of this thesis is how the presence of such singular points influences certain analytic objects on the variety.

The zero sets of both these polynomials are curves, i.e., they are one-dimensional. A more precise way of expressing this is that they can be described by precisely one parameter. The unit circle has for example the familiar parametrization $(x, y) = (\cos t, \sin t)$. For the cusp, one verifies, by inserting it into the equation, that $(x, y) = (t^2, t^5)$ lies on the curve, and with

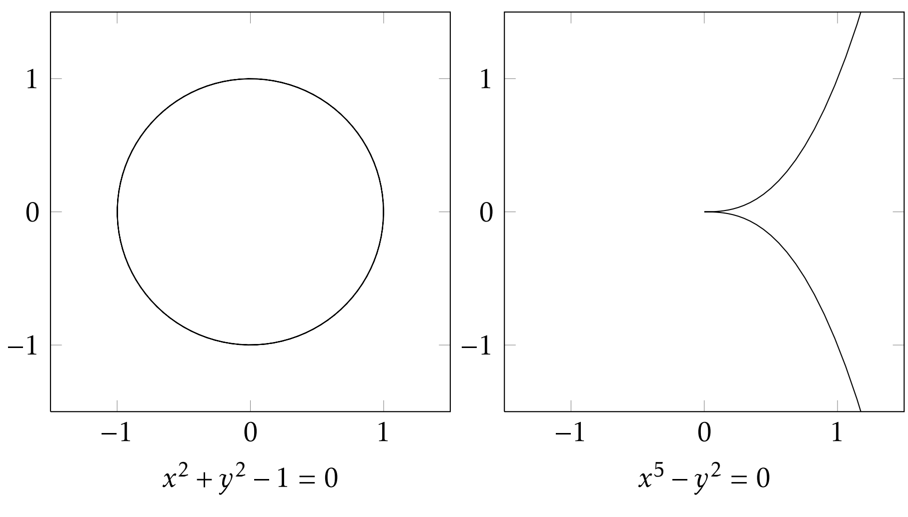

Figure 1. Examples of varieties

a short calculation, one can show that this describes all such solutions, so this gives indeed a parametrization of the cusp.

The general principle here is that considering the zero set of one equation should decrease the dimension by one, as in these examples. However, if we look at the real solutions to $x^2 + y^2 = 0$, it consists of only the origin $(0,0)$, which is zero-dimensional, so the dimension has decreased by two. In order for the general principle to hold, one needs to look instead at all complex solutions to this equation. Remember that by introducing the number i such that $i^2 = -1$, we get the solution $x = i$ to the equation $x^2 + 1 = 0$, which has no real solutions, and that in fact all polynomial equations in one variable has a complex solution of the form $x = a + bi$. Looking over the complex numbers, the equation $x^2 + y^2 = 0$ factors as $(x + iy)(x - iy) = 0$, which then has the solutions $(x, y) = (it, t)$ and $(x, y) = (-it, t)$. Thus, the set of solutions consists of two parts, each of which separately can be described by one (complex) parameter, so it is reasonable to say that the solution set is one-dimensional, i.e., a complex curve. Note however, that since the complex plane has two real dimensions, a complex curve will also have two real dimensions, i.e., it is a real surface.

Just as introducing the complex numbers made polynomial equations in one variable better behaved, in the sense that it made sure a solution always existed, if we look at a single polynomial equation in several variables, the solutions will always be of one dimension less than the number of variables. So for example, looking at a single equation in two variables as above, the complex solutions will always be a complex curve.

This improved behaviour over the complex numbers manifests itself in many ways for varieties, both algebraically and geometrically, and in this thesis, we will always consider varieties over the complex numbers.

In the first part of the introduction, we present a "crash course" in complex analysis and residue theory in one variable. In principle, this part should only require familiarity with basic calculus in two variables, although this might be a bit optimistic. In the next sections, we discuss residue theory in several variables, and try to give a bit of historical background to the contents of this thesis. Here, much more background is assumed, although we have tried to facilitate the reading by relating it to the one-variable case. Then we discuss holomorphic functions on singular varieties, a key concept in this thesis. We try to rather elaborately describe different classes of such functions on a cusp, one of the basic examples of a singular variety, and at the same time discuss special cases of the constructions of residue currents from at least three of the articles. We then conclude with a brief summary of the various articles.

1.1. The division algorithm for polynomials. The main theme (and title!) of this thesis is residue currents on singular varieties. Singular varieties, as we described above, are concrete geometric objects. Residue currents on the other hand are much more abstract objects. Currents in general are a sort of generalized functions, which, although the name suggests so, have no obvious connection to concrete currents like electric currents or ocean currents and so on. Residue currents are a specific type of such currents, which could be seen as a generalization of residues in one-variable complex analysis. We will discuss how such objects can arise by considering a concrete algebraic question.

Let $f(z)$ and $g(z)$ be two polynomials in one complex variable. The question we consider is: *How to determine whether f divides g?* This has a simple algebraic answer, coming from the division algorithm for polynomials. By the division algorithm for polynomials (i.e., performing polynomial division) there exist unique polynomials q and r such that

$$g(z) = f(z)q(z) + r(z), \tag{1.1}$$

where the degree of r, $\deg r$, is strictly smaller than $\deg f$. The polynomial q is thus the "quotient", and r is the remainder term. By the uniqueness in the division algorithm, we get a simple answer to the question above: f divides g if and only if the remainder term r is 0. We will write $f \mid g$ if f divides g. From the algebraic point of view, this thus has a simple answer. We will see how we can express an answer also from an analytic point of view, i.e., in terms of derivatives and integrals and so on.

1.2. Holomorphic functions and Cauchy's integral formula. In order to express divisibility in an analytic way, we will do a very quick "rush" through parts of the basics of complex analysis in one variable. This will of course be very selectively presented, and sketchy at parts. For a nice introduction to complex analysis in one variable, see [**StSh**].

To begin with, what are "complex differentiable" functions? Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-valued function. One way of expressing f to be complex differentiable is that the limit of the difference quotient $\lim_{h\to 0}(f(z+h) - f(z))/h$ exists, just as we would require for a real-valued function, but now with the difference that h is a complex number tending to 0. In case this limit exists, we denote it by $f'(z)$ or $\partial f/\partial z$. By exactly the same argument as for the real function $x \mapsto x^n$, one sees that $z \mapsto z^n$ is complex differentiable, and $\partial z^n/\partial z = nz^{n-1}$.

From our point of view, it will be better to formulate this in another way. We consider instead of $f : \mathbb{C} \to \mathbb{C}$ as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, and assume that f is (real-) differentiable at the origin as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. If we consider its first order Taylor expansion,

$$f(x,y) - f(0,0) = f_x(0)x + f_y(0)y + \mathcal{O}(|(x,y)|^2),$$

and regroup it in terms of $z = x + iy$ and $\bar{z} = x - iy$ instead (or equivalently $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/2i$), we get

$$\begin{aligned} f(x,y) - f(0,0) &= f_x(0,0)\frac{z+\bar{z}}{2} + f_y(0,0)\frac{z-\bar{z}}{2i} + \mathcal{O}(|z|^2) \\ &= \frac{f_x(0,0) - if_y(0,0)}{2} z + \frac{f_x(0,0) + if_y(0,0)}{2}\bar{z} + \mathcal{O}(|z|^2). \end{aligned}$$

In line with the Taylor expansion in terms of x and y, it is thus reasonable to denote the parts here in front of z and $\bar{z}$ as the derivatives of f with respect to z and $\bar{z}$, i.e., we let

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f \text{ and } \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f.$$

The complex differentiable functions will then be the ones "depending only on z and not on $\bar{z}$". Such functions are also called holomorphic, or analytic, functions.

Definition 1. A real-differentiable function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* if $\partial f/\partial \bar{z} = 0$.

In case f is holomorphic, then the limit of the complex difference quotient described above exists, and equals $\partial f/\partial z$, so this is just another way of expressing this "complex differentiability".

The prime examples of holomorphic functions are polynomials in z, like $p(z) = 3z^2 + 2$, and more generally convergent power series, i.e., "polynomials of infinite degree in z". On the other hand, for example $|z|^2 = z\bar{z}$ is not holomorphic. In addition, sums and products of holomorphic functions are holomorphic, so the holomorphic functions form a ring. The quotient of two holomorphic functions is also holomorphic, as long as the denominator does not vanish, i.e., a holomorphic function is invertible if and only if it is non-vanishing.

In order to study holomorphic functions, we will use integral formulas, and to do so we begin with a quick reminder of a theorem from 2-variable calculus.

Theorem 1.1 (Green's formula)**.** *Let $D \subseteq \mathbb{R}^2$ be a domain with smooth positively oriented boundary γ. If P and Q are differentiable on $\overline{D}$, then*

$$\int_\gamma P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

This formula holds with no change if we allow P and Q to be complex-valued. If we introduce $dz = dx + i\,dy$ and $d\bar{z} = dx - i\,dy$, and express Green's formula in terms of the complex derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$, the complex Green's formula takes the following form:

$$\int_\gamma P\,dz + Q\,d\bar{z} = -2i\int_D \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial \bar{z}}\right) dx\,dy.$$

This looks more natural if we denote $-2idxdy$ by $dz \wedge d\bar{z}$:

$$\int_\gamma Pdz + Qd\bar{z} = \int_D \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial \bar{z}}\right) dz \wedge d\bar{z}$$

($dz \wedge d\bar{z}$ does indeed have a meaning, which we refrain from discussing here, and just use as a notation for $-2idxdy$). In particular, if f is holomorphic on $\overline{D}$, then

$$\int_\gamma f(z)dz = -\int_D \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z} = 0.$$

This formula is called the *Cauchy integral theorem*. A consequence of this theorem, and a cornerstone in complex analysis in one variable is the following formula.

Theorem 1.2 (Cauchy's integral formula)**.** *Let $D \subseteq \mathbb{C}$ be a domain with smooth positively oriented boundary γ. Let $z \in D$, and assume that f is holomorphic on $\overline{D}$. Then*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}.$$

In particular, this demonstrates that holomorphic functions are rather rigid objects; their values in the interior of D are completely determined by their values on the boundary.

Proof. We let D_ϵ be D with a disc of radius ϵ around z removed. We let γ_ϵ be the positively oriented circle around z of radius ϵ so that $\gamma - \gamma_\epsilon$ is the boundary of D_ϵ. Then $f(\zeta)/(\zeta - z)$ is holomorphic in ζ for $\zeta \in D_\epsilon$. Thus, by Cauchy's integral theorem,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{f(\zeta)d\zeta}{\zeta - z}. \tag{1.2}$$

By writing $f(\zeta) = f(z) + \mathcal{O}(|\zeta - z|)$, we get that the right-hand side of (1.2) equals

$$f(z)\frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{\mathcal{O}(|\zeta - z|)d\zeta}{\zeta - z}.$$

We are thus finished if we can prove that the first integral is 1, and the second tends to 0 when ϵ tends to 0. To see that the first integral is 1, we note that on $|\zeta - z| = \epsilon$, $1/(\zeta - z) = \overline{\zeta - z}/\epsilon^2$, and applying the complex Green's formula, the integral equals

$$\frac{1}{2\pi i \epsilon^2} \int_{\gamma_\epsilon} \overline{\zeta - z} d\zeta = \frac{1}{\pi \epsilon^2} \int_{|\zeta - z| < \epsilon} dxdy = 1,$$

since the area of the disk of radius ϵ is $\pi\epsilon^2$. Since the integrand in the second term is bounded, and since we integrate over a curve of length $2\pi\epsilon$, this integral will tend to 0 if we let ϵ tend to 0. $\square$

Many of the important properties of holomorphic functions can be derived from this integral formula. We just give one example of that. Since we integrate over the boundary γ of D, the denominator $\zeta - z$ does not vanish if $\zeta \in \gamma$ and $z \in D$, so it is legitimate to differentiate with respect to z under the integral sign. We thus get that

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^2}. \tag{1.3}$$

Since the integrand is holomorphic in z on the set where we integrate, applying $\partial/\partial\bar{z}$ to both sides shows that also $f'(z)$ is holomorphic. By repeating this argument, if f is once complex differentiable, it is infinitely complex differentiable, in stark contrast to real-differentiable functions. In addition, we get similar formulas as (1.3) for higher order derivatives.

1.3. The division algorithm and residues. We now return to expressing the division formula using Cauchy's integral formula. As before, let $f(z)$ and $g(z)$ be polynomials. Using the identity

$$\frac{1}{\zeta - z} = \frac{f(z) + f(\zeta) - f(z)}{f(\zeta)(\zeta - z)}$$

and inserting this into Cauchy's integral formula, we get

$$g(z) = f(z)\frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)d\zeta}{f(\zeta)(\zeta - z)} + \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)(f(\zeta) - f(z))d\zeta}{f(\zeta)(\zeta - z)}, \tag{1.4}$$

where γ is the positively oriented circle of radius R around 0, and R is chosen such that all zeros of $f(\zeta)$ are contained inside this circle, and $|z| < R$. We denote the first and second integral, which are both functions of z, by $q(z)$ and $r(z)$, respectively, so that (1.4) says that

$$g(z) = f(z)q(z) + r(z). \tag{1.5}$$

Note that this has exactly the same form as the division algorithm in (1.1), and we claim that these q and r are in fact the same functions as the polynomials in the division algorithm.

We consider first divisibility as holomorphic functions. For a holomorphic function h on $\overline{D}$, we let

$$R_f(h) := \frac{1}{2\pi i}\int_\gamma \frac{h(\zeta)d\zeta}{f(\zeta)}, \tag{1.6}$$

and if g is holomorphic, we let $gR_f(h) := R_f(gh)$. We claim that $p(\zeta, z) := (f(\zeta) - f(z))/(\zeta - z)$ is holomorphic in ζ. To see this, we use the fact that if $u(\zeta)$ is holomorphic and $u(z) = 0$, then $u(\zeta) = (\zeta - z)u_0(\zeta)$, where u_0 is holomorphic, and apply this to $u(\zeta) = f(\zeta) - f(z)$. Then, if $gR_f = 0$, we get from (1.5) that $g \mid f$, since $r(z) = gR_f(p(\zeta, z)) = 0$, and $q(z)$ is holomorphic in z. Conversely, if $f \mid g$, then $gR_f = 0$ by Cauchy's integral theorem, so we have proved the following.

Proposition 1.3. *Let $f(\zeta)$ and $g(\zeta)$ be holomorphic functions. Then $f \mid g$ as holomorphic functions if and only if $gR_f = 0$.*

This result will in fact also hold for the original problem we considered, i.e., concerning divisibility of polynomials.

Proposition 1.4. *Let $f(\zeta)$ and $g(\zeta)$ be polynomials. Then $f \mid g$ if and only if $gR_f = 0$.*

The only addition needed compared to the proof of Proposition 1.3 is the following lemma.

Lemma 1.5. *Let $g(z)$ and $f(z)$ be polynomials, and let*

$$q(z) := \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)d\zeta}{f(\zeta)(\zeta - z)} \text{ and } r(z) := \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)(f(\zeta)-f(z))d\zeta}{f(\zeta)(\zeta - z)}.$$

Then $q(z)$ and $r(z)$ are polynomials, and $\deg r(z) < \deg f(z)$.

By the uniqueness in the division algorithm, this then says that $q(z)$ and $r(z)$ here coincide with the ones in the division algorithm.

Proof. We first verify that r is a polynomial of degree $< \deg f$. The integrand in r is $p(\zeta,z) = (f(\zeta)-f(z))/(\zeta - z)$ times a function only depending on ζ. Hence, if we show that $p(\zeta,z)$ with ζ fixed is a polynomial in z of degree $< \deg f$, we are done. To see this, we note that the numerator is a polynomial of degree $\deg f$ in z, and since for $z = \zeta$, $f(\zeta)-f(z)$ vanishes, so it is divisible by $\zeta - z$, and $p(\zeta,z)$ will thus be a polynomial in z of degree $< \deg f$.

To see that $q(z)$ is a polynomial, we show that $q^{(k)}(z) \equiv 0$ if $k = \deg g - \deg f + 1$, since then we can just integrate $q^{(k)}(z)$, k times, and see that $q(z)$ is a polynomial of degree $\le k$. We differentiate the definition of $q(z)$ with respect to z, k times, and get that

$$q^{(k)}(z) = k!\int_\gamma \frac{g(\zeta)d\zeta}{f(\zeta)(\zeta - z)^{k+1}}. \tag{1.7}$$

The denominator of the integrand will have degree $\deg f + k + 1 = \deg g + 2$, i.e., 2 higher than the degree of the numerator, so the integrand will be of $\mathcal{O}(|\zeta|^{-2})$, when $|\zeta|$ tends to infinity. Since the integrand is holomorphic in ζ for $|\zeta| > R$, by Cauchy's integral theorem, we can replace the integral over γ by integrating over a circle of larger radius without changing the integral. Letting R tend to infinity, we see that $q^{(k)}(z) \equiv 0$, since we are integrating something $\mathcal{O}(R^{-2})$ over a curve of length $2\pi R$. □

Example 1. Take $f(z) = z^2$. Then,

$$R_f(h) = \frac{1}{2\pi i}\int_\gamma \frac{h(\zeta)d\zeta}{\zeta^2} = \frac{\partial h}{\partial z}(0)$$

by (1.3). We now check what the condition $gR_f = 0$ becomes. Let h be a polynomial, and write $h(\zeta) = a + b\zeta + \zeta^2 c(\zeta)$. Then

$$gR_f(h) = R_f(gh) = \frac{\partial}{\partial\zeta}(g(\zeta)h(\zeta))\Big|_{\zeta=0} = g'(0)a + g(0)b.$$

If this is 0 for all h, i.e., for all choices of a and b, we must thus have that $g(0) = g'(0) = 0$. So by Proposition 1.3, we get the expected condition $z^2 \mid g(z)$ if and only if $g(0) = g'(0) = 0$.

To describe residues, we will first need two facts about zeros of holomorphic functions, see for example [**StSh**], Section 3.1. First of all, the zero set of a holomorphic function is discrete, so in particular, there is a small punctured disk around each zero of a holomorphic function such that the function is non-vanishing on this punctured disk. Secondly, we will need that if f has a zero at z_0, then there is a unique integer m (the order of the zero), such that

$$f(z) = (z - z_0)^m u(z), \tag{1.8}$$

where u is holomorphic and $u(z_0) \neq 0$.

A *meromorphic* function $\psi(z)$ is the quotient $\psi(z) = f(z)/g(z)$ of two holomorphic functions. The *residue* $\operatorname{Res}_{z=z_0}\psi(z)$ of a meromorphic function $\psi(z)$ at a point z_0 is defined as

$$\operatorname{Res}_{z=z_0}\psi(z) := \frac{1}{2\pi i}\int_{|z-z_0|=\epsilon}\psi(z)dz,$$

where $\epsilon > 0$ is small enough such that $\psi(z)$ has no poles, i.e., zeros of the denominator, in the punctured disk $\{0 < |z - z_0| \leq \epsilon\}$. That there exists such an ϵ follows from the discreteness of the zero sets of holomorphic functions as described above. Assume now that $\psi(z)$ has a pole order n at z_0, i.e., if $\psi(z) = f(z)/g(z)$, where f has a zero of order k at z_0 and g has a zero of order ℓ at z_0, then $\psi(z)$ has a pole of order $n = \ell - k$ at z_0. Note then, that $(z - z_0)^n\psi(z)$ is holomorphic near z_0. Then,

$$\operatorname{Res}_{z=z_0}\psi(z) := \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\Big((z - z_0)^n\psi(z)\Big)\Big|_{z=z_0}, \tag{1.9}$$

which can be seen from Cauchy's integral formula and its versions for derivatives like (1.3). We consider now

$$\frac{1}{2\pi i}\int_\gamma \psi(\zeta)d\zeta,$$

where γ is the positively oriented boundary of some domain D, and ψ is meromorphic on D and without any poles on γ. By Cauchy's integral theorem, we can replace the integral over γ by integrals around small circles around the poles, cf., the proof of Theorem 1.2, so together with the definition of residue, we get that

$$\frac{1}{2\pi i}\int_\gamma \psi(\zeta)d\zeta = \sum \operatorname{Res}_{z=z_0}\psi(z),$$

where the sum is over all poles of ψ in D. This is called the *residue theorem*. Combining Proposition 1.4 with the residue theorem and (1.9), we will arrive at the expected condition: If f and g are polynomials, then $f \mid g$ if and only if g vanishes at the zeros of f to at least the same order, i.e., as in Example 1.

Remark 1. That the integral (1.7) is 0 in the proof of Lemma 1.5 can by the residue theorem be reformulated as that the total sum of all residues of the integrand is 0. The argument used in that proof, that this total sum of residues is 0 due to the difference in degrees between the numerator and denominator is a special case of what is called *Jacobi's residue formula*.

1.4. Residue currents in one variable. We have seen how residues can appear in for example divisibility problems. We will now discuss how this relates to the main object of study in this thesis, residue currents, and what they are in one complex variable.

Let ψ be a smooth function on the domain D, with smooth positively oriented boundary γ. If h is a holomorphic function, then the complex Green's formula says that

$$\int_\gamma \psi h d\zeta = -\int_D \frac{\partial\psi}{\partial\bar\zeta} h d\zeta\wedge d\bar\zeta.$$

We will in the rest of this section use the short-hand notation $\bar\partial = -\partial/\partial\bar\zeta$ (the standard notation is that $\bar\partial\varphi = (\partial/\partial\bar\zeta)\varphi d\bar\zeta$, but it will be more convenient to use the notation above in this section). Let now f be a holomorphic function, with zero set Z. If we consider R_f from (1.6), then formally we would get that

$$R_f(h) = \frac{1}{2\pi i}\int_D \bar\partial\left(\frac{1}{f}\right) h d\zeta\wedge d\bar\zeta. \tag{1.10}$$

This formula as it stands of course does not have meaning, since $1/f$ is not differentiable on Z. However, there is a way to interpret $\bar\partial(1/f)$ as a sort of "generalized function". Assume first that we replace $(1/2\pi i)h$ by a smooth function φ on D, which has compact support, i.e., is 0 close to the boundary of D. Formally, we would have

$$\int_D \bar\partial\left(\frac{1}{f}\right)\varphi d\zeta\wedge d\bar\zeta = \int_D \bar\partial\left(\frac{1}{f}\varphi\right) d\zeta\wedge d\bar\zeta - \int_D \frac{1}{f}\bar\partial\varphi d\zeta\wedge d\bar\zeta.$$

By formally applying Green's theorem to the first integral on the right, we would get

$$\int_D \bar\partial\left(\frac{1}{f}\right)\varphi d\zeta\wedge d\bar\zeta = \int_\gamma \frac{1}{f}\varphi d\zeta = 0,$$

since $\varphi = 0$ close to the boundary γ of D. Thus, we would formally have that

$$\int_D \bar\partial\left(\frac{1}{f}\right)\varphi d\zeta\wedge d\bar\zeta = -\int_D \frac{1}{f}\bar\partial\varphi d\zeta\wedge d\bar\zeta. \tag{1.11}$$

The right-hand side of (1.11) exists as an improper integral, called the *principal value*.

Proposition 1.6. *Let f be holomorphic on D, and let η be a smooth function with compact support. Then the limit*

$$\lim_{\epsilon\to 0^+}\int_{D\setminus\{|f(\zeta)|<\epsilon\}} \frac{\eta}{f} d\zeta\wedge d\bar\zeta \tag{1.12}$$

exists.

We postpone the proof and first discuss how this relates to the discussion above. A smooth function with compact support is called a *test function*. Hence, even though $1/f$ is not defined as a function everywhere, we can give it meaning as something which can be integrated against test functions. Such objects are called *distributions* or *currents*. We will denote this integration, i.e., (1.12), by $(1/f).\eta$. Considering $1/f$ as an object one can integrate is maybe not so surprising, as one studies improper integrals also in single-variable calculus. What may be more surprising is that from this viewpoint, one can also differentiate $1/f$ on all of D. If f has no zeroes, so that $1/f$ is smooth on all of D, then (1.11) holds, and if not, one uses it as a definition of the distribution $\bar\partial(1/f)$, i.e., we let $\bar\partial(1/f).\varphi$ be defined as the right-hand side of (1.11), which thus exists by Proposition 1.6.

Note that outside of Z, the zero set of f, it would be reasonable to say that $\bar\partial(1/f) = 0$ since $1/f$ is holomorphic there. This also holds in the following sense: If φ has compact support, and φ is 0 in a neighbourhood of Z, then we can in fact go backwards not just formally in the derivation of (1.11), but with the integral on the left-hand side being only over the set where $\varphi \neq 0$, and hence, the integral is 0 since $\bar\partial(1/f) = 0$ there. We then say that $\bar\partial(1/f)$ has its *support* on Z, i.e., integrated against φ it only depends on the behaviour on arbitrary small neighbourhoods of Z.

Now, to come back to where we started, we have thus given meaning to $\bar\partial(1/f)$ as a distribution, and it is natural to ask if now (1.10) has a meaning and if the equation then holds. Note that although being smooth, h is not a test function since it does not have compact support. However, since $\bar\partial(1/f)$ has support on Z, we can let it act on χh, where χ is a cut-off function which is identically 1 close to Z and has compact support on D. Incorporating this χ will then make (1.10) hold.

To conclude this discussion, we will reformulate Proposition 1.3 in a way that will be generalized in this thesis. We consider, as in Proposition 1.3, the set of holomorphic functions g such that $gR_f = 0$, which is called the *annihilator* of R_f, and is denoted $\operatorname{ann} R_f$. Since the action of R_f on h equals the action of the current $\bar\partial(1/f)$ on h, we can just as well consider the annihilator of $\bar\partial(1/f)$ instead, i.e., $\operatorname{ann}\bar\partial(1/f)$. In addition, we note that the holomorphic functions g such that $f \mid g$ are exactly the functions $g = hf$, where h is a holomorphic function. This is called the *ideal generated by* f, and which we denote by $\mathcal{J}(f)$, i.e., $\mathcal{J}(f) := \{g | g = hf, \text{where } h \text{ is holomorphic}\}$. Formulated in these terms, Proposition 1.3 becomes the following.

Proposition 1.7. *Let f be a holomorphic function. Then*

$$\operatorname{ann}\bar\partial\left(\frac{1}{f}\right) = \mathcal{J}(f).$$

This is a special case of the *duality theorem* for Coleff-Herrera products, which will discuss later on. We remark also that Proposition 1.7 can be proved in a different way.

Alternative proof of Proposition 1.7. Note that if $g \in \operatorname{ann}\bar\partial(1/f)$, then $g\bar\partial(1/f) = 0$. Since g is holomorphic, we get that $\bar\partial(g/f) = 0$. If g/f would be a smooth function, this would mean that it is holomorphic, i.e., $g/f = h$, where h is holomorphic, so $g = hf$, i.e., $g \in \mathcal{J}(f)$. Any distribution T satisfying $\bar\partial T = 0$ will in fact be a smooth function, a result called Weyl's lemma, see for example [**V**], 1.7.6. Thus, $\operatorname{ann}\bar\partial(1/f) \subseteq \mathcal{J}(f)$. The other inclusion follows easily by going backwards in this argument (without the need of using Weyl's lemma).

□

Proof of Proposition 1.6. Here, we will need the two facts about zeros of holomorphic functions, as described in the paragraph around (1.8). First of all, the fact that the zero set of a holomorphic function is discrete implies that if f is holomorphic on $\overline{D}$, then f has only a finite number of zeros on $\overline{D}$, and we denote them by $z_1, \dots, z_n$. Secondly, by the factorization (1.8), we can factorize f as $f(z) = (z-z_i)^m u_i(z)$, where u_i is holomorphic and $u_i(z) \neq 0$ near z_i.

For $z_i \in Z$, we choose a cut-off function χ_i, which is identically 1 in a neighbourhood of z_i, and which is identically 0 in a neighbourhood of all $z_j \in Z$, $j \neq i$. By writing $1 = \chi_1 + \dots + \chi_n + (1 - \chi_1 - \dots - \chi_n)$, the integral (1.12) splits into $n+1$ different integrals. In the last one, there will be no problem letting ϵ tend to 0 since $(1-\chi_1-\dots-\chi_n)$ is identically 0 near Z, so the integrand is smooth everywhere. It thus remains to prove that

$$\lim_{\epsilon\to 0^+}\int_{\{|f|>\epsilon\}} \frac{\chi_i \eta}{f} d\zeta \wedge d\bar\zeta$$

exists. Since by the choice of χ_i, f has only one zero on the set where $\chi_i \neq 0$, we can by translation and the description above assume that $f(\zeta) = \zeta^m u(\zeta)$, where $u(0) \neq 0$. In addition, we could have chosen χ_i such that it is non-zero only on a small ball B around 0 so that we can define $(u(\zeta))^{1/m}$. We then make the change of variables $z = \zeta u(\zeta)^{1/m}$. Then the integral becomes

$$\lim_{\epsilon \to 0^+} \int_{B \cap \{|z^m| > \epsilon\}} \frac{\chi_i \tilde{\eta}}{z^m} dz \wedge d\bar{z}.$$

We then make a complex Taylor expansion of $\chi_i \tilde{\eta}$ of order $m-1$, so that the integral becomes

$$\sum_{\alpha+\beta<m} \int_{B \cap \{|z^m| > \epsilon\}} \frac{a_{\alpha,\beta} z^\alpha \bar{z}^\beta}{z^m} dz \wedge d\bar{z} + \int_{B \cap \{|z^m| > \epsilon\}} \frac{\mathcal{O}(|z|^m)}{z^m} dz \wedge d\bar{z}.$$

In the last integral, the integrand is bounded, so there is no problem letting ϵ tend to 0. It thus remains to see that the integrals in the sum in fact vanish. Switching to polar coordinates, the integrands becomes $-2ia_{\alpha,\beta} e^{i\theta(\alpha-\beta-m)} r^{\alpha+\beta-m+1}$, and in particular, since $\alpha + \beta < m$, we have that $\alpha - \beta - m \neq 0$, so the integral with respect to θ is 0, and all the integrals in the sum then vanish.

□

Following the argument in the proof of Proposition 1.6, or reasoning as in the proof of Cauchy's integral formula and (1.3), one gets that

$$\frac{m!}{2\pi i} \bar{\partial} \frac{1}{z^{m+1}} . \varphi = \varphi^{(m)}(0), \tag{1.13}$$

and for more general functions, similar formulas exist, cf. (1.9).

In might not be apparent what we have gained here by expressing the division algorithm in this form, when we have taken a simple algebraic problem, and transformed it into a more complicated analytic problem. In several variables, a natural generalization is to consider a tuple $f = (f_1, \ldots, f_p)$ of polynomials, and express a function g in terms of f, as $g = h_1 f_1 + \cdots + h_p f_p$ (in one variable, these problems are equivalent due to the polynomial ring on one variable being a principal ideal domain). Algebraically, this problem becomes more involved, while several of the key ingredients in the analytic formulation have already been introduced (although the analytic formulation of course also becomes more involved).

We mention that except for the part about residue currents, the rest of this section only uses facts which are standard in almost any book in one-variable complex analysis. The parts about the division algorithm and residue currents are more specialised, and are usually not treated in such books. However, see for example the beginnings of [**CD**] and [**BGVY**], where residue theory in the one-variable case is discussed rather thoroughly before treating the case of several variables.

2. Residue currents of complete intersections

One-variable complex analysis is a classical area in mathematics, with fundamental work done by Cauchy, Riemann and others during the 19th century. During the beginning of the 20th century, Hartogs and Levi made important discoveries regarding phenomena occurring for holomorphic functions in dimension greater than one, and important contributions to questions arising from the works of Hartogs and Levi were made by Oka and Cartan during the 40's and 50's. For a nice discussion about similarities and differences between one and several complex variables, see [**Ra2**]. After the work of Oka and Cartan, the subject of several complex variables really began to flourish, see for example [**Do6**] for a survey about important results in several complex variables during the second half of the 20th century.

In one complex variable, there is a rather standard curriculum of concepts covered in an introductory book. In several complex variables, the contents of introductory books tend to diverge more quickly, depending on from which point of view the book is written. Classical introductions would be for example [**Ra1**], [**Hö**] and [**GuRo**], all written from rather different viewpoints. For our purposes, the point of view of studying properties of analytic varieties will be of particular importance, which is presented nicely in [**Ta**] and [**Gu2**] (see also the other books in the same series, [**Gu1**] and [**Gu3**]) and also for example [**KK**] and [**dJP**]. More advanced references would be for example [**Dem**] and [**GH**].

We now turn to residue theory in several complex variables. Let f be a holomorphic function on $\Omega \subseteq \mathbb{C}^n$. The *principal value current* $1/f$ can be defined in the same way as in one variable, by

$$\frac{1}{f}.\varphi := \lim_{\epsilon \to 0^+} \int_{\{|f|>\epsilon\}} \frac{\varphi}{f}. \tag{2.1}$$

In one dimension, the existence of this limit was rather elementary, relying on the fact that the zero set consisted of isolated points and that we could factorize the function near the points on the zero set. In several variables, the existence of this limit is not an elementary matter anymore. Assume first that f is an invertible holomorphic function times a monomial; f is then said to have *normal crossings* singularities. In this case, existence of the limit (2.1) can essentially be reduced to the one-variable case. The difficulty of the existence of the limit in (2.1) in the general case is to be able to reduce it to the normal crossings case, which is handled by the deep theorem of Hironaka on resolution of singularities. See [**Ko2**] for a nice presentation of resolution of singularities in the algebraic setting. The existence of the principal value current was proven independently by Herrera and Lieberman in [**HL**] and Dolbeault in [**Do1**, **Do2**, **Do3**]. The form of resolution of singularities used for proving the existence of the principal value current can be found in [**Ko2**], Theorem 3.26.

Now that $1/f$ is defined, one can also define the *residue current* $\bar{\partial}(1/f)$ in the sense of currents, and by the same argument as in the alternative proof of Proposition 1.7, one gets that $\operatorname{ann}\bar{\partial}(1/f) = \mathcal{J}(f)$, just as in the one-variable case.

Note that in this section and in future sections, we consider the usual form-valued $\bar{\partial}$-operator acting on forms, which for functions is

$$\bar{\partial}\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial \bar{z}_i} d\bar{z}_i,$$

in contrast to the non-standard function-valued operator which we used in the previous section.

2.1. Coleff-Herrera products and complete intersection ideals. We now consider a tuple $f = (f_1, \dots, f_p)$ of holomorphic functions. Although it is in general problematic to give meaning to products of currents, Coleff and Herrera showed in [**CH**] that one can give a reasonable meaning to an iterative product of the residue currents $\bar{\partial}(1/f_i)$, i.e., the current

$$\mu^f := \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1}. \tag{2.2}$$

This current is nowadays called the *Coleff-Herrera product* of f. By altering the definition used for the principal value current, we describe one way to define such a product. This way of defining the Coleff-Herrera product is different from the original one, but the current we define here will coincide with the current defined in [**CH**], see Section 2.3. The principal value current can be defined by

$$\frac{1}{f} := \lim_{\epsilon \to 0+} \frac{\chi(|f|^2/\epsilon)}{f},$$

where χ is a smooth approximation of the characteristic function $\chi_{[1,\infty)}$. Thus, $\bar{\partial}(1/f_1)$ is defined. Then, one defines $(1/f_2)$ "on" $\bar{\partial}(1/f_1)$, by

$$\frac{1}{f_2}\bar{\partial}\frac{1}{f_1} := \lim_{\epsilon \to 0+} \frac{\chi(|f_2|^2/\epsilon)}{f_2}\bar{\partial}\frac{1}{f_1}.$$

Taking $\bar{\partial}$ of this current, we define $\bar{\partial}(1/f_2) \wedge \bar{\partial}(1/f_1)$ by

$$\bar{\partial}\frac{1}{f_2} \wedge \bar{\partial}\frac{1}{f_1} = \lim_{\epsilon \to 0+} \frac{\bar{\partial}\chi(|f_2|^2/\epsilon)}{f_2} \wedge \bar{\partial}\frac{1}{f_1},$$

and continuing in the same way, one defines the general product (2.2).

This product behaves "nicely" if $f = (f_1, \dots, f_p)$ defines what is called a *complete intersection*, i.e., if $\operatorname{codim} Z(f) = p$, where $Z(f)$ is the common zero set of f. In other words, f defines a complete intersection if for each f_i we add, the dimension of the common zero set decreases by 1, so that $Z(f)$ is as small as it possibly could. An example of a tuple which is not a complete intersection is $f = (z^2, zw)$ on $\mathbb{C}^2$, which has $\{z = 0\} \subseteq \mathbb{C}^2$ as common zero set, which has codimension 1. The most important aspect of how the Coleff-Herrera product behaves "nicely" in the case of a complete intersection is the following *duality theorem* for Coleff-Herrera products.

Theorem 2.1. *Let $f = (f_1, \dots, f_p)$ be a tuple of holomorphic functions on $\Omega \subseteq \mathbb{C}^n$. Then locally,*

$$\operatorname{ann} \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} = \mathcal{J}(f).$$

Here, $\mathcal{J}(f)$ is the ideal generated by f, i.e.,

$$\mathcal{J}(f) = \{h \mid h = a_1 f_1 + \cdots + a_p f_p, \text{ where } a_i \text{ are holomorphic}\}.$$

This theorem generalizes the result above about the annihilator of one single residue current $\bar{\partial}(1/f)$. The duality theorem was proven independently by Dickenstein and Sessa in [**DS1**], and by Passare in [**P2**]. The proof in [**P2**] is done by means of integral formulas, i.e., like generalizations of (1.4), while the proof in [**DS1**] relies on more abstract algebraic machinery, like sheaf theory and homological algebra. The main analytic content in the proof in [**DS1**] boils down to solving the $\bar{\partial}$-equation locally, i.e., the proof shares more similarities with the alternative proof of Proposition 1.7.

Another aspect in which the Coleff-Herrera product behaves nicely is the following *transformation law* for Coleff-Herrera products.

Theorem 2.2. *Let $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ be two tuples of holomorphic functions defining complete intersections, and assume that there exists a matrix A of holomorphic functions such that $f = gA$. Then*

$$\bar{\partial}\frac{1}{g_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{g_1} = (\det A)\bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1}.$$

The transformation law was proven by Dickenstein and Sessa in [**DS1**]. The proof relied on the earlier known absolute case, i.e., when $p = n$, for cohomological Grothendieck residues, as described in for example [**GH**].

A consequence of the transformation law is that one can view the Coleff-Herrera product essentially as an object associated to a complete intersection ideal, and not to a specific choice of generators, since choosing a different minimal set of generators will only change the current by an invertible holomorphic function. In particular, up to change of signs, it does not depend on the order of the functions (which was already known from [**CH**] by other means). Another consequence is that one can allow $f = (f_1, \dots, f_p)$ to be sections of a vector bundle E of rank p, and the Coleff-Herrera product will then be a section of $\det E^*$, cf., for example [**DP**].

2.2. Applications of Coleff-Herrera products and residues. We discuss here some applications, motivations and developments in the theory of residues and residue currents. We try to provide a rather extensive list of references, although this list will of course be far from exhaustive.

Much of the early work on residue currents by Herrera and others revolved around issues regarding homology and cohomology of complex spaces. The residue currents provided concrete representatives of cohomology classes and realizations of constructions in cohomology theory. See [**HL**], [**Do1**], [**Do2**], [**Do3**], [**Her**], [**CH**], [**RR**], [**CHL**], [**DS1**] and [**DS2**]. Later work in a similar spirit can be found in [**Fa1**].

This work was inspired by the introduction by Grothendieck of a theory of residues in algebraic geometry, as presented by Hartshorne in [**Hart**], which corresponds to the Coleff-Herrera product in case of a complete intersection with discrete zeros. The work by Grothendieck has inspired enormous amounts of work in algebraic geometry and other parts of mathematics. We content ourselves with referring to some developments related to cohomological residues. Concrete representations of the cohomological point residue in the analytic setting is described, for example in [**Ton2**] and [**GH**], see also [**Harv**]. Other algebraic and analytic realizations can be found for example in [**Be**], [**Li**], [**ScSt**], [**Ku1**], [**Hop**], [**HK**] and a comparison between various notions of algebraic residues can be found in [**Bo**].

Early applications of such cohomological residues included for example work on zeros of vector fields and fixed points of holomorphic maps, extending earlier work by Atiyah-Baum-Bott, see for example [**CL**], [**Ton1**], [**Tol**] and [**O**].

Other work in the algebraic setting has been centered around *toric residues*, see for example [**GK**], [**C**] and [**CCD**]. For additional references for cohomological residues in the algebraic setting, see [**Ku2**] and [**CD**].

Much work on residue currents have centered around *effectivity questions in division problems*, i.e., in a theorem regarding ideal membership for polynomials, whether one can say something about the degrees of the terms appearing, see [**BY1**] and [**BGVY**]. Earlier results of that kind, not involving residue currents had been obtained by [**Br**] and [**Ko1**], which both are of more algebraic nature, although [**Br**] also relied on methods from complex analysis.

Another area has been questions related to *intersection theory*, see for example [**BY2**], [**DP**], and earlier results in [**ALJ**], [**LJ**], [**Ba2**], [**EZ**].

Coleff-Herrera products have also been used in providing explicit versions of the *Ehrenpreis-Palamodov fundamental principle*, describing solutions to systems of linear partial differential equations with constant coefficients, see [**BP**], [**Y**] and [**Ri**]. See also the survey [**BeSt**]. The fundamental principle is a generalization of the result in one-variable calculus that the solutions of a linear ordinary differential equation with constant coefficients are linear combinations of functions of the form $y(t) = e^{rt}$, where r are roots of the characteristic polynomial of the differential equation (in case it has simple roots). In addition, Coleff-Herrera products have been used in relation to the *$\bar{\partial}$-equation on singular varieties*, [**HePo1**], and questions related to the *Abel transform* and *complex Radon transform*, see [**Hen**], [**HePa**], [**Fa2**], [**We**] and [**HePo2**]. Inspiration for such work can be found in [**Gr**] and [**GH**].

We finally also mention connections between residue currents and $\mathcal{D}$-*module theory*, see [**Bj2**] and [**Bj1**]. In addition, we should also mention the introduction of Bochner-Martinelli type residue currents in [**PTY**] and the construction of residue currents in [**AW1**], and the work developed after those articles. However, since this will be the topic of Section 3, it will be discussed there.

For more thorough references regarding residue currents, see for example the books [**AY**], [**BGVY**], [**Ts**] and the surveys [**Do4**], [**Do5**] and [**TY**].

2.3. Various definitions of Coleff-Herrera products. Some remarks are in order about different ways of defining the Coleff-Herrera product. In the main case of interest, in case of a complete intersection, essentially all reasonable definitions coincide. Here, we describe in a bit more detail the different definitions, and how they are related to each other.

The original definition in [**CH**] is to define the Coleff-Herrera product as

$$\mu^f.\varphi := \lim_{\delta\to 0^+} \int_{\cap\{|f_i|=\epsilon_i(\delta)\}} \frac{\varphi}{f_1\cdots f_p}, \tag{2.3}$$

where $\epsilon(\delta) = (\epsilon_1(\delta),\dots,\epsilon_p(\delta))$ is an admissible path, which means that $\epsilon_i(\delta)$ tends to 0 "much faster" than $\epsilon_{i+1}(\delta)$ in the sense that there exist constants $C_{i,k}$ such that $\epsilon_i(\delta) < C_{i,k}\epsilon_{i+1}(\delta)^k$ for $k = 1,2,\dots$. Intuitively, this should more or less correspond to letting the ϵ_i tend to zero one at a time. One way of arriving at (2.3) as a reasonable definition for the product is that by Stokes' theorem, for φ a $(n,n-1)$ test form,

$$\bar\partial\frac{1}{f}.\varphi = -\lim_{\epsilon\to 0+}\int_{|f|\geq\epsilon}\frac{\bar\partial\varphi}{f} = \lim_{\epsilon\to 0+}\int_{|f|=\epsilon}\frac{\varphi}{f},$$

i.e., $\bar\partial(\chi_{[1,\infty)}(|f|/\epsilon)/f) = [\{|f| = \epsilon\}]/f$, where $[\{|f| = \epsilon\}]$ denotes the integration current on $\{|f| = \epsilon\}$ (by Sard's theorem, this set is smooth for almost every ϵ). Hence, considering products of such currents should correspond to exactly the integral in (2.3), which one then takes as the definition.

The use of the smooth regularization of $\chi_{[1,\infty)}(|f|/\epsilon)$ was introduced in [**P3**]. It has the advantage that multiplication of a residue current with $\chi(|f|/\epsilon)/f$ or $\bar\partial\chi(|f|/\epsilon)/f$ is well-defined, while multiplying a current with a characteristic function (or derivative of a characteristic function) does not a priori have any meaning (although it can be given meaning here, at least if f defines a complete intersection, see [**Bj1**], Section 6, see also [**RG**]). In [**P3**], averages of integrals of the form

$$\lim_{\delta\to 0+}\int\frac{\bar\partial\chi(\frac{|f_1|}{\epsilon_1(\delta)})\wedge\cdots\wedge\bar\partial\chi(\frac{|f_p|}{\epsilon_p(\delta)})}{f_1\cdots f_p}\wedge\varphi, \tag{2.4}$$

where $\epsilon(\delta)$ ranged over parabolic paths, was considered, and it was shown that this average equals the Coleff-Herrera product in case f defines a complete intersection. By carefully studying the proofs of the existence of both limits, one can see that if one considers (2.4), where the limit is taken by letting ϵ_i to 0 one at a time, this limit equals the Coleff-Herrera product defined by (2.3). An elaboration of this argument can be found in [**LSK**].

In the original definition of the Coleff-Herrera product, it is essential that the limit is taken over an admissible path, as shown by Passare and Tsikh in [**PT**], where the limit of residue integrals (2.3) were shown not to exist for a complete intersection of two functions in $\mathbb{C}^2$, when the limit is taken over a path which is not an admissible path. Later an even simpler family of examples was found by Björk in [**Bj1**], where one of the functions was even a coordinate function.

Using the regularization of the characteristic function, this phenomenon does not occur, as proved by Björk and Samuelsson in [**BjSa**], showing that the residue integral (2.4) is in fact continuous in $(\epsilon_1,\dots,\epsilon_p)$ if f defines a complete intersection.

We will in this thesis mainly use another way of defining the Coleff-Herrera product stemming from a different definition of the principal value current. In [**At**] and [**BG**], Atiyah and Bernšteĭn-Gel′fand defined the principal value current by

$$\frac{1}{f}.\varphi := \int \frac{|f|^{2\lambda}\varphi}{f}\bigg|_{\lambda=0}.$$

Here, by $|_{\lambda=0}$, we mean that the integral on the right-hand side is an analytic function in λ for $\operatorname{Re}\lambda \gg 0$, and by $|_{\lambda=0}$, we denote the analytic continuation of this function to $\lambda = 0$. The existence of this analytic continuation is proved in a similar way as the existence of the principal value current defined in terms of cut-off functions. Using this regularization, one would arrive at the following integral for defining the Coleff-Herrera product of a tuple $(f_1,\dots,f_p)$:

$$\int \frac{\bar\partial|f_1|^{2\lambda_1}\wedge\cdots\wedge\bar\partial|f_p|^{2\lambda_p}}{f_1\cdots f_p}\wedge\varphi.$$

In [**Y**], Yger considered when $(\lambda_1,\dots,\lambda_p) = \lambda(t_1,\dots,t_p)$, where $t_i > 0$ were fixed, and the current was defined as the analytic continuation to $\lambda = 0$. In [**P1**], it was proven that this current equals the average of the integrals in (2.4), as introduced in [**P3**]. In particular, it equals the original Coleff-Herrera product if f defines a complete intersection. In [**LSK**], we show that if one instead lets $\lambda_i = 0$ one at a time, this corresponds to the original Coleff-Herrera product, irrespectively of whether f defines a complete intersection or not.

We also mention that Mazzilli defined in [**Ma**] in the complete intersection case residue currents which satisfy the duality theorem. This construction is more elementary than the one of Coleff and Herrera; it avoids the use of Hironaka's theorem of resolution of singularities, and relies instead only on Weierstrass preparation theorem. However, it is not in general equal to the Coleff-Herrera product.

3. Residue currents of arbitrary ideals

In this section, we will focus on some of the more recent developments in the theory of residue currents.

3.1. Bochner-Martinelli type residue currents. The Coleff-Herrera product is in general not well-behaved when the tuple f does not define a complete intersection. The essential properties like the duality theorem and the transformation law fail, and the definition of the Coleff-Herrera product depends in a very essential way on the order of the functions. Motivated by such issues, Passare, Tsikh and Yger introduced the following object in [**PTY**].

Definition 2. Let $f = (f_1, \dots, f_m)$ be a tuple of holomorphic functions. For $I = \{i_1, \dots, i_p\} \subseteq \{1, \dots, m\}$, the *Bochner-Martinelli type residue current* associated to f is defined as

$$R^f_I.\varphi := \sum_{k=1}^{p} \int \bar\partial |f|^{2\lambda} \wedge \frac{(-1)^{k-1} \bar f_{i_k} d\bar f_{i_1} \wedge \cdots \widehat{d\bar f_{i_k}} \cdots \wedge d\bar f_{i_p}}{|f|^{2p}} \wedge \varphi \Bigg|_{\lambda=0},$$

where $\widehat{d\bar f_{i_k}}$ means that this term is removed.

The reason for the name is that the kernel here is very much related to the classical Bochner-Martinelli kernel, in fact, if $f = (z_1, \dots, z_n)$ and $I = (1, \dots, n)$, then the integral becomes exactly the Bochner-Martinelli formula (in the classical formulation, the factor $\bar\partial |z|^{2n}$ should be replaced by instead integration over $\{|z| = r\}$).

If $m = n$ and f defines a complete intersection, then it is classical that $R^f = R^f_{\{1,\dots,n\}}$ provides a concrete representation of the cohomological Grothendieck residue, see [**Harv**], [**Ton2**] and [**GH**]. In particular, its action on a test form which is $\bar\partial$-closed close to the zero set of f equals the action of the Coleff-Herrera product. This in fact holds more generally. The following is Theorem 4.1 in [**PTY**].

Theorem 3.1. *Let $f = (f_1, \dots, f_p)$ be a tuple of holomorphic functions defining a complete intersection. Then,*

$$R^f_{\{1,\dots,p\}} = \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}.$$

An alternative proof of this can be found in [**An6**].

The definition of Bochner-Martinelli type residue currents had also appeared earlier, for example in the book by [**BGVY**], although it was more extensively developed in [**PTY**]. Using formalism and techniques from previous work on integral formulas, see [**An1**], Andersson proved in [**An3**] that the annihilator of the Bochner-Martinelli type residue current is included in the ideal generated by f. In particular, this provided a proof of the Briançon-Skoda theorem, as had previously also been proved by means of residue currents in [**BGVY**].

The annihilator of R^f is contained in $\mathcal{J}(f)$, but without equality in general. However, as described above, it can be used to solve *division problems* like the *Briançon-Skoda theorem*, and it has been used in several ways in relation to division problems, see for example [**An4**], [**AG**] and [**Wu3**]. For a survey about such techniques, see [**AW3**]. A more detailed study of the annihilator of the Bochner-Martinelli current can be found in [**Wu1**] and [**JW**].

In [**VY**], Bochner-Martinelli type residue currents were used to prove generalizations of *Jacobi's residue formula* (cf., Remark 1) in the projective and more generally the toric setting. Further generalizations to singular projective subvarieties were considered in [**BVY**], while introducing Bochner-Martinelli type residue currents also on singular varieties. Considering the Bochner-Martinelli current as an object associated to global sections of line bundles in [**An3**] allowed for simplifications of the arguments in [**VY**] in the projective setting, something which was also developed in the toric setting in [**Sh**].

In [**BY3**], a weighted version of the Bochner-Martinelli current was introduced in the spirit of Lipman, [**Li**], and it was used to construct *Green currents* in *intersection theory*, in the case of non-proper intersections. Later work in a similar spirit can be found in [**An5**], [**Mé**], [**ASWY**]. More detailed investigations related to weighted Bochner-Martinelli currents can be found [**Wu4**].

3.2. Free resolutions in analytic geometry. Modern complex analysis relates to many other areas of mathematics. A cornerstone is of course analysis, but of great importance are also methods of sheaf theory, homological algebra and commutative algebra. Inspiration for such connections can be found in differential topology, with de Rham cohomology connecting topology and analysis of manifolds, see for example [**Mo**] and [**BT**]. In complex analysis, there is a corresponding connection given by Dolbeault cohomology, see for example [**Gu3**], [**Ar**] and [**GH**].

What we will be concerned with is more regarding the connection between homological algebra and commutative algebra, which in turn is intimately connected with algebraic geometry. One of the pioneering results in this direction was the result of Auslander, Buchsbaum and Serre, characterizing regular local rings as precisely those having finite global dimension. A canonical reference for such questions is the extensive book by Eisenbud, [**E1**]. See also [**E2**] for both classical and modern examples of the utility of free and projective resolutions in the study of algebro-geometric questions, and also [**ILL$^+$**] for other examples.

We will consider here an elementary example of how to answer a geometrical question by using homological algebra. For this part, we consider the ring $\mathcal{O} = \mathcal{O}_0$ of *germs of holomorphic functions* at the origin in $\mathbb{C}^n$, i.e., holomorphic functions defined in some neighbourhood of 0. In this ring, we have unique factorization, i.e., any holomorphic functions factors uniquely as a product of irreducible functions (up to invertible holomorphic functions), see [**Dem**], Theorem 2.10.

Example 2. Consider a complete intersection $f = (f_1, f_2)$ in $\mathcal{O}_0$, i.e., assume that $\operatorname{codim} Z(f_1, f_2) = 2$. This will be the case for any two holomorphic functions vanishing at the origin chosen "at random" (without making this more precise). If we factorize f_1 and f_2, then the complete intersection assumption means precisely that f_1 and f_2 have no factors in common, since if they did have a common factor, then the zero set of that common factor would have codimension 1.

Assume now that we have holomorphic functions a_1, a_2 such that $a_1 f_1 + a_2 f_2 = 0$, i.e., $a_1 f_1 = -a_2 f_2$. By the assumptions above, and by unique factorization, all irreducible factors of f_2 must be factors of a_1, and similarly for f_1 and a_2. We can thus divide by f_1 and f_2 and get that $a_1/f_2 = a_0 = -a_2/f_1$ for some holomorphic function a_0, i.e., $(a_1, a_2) = (-f_2, f_1)a_0$. This means that we get an exact complex

$$0 \to \mathcal{O} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 2} \xrightarrow{\varphi_1} \mathcal{O} \to \mathcal{O}/\mathcal{J}(f_1, f_2) \to 0, \tag{3.1}$$

where

$$\varphi_2 = \begin{bmatrix} -f_2 \\ f_1 \end{bmatrix} \text{ and } \varphi_1 = \begin{bmatrix} f_1 & f_2 \end{bmatrix},$$

i.e., at each step in (3.1), the kernel of the morphism to the right equals the image of the morphism to the left (we verified exactness only at $\mathcal{O}^{\oplus 2}$, the other ones are easier). Since the complex is exact and all the modules here (except for the right-most one) are free, (3.1) is called a *free resolution* of $\mathcal{O}/\mathcal{J}(f_1, f_2)$.

For a complete intersection $f = (f_1, \dots, f_p)$ of codimension $p > 2$, there is a similar explicit construction of a free resolution of $\mathcal{O}/\mathcal{J}(f)$ of length p, called the *Koszul complex*, see for example [**E1**], Chapter 17.

Example 3. Let $\mathcal{J}$ be the ideal $\mathcal{J} = \mathcal{J}(xy, xz, yz)$ in $\mathcal{O} = \mathcal{O}_{\mathbb{C}^3,0}$. Geometrically, $\mathcal{J}$ corresponds to the union of the coordinate axes, i.e.,

$$Z = Z(\mathcal{J}) = \{x = y = 0\} \cup \{x = z = 0\} \cup \{y = z = 0\}. \tag{3.2}$$

Hence, Z has codimension 2, but we used 3 functions above to define $\mathcal{J}$. We saw above, related to the Coleff-Herrera product, that complete intersection ideals were in general better behaved, so it is natural to ask whether $\mathcal{J}$ is also a complete intersection ideal, i.e., can we make a better choice of generators than the above ones, so that $\mathcal{J}$ is generated by only 2 functions?

To answer this, we construct first a free resolution of $\mathcal{O}/\mathcal{J}$. The first part of the free resolution is just the generators $\varphi_1 : \mathcal{O}^{\oplus 3} \to \mathcal{O}$, $\varphi_1 = [xy \quad xz \quad yz]$.

Assume now that $a = [a_1 \quad a_2 \quad a_3]^t$ lies in $\ker \varphi_1$, i.e.,

$$a_1 xy + a_2 xz + a_3 yz = 0. \tag{3.3}$$

We then get that

$$a_1 xy = (-a_2 x - a_3 y)z,$$

and by unique factorization, a_1 must be divisible by z, i.e., $a_1 = z a_{10}$ for some holomorphic function a_{10}. In the same way, one gets that $a_2 = y a_{20}$ and $a_3 = x a_{30}$ for some holomorphic functions a_{20} and a_{30}. Inserting this into (3.3) and dividing by xyz, we then get that $a_{30} = -a_{10} - a_{20}$, so that

$$a = a_{10} \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix} + a_{20} \begin{bmatrix} 0 \\ y \\ -x \end{bmatrix}. \tag{3.4}$$

It is also directly verified that the vectors in (3.4) both lie in $\ker \varphi_1$, so they generate $\ker \varphi_1$, and also that they are linearly independent, i.e.,

$$0 \to \mathcal{O}^{\oplus 2} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 3} \xrightarrow{\varphi_1} \mathcal{O} \to \mathcal{O}/\mathcal{J}, \tag{3.5}$$

is a free resolution of $\mathcal{O}/\mathcal{J}$, where

$$\varphi_2 = \begin{bmatrix} z & 0 \\ 0 & y \\ -x & -x \end{bmatrix} \text{ and } \varphi_1 = \begin{bmatrix} xy & xz & yz \end{bmatrix}.$$

Now, there is a concept of minimality of free resolutions over a local ring like $\mathcal{O}_{\mathbb{C}^3,0}$, which has the unique maximal ideal $\mathfrak{m} = \mathcal{J}(x,y,z)$. A free resolution is *minimal* if all the entries of the morphisms in the free resolution lie in $\mathfrak{m}$. In particular, the free resolution (3.5) is minimal. Minimal free resolutions are unique up to isomorphism, i.e., up to multiplication with invertible holomorphic matrices, see [**E1**], Theorem 20.2. In particular, the ranks of the modules appearing, in this case, 2, 3 and 1 are uniquely determined. Hence, $\mathcal{O}/\mathcal{J}$ can not have the Koszul complex of a tuple (f_1, f_2) of holomorphic functions defining a complete intersection as a free resolution, since then, the ranks would have been 1, 2 and 1 as in the previous example. Hence, $\mathcal{J}$ is not a complete intersection ideal, i.e., the ideal $\mathcal{J}$ can not be defined by fewer than 3 functions although the codimension is 2.

3.2.1. *Local, semi-global, and global resolutions.* In this thesis, the questions we study are essentially of local nature, i.e., the questions take place in the ring of germs of holomorphic functions at a point. However, much of the constructions regarding residue currents work globally or at least semi-globally, but then, other issues would come into the picture, like for example the existence of global resolutions. In addition, when going from local to global duality, issues like solving the $\bar{\partial}$-equation globally come into the picture. We give some comments here regarding questions of existence of resolutions and some of their properties.

In this part, we will need to use the language of sheaves, and various connected concepts in complex analysis like coherent sheaves and Stein manifolds. The necessary concepts and theorems described and used here can be found in most introductory books in complex analysis in several variables, for example [**Fi**].

What we discussed above was stalkwise exactness, i.e., exactness at the level of germs, and exactness of sheaves is then exactness at each stalk. Let now X be a complex manifold of dimension n, with structure sheaf $\mathcal{O}_X$, and let $\mathcal{O}_x$ be the ring of germs of holomorphic functions at a point $x \in X$. By the fact that $\mathcal{O}_x$ is Noetherian, we get that for any ideal $\mathcal{J}_x \subseteq \mathcal{O}_x$, there exists a free resolution of $\mathcal{O}_x/\mathcal{J}_x$. An important fact is *Hilbert's syzygy theorem*, which says that this free resolution can in fact be chosen of length $\leq n$.

In addition, we mention that if $\operatorname{codim} Z(\mathcal{J}) = p$, then any free resolution must have length $\geq p$. This will follow from an inequality between depth and codimension of an ideal, together with the Auslander-Buchsbaum formula relating depth and the length of a minimal free resolution, see [**E1**], Proposition 18.2 and Theorem 19.9. Ideals for which this inequality is in fact an equality, i.e., when $\mathcal{O}/\mathcal{J}$ has a free resolution of length equal to $\operatorname{codim} Z(\mathcal{J})$, are in many ways more nicely behaved. Such ideals are called *Cohen-Macaulay* ideals. In particular, if $f = (f_1, \dots, f_p)$ defines a complete intersection, then $\mathcal{J} = \mathcal{J}(f)$ is Cohen-Macaulay, since it has the Koszul complex as a free resolution. Another example would be any 0-dimensional ideal by the Hilbert syzygy theorem.

In order for free resolutions to exist not just stalkwise, one needs to consider *coherent ideal sheaves*. If $\mathcal{J}$ is a coherent ideal sheaf, then by Oka's lemma, the local resolution over $\mathcal{O}_x$ will induce a free resolution over $\mathcal{O}_y$ for y in a whole neighbourhood of x. For an ideal sheaf, coherence can be described by that for any $x \in X$, if $f_1, \dots, f_m$ generate $\mathcal{J}_x$, then they should generate $\mathcal{J}_y$ also for y in a neighbourhood of x. By Cartan's theorem, if $\mathcal{J}$ is the ideal sheaf $\mathcal{J}_Y$ of holomorphic functions vanishing on an analytic subvariety $Y \subseteq X$, then $\mathcal{J}$ is coherent.

If X is a *Stein manifold*, a free resolution exists in an even larger domain: For any compact $K \subseteq X$, there exists a neighbourhood $V \supset K$ such that $\mathcal{O}/\mathcal{J}$ has a free resolution on V, see for example [**Fi**], Theorem 7.2.6. One then says that $\mathcal{O}/\mathcal{J}$ has a free resolution *semi-globally*.

If X is instead compact, then there can only exist free resolutions in trivial cases due to the fact that all global holomorphic functions on X are constant. On the other hand, what is in fact of interest is locally free resolutions, i.e., resolutions by vector bundles, which in local trivialisations reduce to free resolutions. Such resolutions can exist globally. For example, on projective space $\mathbb{P}^n$, any coherent sheaf has a resolution by vector bundles of length $\leq n$, see [**Fi**], Theorem 7.5.6.

3.3. Andersson-Wulcan currents. As described above, Bochner-Martinelli type residue currents provide one generalization of the Coleff-Herrera product to the case of non complete intersections, which has turned out to be much more useful than the Coleff-Herrera product in that setting.

However, one aspect which is not generalized is the duality theorem. In [**AW1**], Andersson and Wulcan constructed currents generalizing this aspect, inspired by earlier work on Bochner-Martinelli type residue currents.

In [**An3**], the Bochner-Martinelli type residue currents were constructed as currents associated to the Koszul complex. In [**AW1**], such a current is constructed associated to any generically exact complex of Hermitian vector bundles (in fact, this was introduced in [**An2**], but the key properties of the construction are elaborated in [**AW1**]). If the Hermitian complex (E,φ) is pointwise exact outside of the analytic variety Z of codimension p, then the associated current R^E is a current of the form

$$R^E = R^E_p + \cdots + R^E_m,$$

where m is the length of the complex and R^E_k is a $\operatorname{Hom}(E_0,E_k)$-valued $(0,k)$-current, i.e., if locally, $E_0 \cong \mathcal{O}^{\oplus r_0}$ and $E_k \cong \mathcal{O}^{\oplus r_k}$, then R^E_k is a $r_k \times r_0$ matrix of $(0,k)$-currents. In addition, $\operatorname{supp} R^E_k \subseteq Z$. Without describing the definition of this current more precisely, we mention that it is constructed by analytic continuation, like in one of the definitions of the Coleff-Herrera product, of forms constructed explicitly from the entries of the Hermitian complex.

The key property of this construction is the following.

Theorem 3.2. *Let (E,φ) be a locally free resolution of $\mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ is a coherent ideal sheaf. Then locally,*

$$\operatorname{ann} R^E = \mathcal{J}.$$

In particular, if $f = (f_1,\dots,f_p)$ is a complete intersection, then the Koszul complex of f is a free resolution of $\mathcal{O}/\mathcal{J}$. Thus, the current associated to (E,φ) equals the Bochner-Martinelli current of f, which in turn equals the Coleff-Herrera product of f. Hence, Theorem 3.2 is a generalization of the duality theorem for Coleff-Herrera products associated with complete intersections to arbitrary ideals.

We describe some non-trivial examples of currents as appearing in Theorem 3.2.

Example 4. Let $\mathcal{J} = \mathcal{J}(z^2,zw,w^2) \subseteq \mathcal{O} = \mathcal{O}_{\mathbb{C}^2}$. Then $\mathcal{O}/\mathcal{J}$ has a free resolution (E,φ) of the form

$$0 \to \mathcal{O}^{\oplus 2} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 3} \xrightarrow{\varphi_1} \mathcal{O} \to \mathcal{O}/\mathcal{J} \to 0,$$

and

$$R^E = R^E_2 = \bar\partial \frac{1}{z^2} \wedge \bar\partial \frac{1}{w} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \bar\partial \frac{1}{z} \wedge \bar\partial \frac{1}{w^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that

$$\begin{aligned} \operatorname{ann} R^E &= \operatorname{ann} \bar\partial \frac{1}{z^2} \wedge \bar\partial \frac{1}{w} \cap \operatorname{ann} \bar\partial \frac{1}{z} \wedge \bar\partial \frac{1}{w^2} \\ &= \mathcal{J}(z^2,w) \cap \mathcal{J}(z,w^2) = \mathcal{J}(z^2,zw,w^2). \end{aligned}$$

This example can be found in [**Wu2**], where further computations of residue currents associated to monomial ideals are made, see also [**LW1**].

In Example 4, we see that R has entries which are Coleff-Herrera products. If $\mathcal{O}/\mathcal{J}$ is Cohen-Macaulay, then it will in fact always be the case that $R^{\mathcal{J}}$ can be written as a vector of holomorphic functions times a Coleff-Herrera product of a complete intersection, see [**Lä2**, Example 2] (Paper III). This complete intersection will however not in general have support on $Z(\mathcal{J})$. On the other hand, one can express $R^{\mathcal{J}}$ in terms of Bochner-Martinelli currents with support on $Z(\mathcal{J})$, see [**An6**], and [**Lä2**] (Paper III) for an explicit construction.

However, it will in general not be possible to express $R^{\mathcal{J}}$ as a vector of holomorphic functions times a Coleff-Herrera product.

Example 5. Let $\mathcal{J} = \mathcal{J}(xz, xw, yz, yw) \subseteq \mathcal{O} = \mathcal{O}_{\mathbb{C}^4}$. Then, $\mathcal{O}/\mathcal{J}$ has a minimal free resolution of the form

$$0 \to \mathcal{O} \xrightarrow{\varphi_3} \mathcal{O}^{\oplus 4} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 4} \xrightarrow{\varphi_1} \mathcal{O} \to \mathcal{O}/\mathcal{I}_Z.$$

Note that $Z(\mathcal{J}) = \{x = y = 0\} \cup \{z = w = 0\}$, i.e., it is the union of two coordinates planes intersecting at the origin. This zero set has codimension 2, while $\mathcal{O}/\mathcal{J}$ has a minimal free resolution of length 3, so $\mathcal{O}/\mathcal{J}$ is not Cohen-Macaulay. Outside of $\{0\}$, we have

$$R_2^E = \frac{1}{|z|^2 + |w|^2} \wedge \bar\partial \frac{1}{y} \wedge \bar\partial \frac{1}{x} \begin{bmatrix} \bar z \\ \bar w \\ 0 \\ 0 \end{bmatrix} + \frac{1}{|x|^2 + |y|^2} \wedge \bar\partial \frac{1}{w} \wedge \bar\partial \frac{1}{z} \begin{bmatrix} 0 \\ 0 \\ \bar x \\ \bar y \end{bmatrix}.$$

This is Example 4 in [**Lä2**] (Paper III). The formula can in fact be given meaning also over $\{0\}$, see the discussion in the example.

A few remarks are in order after these examples. First of all, by a rather simple calculation, one notes that

$$R = \bar\partial \frac{1}{y} \wedge \bar\partial \frac{1}{x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \bar\partial \frac{1}{w} \wedge \bar\partial \frac{1}{z} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has annihilator $\operatorname{ann} R = \mathcal{J}(x, y) \cap \mathcal{J}(z, w) = \mathcal{J}(xz, xw, yz, yw)$. Hence, there are in general "simpler" currents which have the same annihilator. However, this more complicated form of R^E can be motivated by that it makes it satisfy additional important properties, so that it fits into the framework of integral formulas as in [**An2**].

We also notice that in the examples above, R^E was vector-valued, i.e., consisted of several different currents, while for the Coleff-Herrera product, one single current was sufficient. However, one current will in general not suffice to have duality, see for example the beginning of Section 6 in [**Lä1**] (Paper II).

Several of the applications described above for Coleff-Herrera products and Bochner-Martinelli type residue currents have allowed for generalizations by using Andersson-Wulcan currents. In [**AW1**], the currents were used to provide explicit realizations of the Ehrenpreis-Palamodov fundamental principle, generalizing previous work using the Coleff-Herrera product. It was also described how such currents could be used in relation to division problems for polynomial ideals, something which has been studied rather extensively recently, see for example [**AnNi**], [**ASS**], [**AW2**], [**Sz1**] and [**Sz2**]. They have also been used in relation to the $\bar{\partial}$-equation on singular varieties, see [**AS1**] and [**AS2**]. In [**Lu**], Lundqvist used ideas from [**AW1**] to prove, by more elementary means, generalizations of the Grothendieck duality theorem, generalizing previous work by Passare, [**P2**].

4. Holomorphic functions on analytic varieties

Much of this thesis concerns residue currents and in particular, duality theorems on singular varieties. An important part in the duality theorem is of course holomorphic functions. On a singular variety, there are different notions of holomorphic functions, and we will discuss these different notions in this section. In particular, we will try to elaborate these definitions by considering one of the most basic examples of a singular variety, a cusp, and finish by discussing some consequences of the different notions of holomorphicity for residue theory on singular varieties.

Throughout this section, we let Z be an *analytic subvariety* of $\Omega \subset \mathbb{C}^n$, where Ω is an open subset of $\mathbb{C}^n$, i.e., Z is defined by equations $Z = \{h_1 = \cdots = h_m = 0\}$, where h_i are holomorphic functions on Ω. (This should hold locally, i.e., the h_i are in generally only locally defined. We will in this section be a bit sloppy regarding issues related to what should hold locally/globally, which is best formulated in the language of sheaves). We denote the ring of holomorphic functions on Ω by $\mathcal{O}_\Omega$. The first part of this section discusses various basic concepts in the theory of singular varieties. Good references for this part would be for example [**Gu2**], [**KK**] and [**Dem**].

Example 6. We let $C \subset \mathbb{C}^2$ be the *cusp* defined by $C = \{(z,w) | z^3 - w^2 = 0\}$. It is rather easily verified that C can be parametrized by

$$\pi : \mathbb{C} \to C,\ \pi(t) = (t^2, t^3).$$

If we let $h = z^3 - w^2$, then, since $dh \neq 0$ outside of 0, we get by the implicit function theorem that C is smooth outside of 0, i.e., $C \setminus \{0\}$ is a complex manifold.

On the *regular part* Z_{reg} of Z, i.e., where Z is a complex manifold, there is a natural notion of holomorphic functions, functions which are holomorphic in local coordinates, while in the *singular part* Z_{sing} of Z, it is not as clear what holomorphic functions should be. One natural definition of holomorphic functions on Z is to consider restrictions to Z of holomorphic functions in the ambient space. Phrased slightly differently, we get the following definition.

Definition 3. The ring of *strongly holomorphic* functions $\mathcal{O}_Z$ on Z is the quotient ring $\mathcal{O}_Z = \mathcal{O}_\Omega/\mathcal{I}_Z$, where $\mathcal{I}_Z$ is the ideal of holomorphic functions on Ω vanishing on Z.

On the regular part, it is easily verified that this definition coincides with the usual one described above. This is the most common notion of holomorphic functions on an analytic variety, and is mostly referred to as just holomorphic functions on Z.

Example 7. Let C be the cusp as above. Note that $h(z) = z^3 - w^2$ vanishes on C. We claim that in fact h generates the ideal $\mathcal{I}_C$ of holomorphic functions on $\mathbb{C}^2$ vanishing on C, i.e., $\mathcal{I}_C = \mathcal{J}(h)$. Consider $\varphi \in \mathcal{I}_C$. By Weierstrass division theorem,

$$\varphi(z,w) = \varphi_1(z) + w\varphi_2(z) + (w^2 - z^3)\varphi_3(z,w).$$

Since φ vanishes on C, we get, by inserting that $(z,w) = (t^2,t^3)$ on C, that $\varphi(t^2,t^3) = \varphi_1(t^2) + t^3\varphi_2(t^2) \equiv 0$, and by a Taylor expansion, we get that $\varphi_1 = \varphi_2 \equiv 0$. Hence, $\varphi(z,w) = (z^3 - w^2)\varphi_3(z,w)$, so $\mathcal{I}_C = \mathcal{J}(z^3 - w^2)$, and we get that

$$\mathcal{O}_C \cong \mathcal{O}_\Omega/\mathcal{J}(z^3 - w^2). \tag{4.1}$$

We can also describe holomorphic functions on C with the help of the parametrization π above. Consider a holomorphic function $\varphi(z,w) \in \mathcal{O}_\Omega/\mathcal{I}_Z$. The pullback $\pi^*\varphi(t) = \varphi(t^2,t^3)$ is independent of the choice of representative $\varphi(z,w) \in \mathcal{O}_\Omega$ since $\pi^*(z^3 - w^2) = 0$, so we get a holomorphic function $\tilde{\varphi}(t) = \varphi(t^2,t^3)$. Thus, $\tilde{\varphi}(t)$ is a holomorphic function in t, with no linear term t, since the term t can not be obtained by products of t^2 and t^3, while all other terms t^k can. Conversely, any such holomorphic function in t will be the pullback of a holomorphic function in (z,w), so

$$\mathcal{O}_C \cong \mathbb{C}\{\{t^2,t^3\}\} = \left\{\tilde{\varphi}(t) = \sum_{k\neq 1} a_k t^k\right\}. \tag{4.2}$$

If $(z,w) \in C$, then $(z,w) = (t^2,t^3)$, so the correspondence between (4.1) and (4.2) is that

$$\varphi(z,w) = \tilde{\varphi}(t), \text{ where } (z,w) = (t^2,t^3) \in C \tag{4.3}$$

This example, and in particular (4.3) suggests another possible class of holomorphic functions on C; the functions corresponding to all holomorphic functions in t, and not just holomorphic in t^2 and t^3 as in (4.2). This gives rise to the following definition.

Definition 4. The ring of *weakly holomorphic* functions $\tilde{\mathcal{O}}_Z$ on Z is the ring of those holomorphic functions on Z_{reg} which are locally bounded near Z_{sing}.

All the statements below about weakly holomorphic functions can be found in [**Dem**], Section II.7.

Example 8. We consider a weakly holomorphic function $\varphi \in \tilde{\mathcal{O}}_C$ on the cusp C. Since the parametrization π is holomorphic and φ is holomorphic on $C_{\text{reg}} = C \setminus \{0\}$, we get in the same way as in (4.3) a holomorphic function $\tilde{\varphi}$ on $\mathbb{C} \setminus \{0\}$. Since φ is bounded near $C_{\text{sing}} = \{0\}$, $\tilde{\varphi}$ will be bounded near $\{0\}$, and by the Riemann theorem on removable singularities, $\tilde{\varphi}$ has a holomorphic extension over $\{0\}$. We thus get that

$$\tilde{\mathcal{O}}_C \cong \mathbb{C}\{\{t\}\} = \left\{ \tilde{\varphi}(t) = \sum_{k=0}^{\infty} a_k t^k \right\},$$

where the correspondence is as in (4.3).

Note that the function corresponding to $\tilde{\varphi}(t) = t$, which is weakly but not strongly holomorphic is the pullback of the meromorphic function w/z.

Proposition 4.1. *For $z \in Z$, there exists a strongly holomorphic function $h \in \mathcal{O}_Z$ near z which does not vanish identically on any irreducible component of Z such that $h\tilde{\mathcal{O}}_Z \subseteq \mathcal{O}_Z$.*

The function h is called a *universal denominator*. The reason is that any function $\varphi \in \tilde{\mathcal{O}}_Z$ can be written as $\varphi = g/h$, where g and h are strongly holomorphic. In particular, any weakly holomorphic function is meromorphic.

We note that on the cusp above, the weakly holomorphic functions on C correspond to holomorphic functions on $\mathbb{C}$. This is a more general construction. First of all, the analytic variety Z, or more generally an analytic space Z, is said to be *normal* if the ring of weakly holomorphic functions on Z coincide with the ring of strongly holomorphic functions. An analytic space is a space which locally looks like a subvariety of an open set in $\mathbb{C}^n$, in the same way as a manifold is a space which locally looks like an open set in $\mathbb{C}^n$.

Definition 5. A *normalization* of an analytic space Z is a normal analytic space $\tilde{Z}$ and a proper finite holomorphic map $\pi : \tilde{Z} \to Z$ such that $\pi|_{\tilde{Z} \setminus \pi^{-1}(Z_{\text{sing}})} : \tilde{Z} \setminus \pi^{-1}(Z_{\text{sing}}) \to Z_{\text{reg}}$ is biholomorphic.

A holomorphic map $\pi : \tilde{Z} \to Z$ is finite if $\pi^{-1}(z)$ consists of a finite number of points, and it is proper if the inverse image of a compact set is compact. One can rather easily verify that $\pi : \mathbb{C} \to C$ in the example above is a normalization of the cusp C. We have formulated the definition of normalization not in terms of analytic subvarieties of Ω, but in terms of analytic spaces. The reason is, as in the example with the cusp above, the normalization of a subvariety of an open set Ω is not in a natural way itself a subvariety of Ω.

Of crucial importance here will be: 1) Normalizations always exist, and 2) Normalizations are unique (up to isomorphism). The following relates the normalization to weakly holomorphic functions: In the language of sheaves, $\pi_* \mathcal{O}_{\tilde{Z}} = \tilde{\mathcal{O}}_Z$, where $\pi : \tilde{Z} \to Z$ is the normalization. Concretely, this means that for any weakly holomorphic function $\varphi \in \tilde{\mathcal{O}}_Z$, there exists a holomorphic function $\tilde{\varphi} \in \mathcal{O}_{\tilde{Z}}$ such that $\tilde{\varphi} = \varphi \circ \pi$, and conversely, any holomorphic function $\tilde{\varphi} \in \mathcal{O}_{\tilde{Z}}$ gives rise to a weakly holomorphic function $\varphi \in \tilde{\mathcal{O}}_Z$.

Having discussed two different classes of holomorphic functions on analytic varieties, we now start to consider residue currents on analytic varieties. First of all, for a strongly holomorphic function $f \in \mathcal{O}_Z$, defining the principal value current $1/f$ and the residue current $\bar{\partial}(1/f)$ will work in exactly the same way as in the smooth case in Section 2. Since the proof relies on resolution of singularities, allowing Z to be singular will make no difference. We will also have that $f(1/f) = 1$ and $g\bar{\partial}(1/f) = 0$ if $f \mid g$, see the alternative proof of Proposition 1.7, i.e., $\mathcal{J}(f) \subseteq \operatorname{ann} \bar{\partial}(1/f)$. As for the other inclusion, we will consider that below.

We consider now how to define such currents also associated to a weakly holomorphic function $f \in \tilde{\mathcal{O}}_Z$. One way of defining the principal value current $1/f$ would be to use the fact above, that f is meromorphic, i.e., $f = g/h$, where g and h are strongly holomorphic, and then define $1/f := h(1/g)$. For our purposes, another equivalent definition will be preferable. By the description above, the pull-back $\tilde{f} := \pi^* f$ is strongly holomorphic, where $\pi : \tilde{Z} \to Z$ is the normalization of Z. Hence, we can define $1/\tilde{f}$ as a current on $\tilde{Z}$, and then we let $1/f := \pi_*(1/\tilde{f})$. So for $f, g \in \tilde{\mathcal{O}}_Z$, we define

$$g\bar{\partial}\frac{1}{f} := \pi_*\left(\tilde{g}\bar{\partial}\frac{1}{\tilde{f}}\right),$$

where $\tilde{f} = \pi^* f$, $\tilde{g} = \pi^* g$ and π_* is the push-forward on currents. In particular, since this holds in the normalization, we get that if $f \mid g$ in $\tilde{\mathcal{O}}$, then $g\bar{\partial}(1/f) = 0$.

Example 9. We can use the discussion above to draw some conclusions about the annihilator of the residue current $\bar{\partial}(1/g)$, also for g strongly holomorphic. Consider the weakly holomorphic function $f = w/z|_C$, on the cusp C as above. If we let $g = z|_C$ and $h = w|_C$, which are both strongly holomorphic, then since $f = h/g \in \tilde{\mathcal{O}}_C$, $g \mid h$ in $\tilde{\mathcal{O}}_C$. Thus, $h\bar{\partial}(1/g) = 0$, while $h \notin \mathcal{J}(g)$ in $\mathcal{O}_C$, since $h/g = f \in \tilde{\mathcal{O}}_C \setminus \mathcal{O}_C$. Hence, the duality theorem does not hold for strongly holomorphic functions on C.

This poses the natural question: Does the duality theorem hold if we consider $\tilde{\mathcal{O}}_C$ instead of $\mathcal{O}_C$? In fact, it does not either. If we consider the meromorphic function $f = g/h$, where g and h are as above, then $\pi^* f = 1/t$, so since $\pi^* f$ is unbounded near 0, f is not weakly holomorphic. However,

$g\bar\partial(1/h) = 0$ since

$$g\bar\partial\frac{1}{h}.\varphi(z,w)dz = \pi_*\left(\bar\partial\frac{1}{t}\right).\varphi(z,w)dz$$
$$= \bar\partial\frac{1}{t}.\varphi(t^2,t^3)dt^2 = (2t\varphi(t^2,t^3))|_{t=0} = 0,$$

by (1.13), and similarly, $g\bar\partial(1/h).\varphi(z,w)dw = 0$ for any test function φ. Since $f = g/h$ is not weakly holomorphic, the duality theorem does not hold over $\tilde{\mathcal{O}}_C$.

This example motivates the definition of a last class of holomorphic functions on a singular variety. On a smooth variety, the holomorphic functions are the $\bar\partial$-closed smooth functions, but by Weyl's lemma (see the alternative proof of Proposition 1.7 when $n = 1$, or [**Dem**], (3.29)), it could equally well be defined as the $\bar\partial$-closed $(0,0)$-currents. As we saw in the example above, the function f corresponding to $1/t$ in the normalization was $\bar\partial$-closed, but not smooth, so in particular, Weyl's lemma does not hold on C.

Definition 6. A meromorphic function ψ on Z is said to be *Barlet-Henkin-Passare holomorphic*, denoted $\psi \in \omega_Z^0$, if $\bar\partial\psi = 0$ in the sense of currents.

Note that we assumed here that ψ was a meromorphic function instead of an arbitrary $(0,0)$-current. However, we could in fact equally well have assumed that ψ was an arbitrary $(0,0)$-current. Since a $\bar\partial$-closed $(0,0)$-current is smooth on the regular part of Z, ψ coincides with a holomorphic function on Z_{reg}. By [**HePa**], Theorem 1, ψ then is the principal value current of a meromorphic function.

The definition we used here is due to Henkin and Passare in [**HePa**]. In [**Ba1**], Barlet had introduced a more algebraic formulation of the same object, and connections to the definition we use were discussed. This relation was elaborated in [**Bj1**], see [**HePa**], Remark 5.

This class of holomorphic functions is however not a ring. For example, letting $f = g/h$ be as the example above, then $f^2 = 1/z|_C =: \beta$, which can easily be verified, is not $\bar\partial$-closed. To see this, we just notice that

$$\bar\partial\beta.\varphi(z,w)dz = \bar\partial\frac{1}{t^2}.\varphi(t^2,t^3)dt^2 = (2\pi i)2\varphi(0).$$

by (1.13). Since ω_C^0 is not a ring, the formulation of the duality theorem over ω_C^0 does not make sense since we can not talk about the ideal generated by a Barlet-Henkin-Passare holomorphic function f.

Hence, we have seen that for the Coleff-Herrera product, the duality theorem does not hold over any of these classes of holomorphic functions on C. We finish this section with some examples of currents which indeed have certain prescribed annihilator ideals over the ring of strongly holomorphic functions.

Proposition 4.2. *Let C be the cusp as above, and let $\pi : \mathbb{C} \to C$ be the normalization $\pi(t) = (t^2, t^3)$ of C. Let $\omega = dz/(2w)|_C$, and let*

$$R^z := \bar\partial\left(\frac{1}{z}\omega\right) = \pi_*\left(\bar\partial\frac{1}{t^4}\wedge dt\right) \text{ and } R^w := \bar\partial\left(\frac{1}{w}\omega\right) = \pi_*\left(\bar\partial\frac{1}{t^5}\wedge dt\right).$$

Then

$$\operatorname{ann}_{\mathcal{O}_C} R^z = \mathcal{J}(z) \text{ and } \operatorname{ann}_{\mathcal{O}_C} R^w = \mathcal{J}(w).$$

It is no coincidence that ω appears in both R^z and R^w, and in fact, we will see that we can replace $(1/z)\omega$ or $(1/w)\omega$ by $(1/f)\omega$ for any strongly holomorphic function f on C, and get a current with annihilator $\mathcal{J}(f)$. However, here, we stick to this case since it can by proven by completely elementary means.

Proof. Note that by (1.13),

$$\bar\partial\frac{1}{t^4}.\phi(t)dt = c\frac{\partial^3}{\partial t^3}\phi(0)$$

for some constant $c \neq 0$. Hence, if $g \in \operatorname{ann}_{\mathcal{O}_C} R^z$, then by taking the test function φ to be identically t^ℓ in a neighbourhood of the origin, we get that

$$\frac{\partial^3}{\partial t^3}\left(t^\ell g(t^2,t^3)\right)\Big|_{t=0} = 0 \tag{4.4}$$

for $\ell = 0, 1, \ldots$. In particular, taking $\ell = 3$, we get that $g(0,0) = 0$. Hence, $g = zg_1 + wg_2$ for some holomorphic functions g_1 and g_2. Inserting this into (4.4), and taking $\ell = 0$, we get

$$0 = 2\frac{\partial}{\partial t}g_1(t^2,t^3) + 6g_2(t^2,t^3)\Big|_{t=0} = 6g_2(0).$$

Thus, $g_2 = zg_{21} + wg_{22}$ for some holomorphic functions g_{21} and g_{22}, so regrouping, we get that $g = zh_1 + w^2h_2$. Since $w^2 = z^3$ in $\mathcal{O}_C$, we thus get that $g = zg_0$, i.e., $g \in \mathcal{J}(z)$. It thus remains to see that $z \in \operatorname{ann}_{\mathcal{O}_C} R^z$. This follows because

$$\bar\partial\frac{1}{t^2}\wedge dt.\varphi(t^2,t^3) = c\frac{\partial}{\partial t}\varphi(t^2,t^3)\Big|_{t=0} = 0. \tag{4.5}$$

To prove that $\operatorname{ann}_{\mathcal{O}_C} R^w = \mathcal{J}(w)$ is done in a similar way as for z. If $g \in \operatorname{ann}_{\mathcal{O}_C} R^w$, then by letting ℓ successively in the equation corresponding to (4.4) be equal to $4, 2$ and 0, one arrives at $g = z^3g_1 + wg_2$, which implies that $g = wg_0$ since $z^3 = w^2$ in $\mathcal{O}_C$. For the converse, that w annihilates R^w, one arrives at exactly (4.5). □

5. Summary of papers

Here we give brief summaries of the contents of the papers in this thesis.

5.1. Paper I. In Paper I, we discuss how to construct Coleff-Herrera products and Bochner-Martinelli type residue currents associated to a tuple of weakly holomorphic functions. This generalizes a previous construction by Denkowski in [**Den**], who defined Coleff-Herrera products associated to c-holomorphic functions, i.e., weakly holomorphic functions which are also continuous.

The main idea behind the construction is described in Section 4, we define the currents as the push-forward of currents constructed in the normalization, and can hence reduce it to the case of strongly holomorphic functions. In that way, we are also able to deduce that many properties that hold for strongly holomorphic functions also will hold for weakly holomorphic functions.

Much of the properties we prove are in fact directly reduced to the strongly holomorphic case, but there are a couple of differences arising. Two of them are the following: 1) Weakly holomorphic functions are in general not smooth (in fact, not even continuous in general), so multiplication of a weakly holomorphic function and a current is not necessary well-defined. 2) Weakly holomorphic functions are not necessarily continuous, so it is not in general meaningful to talk about the value of a weakly holomorphic function at a singular point on the variety. In particular, it is not clear what the zero set of a weakly holomorphic function should be. In the article, we illustrate by examples how these issues manifests themselves, and propose solutions to both of them.

5.2. Paper II. As we noted in Example 9, the duality theorem does not hold in general for $\bar{\partial}(1/f)$ on the cusp C. However, it is relatively easy to see that on a normal variety Z, the duality theorem holds for $\bar{\partial}(1/f)$, when f is a strongly holomorphic function on Z. In Paper II, we discuss conditions of which Coleff-Herrera products satisfy the duality principle.

In the case of a single strongly holomorphic function f, we get, with the help of results from Paper I, that the duality theorem holds on Z for all currents $\bar{\partial}(1/f)$ if and only if Z is normal. By Serre's criterion, normality can be described as a condition that certain "singularity subvarieties" associated to Z are sufficiently small. The first such singularity subvariety Z^0 is exactly the singular set, and in general there are more singularity subvarieties $Z^k \subseteq Z_{\text{sing}}$ describing the "complexity" of the singularities of Z.

We get that whether the duality theorem holds for the Coleff-Herrera product of a tuple f or not depends on how the zero set of f intersects these singularity subvarieties associated to the variety. We also get that the duality theorem holds for all complete intersections of codimension q if and only if the variety satisfies a generalization of the normality condition. In particular, we get that the duality theorem holds for all Coleff-Herrera products of complete intersections f on a variety Z if and only if Z is smooth.

5.3. Paper III. The transformation law, Theorem 2.2, is an important tool in the study of Coleff-Herrera products. As described after Theorem 2.2, we can then essentially view the Coleff-Herrera product as a current associated to a complete intersection ideal. From this point of view, the condition in the transformation law, that $f = gA$, can then equivalently be described as that $\mathcal{J}(f) \subseteq \mathcal{J}(g)$.

In Paper III, we discuss a generalization of the transformation law to Andersson-Wulcan currents. The starting point is the idea that if we have two ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$, then one would hope to express the current associated to $\mathcal{J}$ in terms of the current associated to $\mathcal{I}$, just as in the transformation law. In order to describe such a formula, we use that the inclusion of ideals $\mathcal{I} \subseteq \mathcal{J}$ guarantees the existence of a morphism of complexes between free resolutions of $\mathcal{O}/\mathcal{I}$ and $\mathcal{O}/\mathcal{J}$. In case both $\mathcal{I}$ and $\mathcal{J}$ are Cohen-Macaulay ideals of the same dimension, we get the *comparison formula* $R^{\mathcal{J}} = a_p R^{\mathcal{I}}$, where a_p is an explicitly described holomorphic matrix. In particular, if $\mathcal{I}$ and $\mathcal{J}$ are complete intersection ideals, this will be exactly the transformation law. When the ideals are of different dimensions, or are not Cohen-Macaulay, our comparison formula will take a more complicated form, with an extra term appearing.

What led to the discovery of the comparison formula was an attempt to construct currents with prescribed annihilator ideals on singular varieties. This construction is described briefly in the article, and elaborated in Paper IV. We also use the comparison formula to give a proof by means of residue currents of a theorem of Hickel related to the Jacobian determinant of a holomorphic mapping. In addition, the comparison formula is an ingredient in the computations in [**LW1**] of residue currents associated to Artinian monomial ideals. In [**LW1**], we also show how this leads to a factorization of the fundamental cycle of an Artinian monomial ideal in terms of the residue current associated to such an ideal. This will be treated in general in the forthcoming article [**LW2**], also using the comparison formula. This factorization is a generalization of the Poincaré-Lelong formula in the case of a complete intersection, and provides a current version of a previous cohomological result due to Lejeune-Jalabert, [**LJ**].

5.4. Paper IV. Due to the failure in general of the duality theorem to hold for the Coleff-Herrera product on a singular variety, we were led to search for another current associated to an ideal, which have this ideal as annihilator ideal. In Paper IV, we do exactly such a construction.

The starting point of this construction is to take a lifting of the ideal to the ambient space, and consider the corresponding Andersson-Wulcan current associated to this lifting. In Paper III, we describe how this induces a current on the variety with the correct annihilator.

The purpose of Paper IV is to elaborate on this construction, and in particular, give an intrinsic description of this current. This is done with the help of the comparison formula from Paper III. Another important ingredient is the notion of a structure form associated to a singular variety, as constructed in [**AS1**], generalizing the classical Poincaré residue on a complete intersection. The currents constructed on the cusp in Proposition 4.2 are examples of this construction.

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Part II

PAPERS

PAPER I

Residue currents associated with weakly holomorphic functions

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PAPER I

Residue currents associated with weakly holomorphic functions

Richard Lärkäng

Abstract. We construct Coleff-Herrera products and Bochner-Martinelli type residue currents associated with a tuple f of weakly holomorphic functions, and show that these currents satisfy basic properties from the (strongly) holomorphic case. This include the transformation law, the Poincaré-Lelong formula and the equivalence of the Coleff-Herrera product and the Bochner-Martinelli type residue current associated with f when f defines a complete intersection.

1. Introduction

The basic example of a residue current, introduced by Coleff and Herrera in [**CH**], is a current called the *Coleff-Herrera product* associated with a strongly holomorphic mapping $f = (f_1, \dots, f_p)$. The Coleff-Herrera product is defined by

$$\bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p}.\varphi = \lim_{\delta \to 0^+} \int_{\cap\{|f_j| = \epsilon_j(\delta)\}} \frac{\varphi}{f_1 \cdots f_p}, \tag{1.1}$$

where φ is a test form and $\epsilon(\delta)$ tends to 0 along a so-called *admissible path*, which means essentially that $\epsilon_1(\delta)$ tends to 0 much faster than $\epsilon_2(\delta)$ and so on, for the precise definition, see [**CH**]. The Coleff-Herrera product was defined over an analytic space, however, most of the work on residue currents thereafter has focused on the case of holomorphic functions on a complex manifold. The theory of residue currents has various applications, for example to effective versions of division problems etc., see for example [**AW1**], [**BGVY**], [**TY**] and the references therein.

On an analytic space Z, with structure sheaf $\mathcal{O}_Z$, the most common notion of holomorphic functions are the *strongly holomorphic* functions, that is, sections of the structure sheaf, or more concretely, functions which are locally the restriction of holomorphic functions in any local embedding. We will throughout the article assume that Z is an analytic space of pure dimension. In some cases, this can be a little too restrictive, and the *weakly holomorphic* functions might be more natural. These are functions defined on Z_{reg}, which are holomorphic on Z_{reg} and locally bounded at Z_{sing}. Two reasons why these are natural: the ring $\tilde{\mathcal{O}}_{Z,z}$ of germs of weakly holomorphic functions at z is the integral closure of $\mathcal{O}_{Z,z}$ in the ring $\mathcal{M}_{Z,z}$ of germs of meromorphic functions at z, and weakly holomorphic functions correspond to strongly holomorphic functions in any normal modification of $(Z, \mathcal{O}_Z)$. A slightly better behaved but more restrictive notion are the *c-holomorphic* functions, denoted $\mathcal{O}_c$, functions which are weakly holomorphic and continuous on all of Z. We will throughout this article assume that Z is an analytic space of pure dimension.

In a recent article [**D**], Denkowski introduced a residue calculus for c-holomorphic functions, and showed that this calculus satisfies many of the basic properties known from the strongly holomorphic or smooth cases. It is then a natural question to ask what happens in the case of weakly holomorphic functions. However, as in the c-holomorphic case, it is not obvious how to define the associated residue currents.

In the strongly holomorphic case, there are various ways to define the Coleff-Herrera product (for the equivalence of various definitions of the Coleff-Herrera product, also in the non complete intersection case, see for example [**LS**]). The definition we will use is based on analytic continuation as in [**Y**], which was inspired by the ideas in [**At**] and [**BG**] that the principal value current $1/f$ of a holomorphic function f can be defined by $(|f|^{2\lambda}/f)|_{\lambda=0}$. If $f = (f_1, \dots, f_p)$ is strongly holomorphic on Z, we define the Coleff-Herrera product of f by

$$\left.\frac{\bar\partial|f_1|^{2\lambda_1} \wedge \dots \wedge \bar\partial|f_p|^{2\lambda_p}}{f_1 \cdots f_p}\right|_{\lambda_p=0,\dots,\lambda_1=0},$$

where we by $|_{\lambda_p=0,\dots,\lambda_1=0}$ mean that we take the analytic continuation in λ_p to $\lambda_p = 0$, then in λ_{p-1} and so on, see Section 4 for details. Recall that a *modification* of an analytic space Z is a proper surjective holomorphic mapping $\pi : Y \to Z$ from an analytic space Y such that there exists a nowhere dense analytic set $E \subset X$ with $\pi|_{Y\setminus\pi^{-1}(E)} : Y \setminus \pi^{-1}(E) \to X \setminus E$ being a biholomorphism. It is easy to see by analytic continuation, that if $\pi : Y \to Z$ is a modification of Z, then the Coleff-Herrera product of f can be defined as the push-forward of the Coleff-Herrera product of $f' := \pi^* f$. For weakly holomorphic functions, we can use this observation to define the Coleff-Herrera product, since the pull-back of a weakly holomorphic function to the normalization is strongly holomorphic. If f is weakly holomorphic, we define the Coleff-Herrera product of f by

$$\mu^f := \bar\partial\frac{1}{f_1} \wedge \dots \wedge \bar\partial\frac{1}{f_p} := \pi_*\left(\bar\partial\frac{1}{f_1'} \wedge \dots \wedge \bar\partial\frac{1}{f_p'}\right), \tag{1.2}$$

where $f' = \pi^* f$. By the observation above, this of course coincides with the usual definition in case of strongly holomorphic functions, and this definition is also consistent with the one in [**D**] in the case of c-holomorphic functions, see Proposition 4.1.

Because of our definition, the properties we prove of the Coleff-Herrera product for weakly holomorphic functions can mostly be reduced (by going back to the normalization) to the strongly holomorphic case. Thus the main part of this article concerns giving a coherent exposition of the basic theory of residue currents in the strongly holomorphic case. This is done based on analytic continuation of currents and the notion of pseudomeromorphic currents as introduced in [**AW2**], which is developed on a complex manifold. We will see that this approach works well also with strongly holomorphic functions on an analytic space, and we believe that this might be of independent interest, although most of the results should be known.

However, even for the statement of these properties in the weakly holomorphic case, two problems occur, namely how is multiplication of a weakly holomorphic function with a current defined, and what is the zero set of a tuple of weakly holomorphic functions? And hence also, what should a complete intersection mean?

With regards to defining multiplication of a weakly holomorphic function with a current, we take a similar approach as for the definition of the Coleff-Herrera product. Assume μ is a current on Z, and that there exists a modification $\pi : Y \to Z$, with a current μ' on Y such that $\mu = \pi_* \mu'$ (the existence of such μ' is guaranteed if μ is pseudomeromorphic and Y is the normalization of Z, see the introduction of Section 5). If g is strongly holomorphic on Z, then

$$g\mu = \pi_*(\pi^* g \mu'). \tag{1.3}$$

The right-hand side of (1.3) still exists if g is weakly holomorphic on Z and Y is normal, so we take this as a definition of $g\mu$. However, that this is well-defined depend on the fact that we have a certain "canonical" representative of the Coleff-Herrera product in the normalization (or any normal modification). We will see in Section 5 that (1.3) depends on the choice of representative μ' and can thus not be used to define a general multiplication of weakly holomorphic with currents on Z.

For the zero set of one weakly holomorphic function, all reasonable definitions should coincide. For the zero set of a weakly holomorphic mapping f, it is natural to take into account that the zero sets of the individual components of f can "belong" to different irreducible components. We introduce in Section 2 a notion of common zero set of f, depending on f as a mapping, and not only on the individual components, which however may differ from the intersection of the respective zero sets.

The Coleff-Herrera product μ^f in (1.2) associated with a strongly holomorphic mapping $f = (f_1, \dots, f_p)$ satisfies

$$\operatorname{supp} \mu^f \subseteq Z_f \quad \text{and} \quad \bar\partial \mu^f = 0,$$

where Z_f is the common zero set of f. In addition, if f forms a complete intersection, the Coleff-Herrera product is alternating in the residue factors and

$$(f_1,\dots,f_p) \subseteq \operatorname{ann}\mu^f, \tag{1.4}$$

where $(f_1,\dots,f_p)$ is the ideal generated by $f_1,\dots,f_p$, and $\operatorname{ann}\mu^f$ is the annihilator of μ^f, i.e., the ideal of holomorphic functions g such that $g\mu^f = 0$. We also have the *transformation law* for residue currents (see [**DS**]), which says that if $f = (f_1,\dots,f_p)$ and $g = (g_1,\dots,g_p)$ define complete intersections, and there exists a matrix A of holomorphic functions such that $g = Af$, then

$$(\det A)\bar\partial\frac{1}{g_1}\wedge\cdots\wedge\bar\partial\frac{1}{g_p} = \bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}.$$

The *Poincaré-Lelong formula* relates the Coleff-Herrera product of f and the integration current $[Z_f]$ on Z_f (with multiplicities) and it says that

$$\frac{1}{(2\pi i)^p}\bar\partial\frac{1}{f_1}\wedge\cdots\wedge\bar\partial\frac{1}{f_p}\wedge df_p\wedge\cdots\wedge df_1 = [Z_f].$$

We will see that in fact all those statements still hold also in the weakly holomorphic case. However, as mentioned above, zero sets of weakly holomorphic functions and multiplication of currents with weakly holomorphic functions need to be interpreted in the right way.

Remark 1. The inclusion (1.4) if f defines a complete intersection is one direction of the *duality theorem* proven in [**DS**] and [**P2**], which says that on a complex manifold, the inclusion is in fact (locally) an equality. However, in [**L**], we show that on any singular variety, one can always find a tuple f of strongly holomorphic functions such that the inclusion (1.4) is strict.

Bochner-Martinelli type residue currents were first introduced in [**PTY**] by Passare, Tsikh and Yger (on a complex manifold) as an alternative way of defining a residue current corresponding to a tuple of holomorphic functions. In [**BVY**], Bochner-Martinelli type residue currents were constructed on an analytic space in order to prove a generalization of Jacobi's residue formula, generalizing previous results in [**VY**] in the smooth case.

The Bochner-Martinelli type residue currents give another reason why our definition of Coleff-Herrera product is a natural one. In the smooth case, it was proved in [**PTY**] that if the functions define a complete intersection, then the Coleff-Herrera product and the Bochner-Martinelli current coincide. It is suggested in [**BVY**] that the same statement holds in the singular case with a similar proof. We will construct Bochner-Martinelli type residue currents associated with a tuple of weakly holomorphic functions, and we will show that the equality between the Coleff-Herrera product and the Bochner-Martinelli type residue current holds both in the strongly and weakly holomorphic cases. An advantage of the Bochner-Martinelli current, compared to the Coleff-Herrera product, in the weakly holomorphic case is that it can be defined intrinsically on Z as the analytic continuation of an arbitrarily smooth (depending on a parameter λ) form on Z. In contrast, the Coleff-Herrera product is only defined as the analytic continuation of an arbitrarily smooth form on the normalization of Z.

2. Zero sets of weakly holomorphic functions

The behavior of the currents we define will depend in a crucial way on the zero sets of the weakly holomorphic functions, and in this section we will define the zero set of a weakly holomorphic mapping.

Definition 1. Let $f \in \tilde{\mathcal{O}}(Z)$. If f is not identically zero on all irreducible components of Z, we define the *zero set* of f by $Z_f := \{z \in Z \mid (1/f)_z \notin \tilde{\mathcal{O}}_z\}$. Let Z_α be the irreducible components of Z where f is identically zero, and let $Z' = \overline{Z \setminus \cup_\alpha Z_\alpha}$. Then f does not vanish identically on any of the irreducible components of Z', and we define Z_f as $\cup_\alpha Z_\alpha \cup Z_{f|_{Z'}}$.

Remark 2. We have $z \in Z_f$ if and only if there exists a sequence $z_i \to z$ with $z_i \in Z_{\text{reg}}$ such that $f(z_i) \to 0$ (since if we cannot find such a sequence, then $1/f$ is weakly holomorphic). Hence, when f is c-holomorphic, Z_f coincides with the usual zero set of f, when f is seen as a continuous function.

We will use the following characterization of the zero set of a weakly holomorphic function. However, since this is a special case of Proposition 2.3, we omit the proof.

Lemma 2.1. *Let $\pi : Z' \to Z$ be the normalization of Z. If $f \in \tilde{\mathcal{O}}(Z)$, then Z_f is an analytic subset of Z, and $Z_f = \pi(Z_{\pi^* f})$.*

We recall that an analytic space Z is *normal* if $\mathcal{O}_{Z,z} = \tilde{\mathcal{O}}_{Z,z}$ for all $z \in Z$, and that the *normalization* Z' of an analytic space Z is the unique normal space Z' together with a proper finite surjective holomorphic mapping $\pi : Z' \to Z$ such that $\pi|_{Z' \setminus \pi^{-1}(Z_{\text{sing}})} : Z' \setminus \pi^{-1}(Z_{\text{sing}}) \to Z_{\text{reg}}$ is a biholomorphism, see for example [**G**].

For any meromorphic function ϕ, there is a standard notion of zero set of ϕ, that we denote by Z'_ϕ, which is defined by $Z'_\phi = \{z \in Z \mid (1/\phi)_z \notin \mathcal{O}_z\}$. Since weakly holomorphic functions are meromorphic, this gives another definition of zero set if f is a weakly holomorphic function. Clearly $Z_f \subseteq Z'_f$, but as we see in the following example, the inclusion is in general strict, so the two definitions do not coincide.

Example 1. Let $Z = \{z^3 - w^2 = 0\} \subseteq \mathbb{C}^2$, which has normalization $\pi(t) = (t^2, t^3)$, and let $f = 1 + w/z$. Since $\pi^* f = 1 + t^3/t^2 = 1 + t$, f is weakly holomorphic on Z. Since $\{\pi^* f = 0\} = \{t = -1\}$, we get by Lemma 2.1 that

$$Z_f = \pi(\{t = -1\}) = \{(1, -1)\}.$$

However,

$$Z'_f = P_{1/f} = Z \cap \{z + w = 0\} = \{(t^2, t^3) \mid t^2 = -t^3\} = \{(0,0), (1,-1)\},$$

so $Z_f \subsetneq Z'_f$.

To study the dimension of zero sets of weakly holomorphic functions, we will need the following lemma, which shows that subvarieties of the normalization correspond to subvarieties of Z of the same dimension, and vice versa.

Lemma 2.2. *Let $\pi : Z' \to Z$ be the normalization of Z. If Y' is a subvariety of Z', then $\pi(Y')$ is a subvariety of Z with $\dim Y' = \dim \pi(Y')$, and if Y is a subvariety of Z, then $\pi^{-1}(Y)$ is a subvariety of Z' with $\dim Y = \dim \pi^{-1}(Y)$.*

Proof. The first part follows from Remmert's proper mapping theorem, when formulated as for example in [**G**], since π is a finite proper holomorphic mapping. We get from the first part that $\dim \pi^{-1}(Y) = \dim \pi(\pi^{-1}(Y)) = \dim Y$, where the second equality holds since π is surjective.

□

If $f \in \tilde{\mathcal{O}}(Z)$ and $f \not\equiv 0$ on any irreducible component of Z, then $\operatorname{codim} Z_f = 1$ or $Z_f = \emptyset$. In fact, if $f' = \pi^* f$ and $Z_{f'} \neq \emptyset$, then f' is strongly holomorphic, and $Z_{f'} = \{f' = 0\}$ has codimension 1, and since $Z_f = \pi(Z_{f'})$ by Lemma 2.1, Z_f has codimension 1 by Lemma 2.2. However, as is well-known, in contrast to the smooth case, subvarieties of codimension 1 cannot in general be defined as the zero set of one single strongly holomorphic function. As we will see in the next example, this is the case in general for zero sets of weakly holomorphic functions, even for c-holomorphic functions on an irreducible space.

Example 2. Let $V = \{z_1^3 - z_2^2 = z_3^3 - z_4^2 = 0\} \subset \mathbb{C}^4$. Then V has normalization $\pi : \mathbb{C}^2 \to V$, $\pi(t_1, t_2) = (t_1^2, t_1^3, t_2^2, t_2^3)$, and hence $f = z_2/z_1 - z_4/z_3$ is c-holomorphic since $\pi^* f = t_1 - t_2$. The set $Z_f = \{(t^2, t^3, t^2, t^3)\}$ has codimension 1 in Z. However, there does not exist a holomorphic function in a neighborhood of 0 such that $f(t_1^2, t_1^3, t_2^2, t_2^3) = 0$ exactly when $t_1 = t_2$, since in that case, we could write $f(t_1^2, t_1^3, t_2^2, t_2^3) = (t_1 - t_2)^m u(t_1, t_2)$ for some $m \in \mathbb{N}$, where $u(0,0) \neq 0$, which is easily seen to be impossible. Hence, Z_f is not the zero set of one single strongly holomorphic function.

Example 3. Let $Z = Z_1 \cup Z_2 \subset \mathbb{C}^6$, where $Z_1 = \mathbb{C}^3 \times \{0\}$ and $Z_2 = \{0\} \times \mathbb{C}^3$. Define the functions f and g by

$$f(z) = \begin{cases} z_1 & z \in Z_1 \setminus \{0\} \\ 1 & z \in Z_2 \setminus \{0\} \end{cases} \text{ and } g(z) = \begin{cases} 1 & z \in Z_1 \setminus \{0\} \\ z_4 & z \in Z_2 \setminus \{0\} \end{cases}.$$

Then $f, g \in \tilde{\mathcal{O}}(Z)$, and $Z_f = Z_1 \cap \{z_1 = 0\}$, and $Z_g = Z_2 \cap \{z_4 = 0\}$ which both have codimension 1 in Z. However, $Z_f \cap Z_g = \{0\}$, which has codimension 3. Hence, zero sets of weakly holomorphic functions do not behave as well as one could hope with respect to intersections. If we let $f_1 = f_2 = f$, $f_3 = g$, then $Z_{f_1} \cap Z_{f_2} \cap Z_{f_3} = \{0\}$ has codimension 3, while $Z_{f_1} \cap Z_{f_2} = Z_f$ has codimension 1 at 0 in Z. Hence, if one defines a complete intersection for zero sets of weakly holomorphic functions $f = (f_1, \cdots, f_p)$ by requiring that

$Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension p in Z, then it will not follow in general that $(Z_{f_1} \cap \cdots \cap Z_{f_k}, z)$ has codimension k for $z \in Z_{f_1} \cap \cdots \cap Z_{f_p}$.

Remark 3. Note that for c-holomorphic functions $f = (f_1, \cdots, f_p)$, if $f' = \pi^* f$, where $\pi : Z' \to Z$ is the normalization, then $\pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'}) = Z_{f_1} \cap \cdots \cap Z_{f_p}$. Thus if we say that $f = (f_1, \cdots, f_p)$, where $f_i \in \mathcal{O}_c(Z)$, forms a complete intersection in Z if $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension p, then this holds if and only if f' forms a complete intersection in Z' by Lemma 2.2.

As we see in Example 3, this remark does not hold for weakly holomorphic functions, because there, $Z_f \cap Z_g = \{0\}$, while $Z_{f'} \cap Z_{g'} = \emptyset$. Thus, the straightforward generalization of complete intersection, where the zero set $Z_{f_1} \cap \cdots \cap Z_{f_p}$ is required to have codimension p does not share the same good properties in the weakly holomorphic case as in the strongly holomorphic (or c-holomorphic) case. Because of this, we will use a different definition of both the common zero set of weakly holomorphic functions and of a complete intersection. It coincides with the usual definitions in case of strongly holomorphic or c-holomorphic functions, and with our definition the problems above disappear.

Definition 2. Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic. We define the *common zero set* of f, denoted by Z_f, as the set of $z \in Z$ such that there exists a sequence $z_i \in Z_{\text{reg}}$ with $z_i \to z$, and $f_k(z_i) \to 0$ for $k = 1, \cdots, p$. We will see that Z_f is an analytic subset of Z, and hence we say that f forms a *complete intersection* if Z_f has codimension p in Z.

Note that by Remark 2, this definition is consistent with the definition of Z_f in the case of one function. We also see that in Example 3, $Z_{(f,g)} = \emptyset$, and hence, (f, g) is not a complete intersection in our sense. Just as for one function, we can give a characterization of the zero set with the help of the normalization.

Proposition 2.3. *Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic, and let $f' = \pi^* f$, where $\pi : Z' \to Z$ is the normalization. Then*

$$Z_f = \pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'}), \tag{2.1}$$

and if Z_f is nonempty, then it is an analytic subset of Z of codimension $\leq p$. In general,

$$Z_f \subseteq Z_{f_1} \cap \cdots \cap Z_{f_p}, \tag{2.2}$$

with equality if f is c-holomorphic. In addition, f is a complete intersection if and only if f' is a complete intersection in the normalization.

Proof. If $z' \in Z_{f_1'} \cap \cdots \cap Z_{f_p'}$, then we can take a sequence $z_i' \to z'$ such that $z_i' \in \pi^{-1}(Z_{\text{reg}})$. Then, if we let $z_i = \pi(z_i')$, we get that $f_k(z_i) \to 0$, and hence we have the inclusion $Z_f \supseteq \pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'})$ in (2.1). For the other inclusion, if we have a sequence $z_i \to z$ such that $z \in Z_f$, since π is proper we can choose a convergent subsequence $z_{k_i}' \to z'$ such that $\pi(z_{k_i}') = z_{k_i}$, and since

$z \in Z_f$, we must have $f'(z') = 0$, so $z = \pi(z')$, with $z' \in Z_{f_1'} \cap \cdots \cap Z_{f_p'}$. Now, the fact that Z_f is an analytic subset of Z follows by (2.1) and Remmert's proper mapping theorem, since $Z_{f_i'}$ are analytic subsets of Z'. Since f' is strongly holomorphic, $Z_{f'}$ has codimension $\leq p$, so by (2.1) combined with Lemma 2.2 we get that Z_f has codimension $\leq p$. If f is c-holomorphic, the equality in (2.2) follows by (2.1) since for any continuous mapping f, $Z_{f_1} \cap \cdots \cap Z_{f_p} = \pi(Z_{\pi^* f_1} \cap \cdots \cap Z_{\pi^* f_p})$, and the general case also follows from (2.1) since $\pi(Z_{f_1'} \cap \cdots \cap Z_{f_p'}) \subseteq \pi(Z_{f_1'}) \cap \cdots \cap \pi(Z_{f_p'}) = Z_{f_1} \cap \cdots \cap Z_{f_p}$. Finally, the fact that f is a complete intersection if and only if f' is a complete intersection follows from (2.1) together with Lemma 2.2. □

We note that if $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension $\geq p$, then either $Z_f = \emptyset$, or Z_f has codimension p since by Proposition 2.3, $Z_f \subseteq Z_{f_1} \cap \cdots \cap Z_{f_p}$, and Z_f has codimension at most p. Thus, we could have taken as definition of a complete intersection, that $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension $\geq p$, and our results about complete intersection would still be true. However, it would in general give weaker statements, since it since it might very well happen that $Z_{f_1} \cap \cdots \cap Z_{f_p}$ has codimension $< p$, while Z_f has codimension p. In addition, results depending on the exact zero set, like the Poincaré-Lelong formula, Proposition 8.1, would of course not be true if one would use $Z_{f_1} \cap \cdots \cap Z_{f_p}$ instead of Z_f.

Note also that, if $f = (f_1, \cdots, f_p)$ is a complete intersection and $f_0 = (f_1, \cdots, f_k)$, then (Z_{f_0}, z) has codimension k for $z \in Z_f$, since if $z' \in \pi^{-1}(z)$, then $(Z_{f_0'}, z')$ has codimension k, and hence since π is a finite proper holomorphic mapping, $(Z_{f_0}, z) = \cup_{z_j' \in \pi^{-1}(z)} \pi((Z_{f_0'}, z_j'))$ has codimension k in Z.

3. Pseudomeromorphic currents on an analytic space

We will in this section introduce pseudomeromorphic currents on an analytic space. Pseudomeromorphic currents on a complex manifold were introduced by Andersson and Wulcan in [**AW2**], inspired by the fact that currents like the Coleff-Herrera product and Bochner-Martinelli type residue currents are pseudomeromorphic. Two important properties of pseudomeromorphic currents in the smooth case are the direct analogues of Proposition 3.1 and Proposition 3.2. Since these hold also in the singular case, many properties of residue currents hold also for strongly holomorphic functions by more or less the same argument as in the smooth case.

The pseudomeromorphic currents are intrinsic objects of the analytic space Z, so we begin with explaining what we mean by a current on an analytic space. We will follow the definitions used in [**BH**] and [**HL**]. To begin with, we assume that Z is an analytic subvariety of Ω, for some open set $\Omega \subseteq \mathbb{C}^n$. Then, we define the set of smooth forms of bidegree (p,q) in Z by $\mathcal{E}_{p,q}(Z) = \mathcal{E}_{p,q}(\Omega)/\mathcal{N}_{p,q,Z}(\Omega)$, where $\mathcal{E}_{p,q}(\Omega)$ are the smooth (p,q)-forms in Ω and $\mathcal{N}_{p,q,Z}(\Omega) \subset \mathcal{E}_{p,q}(\Omega)$ are the smooth forms φ such that $i^*\varphi \equiv 0$, where

$i : Z_{\rm reg} \to \Omega$ is the inclusion map. The set of test forms on Z, $\mathcal{D}_{p,q}(Z)$, are the forms in $\mathcal{E}_{p,q}(Z)$ with compact support. With the usual topology on $\mathcal{D}_{p,q}(\Omega)$ by uniform convergence of coefficients of differential forms together with their derivatives on compact sets, we give $\mathcal{D}_{p,q}(Z)$ the quotient topology from the projection $\mathcal{D}_{p,q}(\Omega) \to \mathcal{D}_{p,q}(Z)$. Then, (p,q)-currents on Z, denoted $\mathcal{D}'_{p,q}$, are the continuous linear functionals on $\mathcal{D}_{k-p,k-q}(Z)$, where $k = \dim Z$. However, more concretely, this just means that if μ is a (p,q)-current on Z, then $i_*\mu$ is a $(n-k+p, n-k+q)$-current in the usual sense on Ω that vanishes on forms in $\mathcal{N}_{k-p,k-q,Z}(\Omega)$. Conversely, if T is a $(n-k+p, n-k+q)$-current on Ω, that vanishes on forms in $\mathcal{N}_{k-p,k-q,Z}(\Omega)$, then T defines a unique (p,q)-current T' on Z such that $i_*T' = T$.

It is easy to see that the definitions of smooth forms, test forms and currents are independent of the embedding, and hence by gluing together in the same way one does on a complex manifold, we can define the sheaves of smooth forms, test forms and currents on any analytic space Z. Note in particular that by a smooth function on Z, we mean a function which is locally the restriction of a smooth function in the ambient space.

In $\mathbb{C}$, one can define the principal value current $1/z^n = |z|^{2\lambda}/z^n|_{\lambda=0}$ by analytic continuation, where $|_{\lambda=0}$ denotes that for $\operatorname{Re}\lambda \gg 0$, we take the action of $|z|^{2\lambda}/z^n$ on a test form and take the value of the analytic continuation to $\lambda = 0$, which is easily seen to exist by a Taylor expansion, or integration by parts. Thus, if α is a smooth form on $\mathbb{C}^n$ and $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$, with i_j disjoint, then one gets a well-defined current

$$\frac{1}{z_{i_1}^{n_1}} \cdots \frac{1}{z_{i_k}^{n_k}} \bar\partial \frac{1}{z_{i_{k+1}}^{n_{k+1}}} \wedge \cdots \wedge \bar\partial \frac{1}{z_{i_m}^{n_m}} \wedge \alpha \tag{3.1}$$

on $\mathbb{C}^n$ by taking $\bar\partial$ in the current sense together with tensor product of currents and multiplication of currents with smooth forms. In [**AW2**], if α has compact support, a current of the form (3.1) is called an *elementary current*. The class of pseudomeromorphic currents on a complex manifold was then introduced as currents that can be written as a locally finite sum of push-forwards of elementary currents. We will use the same definition on an analytic space Z.

Definition 3. A current μ on Z is said to be *pseudomeromorphic*, denoted $\mu \in \mathcal{PM}(Z)$, if μ can be written as a locally finite sum

$$\mu = \sum (\pi_\alpha)_* \tau_\alpha,$$

where $\pi_\alpha : Z_\alpha \to Z$ is a family of compositions of modifications and open inclusions, and τ_α are elementary currents on Z_α.

Note in particular that, if $\pi : \tilde{Z} \to Z$ is a resolution of singularities of Z, and if $\mu \in \mathcal{PM}(\tilde{Z})$, then $\pi_*\mu \in \mathcal{PM}(Z)$. All the currents introduced in this article are pseudomeromorphic, as we will see directly from the proofs that the currents exist. In [**AW2**], it is shown that if f is holomorphic on a complex manifold X, and $T \in \mathcal{PM}(X)$, one can define a multiplication $(1/f)T$

and $\bar\partial(1/f)\wedge T$. The same idea works equally well for strongly holomorphic functions on an analytic space.

Proposition 3.1. *Let f be strongly holomorphic on Z, such that f does not vanish on any irreducible component of Z, and let $T \in \mathcal{PM}(Z)$. Then the currents*

$$\frac{1}{f}T := \frac{|f|^{2\lambda}}{f}T\bigg|_{\lambda=0} \quad \textit{and} \quad \bar\partial\frac{1}{f}\wedge T := \frac{\bar\partial|f|^{2\lambda}}{f}\wedge T\bigg|_{\lambda=0},$$

where the right-hand sides are defined originally for $\operatorname{Re}\lambda \gg 0$*, have current-valued analytic continuations to* $\operatorname{Re}\lambda > -\epsilon$ *for some* $\epsilon > 0$*, and the values at* $\lambda = 0$ *are pseudomeromorphic. The currents satisfies the Leibniz rule*

$$\bar\partial\left(\frac{1}{f}T\right) = \bar\partial\frac{1}{f}\wedge T + \frac{1}{f}\bar\partial T,$$

and $\operatorname{supp}(\bar\partial(1/f)\wedge T) \subseteq Z_f \cap \operatorname{supp} T$*. If $f \neq 0$, then $(1/f)T$ defined in this way coincides with the usual multiplication of T with the smooth function $1/f$.*

Proof. If Z is smooth, this is Proposition 2.1 in [**AW2**], except for the last statement. However, if $f \neq 0$, then $|f(z)|^{2\lambda}/f(z)$ is smooth in both λ and z, and analytic in λ, so if ξ is a test form, $T.((|f|^{2\lambda}/f)\xi)$ is analytic in λ, and hence the analytic continuation to $\lambda = 0$ coincides with the value $T.((1/f)\xi)$ at $\lambda = 0$. The proof in the general case goes through word for word as in the smooth case in Proposition 2.1 in [**AW2**]. □

The crucial point in the proof of the following proposition is that for any analytic subset $W \subseteq Z$ and any $T \in \mathcal{PM}(Z)$, there exist natural restrictions

$$\mathbf{1}_{W^c}T := |h|^{2\lambda}T|_{\lambda=0} \quad \text{and} \quad \mathbf{1}_W T := T - \mathbf{1}_{W^c}T \tag{3.2}$$

where h is a tuple of holomorphic functions such that $W = \{h = 0\}$. The restrictions are independent of the choice of such h, and are such that $\operatorname{supp}\mathbf{1}_W T \subseteq W$. This is Proposition 2.2 in [**AW2**], and the proof will go through in exactly the same way when Z is an analytic space.

Proposition 3.2. *Assume that $\mu \in \mathcal{PM}(Z)$, and that μ has support on a variety V. If I_V is the ideal of holomorphic functions vanishing on V, then $\bar I_V\mu = 0$. If μ is of bidegree $(*,p)$, and V has codimension $\geq p+1$ in Z, then $\mu = 0$.*

In the case that Z is a complex manifold, this is Proposition 2.3 and Corollary 2.4 in [**AW2**], and the proof there will go through in the same way also when Z is an analytic space. The final step in the proof that $\mu = 0$ in the smooth case is to prove that $\mu = 0$ on V_{reg}, which is proved with the help of the previous part of the proposition, and by degree reasons, and then by induction over the dimension of V, $\mu = 0$. In the singular case, this is done in the same way. Since this is a local statement, we can assume that $Z \subseteq \Omega \subseteq \mathbb{C}^n$, and consider V as a subvariety of Ω. Then, for the same reasons as in the smooth case, we get that $i_*\mu = 0$ on V_{reg}, and by induction over the dimension of V that $i_*\mu = 0$, and hence $\mu = 0$.

4. Coleff-Herrera products of weakly holomorphic functions

Let $f_1, \cdots, f_{q+p} \in \tilde{\mathcal{O}}(Z)$. We want to define the Coleff-Herrera product

$$T = \frac{1}{f_1} \cdots \frac{1}{f_q} \bar\partial \frac{1}{f_{q+1}} \wedge \cdots \wedge \bar\partial \frac{1}{f_{q+p}}.$$

If f is strongly holomorphic, one way to define it is by

$$T = \frac{|f_1|^{2\lambda_1} \cdots |f_q|^{2\lambda_q}}{f_1 \cdots f_q} \frac{\bar\partial |f_{q+1}|^{2\lambda_{q+1}} \wedge \cdots \wedge \bar\partial |f_{q+p}|^{2\lambda_{q+p}}}{f_{q+1} \cdots f_{q+p}} \Bigg|_{\lambda_{q+p}=0,\ldots,\lambda_1=0}, \tag{4.1}$$

which a priori is defined only when $\operatorname{Re}\lambda_i \gg 0$; however, by Proposition 3.1 it has an analytic continuation in λ_{q+p} to $\operatorname{Re}\lambda_{q+p} > -\epsilon$ for some $\epsilon > 0$, and the value at $\lambda_{q+p} = 0$ is pseudomeromorphic. Again, by Proposition 3.1, it has an analytic continuation in λ_{q+p-1} to $\lambda_{q+p-1} = 0$ and so on, and hence the value at $\lambda_{q+p} = 0, \cdots, \lambda_1 = 0$ exists.

Note that if $\pi : Y \to Z$ is any modification of Z, we can define the corresponding Coleff-Herrera product of $f' = \pi^* f$ in Y. Taking the push-forward of this current to Z will in fact give the Coleff-Herrera product of f on Z. To see this, let T^λ denote the form on the right-hand side of (4.1), with $\operatorname{Re}\lambda_i \gg 0$ fixed, and let T'^λ denote the corresponding form on Y with f' instead of f. If $\operatorname{Re}\lambda_i \gg 0$, then T^λ and T'^λ are smooth, and $\pi^* T^\lambda = T'^\lambda$, so $\pi_* T'^\lambda = T^\lambda$, since π is a modification. Thus, by analytic continuation, $T = T^\lambda|_{\lambda_{q+p}=0,\cdots,\lambda_1=0} = \pi_* T'^\lambda|_{\lambda_{q+p}=0,\cdots,\lambda_1=0} = \pi_* T'$.

Now, if f is weakly holomorphic, let $\pi : Z' \to Z$ be the normalization of Z, and $f' = \pi^* f$ which is strongly holomorphic on Z'. Hence, the current

$$T' = \frac{1}{f'_1} \cdots \frac{1}{f'_q} \bar\partial \frac{1}{f'_{q+1}} \wedge \cdots \wedge \bar\partial \frac{1}{f'_{q+p}} \tag{4.2}$$

exists.

Definition 4. If $f = (f_1, \ldots, f_{q+p})$ is weakly holomorphic, we define the *Coleff-Herrera product*

$$T = \frac{1}{f_1} \cdots \frac{1}{f_q} \bar\partial \frac{1}{f_{q+1}} \wedge \cdots \wedge \bar\partial \frac{1}{f_{q+p}} \tag{4.3}$$

of f as $\pi_* T'$, where T' is defined by (4.2).

If f is strongly holomorphic, this definition will be the same as the definition in (4.1) since by the remark above, T can be defined as the push-forward from any modification. In addition, if f is weakly holomorphic, it can be defined by the push-forward of the corresponding current in any normal modification, since any normal modification factors through the normalization.

We will call the factors $1/f_i$ the *principal value factors*, and $\bar\partial(1/f_i)$ the *residue factors*.

Remark 4. Note that even though here, the principal value factors are to the left of the residue factors, we could equally well have the residue and principal value factors mixed. However, changing the order will in general give a different current, but as we will see in Theorem 4.3, if f_i define a complete intersection, the current will not depend on the order (up to change of signs).

Remark 5. The Coleff-Herrera product for $f = (f_1, \dots, f_p)$ strongly holomorphic is originally defined in [**CH**] as the limit of integrals over $\cap\{|f_i| = \epsilon_i(\delta)\}$ as $\epsilon \to 0$, where $\epsilon(\delta)$ tends to 0 along an admissible path, cf., (1.1). When $\epsilon(\delta)$ tends to 0 along an admissible path, this will correspond to taking the analytic continuation to $\lambda = 0$ in the order as in (4.1), and in fact, for arbitrary f, the definition in (1.1) is equal to the one in (4.3) defined by analytic continuation, see [**LS**].

In [**D**] Denkowski gave a definition of the Coleff-Herrera product of f, for f c-holomorphic, and we will see below that his definition coincides with ours in that case. The idea in [**D**] was to consider the graph of f,

$$\Gamma_f = \{(z, f(z)) \in Z \times \mathbb{C}^p_w | z \in Z\},$$

and even though f is only c-holomorphic, the graph will be analytic. If $(z, w) \in \Gamma_f$, then $w = f(z)$, and hence on the graph $f_i = w_i$ is a strongly holomorphic function. If Π is the projection from the graph to Z, since f is continuous, Π is a homeomorphism and in particular proper. The Coleff-Herrera product of f was then defined by

$$\bar\partial\frac{1}{f_1} \wedge \cdots \wedge \bar\partial\frac{1}{f_p} = \Pi_*\left(\bar\partial\frac{1}{w_1} \wedge \cdots \wedge \bar\partial\frac{1}{w_p}\right), \tag{4.4}$$

and since $f_i = w_i$ on Γ_f, this should be a reasonable definition of the Coleff-Herrera product of f. The next proposition shows, as one might hope, that the definition of Denkowski coincides with ours.

Proposition 4.1. *If $f = (f_1, \dots, f_p)$ is c-holomorphic, then the definition of the Coleff-Herrera product of f in* (4.3) *and in* (4.4) *coincide.*

Proof. In [**D**] the definition used for the Coleff-Herrera product of strongly holomorphic functions was the one from [**CH**]. However, by Remark 5 we can assume that the definition by analytic continuation is used instead. Let $\pi : Z' \to Z$ be the normalization of Z and $f' = \pi^* f$. We have projections $\Pi : \Gamma_f \to Z$ and $\Pi' : \Gamma_{f'} \to Z'$, where $\Gamma_f \subseteq Z \times \mathbb{C}^p_w$ and $\Gamma_{f'} \subseteq Z' \times \mathbb{C}^p_{w'}$ are the graphs of f and f'. Thus we have a commutative diagram

$$\begin{array}{ccc} \Gamma_{f'} & \xrightarrow{(\pi\times\mathrm{Id})|_{\Gamma_{f'}}} & \Gamma_f \\ \Big\downarrow{\scriptstyle \Pi'} & & \Big\downarrow{\scriptstyle \Pi} \\ Z' & \xrightarrow{\pi} & Z. \end{array} \tag{4.5}$$

We will denote the current $\bar\partial(1/f_1')\wedge\cdots\wedge\bar\partial(1/f_p')$ on Z' by $\mu^{f'}$, and similarly for μ^w and $\mu^{w'}$ defined on Γ_f and $\Gamma_{f'}$ respectively. Then $\bar\partial(1/f_1)\wedge\cdots\wedge\bar\partial(1/f_p)$ is defined in (4.3) by $\pi_*\mu^{f'}$, and in (4.4) by $\Pi_*\mu^w$. Now, $(\pi\times\mathrm{Id})|_{\Gamma_{f'}}:\Gamma_{f'}\to\Gamma_f$ is a modification of Γ_f so we have $\mu^w=(\Pi\times\mathrm{Id})_*\mu^{w'}$, and since $\Pi':\Gamma_{f'}\to Z'$ is a biholomorphism and $w_i'=f_i'$ on $\Gamma_{f'}$ we also have $\mu^{f'}=\Pi'_*\mu^{w'}$. Thus both are the push-forward of the same current in $\Gamma_{f'}$, and since the diagram (4.5) commutes, both will have the same push-forward to Z. □

The next two theorems are extensions to the case of weakly holomorphic functions of well-known results of the Coleff-Herrera product of strongly holomorphic functions (in the case $q=0$ or $q=1$), see [**CH**], or the case of holomorphic functions on a complex manifold, see [**P1**].

Theorem 4.2. *If $f=(f_1,\cdots,f_{q+p})$ is weakly holomorphic, then T, defined by* (4.3), *satisfies the Leibniz rule*

$$\bar\partial T=\sum_{j=1}^{q}\frac{1}{f_1}\cdots\frac{1}{f_{j-1}}\bar\partial\frac{1}{f_j}\wedge\frac{1}{f_{j+1}}\cdots\frac{1}{f_q}\bar\partial\frac{1}{f_{q+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_{q+p}},$$

and $\operatorname{supp}T\subseteq Z_{(f_{q+1},\cdots,f_{q+p})}$.

Proof. First we assume that f is strongly holomorphic. Then the Leibniz rule follows by analytic continuation, since if $\operatorname{Re}\lambda\gg0$, we have

$$\bar\partial\left(\frac{|f|^{2\lambda}}{f}\right)=\frac{\bar\partial|f|^{2\lambda}}{f}\text{ and }\bar\partial\left(\frac{\bar\partial|f|^{2\lambda}}{f}\right)=0.$$

The weakly holomorphic case follows by taking push-forward from the normalization. For the last part, let T' be the current corresponding to T in the normalization, and $f'=\pi^*f$ be the pull-back of f to the normalization. Then by Proposition 3.1, $T'=0$ outside of $Z_{f_i'}$, $i\geq q+1$, and hence $\operatorname{supp}T\subseteq\pi(\operatorname{supp}T')\subseteq\pi(Z_{(f'_{q+1},\cdots,f'_{q+p})})=Z_{(f_{q+1},\cdots,f_{q+p})}$, where the last equality follows from Proposition 2.3. □

It is natural in this context to ask how to define a reasonable multiplication of a weakly holomorphic function with a current, something which we will need in the case that the current is a Coleff-Herrera product to be able to state the next theorem. If $g\in\tilde{\mathcal{O}}(Z)$, and T is the Coleff-Herrera product in (4.3), we define gT by

$$gT=\pi_*(\pi^*gT'),\tag{4.6}$$

where $\pi:Z'\to Z$ is the normalization of Z, and T' is the corresponding Coleff-Herrera product of $f'=\pi^*f$. In the case that both f and g are c-holomorphic, Denkowski gives a definition of multiplication of g and the Coleff-Herrera product of f in [**D**], and by a similar argument as that in Proposition 4.1, one sees that our definition coincides with the one in [**D**] in that case. Note however, that we do not define a multiplication of a weakly holomorphic function with an arbitrary current, and as we will see in Section 5, this will not be possible if we require it to satisfy certain natural properties.

Theorem 4.3. *Let $f = (f_1,\dots,f_{q+p})$ be weakly holomorphic, such that $(f_{q+1},\dots,f_{q+p})$ defines a complete intersection, and that $(f_i,f_{q+1},\dots,f_{q+p})$ defines a complete intersection for $1 \le i \le q$. Then the principal value factors in*

$$T = \frac{1}{f_1}\cdots\frac{1}{f_q}\bar\partial\frac{1}{f_{q+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_{q+p}}$$

commute with other principal value factors or residue factors (see Remark 4), and the residue factors anticommute. In addition, if $1 \le k \le q$, we have

$$f_k T = \frac{1}{f_1}\cdots\widehat{\frac{1}{f_k}}\cdots\frac{1}{f_q}\bar\partial\frac{1}{f_{q+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_{q+p}}, \tag{4.7}$$

and if $q+1 \le j \le q+p$, then

$$f_j T = 0. \tag{4.8}$$

Note that in case $f_i \in \tilde{\mathcal{O}}(Z)$, then the left-hand sides of (4.7) and (4.8) are defined by (4.6).

Remark 6. In the smooth case, the first part of Theorem 4.3 (about permuting the factors) follows from the theorem of Samuelsson in [**S**], about the analyticity of the residue integral (4.1). In fact, his theorem holds also for strongly holomorphic functions on an analytic space, cf., [**LS**]. Since the proof of the first part of Theorem 4.3 reduces to the strongly holomorphic case, one could thus refer to the results of Samuelsson. However, since the proof of this deep theorem of Samuelsson is quite involved, we still prefer to give a direct proof of the first part of Theorem 4.3, since it can be done by much more elementary means.

Note that in the following lemmas, which we will use to prove Theorem 4.3, we assume that the functions are strongly holomorphic.

Lemma 4.4. *Assume that $f_1,f_2 \in \mathcal{O}(Z)$ and that $T \in \mathcal{PM}(Z)$ is of bidegree $(*,p)$. If $Z_{f_1}\cap Z_{f_2}\cap \operatorname{supp} T \subseteq V$, for some analytic set $V \subseteq Z$ of codimension $\ge p+1$ in Z, then*

$$\frac{1}{f_1}\frac{1}{f_2}T = \frac{1}{f_2}\frac{1}{f_1}T. \tag{4.9}$$

If $Z_{f_1}\cap Z_{f_2}\cap \operatorname{supp} T \subseteq V'$, for some analytic set V' of codimension $\ge p+2$ in Z, then

$$\frac{1}{f_1}\bar\partial\frac{1}{f_2}\wedge T = \bar\partial\frac{1}{f_2}\wedge\frac{1}{f_1}T, \tag{4.10}$$

and if in addition $Z_{f_1}\cap Z_{f_2}\cap \operatorname{supp}\bar\partial T \subseteq V''$, for some analytic set V'' of codimension $\ge p+3$, then

$$\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{f_2}\wedge T = -\bar\partial\frac{1}{f_2}\wedge\bar\partial\frac{1}{f_1}\wedge T. \tag{4.11}$$

Proof. We have by Proposition 3.1 that

$$\frac{1}{f_1}\frac{1}{f_2}T - \frac{1}{f_2}\frac{1}{f_1}T, \tag{4.12}$$

is zero outside of Z_{f_1}, since both terms are just multiplication of $(1/f_2)T$ with the smooth function $(1/f_1)$, and similarly it is zero outside of Z_{f_2}. Thus (4.12) is a pseudomeromorphic current on Z of bidegree $(*,p)$ with support on $Z_{f_1} \cap Z_{f_2} \cap V$, which has codimension $\geq p+1$, so (4.9) follows by Proposition 3.2. Similarly outside of Z_{f_1}, we get that

$$\frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge T - \bar\partial\frac{1}{f_2} \wedge \frac{1}{f_1}T \tag{4.13}$$

is zero, so (4.13) is a pseudomeromorphic current on Z of bidegree $(*,p+1)$ and has support on $Z_{f_1} \cap Z_{f_2} \cap \operatorname{supp} T$, so (4.10) follows by Proposition 3.2. For (4.11), we get by Theorem 4.2 and (4.10) that

$$\begin{aligned}
&\bar\partial\frac{1}{f_1} \wedge \bar\partial\frac{1}{f_2} \wedge T = \bar\partial\left(\frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge T\right) + \frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge \bar\partial T \\
&= \bar\partial\left(\bar\partial\frac{1}{f_2} \wedge \frac{1}{f_1}T\right) + \frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge \bar\partial T \\
&= -\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1} \wedge T - \bar\partial\frac{1}{f_2} \wedge \frac{1}{f_1}\bar\partial T + \frac{1}{f_1}\bar\partial\frac{1}{f_2} \wedge \bar\partial T = -\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1} \wedge T
\end{aligned}$$

where the last equality holds because of (4.10) and the assumption of the support of $\bar\partial T$. □

Lemma 4.5. *Assume $f, g \in \mathcal{O}(Z)$, and $f/g \in \mathcal{O}(Z)$. If $T \in \mathcal{PM}(Z)$ has bidegree $(*,p)$ and $Z_g \cap \operatorname{supp} T \subseteq V$, for some analytic subset V of codimension $\geq p+1$, then*

$$f\left(\frac{1}{g}T\right) = \frac{f}{g}T.$$

Proof. Outside of Z_g, we can see $(1/g)T$ as multiplication by the smooth function $1/g$ by Proposition 3.1. Hence we have $f(1/g)T = (f/g)T$ since their difference is a pseudomeromorphic current with support on $Z_g \cap \operatorname{supp} T$, so it is 0 by Proposition 3.2. □

Proof of Theorem 4.3. First we observe that it is enough to prove the theorem in case f_i are strongly holomorphic, since if $\pi : Z' \to Z$ is the normalization of Z, and $f' = \pi^* f$, then f' is a complete intersection, and if the theorem holds in Z', it holds in Z by taking push-forward of the corresponding currents. Hence, we can assume that $f_i \in \mathcal{O}(Z)$, and the commutativity properties will then follow from Lemma 4.4. For example, if we want to see that $1/f_i$ and $1/f_{i+1}$ commute, we can apply Lemma 4.4 with

$$T = \frac{1}{f_{i+2}} \cdots \frac{1}{f_q}\bar\partial\frac{1}{f_{q+1}} \wedge \cdots \wedge \bar\partial\frac{1}{f_{q+p}},$$

and then multiply with $(1/f_1)\cdots(1/f_{i-1})$ from the left. In case some of the residue factors, say $f_{q+1},\dots,f_{q+k}$, are to the left of the principal value factors, then $Z_{(f_{q+k+1},\dots,f_{q+p})}$ has codimension $p-k$ in a neighborhood of $Z_f \supseteq \operatorname{supp} T$ and the result follows in the same way from Lemma 4.4. The other cases follow similarly from Lemma 4.4.

The equality (4.7) follows from Lemma 4.5 since Z_f has codimension p. By the first part of the theorem, we can assume that $j=q+1$ in (4.8). Then

$$\begin{aligned} f_{q+1}\left(\bar\partial\frac{1}{f_{q+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_{q+p}}\right) &= \bar\partial\left(f_{q+1}\frac{1}{f_{q+1}}\wedge\bar\partial\frac{1}{f_{q+2}}\wedge\cdots\wedge\bar\partial\frac{1}{f_{q+p}}\right)\\ &= \bar\partial\left(\bar\partial\frac{1}{f_{q+2}}\wedge\cdots\wedge\bar\partial\frac{1}{f_{q+p}}\right)=0 \end{aligned}$$

by (4.7), and Theorem 4.2. □

5. Multiplication of currents with weakly holomorphic functions

Now, we will return to the issue of multiplication of currents with weakly holomorphic functions. Assume $g \in \tilde{\mathcal{O}}(Z)$ and $S \in \mathcal{PM}(Z)$. Since $S \in \mathcal{PM}(Z)$, we have $S=\sum(\pi_\alpha)_*\tau_\alpha$, where τ_α are elementary currents on the complex manifolds Z_α. Given such a decomposition, since any normal modification of Z factors through the normalization, that is, $\pi_\alpha = \pi\circ\nu_\alpha$, for some $\nu_\alpha : Z_\alpha \to Z'$, we get a current S' in the normalization Z' of Z such that $\pi_* S' = S$ by taking the push-forward of τ_α to Z', i.e., $S'=\sum(\nu_\alpha)_*\tau_\alpha$. To define multiplication of the Coleff-Herrera product with the weakly holomorphic function g in (4.6), we defined it as the push-forward of $\pi^* g S'$. In general, the current S' will depend on the decomposition $S=\sum(\pi_\alpha)_*\tau_\alpha$. However, in (4.6) we had a canonical representative in the normalization, and hence the multiplication was well-defined. The following example however shows that this multiplication depends on this choice of representative.

Example 4. Let $\pi : \mathbb{C}^n \to \mathbb{C}^{2n}$ be defined by

$$\pi(t_1,\cdots,t_n)=(t_1,\cdots,t_{n-1},t_1^2t_n,\cdots,t_{n-1}^2t_n,t_n^2,t_n^5).$$

Then π is proper and injective, so $\pi(\mathbb{C}^n)=Z$ is an analytic variety of dimension n. Since $(\partial\pi_j/\partial z_i)_{i,j}$ has full rank outside of $\{0\}$, $Z_{\text{sing}}\subseteq\{0\}$, and we will see below that actually $Z_{\text{sing}}=\{0\}$. Let

$$\tilde S=\bar\partial\frac{1}{t_1}\wedge\cdots\wedge\bar\partial\frac{1}{t_{n-1}}\wedge\bar\partial\frac{1}{t_n^3}$$

and $S=\pi_*\tilde S$. Then, since $d(t_nt_i^2)=t_i(2t_ndt_i+t_idt_n)$ and $dt_n^5=5t_n^4dt_n$, $dz_k\wedge S=0$ for $k=n,\dots,2(n-1)$, and $dz_{2n}\wedge S=0$. Hence if $S.\xi\neq 0$, then ξ must be of the form $\xi = \xi_0 dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dz_{2n-1}$. We have

$$S.\xi = \tilde{S}.\xi_0 dt_1 \wedge \cdots \wedge dt_{n-1} \wedge 2t_n dt_n =$$

$$2\cdot(2\pi i)^n \left(\sum_{i=1}^{n-1} t_i^2 \frac{\partial}{\partial z_{n-1+i}} \xi_0 + 2t_n \frac{\partial}{\partial z_{2n-1}} \xi_0 + 5t_n^4 \frac{\partial}{\partial z_{2n}} \xi_0 \right)\Bigg|_{t=0} = 0,$$

and thus $S = 0$. However,

$$t_n \tilde{S}.\xi dt_1 \wedge \cdots \wedge dt_n^2 = 2(2\pi i)^n \xi(0)$$

so $\pi_*(t_n\tilde{S}) = \pi_*(\pi^* g\tilde{S}) \neq 0$, where $g \in \mathcal{O}_c(Z)$ is such that $\pi^* g = t_n$. Note that g is not strongly holomorphic at 0, and hence $Z_{\text{sing}} = \{0\}$. Thus, since $S = \pi_*\tilde{S} = 0$, while $\pi_*((\pi^*g)\tilde{S}) = 0$, it is impossible to define a multiplication of currents with weakly holomorphic functions in a way compatible with push-forwards, i.e., that gS only depends on g and S, and such that $gS = \pi_*((\pi^*g)S')$ if $S = \pi_* S'$.

Hence, the multiplication in (4.6) does not depend only on g and S, but also on the functions f defining S. Recall that the *pole set*, P_ϕ, of a meromorphic function ϕ is the set where ϕ is not strongly holomorphic. Recall also the definitions of the restriction operators $\mathbf{1}_V$ and $\mathbf{1}_{V^c}$ in (3.2). If we require that the current we get in the multiplication has restriction 0 to P_ϕ, the multiplication is in fact uniquely defined in $\mathcal{PM}(Z)$, as the following proposition shows. This can in some cases be a natural condition, and in fact even automatic in some cases, see Corollary 5.2. However, in Example 4, since the common zero set of the functions defining S equals the pole set of g, we expect S and gS to have its support on P_g, and hence the condition is not very natural then.

Proposition 5.1. *Let $\mu \in \mathcal{PM}(Z)$ and $\phi \in \tilde{\mathcal{O}}(Z)$. Then, there exists a unique current, denoted $\phi\mu$, in $\mathcal{PM}(Z)$, such that $\phi\mu$ is just multiplication of the smooth function ϕ with the current μ outside of P_ϕ, and $\mathbf{1}_{P_\phi}(\phi\mu) = 0$. If $\mu = \pi_*\mu'$, where $\pi : Z' \to Z$ is the normalization of Z and $\mu' \in \mathcal{PM}(Z')$, then*

$$\phi\mu = \pi_*((\pi^*\phi)\mathbf{1}_{(\pi^{-1}(P_\phi))^c}\mu'). \tag{5.1}$$

Proof. First, we prove the uniqueness. Assume that T_1 and T_2 are two such currents, so that $T_1 - T_2$ has support on P_ϕ. Hence, $\mathbf{1}_{P_\phi^c}(T_1 - T_2) = 0$. But then,

$$T_1 - T_2 = \mathbf{1}_{P_\phi^c}(T_1 - T_2) + \mathbf{1}_{P_\phi}(T_1 - T_2) = 0,$$

since $\mathbf{1}_{P_\phi} T_1 = \mathbf{1}_{P_\phi} T_2 = 0$. Thus, we only need to prove that $\phi\mu$ in (5.1) satisfies the conditions in the proposition. It is clear that the right-hand side of (5.1) is just multiplication of ϕ with μ outside of P_ϕ. Hence, it remains to prove that $\mathbf{1}_{P_\phi}(\phi\mu) = 0$. However,

$$\mathbf{1}_{P_\phi}(\phi\mu) = \pi_*(\mathbf{1}_{\pi^{-1}(P_\phi)}(\pi^*\phi)\mathbf{1}_{(\pi^{-1}(P_\phi))^c}\mu') = 0,$$

since $\mathbf{1}_V\mathbf{1}_{V^c} = \mathbf{1}_V(1 - \mathbf{1}_V) = 0$ because $\mathbf{1}_V\mathbf{1}_V = \mathbf{1}_V$, and $\mathbf{1}_V$ commutes with multiplication with smooth functions. $\square$

Corollary 5.2. *Assume that $\mu \in \mathcal{PM}(Z)$ is of bidegree $(*,p)$ and $\phi \in \tilde{\mathcal{O}}(Z)$ is such that P_ϕ has codimension $\geq p+1$ in Z. Then there exists a unique current $\phi\mu \in \mathcal{PM}(Z)$ such that $\phi\mu$ coincides with the usual multiplication of ϕ with μ outside of P_ϕ. If $\mu = \pi_*\mu'$, where $\pi : Z' \to Z$ is the normalization of Z and $\mu' \in \mathcal{PM}(Z')$, then*

$$\phi\mu = \pi_*((\pi^*\phi)\mu'). \tag{5.2}$$

Proof. By Proposition 5.1, the only thing we need to prove is that for any $T \in \mathcal{PM}(Z)$ and $T' \in \mathcal{PM}(Z')$ of bidegree $(*,p)$, we have $\mathbf{1}_{P_\phi}T = 0$ and $\mathbf{1}_{\pi^{-1}(P_\phi)}T' = 0$. However, since P_ϕ has codimension $\geq p+1$, $\pi^{-1}(P_\phi)$ has codimension $\geq p+1$ by Lemma 2.2. Hence, $\mathbf{1}_{P_\phi}T = 0$ and $\mathbf{1}_{\pi^{-1}(P_\phi)}T' = 0$ by Proposition 3.2, since the currents have support on P_ϕ and $\pi^{-1}(P_\phi)$ respectively. □

Note, in particular that if Z_{sing} has codimension $\geq p+1$, the condition of the codimension of P_g is automatically satisfied for any weakly holomorphic function $g \in \tilde{\mathcal{O}}(Z)$.

Another question is whether the Coleff-Herrera product could be defined as the analytic continuation of an integral on Z rather than Z'. A natural way to do this would be to try to regularize in (4.3) by factors $\bar\partial|F_i|^{2\lambda_i}$ instead of $\bar\partial|f_i|^{2\lambda_i}$, where F_i is a tuple of strongly holomorphic functions such that $Z_{F_i} = P_{1/f_i}$. However, the analytic continuation to $\lambda = 0$ will in general not coincide with our definition, even if f defines a complete intersection, as the following example shows.

Example 5. Let $Z = \{z \in \mathbb{C}^3 \mid z_1^3 = z_2^2\} = V \times \mathbb{C}$, which has normalization $\pi(s,t) = (s^2, s^3, t)$, and let $\pi^* f_1 = (1+s)t$ and $\pi^* f_2 = s^2$. Then $Z_f = \{0\}$, so f is a complete intersection. Note that $\pi^*(1/f_1) = (1/t)(1-s+O(s^2))$ for $|s| < 1$, and that holomorphic functions in s at the origin correspond to strongly holomorphic functions on V at the origin precisely when the Taylor expansion at the origin contains no term s. Thus $P_{1/f_1} = \pi(\{s=0\} \cup \{s=-1\} \cup \{t=0\})$, so if $\{F = 0\} \supseteq P_{1/f_1}$, then $\{F = 0\} \supseteq Z_{f_2}$. Thus $(\bar\partial|F|^{2\lambda}/f_1) \wedge \bar\partial(1/f_2) = 0$ for $\operatorname{Re}\lambda \gg 0$. However, we have

$$\bar\partial\frac{1}{f_1} \wedge \bar\partial\frac{1}{f_2}.\varphi dz_1 \wedge dz_3 = \frac{1}{1+s}\bar\partial\frac{1}{t} \wedge \bar\partial\frac{1}{s^2}.\varphi(s^2,s^3,t)ds^2 \wedge dt = 4\pi i\varphi(0),$$

so $\bar\partial(1/f_1) \wedge \bar\partial(1/f_2)$ is non-zero.

6. Bochner-Martinelli type residue currents

We will show that we can define a Bochner-Martinelli type residue current associated with a tuple of weakly holomorphic functions, either by using a similar approach as for the Coleff-Herrera product with the help of the normalization, or by defining it intrinsically on Z by means of analytic continuation. In view of Example 5, it is not clear how to do this directly for the Coleff-Herrera product. In addition, we will show that for weakly holomorphic functions defining a complete intersection, the Coleff-Herrera product and the Bochner-Martinelli current coincide, Theorem 6.3.

Let $f = (f_1, \dots, f_p)$ be weakly holomorphic. We will follow the approach by Andersson from [**An1**], and make the identification $f = \sum f_i e_i^*$, where $(e_1, \cdots, e_p)$ is a frame for a trivial vector bundle E over Z, and $(e_1^*, \dots, e_p^*)$ is the dual frame. Since we will only use the case of trivial vector bundles, this identification is not strictly necessary. However, we use this since it greatly simplifies the notation in the proof of Lemma 7.3. Then, on the set where f is strongly holomorphic, $\nabla_f := \delta_f - \bar\partial$ induces a complex on currents on Z with values in $\bigwedge E$, where δ_f is interior multiplication with f. To construct the Bochner-Martinelli current we define

$$\sigma = \sum \frac{\bar f_i e_i}{|f|^2} \quad \text{and} \quad u = \sum_{k=0}^{p-1} \sigma \wedge (\bar\partial\sigma)^k. \tag{6.1}$$

Note that outside of $Z_f \cup P_{f_1} \cup \cdots \cup P_{f_p}$, both u and σ are smooth, and $\nabla_f u = 1$.

Recall that a *universal denominator* at a germ (Z,z) is a strongly holomorphic function h, not vanishing on any irreducible component of (Z,z) such that $h\tilde{\mathcal{O}}_{Z,z} \subseteq \mathcal{O}_{Z,z}$. For each $z \in Z$, there always exist a universal denominator h, such that h is a universal denominator in a neighborhood of z, see for example [**G**], Theorem Q.2.

Proposition 6.1. *Assume that $f = (f_1, \cdots, f_p)$ is weakly holomorphic on Z. Let F be a tuple of strongly holomorphic functions, such that $\{F = 0\} \supseteq Z_f$, and $\{F = 0\}$ does not contain any irreducible component of Z, and let h be a universal denominator on Z. Then the forms $|hF|^{2\lambda} u$ and $\bar\partial |hF|^{2\lambda} \wedge u$ are arbitrarily smooth if* $\operatorname{Re}\lambda \gg 0$, *and have current-valued analytic continuations to* $\operatorname{Re}\lambda > -\epsilon$ *for some $\epsilon > 0$. The currents*

$$U^f := |hF|^{2\lambda} u|_{\lambda=0} \quad \textit{and} \quad R^f := \bar\partial |hF|^{2\lambda} \wedge u|_{\lambda=0} \tag{6.2}$$

are independent of the choice of F and h, and if $\pi : Y \to Z$ is a modification of Z, then $U^f = \pi_* U^{\pi^* f}$ and $R^f = \pi_* R^{\pi^* f}$.

Proof. We first show that $|hF|^{2\lambda} u$ and $\bar\partial |hF|^{2\lambda} \wedge u$ are arbitrarily smooth when $\operatorname{Re}\lambda \gg 0$. Since $\bar\partial |hF|^{2\lambda} = |hF|^{2(\lambda-1)} \bar\partial |hF|^2$, it is enough to prove this for $|hF|^{2\lambda} u$. We let $g_i := h f_i$, where $g_i \in \mathcal{O}(Z)$ since h is a universal denominator. If we differentiate u outside of $\{h = 0\} \cup Z_f$, we get terms of the form $\xi/(h^k |f|^{2n})$, where ξ is smooth, since if $f_i = g_i/h$, the terms in u are smooth except for factors h and $|f|^2$ in the denominators. Thus, we only need to see that $|hF|^{2\lambda}/(h^k |f|^{2n})$ tends to 0 on $\{h = 0\} \cup Z_f$. This is clear outside of Z_f if $\operatorname{Re}\lambda \gg 0$, so we need to prove that $|hF|^{2\lambda}/|f|^{2n}$ tends to 0 on Z_f. If we multiply the numerator and denominator by $|h|^{2n}$, we get

$$|h|^{2n} |hF|^{2\lambda} / (|hf|^{2n}). \tag{6.3}$$

We note that hf is strongly holomorphic, and in fact, $\{hF = 0\} \supseteq \{hf = 0\}$ because

$$Z_{hf} = \pi(Z_{\pi^*(hf)}) = \pi(Z_{\pi^*h}) \cup \pi(Z_{\pi^*f}) = Z_h \cup \pi(Z_{\pi^*f}) = \{h = 0\} \cup Z_f,$$

by Proposition 2.3 and the fact that π is surjective. Thus, (6.3) will tend to 0 on Z_f by the Nullstellensatz if $\operatorname{Re}\lambda \gg 0$.

Now, we assume that Z is smooth. Then we can take $F = f$ and $h \equiv 1$, and in that case, the proposition is the existence part of Theorem 1.1 in [**An1**], except for the fact that $U^f = \pi_* U^{\pi^* f}$ and $R^f = \pi_* R^{\pi^* f}$, which however easily follows by analytic continuation. To see that the definition of R^f is independent of the choice of F, we see from the proof of Theorem 1.1 in [**An1**] that $\bar\partial |F|^{2\lambda} \wedge u$ acting on a test form φ becomes, with a suitable resolution of singularities $\pi : \tilde X \to X$, a finite sum of terms of the kind

$$\int \frac{\bar\partial |u\mu_1|^{2\lambda}}{\mu_2} \wedge \sigma' \wedge \pi^*\varphi, \tag{6.4}$$

where μ_1 and μ_2 are monomials such that $\{\mu_1 = 0\} \supseteq \{\mu_2 = 0\}$, u is non-zero and σ' is smooth. Thus, it is enough to observe that the value at $\lambda = 0$ of (6.4) is independent of μ_1 (where $u\mu_1$ is the pull-back of F), as long as $\{\mu_1 = 0\} \supseteq \{\mu_2 = 0\}$. In the same way, one sees that the definition of U^f is independent of the choice of F.

Now, if f is weakly holomorphic, and $\pi : \tilde Z \to Z$ is a resolution of singularities, from the smooth case we know that $\bar\partial |\pi^*(hF)|^{2\lambda} \wedge \pi^* u$ has a current-valued analytic continuation to $\lambda = 0$ independent of the choice of hF. Hence, the weakly holomorphic case follows by taking push-forward, since $\bar\partial |hF|^{2\lambda} \wedge u = \pi_*(\bar\partial |\pi^*(hF)|^{2\lambda} \wedge \pi^* u)$ for $\operatorname{Re}\lambda \gg 0$. □

In fact, to prove the existence of U^f and R^f, defined by (6.2), it is sufficient to use $|F|^{2\lambda} u$ and $\bar\partial |F|^{2\lambda} \wedge u$, which can be seen are integrable on Z if $\operatorname{Re}\lambda \gg 0$ by going back to the normalization. However, the addition of the universal denominator h ensures that the forms are (arbitrarily) smooth if $\operatorname{Re}\lambda \gg 0$.

The following properties of the Bochner-Martinelli current, R^f, are well-known in the smooth case, see [**PTY**] and [**An1**].

Proposition 6.2. *Let $f = (f_1, \cdots, f_p)$ be weakly holomorphic, and assume that $p' = \operatorname{codim} Z_f$. The current R^f has support on $V = Z_f$, and there is a decomposition $R^f = \sum_{k=p'}^p R_k$, where $R_k \in \mathcal{PM}(Z)$ is a $(0,k)$-current with values in $\bigwedge^k E$. In addition, if f is strongly holomorphic, then $R^f = 1 - \nabla_f U^f$.*

Proof. In case Z is a complex manifold, this is parts of Theorem 1.1 in [**An1**], except for the fact that $R_k \in \mathcal{PM}(Z)$. However, that R_k is pseudomeromorphic can, as was noted in [**AW2**], easily be seen from the proof of Theorem 1.1 in [**An1**]. The proposition then follows in case of an analytic space, by taking push-forward from a resolution of singularities, except for the fact that $R^f = \sum_{k=p'}^p R_k$, where $p' = \operatorname{codim} Z_f$, since modifications does not in general preserve codimensions of subvarieties. However, we get that $R^f = \sum_{k=0}^{p} R_k$, where $R_k \in \mathcal{PM}(Z)$ is a $(0,k)$-current, and R_k has support on Z_f. Thus, by Proposition 3.2, $R_k = 0$ for $k < \operatorname{codim} Z_f = p'$. $\square$

Remark 7. If the mapping f is weakly holomorphic, as we saw in Example 4, we do not have a well-defined multiplication of weakly holomorphic functions with pseudomeromorphic currents on Z. Hence, the formula $R^f = 1 - \nabla_f U^f$ in the strongly holomorphic case does not necessarily have any meaning if f is weakly holomorphic. However, one can give this multiplication meaning by Proposition 5.1. With this definition of multiplication, one can verify that

$$R^f = 1 - \nabla_f U^f,$$

if f is weakly holomorphic. This can be seen by using that this formula holds in the normalization, together with the fact that $U^{f'}$ has the standard extension property, SEP, i.e., that $\mathbf{1}_{\{h=0\}} U^f = 0$ for any tuple h of strongly holomorphic functions not vanishing on any irreducible component of Z. This follows from that $U^{f'}$ is a principal value current, i.e., when $U^{f'}$ is written as a sum of push-forwards of elementary currents, the elementary currents contain no residue factors, and hence have the SEP.

Theorem 6.3. *If $f = (f_1, \dots, f_p)$ is weakly holomorphic forming a complete intersection and $R^f = \mu \wedge e$, where $e = e_p \wedge \dots \wedge e_1$, then*

$$\mu = \mu^f := \bar\partial \frac{1}{f_1} \wedge \dots \wedge \bar\partial \frac{1}{f_p}.$$

Proof. To begin with, we will assume that f is strongly holomorphic. The proof will follow the same idea as the proof in the smooth case in [**An2**], Theorem 3.1. Let

$$V = \frac{1}{f_p} e_p + \frac{1}{f_{p-1}} \bar\partial \frac{1}{f_p} \wedge e_p \wedge e_{p-1} + \dots + \frac{1}{f_1} \bar\partial \frac{1}{f_2} \wedge \dots \wedge \frac{1}{f_p} \wedge e_p \wedge \dots \wedge e_1.$$

Then, by Proposition 4.3, V satisfies

$$\nabla_f V = 1 - \bar\partial \frac{1}{f_1} \wedge \dots \wedge \bar\partial \frac{1}{f_p} \wedge e.$$

Following the proof of Theorem 3.1 in [**An2**], locally, assume $Z \subseteq \Omega \subseteq \mathbb{C}^n$, ω is an arbitrary neighborhood of Z_f in Ω and χ is a smooth function with support on ω which is $\equiv 1$ in a neighborhood of Z_f. Let $i : Z \to \Omega$ be the inclusion, and let $g = i^*\chi - i^*(\bar\partial\chi) \wedge u$. Then, since $\nabla_f u = 1$ on $\operatorname{supp} \bar\partial\chi$, $\nabla_f g = 0$, and hence

$$\nabla_f (g \wedge (U^f - V)) = g \wedge \nabla_f (U^f - V) = g_0 (\mu^f - \mu) \wedge e = (\mu^f - \mu) \wedge e, \tag{6.5}$$

where $g_0 = \chi$ is the component of bidegree $(0,0)$ in g, which is 1 in a neighborhood of $\operatorname{supp}(\mu^f - \mu)$. A current T is said to have the standard extension property, SEP, with respect to an analytic variety W if for any holomorphic function h such that h is not identically 0 on any irreducible component of

W, then $|h|^{2\lambda}T|_{\lambda=0} = T$. Since μ and μ^f are currents in $\mathcal{PM}(Z)$ of bidegree $(0,p)$, with support on $W = \{f = 0\}$, μ and μ^f have the SEP, since if h does not vanish on any irreducible component of W, $\mu - |h|^{2\lambda}\mu|_{\lambda=0}$ has support on $W \cap \{h = 0\}$, which has codimension $\geq p+1$, and by Proposition 3.2 it is 0. Also, μ and μ^f are $\bar\partial$-closed and are annihilated by $\bar{I}_W$, see Proposition 3.2, so $i_*\mu, i_*\mu^f \in CH_W$, where CH_W denotes $\bar\partial$-closed $(0, \operatorname{codim} W)$-currents with support on W satisfying the SEP. By Lemma 3.3 in [**An2**], we know that a $\bar\partial$-closed current in CH_W cannot be equal to $\bar\partial v$, where v can be chosen with support arbitrarily close to W, unless it is 0. Hence, by looking at the components of top degree in (6.5), we have $i_*(\mu - \mu^f) = 0$, so $\mu = \mu^f$.

Now, if f_i are weakly holomorphic, then the current R^f will be the push-forward of the corresponding current $R^{\pi^* f}$, where $\pi : Z' \to Z$ is the normalization of Z, and the same holds for the Coleff-Herrera product μ^f. Hence, equality holds in the normalization, and taking push-forward we get equality in the general case. □

7. The transformation law

With the Bochner-Martinelli type currents developed in the previous section, we will now prove the transformation law for Coleff-Herrera products of weakly holomorphic functions.

Theorem 7.1. *Assume that $f = (f_1, \cdots, f_p)$ and $g = (g_1, \cdots, g_p)$ are weakly holomorphic, defining complete intersections, and that there exists a matrix A of weakly holomorphic functions such that $g = Af$. Then*

$$\bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} = (\det A) \bar\partial \frac{1}{g_1} \wedge \cdots \wedge \bar\partial \frac{1}{g_p}.$$

If A is invertible, one can prove the transformation law with the help of Theorem 6.3 together with the fact that the Bochner-Martinelli current is independent of the metric chosen to define σ^f (here, in (6.1), σ^f is defined with respect to the trivial metric on E), see [**An1**]. We will see that we can use a similar idea even in the case that A is not invertible. In [**D**] Denkowski proved the transformation law for c-holomorphic functions based on a more direct approach.

To begin with, we assume that f, g and A are strongly holomorphic. As in the previous section, we will identify f and g with sections of vector bundles, however we will here identify them with sections of two different vector bundles. Let E and E' be trivial holomorphic vector bundles over Z with frames e and e', and make the identifications $f = \sum f_i e_i^*$, $g = \sum g_i e_i'^*$ and $A \in \operatorname{Hom}(E', E)$ such that $g = fA$.

Lemma 7.2. *Let $\bigwedge A : \bigwedge E' \to \bigwedge E$ denote the linear extension of the mapping $(\bigwedge A)(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$. Then $\delta_f(\bigwedge A) = (\bigwedge A)\delta_g$.*

Proof. Note first that $\delta_f A e'_j = g_j = \delta_g e'_j$. Hence, we have

$$\begin{aligned}
&\delta_f(\bigwedge A)(e'_{i_1} \wedge \cdots \wedge e'_{i_k}) = \delta_f(Ae'_{i_1} \wedge \cdots \wedge Ae'_{i_k}) \\
&= \sum (-1)^{j-1} Ae'_{i_1} \wedge \cdots \wedge \delta_f(Ae'_{i_j}) \wedge \cdots \wedge Ae'_{i_k} \\
&= \sum (-1)^{j-1} (\bigwedge A)(e'_{i_1} \wedge \cdots \wedge \delta_g e'_{i_j} \wedge \cdots \wedge e'_{i_k}) = (\bigwedge A)\delta_g(e'_{i_1} \wedge \cdots \wedge e'_{i_k}).
\end{aligned}$$

□

To relate the currents μ^f and μ^g, we will first derive a relation between the currents U^f and U^g as defined by (6.2).

Lemma 7.3. *If f and g are strongly holomorphic and defining complete intersections, then there exists a current R_1 such that $U^f - (\bigwedge A)U^g = \nabla_f R_1$.*

Proof. Let σ, u, σ' and u' be the forms defined by (6.1) corresponding to f and g. Since A is holomorphic, $(\bigwedge A)\bar\partial\sigma' = \bar\partial(A\sigma')$ outside of $\{g = 0\}$, and hence if we let $u'_A = \sum(A\sigma') \wedge (\bar\partial A\sigma')^{k-1}$, then $\nabla_f u'_A = 1$ outside of $\{g = 0\}$ by Lemma 7.2. Thus, if $\operatorname{Re}\lambda \gg 0$,

$$\nabla_f(|g|^{2\lambda} u'_A \wedge u) = |g|^{2\lambda} u - |g|^{2\lambda} u'_A - \bar\partial |g|^{2\lambda} \wedge u'_A \wedge u. \tag{7.1}$$

We want to see that all the terms in (7.1) have current-valued analytic continuations to $\lambda = 0$. First, we note that since $\{g = 0\} \supseteq \{f = 0\}$, $|g|^{2\lambda} u|_{\lambda=0} = U^f$ by Proposition 6.1, and since $u'_A = (\bigwedge A)u'$ we get that $|g|^{2\lambda} u'_A|_{\lambda=0} = (\bigwedge A)U^g$. Thus it remains to see that the left-hand side of (7.1) has an analytic continuation to $\lambda = 0$, and that the analytic continuation of the last term vanishes at $\lambda = 0$. To see that those terms have analytic continuations to $\lambda = 0$ is similar to showing the existence of the Bochner-Martinelli currents U^f and R^f. If we recall briefly the proof of the existence of U^f and R^f in [**An1**], the key step was that $\sigma \wedge (\bar\partial\sigma)^{k-1}$ is homogeneous with respect to f in the sense that if $f = f_0 f'$, then $\sigma \wedge (\bar\partial\sigma)^{k-1} = (1/f_0^k)\sigma_0 \wedge (\bar\partial\sigma_0)^{k-1}$, where σ_0 is smooth if $|f'| \neq 0$. By blowing up along the ideals $(f_1, \dots, f_p)$ and $(g_1, \dots, g_p)$ followed by a resolution of singularities, see [**AHV**], we can assume that locally $\pi^* f = f_0 h$ and $\pi^* g = g_0 g'$, where $h \neq 0$, $g' \neq 0$, and by a further resolution of singularities, we can assume that locally f_0, g_0 are monomials. Since $\{g = 0\} \supseteq \{f = 0\}$, we get that $\{g_0 = 0\} \supseteq \{f_0 = 0\}$. Thus, by the homogeneity of $\sigma' \wedge (\bar\partial\sigma')^{k-1}$ and $\sigma \wedge (\bar\partial\sigma)^{l-1}$ with respect to f and g, we get, since $u'_A = (\bigwedge A)u'$, that $|g|^{2\lambda} u'_A \wedge u$ and $\bar\partial|g|^{2\lambda} \wedge u'_A \wedge u$ acting on a test form φ becomes finite sums of the form

$$\int \frac{|v|^{2\lambda}|g_0|^{2\lambda}}{(g_0)^k f_0^l} \xi_{k,l} \wedge \pi^*\varphi \quad \text{and} \quad \int \frac{\bar\partial(|v|^{2\lambda}|g_0|^{2\lambda})}{(g_0)^k f_0^l} \wedge \xi_{k,l} \wedge \pi^*\varphi,$$

where $\xi_{k,l}$ are smooth $(0, k + l - 2)$-forms. Thus both have analytic continuations to $\lambda = 0$, and $R_2 := \bar\partial|g|^{2\lambda} \wedge u'_A \wedge u|_{\lambda=0}$ has support on $\{g = 0\}$. Since $R_2 \in \mathcal{PM}(Z)$ and consists of terms of bidegree $(0, k + l - 1)$, where $k + l \leq p$, with support on $\{g = 0\}$ which has codimension p, we get that

$R_2 = 0$ by Proposition 3.2. Thus, if we let $R_1 := |g|^{2\lambda} u'_A \wedge u|_{\lambda=0}$, we get that $\nabla_f R_1 = U^f - (\bigwedge A)U^g$. □

Now we are ready to prove the transformation law.

Proof of Theorem 7.1. Assume first that f, g and A are strongly holomorphic, and make the same identifications as after the statement of Theorem 7.1. Since $(\bigwedge A)R^g = (\bigwedge A)(1 - \nabla_g U^g) = 1 - \nabla_f (\bigwedge A)U^g$ by Lemma 7.2, we get from Lemma 7.3 that

$$(\bigwedge A)R^g - R^f = \nabla_f \left((\bigwedge A)U^g - U^f \right) = \nabla_f^2 R_1 = 0,$$

so

$$(\bigwedge A)R^g = R^f.$$

Thus, we get by Theorem 6.3 that

$$(\bigwedge A)\left(\mu^g \wedge e'_p \wedge \cdots \wedge e'_1\right) = \mu^f \wedge e_p \wedge \cdots \wedge e_1,$$

and since the left-hand side is equal to

$$(\det A)\mu^g \wedge e_p \wedge \cdots \wedge e_1,$$

the transformation law follows. Now, if f, g and A are weakly holomorphic, the transformation law follows since equality must hold in the normalization because the pullback of f and g define complete intersections in the normalization. Hence, equality must hold also in Z by taking push-forward. □

8. The Poincaré-Lelong formula

Let $f_1, \cdots, f_p$ be strongly holomorphic functions forming a complete intersection. The Poincaré-Lelong formula says that

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} \wedge df_p \wedge \cdots \wedge df_1 = [Z_f] = \sum \alpha_i [V_i], \tag{8.1}$$

where V_i are the irreducible components of Z_f and $[Z_f]$ is the integration current on Z_f with multiplicities. In case $p = \dim Z$ the multiplicity α_i at a point $x_i \in Z_f$ is given as the number of elements near x_i of a generic fiber of f. In case $p < \dim Z$ the multiplicity is given as the intersection multiplicity of Z_f with L, where L is a plane of dimension $\dim Z - p$ transversal to Z_f. For a thorough discussion of the multiplicities see [**C**], and for a proof of the Poincaré-Lelong formula see Section 3.6 in [**CH**].

Now, if f_i are weakly holomorphic functions defining a complete intersection, we can give a relatively short proof that a formula similar to (8.1) holds in Z. In the strongly holomorphic case, assuming $Z \subseteq \Omega \subseteq \mathbb{C}^n$, $i_*[Z_f]$ can be seen either as the intersection of the holomorphic chains Z_{F_i} with Z, where F_i are some holomorphic extensions of f_i to Ω, or as a product of closed positive currents, see [**C**], that is

$$i_*[Z_f] = [Z_{F_1} \cdot \cdots \cdot Z_{F_p} \cdot Z] = [Z_{F_1}] \wedge \cdots \wedge [Z_{F_p}] \wedge [Z].$$

However, these types of products are in general only defined in case $Z_{F_1} \cap \cdots \cap Z_{F_p} \cap Z$ has codimension equal to $\operatorname{codim} Z + \sum \operatorname{codim} Z_{F_i}$. Since zero sets of weakly holomorphic functions are in general not zero sets of strongly holomorphic functions, as we saw in Example 2, we cannot expect to have a similar interpretation for weakly holomorphic functions, since there are no natural counterparts to the holomorphic $(n-1)$-chains Z_{F_i} or closed positive $(1,1)$-currents $[Z_{F_i}]$.

From now on, we assume that $f = (f_1, \dots, f_p)$ is weakly holomorphic defining a complete intersection. Let $\pi : Z' \to Z$ be the normalization of Z, so that in particular, π is a finite proper holomorphic map. Since $f' = \pi^* f$ forms a complete intersection, (8.1) holds for f' in the normalization. Note that, if V_i are the irreducible components of $Z_{f'}$, then $W_i := \pi(V_i)$ are irreducible in Z. If $f : V \to W$ is a branched holomorphic cover with exceptional set E, we say that f is a **-covering* if $W \setminus E$ is a connected manifold. In particular, this means that the sheet-number of f is constant outside the exceptional set. By the Andreotti-Stoll theorem, see [**Ł**], if $f : V \to W$ is a finite proper holomorphic map, V has constant dimension and W is irreducible, then f is a *-covering. If $V \subset Z'$ is an irreducible component of $Z_{f'}$ and we consider $\pi|_V : V \to W$, where $W = \pi(V)$, it is a finite proper holomorphic map satisfying the conditions required for the Andreotti-Stoll theorem. Hence, there exists an integer k such that $\pi|_V$ is a k-sheeted finite branched holomorphic covering. Thus $\pi_* \alpha[V] = k\alpha[W]$. For $f = (f_1, \cdots, f_p)$ a weakly holomorphic mapping forming a complete intersection, we define the left-hand side of (8.1) as the push-forward of the corresponding current in the normalization. Thus, since we have by (8.1) that

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} \wedge df_p \wedge \cdots \wedge df_1 = \pi_*[Z_{f'}],$$

we have proved the following.

Theorem 8.1. *Let $f = (f_1, \cdots, f_p)$ be a weakly holomorphic mapping forming a complete intersection. Then*

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} \wedge df_p \wedge \cdots \wedge df_1 = \sum \beta_i [W_i] \tag{8.2}$$

where $\beta_i \in \mathbb{N}$ and W_i are the irreducible components of $W = Z_f$. More explicitly, if $[Z_{f'}] = \sum \alpha_i [V_i]$ and say $V_{i_1}, \cdots, V_{i_k}$ are the sets V_j such that $\pi(V_j) = W_i$, then $\beta_i = \sum k_{i_j} \alpha_{i_j}$, where k_j is the number of elements in a generic fiber of $\pi|_{V_j}$.

Remark 8. In [**D**] Denkowski proves the Poincaré-Lelong formula for $f = (f_1, \dots, f_p) \in \mathcal{O}_c^{\oplus p}(Z)$ (based on his construction on Γ_f, however as for the Coleff-Herrera product in Proposition 4.1 our definition coincides with his). In that case, it gives a different interpretation of the multiplicities as the intersection cycle

$$\frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_1} \wedge \cdots \wedge \bar\partial \frac{1}{f_p} \wedge df_p \wedge \cdots \wedge df_1 = \pi_*([\Gamma_f] \cdot [Z \times \{0\}]),$$

where $\pi : Z \times \mathbb{C}^p \to Z$ is the projection.

Note that if f is weakly holomorphic, since f is in general not smooth on Z_{sing}, df is not in general defined on all Z (although its pullback to the normalization has a smooth extension to all of Z') so, as for multiplication with weakly holomorphic functions in Example 4, it might for example happen that $\bar\partial(1/f) = 0$ while $\bar\partial(1/f) \wedge df \neq 0$. For example, if $Z = \{z^3 = w^2\}$, $\pi(t) = (t^2, t^3)$ and $f = w/z \in \tilde{\mathcal{O}}(Z)$, that is $\pi^* f = t$, then $\bar\partial(1/f) = 0$ while $\bar\partial(1/f) \wedge df = 2\pi i[0]$, as expected, since $Z_f = \{0\}$.

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PAPER II

On the duality theorem on an analytic variety

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PAPER II

On the duality theorem on an analytic variety

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Abstract. The duality theorem for Coleff-Herrera products on a complex manifold says that if $f = (f_1,\dots,f_p)$ defines a complete intersection, then the annihilator of the Coleff-Herrera product μ^f equals (locally) the ideal generated by f. This does not hold unrestrictedly on an analytic variety Z. We give necessary, and in many cases sufficient conditions for when the duality theorem holds. These conditions are related to how the zero set of f intersects certain singularity subvarieties of the sheaf $\mathcal{O}_Z$.

1. Introduction

Let $f = (f_1,\dots,f_p)$ be a tuple of holomorphic functions on an analytic variety Z, where we throughout the article will assume that Z has pure dimension. The *Coleff-Herrera product* of f, as introduced in [**CH**], can be defined by

$$(1.1)\qquad \mu^f = \bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.\varphi := \int_Z \frac{\bar\partial|f_p|^{2\lambda}\wedge\cdots\wedge\bar\partial|f_1|^{2\lambda}}{f_p\cdots f_1}\wedge\varphi\bigg|_{\lambda=0}.$$

Here, φ is a test form, and the integral on the right-hand side is analytic in λ for $\operatorname{Re}\lambda \gg 0$, and has an analytic continuation to $\lambda = 0$, and $|_{\lambda=0}$ denotes this value. We denote the Coleff-Herrera product of f either by μ^f, or by $\bar\partial(1/f_1)\wedge\cdots\wedge\bar\partial(1/f_p)$. The definition (1.1) is different from the original one, but in the case we focus on here, that f defines a *complete intersection*, i.e., that $\operatorname{codim} Z_f = p$, various different definitions including this definition and the original definition by Coleff and Herrera coincide, also on a singular variety, see [**LS**].

If f defines a complete intersection, the *duality theorem*, proven by Dickenstein and Sessa, [**DS**], and Passare, [**P**], gives a close relation between the Coleff-Herrera product of f and the ideal $\mathcal{J}(f_1,\dots,f_p)$ generated by f. This is done by means of the annihilator, $\operatorname{ann}\mu^f$, of μ^f, i.e., the holomorphic functions g such that $g\mu^f = 0$.

Theorem 1.1. *Let $f = (f_1,\dots,f_p)$ be a holomorphic mapping on a complex manifold defining a complete intersection. Then locally,*

$$\mathcal{J}(f_1,\dots,f_p) = \operatorname{ann}\mu^f.$$

The Coleff-Herrera product of a holomorphic mapping is a current on Z. Currents on singular varieties can be defined in a similar way as on manifolds, i.e., as linear functionals on test forms, see for example [**L**]. However, currents on Z also has a characterization in terms of currents in the ambient space: If $i : Z \to \Omega$ is the inclusion, $\operatorname{codim} Z = k$, and μ is a (p,q)-current on Z, then $i_*\mu$ is a $(k+p, k+q)$-current on Ω that vanishes on all forms that vanish on Z. Conversely, if T is a $(k+p,k+q)$-current on Ω, that vanishes on all forms that vanish on Z, then T defines a unique (p,q)-current T' on Z such that $i_*T' = T$. When we consider the Coleff-Herrera product in the ambient space, i.e., $i_*\mu^f$, we will denote it by

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge[Z],$$

and in fact, by analytic continuation, it can be defined by

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge[Z] = \left.\frac{\bar\partial|f_p|^{2\lambda}\wedge\cdots\wedge\bar\partial|f_1|^{2\lambda}}{f_p\cdots f_1}\wedge[Z]\right|_{\lambda=0}. \tag{1.2}$$

On an analytic variety, one can find rather simple examples of functions annihilating the Coleff-Herrera product of a complete intersection without lying in the ideal. However, we have an inclusion in one of the directions, see [**CH**], Theorem 1.7.7.

Theorem 1.2. *If $f = (f_1,\dots,f_p)$ are holomorphic on Z, defining a complete intersection, then $\mathcal{J}(f_1,\dots,f_p) \subseteq \operatorname{ann}\mu^f$.*

In this article, we discuss this inclusion, and give conditions for when the inclusion is an equality, and when the inclusion is strict.

Throughout this article, we will only discuss the duality theorem for strongly holomorphic functions on Z, i.e., functions f on Z, which are locally the restriction of holomorphic functions in the ambient space, denoted $f \in \mathcal{O}(Z)$. When we say holomorphic functions, we refer to strongly holomorphic functions. However, we will sometimes refer to them as strongly holomorphic functions, to make a distinction to weakly holomorphic, which we use in the introduction to provide examples. Recall that a function $f : Z_{\text{reg}} \to \mathbb{C}$ is *weakly holomorphic* on Z, denoted $f \in \tilde{\mathcal{O}}(Z)$, if f is holomorphic on Z_{reg}, and f is locally bounded at Z_{sing}. Recall also that a germ of a variety, (Z,z), is said to be *normal* if $\mathcal{O}_{Z,z} = \tilde{\mathcal{O}}_{Z,z}$, and that the *normalization* of a variety Z is the unique (up to analytic isomorphism) normal variety Z' together with a finite proper surjective holomorphic map $\pi : Z' \to Z$ such that $\pi|_{Z'\setminus\pi^{-1}(Z_{\text{sing}})} : Z'\setminus\pi^{-1}(Z_{\text{sing}}) \to Z_{\text{reg}}$ is a biholomorphism, see for example [**D**], Section II.7.

One of the reasons we do not have equality in Theorem 1.2 is because of weakly holomorphic functions, namely if $f = (f_1,\dots,f_p)$ is strongly holomorphic and defining a complete intersection, and $g = \sum a_i f_i$ is strongly holomorphic while the functions a_i are only weakly holomorphic, then by Theorem 4.3 in [**L**] (the analogue of Theorem 1.2 for weakly holomorphic functions), $g\mu^f = 0$, but it might very well happen that the a_i cannot be chosen to be strongly holomorphic. For example, let $Z = \{z^3 = w^2\} \subseteq \mathbb{C}^2$, which has normalization $\pi(t) = (t^2, t^3)$, and let $f \in \tilde{\mathcal{O}}(Z)$ be such that $\pi^* f = t$. Then $f^2 = z$ and $f^3 = w$ on Z, so that $f^2, f^3 \in \mathcal{O}(Z)$ and $f^3\bar\partial(1/f^2) = 0$ (note that since f^2 is strongly holomorphic on Z, we see this as a current on Z, as explained above), while $f^3 \neq gf^2$ for any $g \in \mathcal{O}(Z)$, since $f \notin \mathcal{O}(Z)$. That $f^3\bar\partial(1/f^2) = 0$ can be seen either by going back to the normalization, where we get $t^3\bar\partial(1/t^2)$, which is 0 by the (smooth) duality theorem, or by seeing it as a current in the ambient space, and using the Poincaré-Lelong formula as in Example 1 below.

Let us now consider a germ of a normal variety (Z,z), and the Coleff-Herrera product of one holomorphic function. Assume that $g \in \operatorname{ann}\bar\partial(1/f)$. Since $\bar\partial(1/f)$ is just $\bar\partial$ of $1/f$ in the current sense and g is holomorphic, we get that

$$\bar\partial\left(g\frac{1}{f}\right) = 0.$$

In the smooth case, by regularity of the $\bar\partial$-operator on 0-currents, $g(1/f)$ would be a holomorphic function. This will not hold in general on a singular space (as the example above shows). However, we get that $g/f \in \mathcal{O}(Z_{\mathrm{reg}})$. If (Z,z) is normal, then $\operatorname{codim}(Z_{\mathrm{sing}}, z) \geq 2$ in Z, and any function holomorphic on an analytic variety outside some subvariety of codimension ≥ 2 is locally bounded, see [**D**], Proposition II.6.1. Thus, g/f is weakly holomorphic, and since (Z,z) is normal, $g/f \in \mathcal{O}_{Z,z}$, i.e., $g \in \mathcal{J}(f)$. Combined with Theorem 1.2, we get that the duality theorem holds for the Coleff-Herrera product of one holomorphic function on (Z,z) if it is normal.

Assume now that (Z,z) is not normal. Then, there exists $\phi \in \tilde{\mathcal{O}}_{Z,z} \setminus \mathcal{O}_{Z,z}$. Since weakly holomorphic functions are meromorphic, we can write $\phi = g/h$ for some strongly holomorphic functions g and h. Then $g\bar\partial(1/h) = 0$, by Theorem 4.3 in [**L**] (the analogue of Theorem 1.2 for weakly holomorphic functions). However, since $g/h = \phi \in \tilde{\mathcal{O}}_{Z,z} \setminus \mathcal{O}_{Z,z}$, $g \notin \mathcal{J}(h)$ (in $\mathcal{O}_{Z,z}$).

Hence, in the case of the Coleff-Herrera product of one single holomorphic function on a germ of an analytic variety (Z,z), we get that the duality theorem holds for all f if and only if (Z,z) is normal. The next example shows that this characterization does not extend to tuples of holomorphic functions.

Example 1. Let $Z = \{z_1^2 + \dots + z_k^2 = 0\} \subseteq \mathbb{C}^k$, where $k \geq 3$. Then Z is normal since Z is a reduced complete intersection with $Z_{\mathrm{sing}} = \{0\}$, and a reduced complete intersection is normal if and only if $\operatorname{codim} Z_{\mathrm{sing}} \geq 2$ (see the discussion after Definition 1). Let $\mu = \bar\partial(1/z_{k-1}) \wedge \dots \wedge \bar\partial(1/z_1)$ (seen as a current on Z). We claim that $z_k\mu = 0$. To see this, we consider this as a current in the ambient space, i.e., $i_*(z_k\mu)$, and use the Poincaré-Lelong formula,

$$i_*(z_k\mu) = z_k\bar{\partial}\frac{1}{z_{k-1}} \wedge \cdots \wedge \bar{\partial}\frac{1}{z_1} \wedge \frac{1}{2\pi i}\bar{\partial}\frac{1}{z_1^2 + \cdots + z_k^2} \wedge d(z_1^2 + \cdots + z_k^2).$$

Then, $z_k dz_i^2 = 2z_i z_k dz_i$ and $z_i z_k \in \mathcal{J}(z_1, \cdots, z_{k-1}, z_1^2 + \cdots + z_k^2)$ for $i = 1, \dots, k$, so each such term annihilates the current by Theorem 1.2. However, $z_k \notin \mathcal{J}(z_1, \cdots, z_{k-1})$ in $\mathcal{O}(Z)$.

We will show that depending on certain singularity subvarieties of the analytic sheaf $\mathcal{O}_Z$, compared to the zero set of f, we can give sufficient (and in many cases necessary) conditions for when the duality theorem holds on an analytic variety. This condition can be seen as a generalization of normality, coinciding with the usual notion of normality in the case $p = 1$.

Given a coherent ideal sheaf $\mathcal{J}$, there exists locally a finite free resolution

$$0 \to \mathcal{O}(E_N) \xrightarrow{\varphi_N} \mathcal{O}(E_{N-1}) \to \dots \xrightarrow{\varphi_1} \mathcal{O}(E_0) \tag{1.3}$$

where $\mathcal{O}(E_k)$ is the sheaf associated to the vector bundle E_k. We define Z_k as the set of points where φ_k does not have optimal rank. If $Z = Z(\mathcal{J})$ and $p = \operatorname{codim} Z$, then $Z_1 = \cdots = Z_p = Z$ and $Z_{k+1} \subseteq Z_k$, see [**E**], Corollary 20.12. If $\mathcal{J} = \mathcal{J}_Z$, the ideal of holomorphic functions vanishing on Z, then we define

$$Z^0 := Z_{\text{sing}} \quad \text{and} \quad Z^k := Z_{p+k} \quad \text{for } k \geq 1, \tag{1.4}$$

where $p = \operatorname{codim} Z$. These sets are in fact independent of the choice of resolution by the uniqueness of minimal free resolutions in a local Noetherian ring, and from Lemma 3.1 and the remark following it in [**AW3**], Z^k are independent of the local embedding of Z into $\mathbb{C}^n$. Hence they are intrinsic subvarieties of Z. We will use the convention that $\operatorname{codim} Z^k$ refers to the codimension in Z, while by $\operatorname{codim} Z_k$, we refer to the codimension in the ambient space.

Theorem 1.3. *Let $f = (f_1, \dots, f_p)$ be a holomorphic mapping on a germ of an analytic variety (Z, z) defining a complete intersection. If* $\operatorname{codim}(Z^k \cap Z_f) \geq k + p + 1$ *for $k \geq 0$, then* $\operatorname{ann} \mu^f = \mathcal{J}(f_1, \dots, f_p)$.

The proof of Theorem 1.3 is in Section 4.

One might conjecture that this equality of the annihilator and the ideal holds if and only if the conditions in the theorem are satisfied. We have not been able to prove this in this generality, but have focused on a slightly weaker formulation of it. To do this, we introduce the notion of p-duality for an analytic variety.

Definition 1. If (Z, z) is a germ of an analytic variety, we say that (Z, z) has *p-duality* if for all $f = (f_1, \dots, f_p) \in \mathcal{O}_{Z,z}^{\oplus p}$ defining a complete intersection, we have $\operatorname{ann} \mu^f = \mathcal{J}(f_1, \dots, f_p)$.

Theorem 1.3 implies the following statement:

$(*)$ $\qquad (Z,z)$ has p-duality if $\operatorname{codim} Z^k \geq p+k+1$, for $k \geq 0$.

We believe that the converse of $(*)$ holds, and we will discuss this throughout the rest of this introduction. We show that indeed, in many cases, the converse of $(*)$ holds, and if the condition in $(*)$ is not a precise condition for p-duality, it is at least very close to being so.

We saw above that 1-duality is equivalent to that Z is normal. The condition $\operatorname{codim} Z^k \geq k+2$ in $(*)$ is exactly the condition that Z is normal. This is proved in [**M**], but can also be seen using the conditions R1 and S2 in Serre's criterion for normality. Indeed, one can verify that the conditions R1 and S2 are equivalent to the condition $\operatorname{codim} Z^k \geq k+2$. Thus, the converse of $(*)$ holds when $p=1$.

Recall that a germ (Z,z) is said to be *Cohen-Macaulay* if the ring $\mathcal{O}/\mathcal{J}_{Z,z}$ is Cohen-Macaulay. More concretely, this means that $\mathcal{O}/\mathcal{J}_{Z,z}$ has a free resolution of length $p = \operatorname{codim}(Z,z)$. Equivalently, $Z^k = \emptyset$ for $k \geq 1$. Hence, if (Z,z) is Cohen-Macaulay, the condition $\operatorname{codim} Z^k \geq p+k$ for $k \geq 0$ becomes just $\operatorname{codim} Z_{\text{sing}} \geq p$. In case (Z,z) is Cohen-Macaulay, the converse of $(*)$ holds.

Proposition 1.4. *Assume that (Z,z) is Cohen-Macaulay and that* $\operatorname{codim} Z_{\text{sing}} = k$. *If $q \geq k$, then there exists $f = (f_1,\ldots,f_q) \in \mathcal{O}_{Z,w}^{\oplus q}$, for some w arbitrarily close to z, defining a complete intersection, and $g \in \mathcal{O}_{Z,w}$ such that $g \in \operatorname{ann}\mu^f$, but $g \notin \mathcal{J}(f_1,\ldots,f_q)$.*

Remark 1. In general, we need to move to a nearby germ in order to find the counterexample, however, if Z_{sing} is a complete intersection in Z, we can take $w = z$.

In particular, if (Z,z) is a reduced complete intersection, then (Z,z) is Cohen-Macaulay since the Koszul complex is a free resolution of length $\operatorname{codim}(Z,z)$, see [**GH**, p. 688].

In Example 1, $(Z,0)$ is Cohen-Macaulay (since it is a reduced complete intersection) and $Z_{\text{sing}} = \{0\}$, which has codimension $k-1$ in $(Z,0)$. Proposition 1.4 then says that there exists a complete intersection $f = (f_1,\ldots,f_{k-1})$ and $g \notin \mathcal{J}(f_1,\ldots,f_{k-1})$ such that $g \in \operatorname{ann}\mu^f$. Then $f = (z_1,\ldots,z_{k-1})$ and $g = z_k$ is exactly such an example, while for any complete intersection of codimension $< k-1$, the duality theorem holds by Theorem 1.3.

If (Z,z) is not Cohen-Macaulay, we get the converse of $(*)$ only for the least p such that the condition in $(*)$ is not satisfied.

Proposition 1.5. *Assume that (Z,z) satisfies* $\operatorname{codim} Z^k \geq k+p$ *for all $k \geq 0$, with equality for some $k \geq 1$. Then there exists $f = (f_1,\ldots,f_p) \in \mathcal{O}_{Z,z}^{\oplus p}$ defining a complete intersection, and $g \in \mathcal{O}_{Z,z}$, such that $g \in \operatorname{ann}\mu^f$, but $g \notin \mathcal{J}(f_1,\ldots,f_p)$.*

If $p=1$, then the weakly holomorphic functions give rise to counterexamples as described above.

The proofs of Proposition 1.4 and Proposition 1.5 are in Section 7 and Section 8 respectively. To prove Proposition 1.4, we use Theorem 6.3, which says that there exists a tuple ξ of holomorphic $(p,0)$-forms such that

$$[Z] = \sum \xi_i \wedge R_i^Z, \tag{1.5}$$

where $[Z]$ is the integration current on Z, and $R^Z = (R_1^Z, \dots, R_N^Z)$ is a tuple of currents such that $\mathcal{J}_Z = \cap_{i=1}^N \operatorname{ann} R_i^Z$, and the current R^Z is defined by means of a free resolution of $\mathcal{O}/\mathcal{J}_Z$, see Section 3. The existence of such ξ_i is proved in [**A3**], but the tuple ξ is not explicitly given. What we prove in Theorem 6.3 is that if R^Z is the current associated with a *minimal* free resolution, then all ξ_i vanish at Z_{sing}. This result can be seen as a generalization of the Poincaré-Lelong formula from the reduced complete intersection case to the Cohen-Macaulay case. In the reduced complete intersection case, the representation (1.5) is given by the Poincaré-Lelong formula, and since in that case, ξ is explicitly given, the fact that ξ vanish at Z_{sing} follows from the implicit function theorem, see Section 2.

Summarizing Theorem 1.3 and Propositions 1.4 and 1.5, we get the following.

Corollary 1.6. *Assume that* $\operatorname{codim} Z^k \geq k+p$ *for all* $k \geq 0$*, with equality for some* k*. Then* (Z,w) *has* q*-duality for* $q < p$ *and all* w *in some neighborhood of* z*, and* (Z,w) *does not have* q*-duality for* $q = p$ *for some* w *arbitrarily close to* z*. In addition, if* $\operatorname{codim} Z_{\text{sing}} = p$*, that is, we have equality for* $k = 0$*, then* (Z,w) *does not have* q*-duality for* $q > p$ *for some* w *arbitrarily close to* z*.*

Proof. The only part that does not follow immediately from Theorem 1.3, Proposition 1.4 and Proposition 1.5 is if $q > p$, (Z,z) is not Cohen-Macaulay but there is equality in $\operatorname{codim} Z^k \geq k+p$ for $k=0$. However, in that case, $\operatorname{codim} Z^0 = p$ and $\operatorname{codim} Z^1 \geq p+1$, so since $Z^0 \supseteq Z^1$, there is some $w \in Z^0$ arbitrarily close to z such that (Z,w) is Cohen-Macaulay (i.e., $w \in Z^0 \setminus Z^1$), and we can apply Proposition 1.4.

□

2. The case of a reduced complete intersection

We begin by showing how to prove Corollary 1.6 in the case when Z is a reduced complete intersection, i.e., that $Z = \{h_1 = \cdots = h_r = 0\}$, where $r = \operatorname{codim} Z$, and $dh_1 \wedge \cdots \wedge dh_r \neq 0$ generically on Z.

Proposition 2.1. *Let* (Z,z) *be a reduced complete intersection and assume that* $\operatorname{codim} Z_{\text{sing}} = p$*. Then, for all* w *in some neighborhood of* z*,* (Z,w) *has* q*-duality for* $q < p$*, and there exists* w *arbitrarily close to* z *such that* (Z,w) *does not have* q*-duality for* $q \geq p$*.*

In this case, the main ideas of the proof in the general case appear, but it only involves the Coleff-Herrera product, and hence we avoid many of the technicalities of the proof in the general case.

By the Poincaré-Lelong formula, see Section 3.6 in [**CH**],

$$\frac{1}{(2\pi i)^r}\bar\partial\frac{1}{h_r}\wedge\cdots\wedge\bar\partial\frac{1}{h_1}\wedge dh_1\wedge\cdots\wedge dh_r=[Z]. \tag{2.1}$$

Now, let $f=(f_1,\dots,f_q)$ be a complete intersection on Z, and consider μ^f as a current in the ambient space, as given by (1.2). By considering the regularization of μ^f in (1.2), using the Poincaré-Lelong formula (2.1) on $[Z]$, and also regularizing μ^h in (2.1), we get

$$i_*\mu^f=\left.\frac{\bar\partial|f_q|^{2\lambda_2}\wedge\cdots\wedge\bar\partial|f_1|^{2\lambda_2}\wedge\bar\partial|h_r|^{2\lambda_1}\wedge\cdots\wedge\bar\partial|h_1|^{2\lambda_1}}{f_q\dots f_1h_r\dots h_1}\wedge\eta\right|_{\lambda_1=0,\lambda_2=0}, \tag{2.2}$$

where $\eta=(2\pi i)^{-r}dh_1\wedge\cdots\wedge dh_r$. Note that f being a complete intersection on Z means that (f,h) is a complete intersection on $\mathbb{C}^n$. In this case, by results of Samuelsson, [**S**], the right-hand side of (2.2) is continuous in (λ_1,λ_2) near $(0,0)$. In particular, we can instead take the analytic continuation where $\lambda_1=\lambda_2=\lambda$ to $\lambda=0$, which equals the Coleff-Herrera product of (f,h), i.e.,

$$i_*\mu^f=\bar\partial\frac{1}{f_q}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\wedge\bar\partial\frac{1}{h_r}\wedge\cdots\wedge\bar\partial\frac{1}{h_1}\wedge\eta. \tag{2.3}$$

The representation (2.3) of the Coleff-Herrera product will be the basis of proving Proposition 2.1. First, we consider the case when $q<p$. Since Z is a reduced complete intersection, $\eta=(2\pi i)^{-r}dh_1\wedge\cdots\wedge dh_r$ is non-vanishing on Z_{reg}. Thus, if $g\in\operatorname{ann}\mu^f$, i.e., by considering g in the ambient space, $gi_*\mu^f=0$, we get from (2.3) that g annihilates the Coleff-Herrera product $\mu^{(f,h)}$ on Z_{reg}. The Coleff-Herrera product belongs to a class of currents called pseudomeromorphic currents, see Section 3. This class of currents is closed under multiplication with smooth functions, and have the property that if T is a pseudomeromorphic $(*,k)$-current with support on a variety of codimension $>k$, then $T=0$, see Proposition 3.3. Thus, the current $g\mu^{(f,h)}$ is in fact 0, since it is a $(0,q+r)$-current with support on Z_{sing} which has codimension $p+r$ (in $\mathbb{C}^n$). By the duality theorem (on $\mathbb{C}^n$), $g\in\mathcal{J}(f,h)$, i.e., $g\in\mathcal{J}(f)$ in $\mathcal{O}_Z=\mathcal{O}/\mathcal{J}(h)$. Hence, Z has q-duality if $q<p$.

We now consider the case when $q\ge p$. We can find w arbitrarily close to z, and a complete intersection $f=(f_1,\dots,f_q)$ on (Z,w) such that $Z(f)\subseteq Z_{\text{sing}}$, see Section 5, and in particular Lemma 5.3. Let $\mathcal{I}=\mathcal{J}(f_1,\dots,f_q)$, and $V=Z(I)$. It follows from the Nullstellensatz that there exists a holomorphic function g such that $g\notin\mathcal{I}$, but $g\in\mathcal{J}_V$ and $g\mathcal{J}_V\subseteq\mathcal{I}$, see the proof of Proposition 1.4 in Section 7.

We claim the g annihilates μ^f, and since $g\notin\mathcal{J}(f)$, this proves the second part of Proposition 2.1. To prove this claim, note first that by the implicit function theorem, $\eta=(2\pi i)^{-r}dh_1\wedge\cdots\wedge dh_r$ vanishes on Z_{sing}, i.e., if $\eta=\sum_{|I|=r}h_Idz_I$, then each $h_I\in\mathcal{J}_{Z_{\text{sing}}}$. Since $V\subseteq Z_{\text{sing}}$, we get that

$\mathcal{J}_{Z_{\text{sing}}} \subseteq \mathcal{J}_V$. Hence, $gh_I \in g\mathcal{J}_{Z_{\text{sing}}} \subseteq g\mathcal{J}_V \subseteq \mathcal{I}$, where the last inclusion follows by the choice of g. Thus, we get from multiplying (2.3) by g that g annihilates μ^f, since each term gh_I from $g\eta$ annihilates the Coleff-Herrera product $\mu^{(f,h)}$.

3. Residue currents and free resolutions

When Z is a reduced complete intersection defined by h, the Coleff-Herrera product μ^h is a natural current associated to Z, and in Section 2, the factorization of the integration current $[Z]$ in terms of μ^h was the starting point of the argument. We want to find a corresponding current R^Z and a factorization of the integration current $[Z]$ also when Z is not a complete intersection, see Theorem 6.3 below. To do this, we use a construction by Andersson and Wulcan of currents associated to free resolutions of ideals, [**AW1**].

Let $\mathcal{J}$ be a coherent ideal sheaf, and let (E,φ) be a locally free resolution of the sheaf $\mathcal{O}/\mathcal{J}$ as in (1.3). Mostly, we will use the case when $\mathcal{J} = \mathcal{J}_Z$, the sheaf of holomorphic functions vanishing on the analytic variety Z.

In particular, if Z is a reduced complete intersection, and $\mathcal{J}_Z = \mathcal{J}(h_1,\dots,h_p)$, then the Koszul complex of h is a free resolution of $\mathcal{O}/\mathcal{J}_Z$. In this case, the current associated to the Koszul complex of h equals the Coleff-Herrera product μ^h, Theorem 3.2.

To construct the current associated to E, one first defines, outside of $Z = Z(\mathcal{J})$, right inverses $\sigma_k : E_{k-1} \to E_k$ to φ_k which are minimal with respect to some metric on E, i.e., $\varphi_k\sigma_k|_{\operatorname{Im}\varphi_k} = \operatorname{Id}_{\operatorname{Im}\varphi_k}$, $\sigma_k = 0$ on $(\operatorname{Im}\varphi_k)^\perp$, and $\operatorname{Im}\sigma_k \perp \ker\varphi_k$. One lets

$$u = \sigma_1 + \sigma_2\bar\partial\sigma_1 + \cdots + \sigma_N\bar\partial\sigma_{N-1}\dots\bar\partial\sigma_1.$$

Then, if $F \not\equiv 0$ is a holomorphic function vanishing at Z, R^E is defined by

$$R^E = \bar\partial|F|^{2\lambda} \wedge u|_{\lambda=0}, \tag{3.1}$$

where for $\operatorname{Re}\lambda \gg 0$, this is a (current-valued) analytic function in λ, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$. See [**AW1**] for more details.

Let R^E_k denote the part of R^E with values in E_k, i.e., R^E_k is a E_k-valued $(0,k)$-current. If $Z = Z(\mathcal{J})$, and $\operatorname{codim} Z = p$, then we will in fact have that

$$R^E = R^E_p + \cdots + R^E_N, \tag{3.2}$$

where N is the length of the free resolution (E,φ).

The current R^E has the following crucial property, [**AW1**], Theorem 1.1.

Theorem 3.1. *Let R^E be the current associated to a free resolution (E,φ) of an ideal $\mathcal{J}$. Then $\operatorname{ann} R^E = \mathcal{J}$.*

If Z is an analytic subvariety, we will denote by R^Z the current associated with a free resolution of $\mathcal{J}_Z$ of minimal length. Note that this current is not in general uniquely defined, as it might depend on the choice of metrics.

In this article, we are only concerned with local (or semi-local) statements, so the reader may very well assume the vector bundles are in fact free modules. However, we still keep the notation of vector bundles, partly to keep a consistent notation, but also since it is advantageous to be able to refer to the specific vector bundle E_k and not just the free module $\mathcal{O}^{\oplus r_k}$.

If $f = (f_1, \dots, f_p)$ defines a complete intersection, the Coleff-Herrera product coincides with the so called *Bochner-Martinelli current* of f, as introduced by Passare, Tsikh and Yger in [**PTY**] in the smooth case. It was also developed in the case of an analytic variety in [**BVY**]. If f defines a complete intersection, the Bochner-Martinelli current of f, denoted R^f, can be defined as the current associated with the Koszul complex of f. In fact, in [**AW1**], currents associated with any generically exact complex of vector bundles are defined, and not only free resolutions as described above, and then the Bochner-Martinelli current for an arbitrary f can be defined as the current associated with the Koszul complex of f, see [**A1**]. This equality of the Coleff-Herrera product and the Bochner-Martinelli current makes the Coleff-Herrera product fit in the framework of residue currents associated with a free resolution, and this substitution will be used throughout the arguments. The theorem below is Theorem 4.1 in [**PTY**] in the smooth case, and Theorem 6.3 in [**L**] in the singular case.

Theorem 3.2. *If $f = (f_1, \dots, f_p)$ defines a complete intersection on Z, then the Bochner-Martinelli current R^f of f equals the Coleff-Herrera product μ^f of f.*

Pseudomeromorphic currents were introduced in [**AW2**]. A current of the form

$$\frac{1}{z_{i_1}^{k_1}} \cdots \frac{1}{z_{i_m}^{k_m}} \bar\partial \frac{1}{z_{i_{m+1}}^{k_{m+1}}} \wedge \cdots \wedge \bar\partial \frac{1}{z_{i_p}^{k_p}} \wedge \alpha,$$

where α is a smooth form with compact support, is called an elementary current. A current T is said to be a *pseudomeromorphic current*, denoted $T \in \mathcal{PM}$, if it is a locally finite sum of push-forwards of elementary currents under compositions of smooth modifications and open inclusions. As can be seen from their construction, the Coleff-Herrera product μ^f and the current R^E associated with a free resolution are pseudomeromorphic. We will need the following two properties of pseudomeromorphic currents, see Proposition 2.3 and Corollary 2.4 in [**AW2**].

Proposition 3.3. *If $T \in \mathcal{PM}$ is of bidegree $(0,p)$ and T has support on a variety of codimension $\geq p+1$, then $T = 0$.*

Proposition 3.4. *If $T \in \mathcal{PM}$ has support on Z, and if f is a holomorphic function vanishing on Z, then $\bar f T = 0$.*

We will use results from [**A2**], that one can define products of the currents R^f and R^Z, and that under certain conditions, the annihilator of the product $R^f \wedge R^Z$ equals the sum of the ideals $\mathcal{J}(f) + \mathcal{J}_Z$. This type of product can be defined more generally for currents R^E and R^F associated with two free resolutions E and F. If R^E is defined by

$$R^E := \bar\partial |G|^{2\lambda} \wedge u|_{\lambda=0},$$

then $R^E \wedge R^F$ can be defined by

$$R^E \wedge R^F := \bar\partial |G|^{2\lambda} \wedge u \wedge R^F|_{\lambda=0}.$$

Remark 2. If we consider $R^f \wedge R^Z$, where $f = (f_1,\dots,f_p)$ is a strongly holomorphic mapping on Z, then this depends a priori on the choice of representatives of f in the ambient space. We will only need that under certain conditions, $\operatorname{ann} R^f \wedge R^Z = \mathcal{J}(f) + \mathcal{J}_Z$, which is independent of the choice of representatives. However, one can in fact show that $R^f \wedge R^Z$ does not depend on the choice of representatives, essentially due to that R^Z is annihilated by both holomorphic and anti-holomorphic functions vanishing on Z.

If

$$0 \to E_n \xrightarrow{\varphi_n} E_{n-1} \to \dots \xrightarrow{\varphi_1} E_0 \to 0$$

and

$$0 \to F_m \xrightarrow{\psi_m} F_{m-1} \to \dots \xrightarrow{\psi_1} F_0 \to 0$$

are two complexes, then one can form the tensor product of the complexes, denoted $(E\otimes F, \varphi\otimes\psi)$, by letting $(E\otimes F)_k = \oplus_{i+j=k} E_i \otimes F_j$ and $(\varphi\otimes\psi)(\xi\otimes\eta) = \varphi_i\xi\otimes\eta + (-1)^i\xi\otimes\psi_j\eta$ if $\xi\otimes\eta \in E_i\otimes F_j$.

The following theorem, Theorem 4.1 and Remark 8 in [**A2**], and its corollary gives conditions for when the annihilator of $R^E \wedge R^F$ coincides with the sum of the annihilators, and when the tensor product of two (minimal) free resolutions is a (minimal) free resolution.

Theorem 3.5. *Let (E,φ) and (F,ψ) be free resolutions of ideal sheaves $\mathcal{I}$ and $\mathcal{J}$, and let $Z_k^{\mathcal{I}}$ and $Z_l^{\mathcal{J}}$ be the associated sets where φ_k and ψ_l does not have optimal rank. If* $\operatorname{codim}(Z_k^{\mathcal{I}} \cap Z_l^{\mathcal{J}}) \geq k+l$ *for all $k,l \geq 1$, then* $\operatorname{ann} R^E \wedge R^F = \mathcal{I} + \mathcal{J}$ *and $(E\otimes F, \varphi\otimes\psi)$ is a free resolution of $\mathcal{I}+\mathcal{J}$. In addition, if both E and F are minimal free resolutions at some point z, then the tensor product is a minimal free resolution.*

To be precise, the last statement is not included in [**A2**]. However, if the tensor product is a free resolution, it follows immediately from the definition of minimality at some z, that $\operatorname{Im}\varphi_k \subseteq \mathfrak{m}_z\mathcal{O}(E_{k-1})$ (where $\mathfrak{m}_z$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,z}$), that it is minimal.

Corollary 3.6. *If $f = (f_1,\dots,f_p)$ is a reduced complete intersection on Z, and* $\operatorname{codim} Z_f \cap Z^l \geq p+l$ *for $l \geq 1$, then* $\operatorname{ann} R^f \wedge R^Z = \mathcal{J}(f) + \mathcal{J}_Z$, *and the tensor product of the Koszul complex of f and a free resolution of $\mathcal{J}_Z$ is a free resolution of $\mathcal{J}(f)+\mathcal{J}_Z$. In addition, if the free resolution of $\mathcal{J}_Z$ is minimal at some point z, then the tensor product is a minimal free resolution.*

Proof. If f is a complete intersection, then the Koszul complex of f is a minimal free resolution, and its associated singularity subvarieties Z_k^f are equal to Z_f for $k \le p$, and empty for $k > p$. Since $Z_l = Z$ for $l \le \operatorname{codim} Z$, the condition $\operatorname{codim} Z_f \cap Z_l \ge p + l$ is automatic for $l \le \operatorname{codim} Z$ since f is a complete intersection on Z. Thus, the condition $\operatorname{codim} Z_k^f \cap Z_l \ge k + l$ becomes just $\operatorname{codim} Z_f \cap Z^l \ge p + l$. $\square$

4. Proof of Theorem 1.3

The inclusion $\mathcal{J}(f_1,\ldots,f_p) \subseteq \operatorname{ann} \mu^f$ follows from Theorem 1.2 (also without the conditions on $Z^k \cap Z_f$), so we only need to prove the reverse inclusion. Assume that $Z \subseteq \Omega \subseteq \mathbb{C}^n$ and that $\operatorname{codim} Z = q$. Then $i_*\mu^f = \mu^f \wedge [Z]$, where $i : Z \to \Omega$ is the inclusion, and by Theorem 3.2, $\mu^f \wedge [Z] = R^f \wedge [Z]$. We will show that $g \in \operatorname{ann}(R^f \wedge [Z])$ implies that $g \in \operatorname{ann}(R^f \wedge R^Z)$ (which does not hold in general, but does under the conditions of the theorem). By (3.21) in [**AS**], outside of Z_{sing} there exists a smooth $(q,0)$-vector field γ such that $\gamma \neg [Z] = R_q^Z$. Then, outside of Z_{sing},

$$gR^f \wedge R_q^Z = gR^f \wedge (\gamma \neg [Z]) = \gamma \neg (gR^f \wedge [Z]) = 0.$$

Hence $gR^f \wedge R_q^Z$ is a $(0,p+q)$-current with support on $Z_f \cap Z_{\text{sing}}$, so by Proposition 3.3, it is 0 since $Z_f \cap Z_{\text{sing}}$ has codimension $\ge p+q+1$.

Outside of Z^{k+1}, there exists a smooth $\operatorname{Hom}(E_{q+k}, E_{q+k+1})$-valued smooth $(0,1)$-form α_{q+k+1} such that $R_{q+k+1}^Z = \alpha_{q+k+1} R_{q+k}^Z$, see [**AW1**]. We will prove by induction that

$$gR^f \wedge R_{q+k}^Z = 0. \tag{4.1}$$

Above we proved this for $k = 0$, so let us assume that it is proved for k. Then, outside of $Z^f \cap Z^{k+1}$,

$$gR^f \wedge R_{q+k+1}^Z = \alpha_{q+k+1}(gR^f \wedge R_{q+k}^Z) = 0.$$

Thus $gR^f \wedge R_{q+k+1}^Z$ has support on $Z^f \cap Z^{k+1}$ which has codimension $\ge p+q+k+2$, and since it is a pseudomeromorphic current of bidegree $(0,p+q+k+1)$, it is 0 by Proposition 3.3. Thus we have proven that $g \in \operatorname{ann}(R^f \wedge R^Z)$. By Corollary 3.6, $\operatorname{ann}(R^f \wedge R^Z) = \mathcal{J}(f) + \mathcal{J}_Z$, and hence we get that $g \in \mathcal{J}(f) + \mathcal{J}_Z$.

5. Complete intersections and choice of coordinates

This section contains several lemmas about choices of coordinates and existence of complete intersections containing a certain variety. They will be used throughout the rest of the sections. This first lemma, which is based on the first lemma in Section 5.2.2 in [**GR**], is the basis for the rest of them.

Lemma 5.1. *Assume that $(V,z) \subseteq (Z,z)$, where (Z,z) has pure dimension, V has codimension ≥ 1 in Z and that there exists $f = (f_1,\ldots,f_m)$ such that $(V,z) = (Z,z) \cap \{f_1 = \cdots = f_m = 0\}$. Then there exists a finite union, E, of proper linear* *subspaces of* $\mathbb{C}^m$*, such that* $(Z,z) \cap \{a \cdot f = 0\}$ *has codimension* 1 *in* (Z,z) *if* $a \in \mathbb{C}^m \setminus E$.

Proof. The set E of $a \in \mathbb{C}^m$ such that $(Z,z) \cap \{a \cdot f = 0\} = (Z,z)$ is a linear subspace of $\mathbb{C}^m$, and since $(Z,z) \cap \{f_1 = \cdots = f_m = 0\}$ has positive codimension, it must be a proper subspace. If (Z,z) is irreducible, there thus exists a proper subspace $E \subseteq \mathbb{C}^m$ such that $(Z,z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z,z) if $a \in \mathbb{C}^m \setminus E$. If (Z,z) is reducible, then there exists such subspaces E_i for each irreducible component (Z_i,z) of (Z,z), and thus we can take $E = \cup E_i$. □

The following two lemmas are about existence of certain complete intersections containing a given variety, and their existence are the basis for the counterexamples to the duality theorem.

Lemma 5.2. *Assume that* $(V,z) \subseteq (Z,z)$*, where* (Z,z) *has pure dimension,* $\operatorname{codim} V = p$ *in* Z*, and let* $f = (f_1,\ldots,f_m)$ *be such that* $(V,z) = (Z,z) \cap \{f_1 = \cdots = f_m = 0\}$*. Then there exists* $f' = (f_1',\ldots,f_p')$*, a complete intersection on* Z*, such that* $(V,z) \subseteq (V',z) := (Z,z) \cap \{f_1' = \cdots = f_p' = 0\}$*, where* $f_i' = \sum a_{i,j} f_j$.

Proof. By Lemma 5.1, there exists $E \subseteq \mathbb{C}^m$ such that $(Z,z) \cap \{a \cdot f = 0\}$ has codimension 1 in (Z,z) for $a \in \mathbb{C}^m \setminus E$. We choose $f_1' = a \cdot f$, for some $a \in \mathbb{C}^m \setminus E$. Proceeding in the same way with $(Z,z) \cap \{f_1' = 0\}$ instead of (Z,z), we get f_2' such that $(Z,z) \cap \{f_1' = f_2' = 0\}$ has codimension 2 in Z. Repeating this, $f' = (f_1',\ldots,f_p')$ will be the desired complete intersection. □

Lemma 5.3. *Assume that* $(V,z) \subseteq (Z,z)$*, where* (V,z) *has codimension* p *in* (Z,z) *and* $\dim(Z,z) = d$*. Then, for some* w *arbitrarily close to* z*, there exists a complete intersection* $f = (f_1,\ldots,f_d) \in \mathcal{O}_{Z,w}^{\oplus d}$ *such that* $(V,w) = (Z,w) \cap \{f_1 = \cdots = f_p = 0\}$.

Proof. By Lemma 5.2, there exists $f = (f_1,\ldots,f_p)$ a complete intersection on (Z,z) such that $(V,z) \subseteq (V',z)$, where $V' = \{f_1 = \cdots = f_p = 0\}$. Since the set where V' is reducible has codimension $> p$, there exists some w arbitrarily close to z such that $(V,w) = (V',w)$. Then we apply Lemma 5.2 again to $(\{w\},w) \subseteq (V,w)$ to find $(f_{p+1},\ldots,f_d)$, a complete intersection on (V,w), so that $f = (f_1,\ldots,f_d)$ is the desired complete intersection. □

This last lemma is about the existence of a certain choice of coordinates, which is used in the proof of Theorem 6.3.

Lemma 5.4. *Let* $(Z,0) \subseteq (\mathbb{C}^n,0)$ *and assume that* Z *has pure dimension* d*. Then we can choose coordinates* w *on* $\mathbb{C}^n$ *such that* $(Z,0) \cap \{w_I = 0\} = \{0\}$ *for all* $I \subseteq \{1,\ldots,n\}$ *with* $|I| = d$.

Proof. We will choose the coordinates w on $\mathbb{C}^n$ inductively. By Lemma 5.1, there exists E such that $(Z,0) \cap \{a \cdot z = 0\}$ has codimension 1 in Z if $a \notin E$, and we choose $w_1 = a \cdot z$ for some $a \notin E$. Now, we assume by induction that we have chosen coordinates $(w_1,\ldots,w_k)$ such that $(Z,0) \cap \{w_I = 0\}$ has codimension $|I|$ for each $I \subseteq \{1,\ldots,k\}$ with $|I| \leq d$. For each $I \subseteq \{1,\dots,k\}$ with $|I| \le d-1$, we can then find E_I by Lemma 5.1 such that $(Z,0) \cap \{w_I = 0\} \cap \{a \cdot z = 0\}$ has codimension 1 in $(Z,0) \cap \{w_I = 0\}$ if $a \notin E_I$. Since each E_I is a finite union of proper subspaces of $\mathbb{C}^n$, we can find $a \in \mathbb{C}^n \setminus \cup E_I$, and we then let $w_{k+1} = a \cdot z$. Proceeding in this way, $w = (w_1,\dots,w_n)$ will be the desired choice of coordinates. $\square$

6. Representations of the integration current in the Cohen-Macaulay case

To prove Proposition 1.4, we will use the following representation of the integration current $[Z]$ on Z in terms of the current R^Z. Assume that Z is Cohen-Macaulay, and that $\operatorname{codim} Z = p$, so that $R^Z = R^Z_p$ by (3.2). By Example 1, [**A3**], there exist holomorphic $(p,0)$-forms ξ_i such that

$$[Z] = \sum \xi_i \wedge R^Z_{p,i}, \tag{6.1}$$

where $R^Z_{p,i}$ are the various components of R^Z, i.e., given a local frame $(e_1,\dots,e_N)$ of $\mathcal{O}(E_p)$, $R^Z_p = \sum R^Z_{p,i} e_i$.

If Z is a reduced complete intersection defined by $f = (f_1,\dots,f_p)$, then $R^Z = \mu^f$ by Theorem 3.2, and by the Poincaré-Lelong formula, see [**CH**], we have

$$[Z] = \frac{1}{(2\pi i)^p} \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge df_1 \wedge \cdots \wedge df_p.$$

Thus, we can take $\xi = df_1 \wedge \cdots \wedge df_p$, and then it is clear by the implicit function theorem that ξ vanishes at Z_{sing}. We will show that this is the case also when Z is Cohen-Macaulay. This is Theorem 6.3, and the proof will use the following lemmas. Recall that the *socle* of module M over a local ring $(R,\mathfrak{m},k)$ is defined as $\operatorname{Hom}_R(k,M)$, see [**BH**]. We will use the following characterization of the socle, which is immediate from the definition:

$$\operatorname{Hom}_R(k,M) \cong \{\alpha \in M \mid \mathfrak{m}\alpha = 0\}. \tag{6.2}$$

Lemma 6.1. *Let $\mathfrak{q}$ be a germ of an ideal at 0 such that $\sqrt{\mathfrak{q}} = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal at 0, and let*

$$0 \to \mathcal{O}(E_n) \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_1} \mathcal{O}(E_0) \to \mathcal{O}/\mathfrak{q} \to 0 \tag{6.3}$$

be a minimal free resolution of $\mathcal{O}/\mathfrak{q}$, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q}) = \operatorname{rank} E_n.$$

Proof. We have

$$\operatorname{rank} E_n = \dim \operatorname{Tor}_n(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$$

since $\operatorname{Tor}_n(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q})$ is just the n:th homology of the complex (6.3) tensored with $\mathcal{O}/\mathfrak{m}$. This is $\mathbb{C}^{\operatorname{rank} E_n}$ since the free resolution is minimal so that if

$$\tilde\varphi_n : \mathcal{O}(E_n) \otimes \mathcal{O}/\mathfrak{m} \to \mathcal{O}(E_{n-1}) \otimes \mathcal{O}/\mathfrak{m},$$

then $\tilde\varphi_n = 0$ since $\operatorname{Im} \varphi_n \subseteq \mathfrak{m} E_{n-1}$ by definition of minimality of a free resolution. However, $\operatorname{Tor}_n(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q})$ can also be computed by taking a free resolution of $\mathcal{O}/\mathfrak{m}$, tensoring it with $\mathcal{O}/\mathfrak{q}$ and taking homology. Since the Koszul complex of $(z_1,\dots,z_n)$ is a free resolution of $\mathcal{O}/\mathfrak{m}$, we get

$$\begin{aligned}\mathrm{Tor}_n(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q}) &\cong \mathrm{Ker}\left(\bigwedge^{n}\mathcal{O}/\mathfrak{q}\xrightarrow{\delta_z}\bigwedge^{n-1}\mathcal{O}/\mathfrak{q}\right)\\ &\cong \{\alpha\in\mathcal{O}/\mathfrak{q}\mid \mathfrak{m}\alpha=0\}\cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q}),\end{aligned}$$

where the last equality is (6.2). □

Lemma 6.2. *Assume that there exist pseudomeromorphic currents $\mu_1,\dots,\mu_N$ such that $\mathfrak{q}=\cap\,\mathrm{ann}\,\mu_i$, where $\mathfrak{q}$ is an ideal such that $\sqrt{\mathfrak{q}}=\mathfrak{m}$. Then*

$$N\geq \dim_{\mathbb{C}}\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q}).$$

Proof. We claim that there exists a $\mathbb{C}$-linear injective mapping

$$\tilde{\mu}:\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q})\to\mathbb{C}^N,$$

which proves the statement. We consider $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q})$ as (6.2). Since $\mathfrak{q}\subseteq\mathrm{ann}\,\mu_i$, the mapping $\alpha\mapsto\alpha\mu_i, \alpha\in\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m},\mathcal{O}/\mathfrak{q})$ is well-defined. Since $\mathfrak{m}\alpha=0$, and $\overline{\mathfrak{m}}\mu_i=0$ by Proposition 3.4, $\alpha\mu_i$ is a current of order 0 with support on $\{0\}$. Thus

$$\alpha\mu_i=a_iR_0, \tag{6.4}$$

for some $a_i\in\mathbb{C}$, where R_0 is the current $\delta_{z=0}d\bar{z}$, that is, $R_0.\alpha dz=\alpha(0)$. We thus get a mapping

$$\tilde{\mu}(\alpha)=(a_1,\dots,a_N),$$

where a_i are defined by (6.4). It only remains to see that $\tilde{\mu}$ is injective. However, if $\tilde{\mu}(\alpha)=0$, then $\alpha\in\cap\,\mathrm{ann}\,\mu_i=\mathfrak{q}$, so $\alpha=0$ in $\mathcal{O}/\mathfrak{q}$. □

Combining Lemma 6.1 and Lemma 6.2, if f is a complete intersection on Z, where Z is Cohen-Macaulay, then none of the components in the decomposition $R^f\wedge R^Z=\sum R^f\wedge R^Z_{p,i}$ are redundant. This will be a crucial step in the proof of the following theorem.

Theorem 6.3. *Let $Z\subseteq\Omega\subseteq\mathbb{C}^n$ be a subvariety of Ω of codimension p, and assume that Z is Cohen-Macaulay. Then there exists holomorphic $(p,0)$-forms ξ_i such that*

$$[Z]=\sum\xi_i\wedge R^Z_{p,i},$$

and if R^Z is defined with respect to a minimal free resolution of $\mathcal{O}_Z$, then all ξ_i vanish at Z_{sing}.

Proof. As mentioned in the introduction of the section, the existence of ξ_i is Example 1 in [**A3**], so we only need to prove that ξ_i vanish at Z_{sing} if R^Z is defined with respect to a minimal free resolution. Assume that $0\in Z_{\mathrm{sing}}$.

We begin by choosing coordinates in $\mathbb{C}^n$ such that $\{w_J = 0\} \cap Z = \{0\}$ for all $J \subseteq \{1,\dots,n\}$ with $|J| = n-p$, which is possible by Lemma 5.4. We have

$$[Z] = \sum_{i,|I|=p} \xi_{I,i} dw_I \wedge R^Z_{p,i}, \tag{6.5}$$

where $\xi_{I,i}$ are holomorphic functions, and we are done if we can prove that $\xi_{I,i}(0) = 0$ for all $\xi_{I,i}$.

Fix some $I \subseteq \{1,\dots,n\}$ with $|I| = p$. Let $w' = (w_{J_1},\dots,w_{J_{n-p}})$, where $J = I^c$. By the Poincaré-Lelong formula applied to w' on Z, see [**CH**], Section 1.9, we have that

$$\frac{1}{(2\pi i)^p} R^{w'} \wedge dw' \wedge [Z] = k[0]$$

for some $k \geq 1$. Combined with the Poincaré-Lelong formula applied to w in $\mathbb{C}^n$, we get

$$R^w \wedge dw = (2\pi i)^n[0] = ((2\pi i)^{n-p}/k) R^{w'} \wedge dw' \wedge [Z].$$

Since by (6.5)

$$dw' \wedge [Z] = \pm \sum_i \xi_{I,i} dw \wedge R^Z_{p,i}$$

we get that

$$R^w = C \sum_i \xi_{I,i} R^{w'} \wedge R^Z_{p,i} \tag{6.6}$$

for some constant $C \neq 0$.

We first consider the case when R^Z consists of one single component R^Z_p. By Corollary 3.6, $\operatorname{ann}(R^{w'} \wedge R^Z_p) = \mathcal{J}(w') + \mathcal{J}_Z$. We claim that the inclusion $\mathcal{J}(w)_0 \supseteq (\mathcal{J}(w') + \mathcal{J}_Z)_0$ is strict. If the inclusion is not strict, then w' generates the maximal ideal $\mathfrak{m}_{Z,0}$ in $\mathcal{O}_{Z,0}$, which is a contradiction by Proposition 4.32 in [**D**], since the number of functions needed to generate the maximal ideal at a singular point must be strictly larger than the dimension. Thus there exists a g in

$$\mathcal{J}(w)_0 \setminus (\mathcal{J}(w') + \mathcal{J}_Z)_0 = (\operatorname{ann} R^w)_0 \setminus (\operatorname{ann}(R^{w'} \wedge R^Z_p))_0.$$

Multiplying (6.6) by g, we get that $g\xi_I \in \operatorname{ann}(R^{w'} \wedge R^Z_p)$, and hence we must have $\xi_I(0) = 0$.

Now we consider the case when R^Z_p consists of more than one component. By Corollary 3.6, the tensor product of the Koszul complex of w' and the minimal free resolution of $\mathcal{J}_Z$ is a minimal free resolution of $\mathfrak{q} := \mathcal{J}(w') + \mathcal{J}_Z$, and the rank N of its left-most non-zero module is equal to the rank of the left-most non-zero module in the free resolution of $\mathcal{J}_Z$ since the left-most non-zero module of the Koszul complex has rank 1. By Corollary 3.6, we have

$$\mathfrak{q} = \cap_{i=1}^N \operatorname{ann}(R^{w'} \wedge R^Z_{p,i}). \tag{6.7}$$

By Lemma 6.1, $N = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}, \mathcal{O}/\mathfrak{q})$ and by Lemma 6.2, if $\mathfrak{q} = \cap_{i=1}^m \operatorname{ann} \mu_i$, then $m \geq N$. Thus, if we remove one term $\operatorname{ann}(R^{w'} \wedge R^Z_{p,i})$ from the intersection in (6.7), we get something strictly larger, i.e., for any i,

$$(\cap_{j\neq i} \operatorname{ann}(R^{w'} \wedge R^Z_{p,j})) \setminus (\operatorname{ann} R^{w'} \wedge R^Z_{p,i}) \neq \emptyset. \tag{6.8}$$

We fix some $i = 1, \dots, n$, and take g_i in (6.8) and multiply (6.6) by g_i. Since $g_i \in \cap_{j\neq i} \operatorname{ann}(R^{w'} \wedge R^Z_{p,j})$, we must have $g_i \in \mathfrak{m}$, so $g_i R^w = 0$. Thus we get

$$g_i \xi_{I,i} R^{w'} \wedge R^Z_{p,i} = 0.$$

Since $g_i \notin \operatorname{ann}(R^{w'} \wedge R^Z_{p,i})$ but $g_i \xi_{I,i} \in \operatorname{ann}(R^{w'} \wedge R^Z_{p,i})$, we must have $\xi_{I,i} \in \mathfrak{m}$, and we are done. □

7. Proof of Proposition 1.4

By moving to a nearby germ (Z, w), we can assume that Z_{sing} has pure codimension k, and that there exists a complete intersection $f = (f_1, \dots, f_q)$ on (Z, w) such that $(Z_{\mathrm{sing}}, w) = \{f_1 = \dots = f_k = 0\} \cap (Z, w)$, see Lemma 5.3. We let $\mathcal{I} = \mathcal{J}(f_1, \dots, f_q)_w$ and $V = Z(\mathcal{I})$, and since $q \geq k$, $V \subseteq Z_{\mathrm{sing}}$. Since $\mathcal{J}_{V,w}$ is finitely generated over $\mathcal{O}_{Z,w}$, we get from the Nullstellensatz that $\mathcal{J}^m_{V,w} \subseteq \mathcal{I}$ for m sufficiently large. Now, we choose m to be minimal such that this inclusion holds. Thus, there exists a function $g \in \mathcal{J}^{m-1}_{V,w} \setminus \mathcal{I}$, such that $g \mathcal{J}_{V,w} \subseteq \mathcal{I}$. Since $g \notin \mathcal{I}$, we are done if we can show that $g\mu^f \wedge [Z] = 0$.

By Theorem 3.2, we can replace μ^f by R^f, and instead show that $gR^f \wedge [Z] = 0$. By Theorem 6.3

$$gR^f \wedge [Z] = g \sum \xi_i \wedge R^f \wedge R^p_i,$$

where ξ_i are holomorphic $(p, 0)$-forms vanishing on Z_{sing}. Thus $\xi_i = \sum \xi_{I,i} dw_I$, where $\xi_{I,i}$ are holomorphic functions vanishing at Z_{sing}. Since $g\mathcal{J}_{V,w} \subseteq \mathcal{I}$ and $\mathcal{J}_{Z_{\mathrm{sing}},w} \subseteq \mathcal{J}_{V,w}$, we get that $g\xi_{I,i} \in \mathcal{I}$ in $\mathcal{O}_{Z,w}$. By Corollary 3.6, $\operatorname{ann} R^f \wedge R^Z = \mathcal{I} + \mathcal{J}_{Z,w}$. Since if $g\xi_{I,i} \in \mathcal{I}$ in $\mathcal{O}_{Z,w}$, then $g\xi_{I,i} \in \mathcal{I} + \mathcal{J}_{Z,w}$ in $\mathcal{O}_{\mathbb{C}^n,w}$, we get that $gR^f \wedge [Z] = 0$.

8. Singularity subvarieties and counterexamples in the non Cohen-Macaulay case

We will recall the notion of singularity subvarieties of analytic sheaves from [**ST**]. Let R be a local Noetherian ring and $M \neq 0$ a finitely generated R-module. A *regular M-sequence* in an ideal $I \subseteq R$ is a sequence $(f_1, \dots, f_p)$ in I such that f_i is not a zero-divisor in $M/(f_1, \dots, f_{i-1})M$ for $i = 1, \dots, p$. The *depth* of an ideal I on a module M, denoted $\operatorname{depth}_I M$ is the maximal length of a regular M-sequence in I. By $\operatorname{depth}_R M$, we will denote the depth of the maximal ideal $\mathfrak{m}$ of R on M. This is also called the homological codimension of M. The *homological dimension* of M, denoted $\mathrm{dh}_R M$, is defined as the minimal length of any free resolution of M.

A *regular local ring* is a local ring R such that the maximal ideal $\mathfrak{m}$ of R is generated by $n = \dim R$ elements, where $\dim R$ is the Krull-dimension of R, that is, the maximal length of a strict chain of prime ideals in R. In particular, if Z is an analytic variety, then $\mathcal{O}_{Z,z}$ is a regular local ring if and only if $z \in Z_{\mathrm{reg}}$, see Proposition 4.32 in [**D**]. The following is Theorem 19.9 in [**E**], the Auslander-Buchsbaum formula.

Proposition 8.1. *If R is a regular local ring, and M is a finitely generated R-module, then* $\mathrm{dh}_R M + \mathrm{depth}_R M = \dim R$.

Let $\mathcal{F}$ be a coherent analytic sheaf on $\Omega \subseteq \mathbb{C}^n$, and let $\mathcal{O}_z$ denote the ring of germs of holomorphic functions at z in Ω. The *singularity subvarieties*, S_m, of $\mathcal{F}$ are defined by

$$S_m(\mathcal{F}) = \{z \in \Omega; \mathrm{depth}_{\mathcal{O}_z} \mathcal{F}_z \leq m\},$$

where we use the convention that $\mathrm{depth}_R M = \infty$ if $M = 0$, so that $S_m \subseteq \mathrm{supp}\, \mathcal{F}$. We will use the following alternative definition of the sets Z_k associated with an analytic sheaf above:

$$Z_k(\mathcal{F}) = \{z \in \Omega; \mathrm{dh}_{\mathcal{O}_z} \mathcal{F}_z \geq k\} \tag{8.1}$$

(in the introduction, we defined the sets Z_k if $\mathcal{F}$ was of the form $\mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ was an coherent ideal sheaf, but the same definition works for any coherent analytic sheaf). To see this, note first that if $\mathrm{rank}\, \varphi_k(z)$ is constant in a neighborhood of some $z_0 \in \Omega$ (i.e., $z_0 \notin Z_k$), then $\mathcal{O}(E_{k-1})/\mathrm{Im}\, \varphi_k$ is free in a neighborhood of z_0, so $\mathcal{O}/\mathcal{J}$ has a free resolution of length $k-1$. Conversely, by the uniqueness of minimal free resolutions, $\mathrm{rank}\, \varphi_k(z)$ must be constant in a neighborhood of z if $k > \mathrm{dh}_{\mathcal{O}_z} \mathcal{F}_z$.

Proposition 8.2. *If $\mathcal{F}$ is coherent analytic sheaf on some open set in $\mathbb{C}^n$, we have $S_k(\mathcal{F}) = Z_{n-k}(\mathcal{F})$.*

Proof. This follows from Proposition 8.1 and (8.1). □

Let $\Omega \subseteq \mathbb{C}^n$ be an open set, A a subvariety of Ω with ideal sheaf $\mathcal{J}_A$, and $\mathcal{F}$ a coherent analytic sheaf in Ω. For $z \in \Omega$, we define

$$\mathrm{depth}_{A,z} \mathcal{F} = \begin{cases} \infty & \text{if } \mathcal{F}_z = 0 \\ \mathrm{depth}_{\mathcal{J}_{A,z}} \mathcal{F} & \text{otherwise} \end{cases}.$$

and

$$\mathrm{depth}_A \mathcal{F} = \inf_{z \in A} \mathrm{depth}_{A,z} \mathcal{F}$$

The following is (part of) Theorem 1.14 in [**ST**].

Theorem 8.3. *Let $\Omega \subseteq \mathbb{C}^n$ be some open set, A a subvariety of Ω, and $\mathcal{F}$ a coherent analytic sheaf in Ω. Then for $q \geq 1$, we have* $\mathrm{depth}_A \mathcal{F} \geq q$ *if and only if* $\dim A \cap S_{k+q}(\mathcal{F}) \leq k$ *for all k.*

In particular, if we let Z be an analytic subvariety of Ω, $\mathcal{F} = \mathcal{O}_Z$, and $A = Z^1$, where the sets Z^k associated with Z are defined as in (1.4), we get the following.

Corollary 8.4. *For $q \geq 1$, we have* $\operatorname{depth}_{Z^1} \mathcal{O}_Z \geq q$ *if and only if* $\operatorname{codim} Z^k \geq q + k$ *in Z for all $k \geq 1$*

Proof. If we apply Theorem 8.3 to $A = Z^1$ and $\mathcal{F} = \mathcal{O}_Z$, then we only need to prove that $\operatorname{codim} Z^k \geq q + k$ for $k \geq 1$ is equivalent to $\dim Z^1 \cap S_{k+q}(\mathcal{O}_Z) \leq k$. We can write the last condition as $\dim(Z^1 \cap Z_{n-k-q}) \leq k$ by Proposition 8.2. If we replace $\dim V$ by $n - \operatorname{codim} V$ and set $k' = n - k - q$, we get $\operatorname{codim}(Z^1 \cap Z_{k'}) \geq q + k'$. Since $Z_k = Z$ for $k \leq p$, where $p = \operatorname{codim} Z$, and $Z^1 = Z_{p+1}$, this condition for $k \leq p$ is equivalent to $\operatorname{codim} Z_{p+1} \geq p + q + 1$ (in Ω), and since $Z_k \subseteq Z_{p+1} = Z^1$ for $k > p+1$, this is equivalent to $\operatorname{codim} Z_{p+k} \geq p + q + k$ for $k \geq 2$. $\square$

In $\mathbb{C}^n$, it is a standard result that a tuple $f = (f_1, \dots, f_p)$ of holomorphic functions is a complete intersection if and only if it is a regular sequence (see for example [**dJP**], Corollary 4.1.20). However, Corollary 8.4 says that this is not always the case on a singular variety. We will illustrate this with an example.

Example 2. Let $\pi(t_1, t_2) = (t_1, t_1 t_2, t_2^2, t_2^3)$, and let $Z = \pi(\mathbb{C}^2)$. Then $Z_{\text{sing}} = \{0\}$, because outside of $\{t_1 = t_2 = 0\}$, one can construct a holomorphic inverse to π, and we will see that Z is not normal at 0, so $0 \in Z_{\text{sing}}$. The function f such that $\pi^* f = t_2$ is weakly holomorphic on Z, since when $t_1 \neq 0$, $f = z_2/z_1$, and when $t_2 \neq 0$, $f = z_4/z_3$, so that $f \in \mathcal{O}(Z_{\text{reg}})$, and it is clear that f is locally bounded near $Z_{\text{sing}} = \{0\}$. However, f is not strongly holomorphic at 0, because if $f = h$ on Z in a neighborhood of 0, where h is holomorphic in a neighborhood of 0 in $\mathbb{C}^4$, then by taking pull-back by π to $\mathbb{C}^2$, we get

$$t_2 = h(t_1, t_1 t_2, t_2^2, t_2^3),$$

which can be seen to be impossible by a Taylor expansion of h at 0.

Since Z has pure dimension, $\operatorname{codim} Z^k \geq k + 1$ for $k \geq 1$ by [**E**], Corollary 20.14b. Hence, $Z^k = \emptyset$ for $k \geq 2$. Since Z is not normal, it does not satisfy the condition

$$\operatorname{codim} Z^k \geq k + 2, \quad k \geq 0 \tag{8.2}$$

for normality (see the introduction). However, since $Z^0 = Z_{\text{sing}} = \{0\}$, the condition (8.2) is satisfied for all $k \neq 1$. Thus, since $Z^1 \subseteq Z_{\text{sing}}$, and $\operatorname{codim} Z^1 \not\geq 3$, we must have $Z^1 = \{0\}$. By Corollary 8.4, there does not exist a regular $\mathcal{O}_Z$-sequence $f = (f_1, f_2)$ in $\mathcal{J}_{Z^1}$, since any such sequence has length ≤ 1. In particular, if we take $f = (z_1, z_3)$, then f is a complete intersection since $Z \cap \{z_1 = z_3 = 0\} = \{0\}$, but f is not a regular sequence. We claim that one can also see this more directly. To begin with, it is clear that

$z_3 \notin (z_1)$ in $\mathcal{O}_Z$ since $Z \cap \{z_1 = 0\} \not\subseteq Z \cap \{z_3 = 0\}$. We also have that $z_2 \notin (z_1)$ in $\mathcal{O}_Z$, since if $z_2 \in (z_1)$, then by taking pull-back to $\mathbb{C}^2$ as above, we get

$$t_1 t_2 = t_1 h(t_1, t_1 t_2, t_2^2, t_2^3),$$

which is easily seen to be impossible. However, since $z_2 z_3 = z_1 z_4$ in $\mathcal{O}_Z$, we get that $z_2 z_3 \in (z_1)$ in $\mathcal{O}_Z$. Thus, z_3 is a zero-divisor in $\mathcal{O}_Z/(z_1)$, i.e., (z_1, z_3) is not a regular $\mathcal{O}_Z$-sequence in $\mathcal{J}_{Z^1}$.

Lemma 8.5. *Let $f = (f_1, \dots, f_k)$ be a complete intersection on (Z, z). If*

$$\operatorname{ann}\left(\bar\partial \frac{1}{f_r} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}\right) = \mathcal{J}(f_1, \dots, f_r) \text{ for all } r < k,$$

then $(f_1, \dots, f_k)$ is a regular $\mathcal{O}_{Z,z}$-sequence.

Proof. If $k = 1$, this is clear since $\mathcal{O}_{Z,z}$ is reduced and f is assumed to be a complete intersection. By induction over k, we can assume that $(f_1, \dots, f_{k-1})$ is a regular $\mathcal{O}_{Z,z}$-sequence. Assume that $(f_1, \dots, f_k)$ is not a regular sequence in $\mathcal{O}_{Z,z}$. Then, since $f_k \notin \mathcal{J}(f_1, \dots, f_{k-1})$, there exist $g \notin \mathcal{J}(f_1, \dots, f_{k-1})$ such that $f_k g \in \mathcal{J}(f_1, \dots, f_{k-1})$. But since $g \in \mathcal{J}(f_1, \dots, f_{k-1})$ outside of $\{f_k = 0\}$, we get that

$$\operatorname{supp}\left(g \bar\partial \frac{1}{f_{k-1}} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}\right) \subseteq \{f_1 = \cdots = f_k = 0\}$$

by Theorem 1.2. But then by Proposition 3.3, we get that

$$g \in \operatorname{ann} \bar\partial \frac{1}{f_{k-1}} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} = \mathcal{J}(f_1, \dots, f_{k-1}),$$

which is a contradiction. □

Proof of Proposition 1.5. By Lemma 5.2, there exists a complete intersection $(f_1, \dots, f_{p+1})$ such that $Z^1 \subseteq \{f_1 = \cdots = f_{p+1} = 0\}$. By Corollary 8.4, $(f_1, \dots, f_{p+1})$ is not a regular $\mathcal{O}_{Z,z}$-sequence in $\mathcal{J}(f_1, \dots, f_{p+1})_z$. Thus by Lemma 8.5, we must have that

$$\operatorname{ann}\left(\bar\partial \frac{1}{f_k} \wedge \cdots \wedge \bar\partial \frac{1}{f_1}\right) \supsetneq \mathcal{J}(f_1, \dots, f_k) \tag{8.3}$$

for some $k \leq p$. However, by Theorem 1.3, we have equality for $k \leq p - 1$. Thus we must have strict inclusion in (8.3) for $k = p$. □

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PAPER III

A comparison formula for residue currents

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PAPER III

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ABSTRACT. Given two ideals $\mathcal{I}$ and $\mathcal{J}$ of holomorphic functions such that $\mathcal{I} \subseteq \mathcal{J}$, we describe a comparison formula relating the Andersson-Wulcan currents of $\mathcal{I}$ and $\mathcal{J}$. More generally, this comparison formula holds for residue currents associated to two generically exact complexes of vector bundles, together with a morphism between the complexes.

We then show various applications of the comparison formula including generalizing the transformation law for Coleff-Herrera products to Andersson-Wulcan currents of Cohen-Macaulay ideals, proving that there exists a natural current $R_Z^{\mathcal{J}}$ on a singular variety Z such that $\operatorname{ann} R_Z^{\mathcal{J}} = \mathcal{J}$, and giving an analytic proof of a theorem of Hickel related to the Jacobian determinant of a holomorphic mapping by means of residue currents.

1. Introduction

Given a tuple $f = (f_1, \dots, f_p)$ of germs holomorphic functions at the origin in $\mathbb{C}^n$ defining a complete intersection, i.e., so that $\operatorname{codim} Z(f) = p$, there exists a current, called the *Coleff-Herrera product* of f,

$$\mu^f = \bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1}, \tag{1.1}$$

associated to it, as introduced in [**CH**]. One of the fundamental properties of the Coleff-Herrera product is the *duality theorem*, which says that $\operatorname{ann} \mu^f = \mathcal{J}(f)$, where $\operatorname{ann} \mu^f$ is the annihilator of μ^f, i.e., the holomorphic functions g such that $g\mu^f = 0$, and $\mathcal{J}(f)$ is the ideal generated by f. The duality theorem was proven independently by Dickenstein and Sessa, [**DS1**], and Passare, [**P**].

Another fundamental property of the Coleff-Herrera product is that it satisfies the *transformation law*. Earlier versions of the transformation law involving cohomological residues (Grothendieck residues) had appeared, see for example [**To**], (4.3), and [**GH**], page 657.

Theorem 1.1. *Let $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ be tuples of holomorphic functions defining complete intersections. Assume there exists a matrix A of* *holomorphic functions such that $f = gA$. Then*

$$\bar\partial\frac{1}{g_p}\wedge\cdots\wedge\bar\partial\frac{1}{g_1} = (\det A)\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.$$

In the setting of Coleff-Herrera products, the transformation law was first stated in [**DS1**], and it was explained that the proof can be reduced to the absolute case (when $p = n$) and cohomological residues, together with the technique from [**CH**] of fibered residues. An elaboration of this proof can be found in [**DS2**].

For cohomological residues as in [**GH**], the idea of the proof is that if $dg_1\wedge\cdots\wedge dg_n$ is non-vanishing, and A is invertible, then the transformation law is essentially the change of variables formula for integrals, and the general case is reduced to the previous case by perturbations of g and A.

In the case when $p = n$, the transformation law combined with the Nullstellensatz allow to express in an explicit fashion the action of μ^f, see for example [**Ts**], page 22. Essentially the same idea is also used in [**GH**] to prove the duality theorem for Grothendieck residues by using the transformation law.

One particular case of the transformation law is when we choose different generators $f' = (f_1',\dots,f_p')$ of the ideal generated by f. Then the Coleff-Herrera product of f' differs from the one of f only by an invertible holomorphic function, and hence, it can essentially be considered as a current associated to the ideal $\mathcal{J}(f)$ rather than the tuple f.

The requirement that $f = gA$ means that $\mathcal{J}(f)\subseteq\mathcal{J}(g)$. Thus, by considering the Coleff-Herrera product of g as a current associated to the ideal $\mathcal{J}(g)$, the transformation law says that inclusion of ideals $\mathcal{J}(f)\subseteq\mathcal{J}(g)$ implies that we can express the Coleff-Herrera product associated to $\mathcal{J}(g)$ in terms of the Coleff-Herrera product associated to $\mathcal{J}(f)$.

1.1. A comparison formula for Andersson-Wulcan currents. Now, consider an arbitrary ideal $\mathcal{J}\subseteq\mathcal{O}=\mathcal{O}_{\mathbb{C}^n,0}$ of holomorphic functions. Throughout this article, we will let $\mathcal{O}$ denote $\mathcal{O}_{\mathbb{C}^n,0}$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$ unless otherwise stated. Let (E,φ) be a Hermitian resolution of $\mathcal{O}/\mathcal{J}$,

$$0\to E_N\xrightarrow{\varphi_N}E_{N-1}\to\dots\xrightarrow{\varphi_1}E_0\to\mathcal{O}/\mathcal{J}\to 0,$$

i.e., a free resolution of $\mathcal{O}/\mathcal{J}$, where the free modules are equipped with Hermitian metrics. Given E, Andersson and Wulcan constructed in [**AW1**] a current R^E such that $\operatorname{ann} R^E = \mathcal{J}$, where $R^E = \sum_{k=p}^N R_k^E$, $p = \operatorname{codim} Z(\mathcal{J})$, and R_k^E are $\operatorname{Hom}(E_0,E_k)$-valued $(0,k)$-currents. We will sometimes denote the current R^E by $R^{\mathcal{J}}$, although it depends on the choice of Hermitian resolution E of $\mathcal{O}/\mathcal{J}$. We refer to Section 2 for a more thorough description of the current R^E. Such currents have been used, for example in the study of various questions related to singular varieties, like division problems in [**ASS**, **AW3**, **S**], and the $\bar\partial$-equation in [**AS1**, **AS2**].

In case $\mathcal{J}$ is a complete intersection defined by a tuple f, then $\mathcal{J}$ has an explicit free resolution; the Koszul complex of f. In that case, the Andersson-Wulcan current associated to the Koszul complex coincides with the Coleff-Herrera product of f, see Section 2.3.

We now consider two ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$, and free resolutions (E,φ) and (F,ψ) of $\mathcal{O}/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}$ respectively. We choose minimal free resolutions, so that in particular $\operatorname{rank} E_0 = \operatorname{rank} F_0 = 1$, i.e., $E_0 \cong \mathcal{O} \cong F_0$, and we let $a_0 : E_0 \to F_0$ be this isomorphism. Since $\mathcal{I} \subseteq \mathcal{J}$, we get the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$, and by the choice of a_0, the diagram

$$\begin{tikzcd} E_0 \arrow[r] & \mathcal{O}/\mathcal{J} \\ F_0 \arrow[r] \arrow[u, "a_0"] & \mathcal{O}/\mathcal{I} \arrow[u, "\pi"'] \end{tikzcd}$$

commutes. Using the fact that the F_k are free, and that (E,φ) is exact, by a simple diagram chase one can show that one can complete this to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E_N \arrow[r, "\varphi_N"] & E_{N-1} \arrow[r, "\varphi_{N-1}"] & \cdots \arrow[r, "\varphi_1"] & E_0 \arrow[r] & \mathcal{O}/\mathcal{J} \arrow[r] & 0 \\ 0 \arrow[r] & F_N \arrow[r, "\psi_N"] \arrow[u, "a_N"] & F_{N-1} \arrow[r, "\psi_{N-1}"] \arrow[u, "a_{N-1}"] & \cdots \arrow[r, "\psi_1"] & F_0 \arrow[r] \arrow[u, "a_0"] & \mathcal{O}/\mathcal{I} \arrow[r] \arrow[u, "\pi"] & 0, \end{tikzcd} \tag{1.2}$$

of the free resolutions, i.e., $a : (F,\psi) \to (E,\varphi)$ is a morphism of complexes.

The main result of this article is a comparison formula for the currents associated to $\mathcal{I}$ and $\mathcal{J}$, obtained from the morphism a. The formula involves forms u^E and u^F, which are certain endomorphism-valued forms on the free resolutions E and F, see Section 2 for details about how they are defined.

Theorem 1.2. *Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{O}$ be two ideals of germs of holomorphic functions such that $\mathcal{I} \subseteq \mathcal{J}$, and let (E,φ) and (F,ψ) be minimal free resolutions of $\mathcal{O}/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}$ respectively. Let $a : (F,\psi) \to (E,\varphi)$ be the morphism in* (1.2) *induced by the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$. Then,*

$$R^{\mathcal{J}} a_0 - a R^{\mathcal{I}} = \nabla_\varphi M, \tag{1.3}$$

where $\nabla_\varphi = \sum \varphi_k - \bar\partial$,

$$M = \bar\partial |G|^{2\lambda} \wedge u^E \wedge a u^F \Big|_{\lambda=0},$$

and G is a tuple of holomorphic functions such that $\{G = 0\}$ contains the set where (E,φ) and (F,ψ) are not pointwise exact.

The theorem in fact holds in a more general setting. First of all, there are Andersson-Wulcan currents associated not just to free resolutions, but to generically exact complexes of vector bundles, and the theorem holds for such residue currents together with arbitrary morphisms of the complexes, Theorem 3.2. To elaborate more precisely how the current M and ∇_φ are defined, more background from the construction of the Andersson-Wulcan currents is required. We refer to Section 2 for the necessary background, and Section 3 for a more precise statement of the comparison formula in the general form.

One of the main applications of the comparison formula will be to construct residue currents with prescribed annihilator ideals on singular varieties, generalizing the construction of Andersson-Wulcan. We will treat one aspect of the construction here, which is a rather direct consequence of the comparison formula, and we will elaborate it in the article [**L3**]. We will also discuss other direct applications and special cases of the comparison formula.

1.2. A transformation law for Andersson-Wulcan currents associated with Cohen-Macaulay ideals. Our first application is a situation in which the current M in (1.3) vanishes. This gives a direct generalization of the transformation law for Coleff-Herrera products to Andersson-Wulcan currents associated with Cohen-Macaulay ideals. We recall that an ideal $\mathcal{J}$ is *Cohen-Macaulay* if $\mathcal{O}/\mathcal{J}$ has a free resolution of length equal to $\operatorname{codim} Z(\mathcal{J})$.

Theorem 1.3. *Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{O}$ be two Cohen-Macaulay ideals of germs of holomorphic functions of the same codimension p such that $\mathcal{I} \subseteq \mathcal{J}$. Let (F,ψ) and (E,φ) be free resolutions of length p of $\mathcal{O}/\mathcal{I}$ and $\mathcal{O}/\mathcal{J}$ respectively. If $a : (F,\psi) \to (E,\varphi)$ is the morphism in* (1.2) *induced by the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$, then*

$$R_p^{\mathcal{J}} a_0 = a_p R_p^{\mathcal{I}}.$$

The proof of Theorem 1.3 is given in Section 4; it is a special case of the more general Theorem 4.1. In Remark 3 in Section 4, we describe how the transformation law for Coleff-Herrera product is a special case of Theorem 1.3.

In the article [**DS2**], two proofs of the transformation law for Coleff-Herrera products are given. One of the proofs can in fact be adapted to give an alternative proof of Theorem 1.3, see Section 4.

See Section 4 for various examples of how one can use Theorem 1.3 or its generalization Theorem 4.1 to express the current $R^{\mathcal{I}}$ for a Cohen-Macaulay ideal $\mathcal{I}$ in terms of other currents in an explicit way. In Section 5, we give an example of a computation when the ideal is not Cohen-Macaulay.

In a forthcoming article joint with E. Wulcan, we use Theorem 1.3 to compute currents like $D\varphi_1 \circ \cdots \circ D\varphi_p \circ R_p$, generalizing the Poincaré-Lelong formula. In another joint article, [**LW**], we use Theorem 1.3 to calculate in a simpler and in some aspects more explicit way residue currents associated to Artinian monomial ideals, compared to earlier work by the second author.

1.3. Andersson-Wulcan currents on an analytic variety. Let $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ be an ideal of holomorphic functions. Then, the Andersson-Wulcan current $R^{\mathcal{J}}$ is a current associated to $\mathcal{J}$ such that its annihilator equals $\mathcal{J}$. It is natural to ask if there exists a similar current on a singular analytic variety $(Z,0) \subseteq (\mathbb{C}^n,0)$ associated to an ideal $\mathcal{J} \subseteq \mathcal{O}_Z$, where $\mathcal{O}_Z = \mathcal{O}/\mathcal{I}_Z$ and $\mathcal{I}_Z$ is the ideal of holomorphic functions vanishing on Z.

There exists one natural candidate. If we consider the Andersson-Wulcan current $R^{\mathcal{J}+\mathcal{I}_Z}$ on $(\mathbb{C}^n,0)$, its annihilator equals $\mathcal{J}+\mathcal{I}_Z$. Thus, since its annihilator contains $\mathcal{I}_Z$, we get a well-defined multiplication of it by holomorphic functions on Z, and its annihilator considered as an ideal in $\mathcal{O}_Z$ equals $\mathcal{J}$.

We first remind briefly how to define currents on analytic varieties. The usual way to define currents on an analytic variety is to first define test forms on analytic varieties, and then define currents as continuous linear functionals on the test forms. However, more concretely, if $(Z,0) \subseteq (\mathbb{C}^n,0)$, and i is the inclusion $i : (Z,0) \to (\mathbb{C}^n,0)$, then T is a current of bidimension (d,e) on Z if i_*T is a current of bidimension (d,e) on $(\mathbb{C}^n,0)$ which vanishes when acting on test forms ϕ such that $\phi|_{Z_{\text{reg}}} = 0$. Conversely, if T' is any such current on $(\mathbb{C}^n,0)$, then T' defines a unique current T on Z such that $i_*T = T'$.

Since $R^{\mathcal{J}+\mathcal{I}_Z}$ is a current of bidimension $(n,n-p)$ on $\mathbb{C}^n$, it cannot be a current on Z for degree reasons, so we consider instead $R^{\mathcal{J}+\mathcal{I}_Z} \wedge dz$, where $dz = dz_1 \wedge \cdots \wedge dz_n$, which has the same annihilator as $R^{\mathcal{J}+\mathcal{I}_Z}$.

Theorem 1.4. *Let $(Z,0)$ be a subvariety of $(\mathbb{C}^n,0)$, and let $\mathcal{J} \subseteq \mathcal{O}_Z$ be an ideal. Then there exists a current $R^{\mathcal{J}}_Z$ on Z such that* $\operatorname{ann} R^{\mathcal{J}}_Z = \mathcal{J}$, *and* $i_*R^{\mathcal{J}}_Z = R^{\mathcal{J}+\mathcal{I}_Z} \wedge dz$.

Note that we do not assume that $(Z,0)$ has pure dimension, i.e., it may consist of components of different dimensions. The proof of Theorem 1.4 is given in Section 6, using the comparison formula, Theorem 1.2. As we described above, $R^{\mathcal{J}+\mathcal{I}_Z} \wedge dz$ will define the current $R^{\mathcal{J}}_Z$ with the correct annihilator if we prove that if ϕ is a test form vanishing on Z_{reg}, then $R^{\mathcal{J}+\mathcal{I}_Z} \wedge dz$ acting on ϕ is zero.

In fact, it will essentially follow from the proof of Theorem 1.4 that the construction of $R^{\mathcal{J}}_Z$ on the singular variety Z can be seen as a generalization of the construction of the Andersson-Wulcan current $R^{\mathcal{J}}$ of an ideal $\mathcal{J}$ on a complex manifold. Since elaborating on this would lead us too far astray, we will treat this topic in a separate article, [**L3**].

Note that by construction, $\mathcal{I}_Z$ annihilates $R^{\mathcal{J}+\mathcal{I}_Z}$, and by properties of pseudomeromorphic currents, $\overline{\mathcal{I}}_Z$, i.e., antiholomorphic functions vanishing on Z, also annihilate $R^{\mathcal{J}+\mathcal{I}_Z}$, Proposition 2.3. If $(Z,0)$ is not irreducible, it is easily seen that $\mathcal{I}_Z$ and $\overline{\mathcal{I}}_Z$ do not generate the ideal of smooth functions vanishing on Z. For example, if $Z = \{zw = 0\} \subseteq \mathbb{C}^2$, then $z\bar{w}$ is smooth and vanishes at Z, but it does not lie in the (smooth) ideal generated by holomorphic and antiholomorphic functions vanishing on Z. However, this can happen also when $(Z,0)$ is irreducible. For example, the variety $Z = \{z_3^3(z_1^2+z_2^2)-z_1^3 = 0\} \subseteq \mathbb{C}^3$ is irreducible at 0, but there exist z arbitrarily close to 0 such that (Z,z) is not irreducible. In this case, the ideal of smooth functions vanishing on Z is strictly larger than the ideal generated by $\mathcal{I}_Z$ and $\overline{\mathcal{I}}_Z$, see [**N**], Proposition 9, Chapter IV and [**M**], Theorem 3.10, Chapter VI. Thus, it is not immediate whether it is possible to prove Theorem 1.4 using only that it is annihilated by $\mathcal{I}_Z$ and $\overline{\mathcal{I}}_Z$.

Remark 1. In case $\mathcal{J}$ is a complete intersection defined by a tuple f on a complex manifold Z, the Coleff-Herrera product of f coincides with the Andersson-Wulcan current of $\mathcal{J}$, and hence is a current on Z such that its annihilator equals $\mathcal{J}$. If Z is singular, the Coleff-Herrera product of f still exists, and is a current on Z. However, in general its annihilator is strictly larger than $\mathcal{J}$, see [**L2**].

Trying to prove Theorem 1.4 was actually how we were lead to discover the comparison formula. Proving that $R^{\mathcal{I}_Z} \wedge dz$ corresponds to a current on Z follows in a rather straightforward way by using properties of pseudomeromorphic currents if Z has pure dimension. Since the holomorphic annihilator of $R^{\mathcal{J}+\mathcal{I}_Z}$ is larger than that of $R^{\mathcal{I}_Z}$, and it has smaller support, it should be easier to annihilate it, and hence, $R^{\mathcal{J}+\mathcal{I}_Z}$ should also be a current on Z. One way of making this into a formal mathematical argument would be to express $R^{\mathcal{J}+\mathcal{I}_Z}$ in terms of $R^{\mathcal{I}_Z}$. In the case of two complete intersections f and g instead of $\mathcal{J}+\mathcal{I}_Z$ and $\mathcal{I}_Z$, the transformation law expresses this relation. Trying to extend this to more general ideals, we arrived at Theorem 1.2.

1.4. The Jacobian determinant of a holomorphic mapping. Let $f = (f_1,\dots,f_n) \in \mathcal{O}^{\oplus n} = \mathcal{O}^{\oplus n}_{\mathbb{C}^n,0}$, such that $0 \in Z(f_1,\cdots,f_n)$, and let J_f the Jacobian determinant of f, i.e.,

$$df_1 \wedge \cdots \wedge df_n = J_f dz_1 \wedge \cdots \wedge dz_n.$$

If f is a complete intersection, it follows from the Poincaré-Lelong formula, [**CH**], Section 3.6, that

$$\bar\partial \frac{1}{f_n} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge J_f dz_1 \wedge \cdots \wedge dz_n = k[0],$$

where k is the multiplicity of f at 0, i.e., the generic number of preimages $f^{-1}(z)$ close to 0 for z close to 0. In particular, since $0 \in Z(f_1,\dots,f_n)$, $k \geq 1$. Thus, J_f does not annihilate μ^f, so by the duality theorem, $J_f \notin \mathcal{J}(f)$. Hickel proved in [**H**], that the converse of this also holds.

Theorem 1.5. *Let $f = (f_1,\dots,f_n)$ be a tuple of germs of holomorphic functions in $\mathcal{O}_{\mathbb{C}^n,0}$, and let J_f be the Jacobian determinant of f. Then $J_f \in \mathcal{J}(f_1,\dots,f_n)$ if and only if* $\operatorname{codim} Z(f_1,\dots,f_n) < n$.

We will use the generalization Theorem 3.2 of Theorem 1.2 to give a proof of this theorem by means of residue currents, the proof is given in Section 7.

The results in [**H**] concern more general rings than just $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$, the ring of germs of holomorphic functions, and generalize previous results by Vasconcelos in the case of the polynomial ring over a field, [**V**]. In the proof in [**H**], as is the case here, residues are used. However, the proof in [**H**] uses Lipman residues, which are very much algebraic in nature, compared to Andersson-Wulcan currents, which are analytic in nature.

In the other applications of our comparison formula, we have considered Andersson-Wulcan currents associated to free resolutions. In the proof of Theorem 1.5, we use the comparison formula when the source complex is the Koszul complex of f, which is generically exact, and exact if and only if f is a complete intersection. The target complex will be a free resolution of the ideal $\mathcal{J}(f)$, and in order to get the induced morphism between the complexes, it is only required that the target complex is exact, see Proposition 3.1.

The current associated to the Koszul complex of f is called the Bochner-Martinelli current, as introduced in [**PTY**]. In fact, Theorem 1.5 was an important tool in the study of annihilators of Bochner-Martinelli currents in [**JW**].

2. Andersson-Wulcan currents and pseudomeromorphic currents

In this section, we recall the construction of residue currents associated to free resolutions of ideals, or more generally, residue currents associated to generically exact complexes, as constructed in [**AW1**] and [**A3**]. This is done in a rather detailed manner, since the construction of the comparison formula, and the properties of the currents appearing in the formula requires rather detailed knowledge of the construction of Andersson-Wulcan currents and their properties.

Let (E,φ) be a Hermitian complex (i.e., a complex of vector bundles equipped with Hermitian metrics), which is generically exact, i.e., the complex is pointwise exact outside some analytic set Z. Mainly, (E,φ) will be a free resolution of a module $\mathcal{O}/\mathcal{J}$, for some ideal $\mathcal{J} \subseteq \mathcal{O}$. When we refer to exactness of the complex, we mean that the induced complex of sheaves of $\mathcal{O}$-modules is exact. When we refer to exactness as vector bundles, we will refer to it as pointwise exactness, and generic exactness means it is pointwise exact outside of some analytic set. This is in contrast to the notation in for example [**AW1**], where the induced complex of sheaves of $\mathcal{O}$-modules is denoted $\mathcal{O}(E)$, and exactness as vector bundles or sheaves depends on if the complex is referred to as E or $\mathcal{O}(E)$.

2.1. The superbundle structure of the total bundle E. The bundle $E = \oplus E_k$ has a natural superbundle structure, i.e., a $\mathbb{Z}_2$-grading, which splits E into odd and even elements E^+ and E^-, where $E^+ = \oplus E_{2k}$ and $E^- = \oplus E_{2k+1}$.

Then also $\mathcal{D}'(E)$, the sheaf of current-valued sections of E inherits a superbundle structure by letting the degree of an element $\mu \otimes \omega$ be the sum of the degrees of μ and ω modulo 2, where μ is a current and ω is a section of E.

Now, also $\operatorname{End} E$ gets a superbundle structure by letting the even elements be the endomorphisms preserving the degree, and the odd elements the endomorphisms switching degree. Given g in $\operatorname{End} E$, we consider it also as en element of $\operatorname{End}\mathcal{D}'(E)$ by the formula

$$g(\mu \otimes \omega) = (-1)^{(\deg g)(\deg \mu)} \mu \otimes g\omega$$

if g is homogeneous. Also, $\bar\partial$ can be considered as an element of $\operatorname{End}\mathcal{D}'(E)$ by the formula $\bar\partial(\mu \otimes \omega) = \bar\partial\mu \otimes \omega$ if ω is a holomorphic section of E.

We let $\nabla := \varphi - \bar\partial$. Note that the action of φ on $\mathcal{D}'(E)$ is defined so that $\bar\partial$ and φ anti-commute, and hence, $\nabla^2 = 0$. Note also that since φ and $\bar\partial$ are odd mappings, ∇ is odd.

The mapping ∇ induces a mapping ∇_{End} on $\mathcal{D}'(\operatorname{End} E)$ by the formula

$$\nabla(\alpha\xi) = \nabla_{\mathrm{End}}(\alpha)\xi + (-1)^{\deg \alpha} \alpha \nabla \xi,$$

where α is a section of $\mathcal{D}'(\operatorname{End} E)$ and ξ is a section of E. By the fact that $\nabla^2 = 0$, and that ∇ is odd, we also get that $\nabla_{\mathrm{End}}^2 = 0$. Note also that if α and β are sections of $\mathcal{D}'(\operatorname{End} E)$, of which at least one of them is smooth, so that $\alpha\beta$ is defined, then

$$\nabla_{\mathrm{End}}(\alpha\beta) = \nabla_{\mathrm{End}}(\alpha)\beta + (-1)^{\deg \alpha} \alpha \nabla_{\mathrm{End}} \beta. \tag{2.1}$$

2.2. The residue current R associated to a generically exact complex of vector bundles. Let Z be the set where E is not pointwise exact. Outside of Z, let $\sigma_k : E_{k-1} \to E_k$ be the right-inverses to φ_k which are minimal with respect to the metrics on E, i.e., $\varphi_k \sigma_k|_{\operatorname{Im} \varphi_k} = \operatorname{Id}_{\operatorname{Im} \varphi_k}$, $\sigma_k = 0$ on $(\operatorname{Im} \varphi_k)^\perp$, and $\operatorname{Im} \sigma_k \perp \ker \varphi_k$. Then,

$$\varphi_{k+1}\sigma_k + \sigma_{k-1}\varphi_k = \operatorname{Id}_{E_k}. \tag{2.2}$$

Let $\sigma := \sum \sigma_k$, considered as an element in $\operatorname{End}(E)$, i.e., $\sigma_k = 0$ on E_l, $l \neq k-1$. Then, $\nabla_{\mathrm{End}}\sigma = \operatorname{Id}_E - \bar\partial\sigma$ by (2.2). Thus, if we let

$$u := \sum \sigma(\bar\partial\sigma)^{k-1}, \tag{2.3}$$

then $\nabla_{\mathrm{End}} u = \operatorname{Id}_E$ by (2.1).

The form u is smooth outside of Z, and we define a current extension U of u over Z, $U := |F|^{2\lambda} u|_{\lambda=0}$, where F is a tuple of holomorphic functions such that $Z(F) \supseteq Z$. By $|_{\lambda=0}$, we mean that for $\operatorname{Re}\lambda \gg 0$, $|F|^{2\lambda}u$ is a (arbitrarily) smooth form, and its action on a test form depends analytically on λ, and the action of U on the test form is defined as the analytic continuation to $\lambda = 0$ of the action of $|F|^{2\lambda}u$ on the test form. The existence of this analytic continuation is non-trivial, and it relies on the theorem of Hironaka on resolutions of singularities, see [**AW1**]. The definition of U is independent of the choice of F, cf., the discussion of a similar statement in the proof of Proposition 6.1 in [**L1**].

Since u and U coincide where u is smooth, $\nabla_{\mathrm{End}} U = \mathrm{Id}_E$ outside of Z, and the residue current R associated to E is defined as the difference between these currents, $R = \mathrm{Id}_E - \nabla_{\mathrm{End}} U$, which thus is a current with support on Z. Applying ∇_{End} to $|F|^{2\lambda} u$, and using (2.1) and that $\nabla_{\mathrm{End}} u = \mathrm{Id}_E$, it follows that one could also define R by

$$R = \bar\partial |F|^{2\lambda} \wedge u|_{\lambda=0}.$$

The current R satisfies the fundamental property that if E is a free resolution of $\mathcal{O}/\mathcal{J}$, then $\operatorname{ann} R = \mathcal{J}$. Note that since $\nabla_{\mathrm{End}} \mathrm{Id}_E = 0$,

$$\nabla_{\mathrm{End}} R = \nabla_{\mathrm{End}} \mathrm{Id}_E - \nabla^2_{\mathrm{End}} U = 0.$$

Since R is a $\mathrm{End}(E)$-valued current, it consists of various components R^l_k, where R^l_k is the part of R taking values in $\mathrm{Hom}(E_l, E_k)$ and R^l_k is a $(0, k-l)$-current. We will denote the part of R taking values in $\mathrm{Hom}(E, E_k)$ by R_k. In case we know more about the complex E, more can be said about which components are non-vanishing. First, if $k - l < \operatorname{codim} Z$, then $R^l_k = 0$, Proposition 2.2 in [**AW1**], and if E is exact, i.e., a free resolution, then $R^l_k = 0$ if $l \neq 0$, Theorem 3.1 in [**AW1**].

In particular, if E is a free resolution of length N of $\mathcal{O}/\mathcal{J}$, where $\operatorname{codim} Z(\mathcal{J}) = p$, then

$$R = \sum_{k=p}^{N} R^0_k.$$

2.3. Residue currents associated to the Koszul complex. Let $f = (f_1, \dots, f_p)$ be a tuple of holomorphic functions. Then there exists a well-known complex associated to f, the Koszul complex $(\bigwedge^k \mathcal{O}^{\oplus p}, \delta_f)$ of f, which is pointwise exact outside of the zero set $Z(f)$ of f. We let e_i be the trivial frame of $\mathcal{O}^{\oplus p}$, and identify f with the section $f = \sum f_i e_i^*$ of $(\mathcal{O}^{\oplus p})^*$, so that δ_f is the contraction with f.

In [**PTY**], a current called the *Bochner-Martinelli* current of a tuple f was introduced, which we will denote by R^f. One way of defining it is as the Andersson-Wulcan current associated to the Koszul complex of f, see [**A1**] for a presentation from this viewpoint.

In case the tuple f defines a complete intersection, the Koszul complex of f is exact, i.e., a free resolution of $\mathcal{O}/\mathcal{J}(f)$, so the annihilator of the Bochner-Martinelli current equals $\mathcal{J}(f)$. Another current with the same annihilator is the Coleff-Herrera product of f, (1.1), which can be defined by analytic continuation,

$$\bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} . \phi := \left. \frac{\bar\partial |f_p|^{2\lambda} \wedge \cdots \wedge \bar\partial |f_1|^{2\lambda}}{f_1 \cdots f_p} . \phi \right|_{\lambda=0} .$$

In fact, these two currents coincide.

Theorem 2.1. *Let $f = (f_1,\dots,f_p)$ be a tuple of holomorphic functions defining a complete intersection. Let R^f be the Bochner-Martinelli current of f, $R^f = \mu\wedge e_1\wedge\cdots\wedge e_p$, and let μ^f be the Coleff-Herrera product of f. Then, $\mu = \mu^f$.*

The theorem was originally proved in [**PTY**], Theorem 4.1, see also [**A4**], Corollary 3.2 for an alternative proof.

2.4. Pseudomeromorphic currents. Many arguments regarding Andersson-Wulcan currents use the fact that they are pseudomeromorphic. Pseudomeromorphic currents were introduced in [**AW2**], based on similarities in the construction of Andersson-Wulcan currents and Coleff-Herrera products.

A current of the form

$$\frac{1}{z_{i_1}^{n_1}}\cdots\frac{1}{z_{i_k}^{n_k}}\bar\partial\frac{1}{z_{i_{k+1}}^{n_{k+1}}}\wedge\cdots\wedge\bar\partial\frac{1}{z_{i_m}^{n_m}}\wedge\alpha,$$

where α is a smooth form with compact support is said to be an *elementary current*, and a current on a complex manifold X is said to be *pseudomeromorphic*, denoted $T\in\mathcal{PM}(X)$, if it can be written as a locally finite sum of push-forwards of elementary currents under compositions of modifications and open inclusions. As can be seen from the construction, Coleff-Herrera products, Andersson-Wulcan currents and all currents appearing in this article are pseudomeromorphic. In addition, as is apparent from the definition, the class of pseudomeromorphic currents is closed under push-forwards of currents under modifications and under multiplication by smooth forms.

An important property of pseudomeromorphic currents is that they satisfy the following *dimension principle*, Corollary 2.4 in [**AW2**].

Proposition 2.2. *If $T\in\mathcal{PM}(X)$ is a (p,q)-current with support on a variety Z, and* $\operatorname{codim} Z > q$, *then* $T=0$.

Another important property is the following, Proposition 2.3 in [**AW2**].

Proposition 2.3. *If $T\in\mathcal{PM}(X)$, and Ψ is a holomorphic form vanishing on* $\operatorname{supp} T$, *then*

$$\overline{\Psi}\wedge T = 0.$$

Pseudomeromorphic currents also have natural restrictions to analytic subvarieties. If $T\in\mathcal{PM}(X)$, and $Z\subseteq X$ is a subvariety of X, and h is a tuple of holomorphic functions such that $Z=Z(h)$, one can define

$$\mathbf{1}_{X\setminus Z}T := |h|^{2\lambda}T|_{\lambda=0} \text{ and } \mathbf{1}_Z T := T - \mathbf{1}_{X\setminus Z}T.$$

This definition is independent of the choice of tuple h, and $\mathbf{1}_Z T$ is a pseudomeromorphic current with support on Z.

2.5. Coleff-Herrera currents. Coleff-Herrera currents were introduced in [**DS1**] (under the name "locally residual currents"), as canonical representatives of cohomology classes in moderate local cohomology. Let Z be a subvariety of pure codimension p of a complex manifold X. A $(*,p)$-current μ on X is a *Coleff-Herrera current*, denoted $\mu \in CH_Z$, if $\bar\partial\mu = 0$, $\overline{\psi}\mu = 0$ for all holomorphic functions ψ vanishing on Z, and μ has the standard extension property, SEP, with respect to Z, i.e., $\mathbf{1}_V\mu = 0$ for any hypersurface V of Z.

This description of Coleff-Herrera currents is due to Björk, see [**B1**], Chapter 3, and [**B2**], Section 6.2. In [**DS1**], locally residual currents were defined as currents of the form $\omega\wedge R^h$, where ω is a holomorphic $(*,0)$-form, and $Z = Z(h)$ (at least if Z is a complete intersection defined by h).

One particular case of Coleff-Herrera currents that will be of interest to us are Andersson-Wulcan currents R^E associated to free resolutions (E,φ) of minimal length of Cohen-Macaulay modules $\mathcal{O}/\mathcal{J}$. Such a current is $\bar\partial$-closed since $\nabla R^E = 0$ implies that $\bar\partial R^E_p = \varphi_{p+1}R^E_{p+1} = 0$ since E is assumed to be of minimal length. The other properties needed in order to be a Coleff-Herrera current are satisfied by the fact that they are pseudomeromorphic, Proposition 2.2 and Proposition 2.3.

2.6. Singularity subvarieties of free resolutions. In the study of residue currents associated to free resolutions of ideals, an important ingredient is certain singularity subvarieties associated to the ideal. Given a free resolution (E,φ) of an ideal $\mathcal{J}$, the variety $Z_k = Z^E_k$ is defined as the set where φ_k does not have optimal rank. These sets are independent of the choice of free resolution. If $\operatorname{codim} Z(\mathcal{J}) = p$, then $Z_k = Z$ for $k \le p$, Corollary 20.12 in [**E**]. In addition, Corollary 20.12 says that $Z_{k+1} \subseteq Z_k$, and $\operatorname{codim} Z_k \ge k$ by Theorem 20.9 in [**E**]. In fact, Theorem 20.9 in [**E**] is a characterization of exactness, the Buchsbaum-Eisenbud criterion, which says that a generically exact complex of free modules is exact if and only if $\operatorname{codim} Z_k \ge k$.

The fact that these sets are important in the study of residue currents associated to free resolutions stems from the following. Outside of Z_k, σ_k is smooth, so by using that $\sigma_{l+1}\bar\partial\sigma_l = \bar\partial\sigma_{l+1}\sigma_l$ (see [**AW1**], (2.3)), $R_k = \bar\partial\sigma_k R_{k-1}$ outside of Z_k. This combined with the dimension principle for pseudomeromorphic currents allows for inductive arguments regarding residue currents, see for example Section 6.

3. A comparison formula for Andersson-Wulcan currents

The starting point of Theorem 1.2 is that the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$, when $\mathcal{I} \subseteq \mathcal{J}$, induces a morphism of complexes $a : (F,\psi) \to (E,\varphi)$, where (F,ψ) and (E,φ) are free resolutions of $\mathcal{O}/\mathcal{I}$ and $\mathcal{O}/\mathcal{J}$ respectively. The existence of such a morphism holds much more generally in homological algebra, of which the following formulation is suitable for our purposes, what is sometimes referred to as the *comparison theorem*.

Proposition 3.1. *Let $\alpha : M \to N$ be a homomorphism of $\mathcal{O}$-modules, and let (F,ψ) be a complex of free $\mathcal{O}$-modules with* $\operatorname{coker}\psi_1 = M$, *and let (E,φ) be a free resolution of N. Then, there exists a morphism $a : (F,\psi) \to (E,\varphi)$ of complexes which extends α. If $\tilde{a}$ is any other such morphism, then there exists a homotopy $s : (F,\psi) \to (E,\varphi)$ of degree -1 such that $a_i - \tilde{a}_i = \varphi_{i+1}s_i - s_{i-1}\psi_i$.*

That a extends α means that the map induced by a_0 on $F_0/(\operatorname{im}\psi_1) \cong M \to N \cong E_0/(\operatorname{im}\varphi_1)$ equals α. Both the existence and uniqueness up to homotopy of a follows from defining a or s inductively by a relatively straightforward diagram chase, see [**E**], Proposition A3.13.

This is the general formulation of our main theorem, Theorem 1.2.

Theorem 3.2. *Let $a : (F,\psi) \to (E,\varphi)$ be a morphism of Hermitian complexes, and let u^E and u^F be the forms associated to E and F as defined in* (2.3), *and let*

$$M := \bar\partial |G|^{2\lambda} \wedge u^E \wedge a u^F \Big|_{\lambda=0}, \tag{3.1}$$

where G is a tuple of holomorphic functions such that $G \not\equiv 0$, and $Z(G)$ contains the set where (E,φ) and (F,ψ) are not pointwise exact. Then

$$R^E a - a R^F = \nabla M, \tag{3.2}$$

where $\nabla = \nabla_{\varphi\oplus\psi}$.

Note that ∇ is defined with respect to the complex $(E\oplus F, \varphi\oplus\psi)$, and the superstructure, as in Section 2.1, of this complex is the grading $(E\oplus F)^+ = E^+ \oplus F^+$, $(E\oplus F)^- = E^- \oplus F^-$.

Proof. To begin with, we should prove the existence M, i.e., that the analytic continuation of the left-hand side (3.1) has a current-valued analytic continuation to $\lambda = 0$. However, we begin by considering the current

$$M' = |G|^{2\lambda} \wedge u^E \wedge a u^F \Big|_{\lambda=0}, \tag{3.3}$$

and proving the existence of this current, the existence of M follows in the same way. The existence of the analytic continuation in (3.3) follows from a straightforward combination of the proof of the existence of the analytic continuation in the definition of U^E and U^F associated to E and F, see Section 2 of [**AW1**], and the proof of the existence of the analytic continuation of a similar current in [**L1**], Lemma 7.3. The main point of the argument is that by principalization of ideals and resolution of singularities, the components of u^E and u^F can respectively locally be written as push-forwards of smooth forms divided by single holomorphic functions, and by further principalization and resolution of singularities, the components of $u^E \wedge a u^F$ can locally be written as the push-forward of a smooth form divided by a monomial, of which the existence of the analytic continuation is elementary.

Now, since a is a morphism of complexes, $\varphi a = a\psi$, and hence, $\nabla a = \varphi a - a\psi = 0$. Thus, since outside of $Z(G)$, $\nabla_\varphi u^E = \operatorname{Id}_E$ and $\nabla_\psi u^F = \operatorname{Id}_F$, and since u^E has odd degree and a even degree, we get by (2.1) that

$$\nabla M' = \left(-\bar\partial |G|^{2\lambda} u^E \wedge a u^F + |G|^{2\lambda} a u^F - |G|^{2\lambda} u^E a\right)|_{\lambda=0}, \tag{3.4}$$

i.e.,

$$\nabla M' = -M + aU^F - U^E a.$$

Applying ∇ to this equation, we get (3.2) since $\nabla^2 = 0$, and

$$\begin{aligned}\nabla(aU^F - U^E a) &= a\nabla U^F - \nabla U^E a = \\ &= a(\mathrm{Id}_F - R^F) - (\mathrm{Id}_E - R^E)a = R^E a - aR^F.\end{aligned}$$

□

The main idea in the proof of Theorem 3.2, to form a ∇-potential essentially of the form $U \wedge U'$ to $U - U'$ appears in various works regarding residue currents, for example in [**A1**] and [**AW1**] in order to prove that under suitable conditions, the definition of the residue currents do not depend on the choice of metrics. This corresponds to applying the comparison formula in the case when (E,φ) and (F,ψ) have the same underlying complex, but are equipped with different metrics.

Another instance where such a construction appear is for example [**L1**], regarding the transformation law for Coleff-Herrera products of (weakly) holomorphic functions, of which its relation to the comparison formula is elaborated in Remark 3. It also appears in [**A2**] and [**W**], regarding products of residue currents, but the relation to the comparison formula is not as apparent.

Remark 2. Note that in Proposition 3.1, the complex (F,ψ) does not have to be exact. For our comparison formula to work, neither the complex (E,φ) has to be exact, as long as the morphism a exists. For example, if we have $f = gA$, for some tuples g and f of holomorphic functions, and a holomorphic matrix A, as in Remark 3, then A induces a morphism between the Koszul complexes of f and g, and we can apply the comparison formula also when the Koszul complex of g is not exact.

4. A transformation law for Andersson-Wulcan currents associated to Cohen-Macaulay ideals

In this section, we state and prove the general version of our transformation law for Andersson-Wulcan currents associated to Cohen-Macaulay ideals.

Theorem 4.1. *Let $\mathcal{J} \subseteq \mathcal{O}$ be a Cohen-Macaulay ideal of codimension p, with a free resolution (E,φ) of length p, and let (F,ψ) be a generically exact complex such that the set Z where (F,ψ) is not pointwise exact has codimension p. If $a : (F,\psi) \to (E,\varphi)$ is a morphism of complexes, then*

$$R^E_p a_0 = a_p R^F_p.$$

Such a morphism $a : (F,\psi) \to (E,\varphi)$ exists if $\operatorname{im}\psi_1 \subseteq \mathcal{J}$.

Proof. The last part about the existence of a follows immediately from Proposition 3.1. Thus, applying Theorem 3.2, since R^E only takes values in $\mathrm{Hom}(E_0, E_p)$, only the term $R^E_p a_0$ of $R^E_p a$ remains, and it will be enough to see that the current M as defined by (3.1) is 0. We write $M = \sum_{k,l} M_{l,k}$, where $M_{l,k}$ is the component of M with values in $\mathrm{Hom}(F_l, E_k)$. The current $M_{l,k}$ is a $(0, k-l-1)$-current with support on Z which has codimension p, so since $k \le p$, $M_{l,k}$ is 0 by the dimension principle, Proposition 2.2.

□

Example 1. Let $\pi : \mathbb{C} \to \mathbb{C}^3$, $\pi(t) = (t^3, t^4, t^5)$, and let Z be the germ at 0 of $\pi(\mathbb{C})$. One can show that the ideal of holomorphic functions vanishing at Z equals $\mathcal{J} = (y^2 - xz, x^3 - yz, x^2y - z^2)$.

The module $\mathcal{O}/\mathcal{J}$ has a minimal free resolution

$$0 \to \mathcal{O}^{\oplus 2} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 3} \xrightarrow{\varphi_1} \mathcal{O} \to \mathcal{O}/\mathcal{J},$$

where

$$\varphi_2 = \begin{bmatrix} -z & -x^2 \\ -y & -z \\ x & y \end{bmatrix} \text{ and } \varphi_1 = \begin{bmatrix} y^2 - xz & x^3 - yz & x^2y - z^2 \end{bmatrix}.$$

To check that this is a resolution, one verifies first that it indeed is a complex. Secondly, since $I_1 = I(\varphi_1) = \mathcal{J}$, and $I_2 = I(\varphi_2) = \mathcal{J}$ (the Fitting ideals of φ_1 and φ_2), the complex is exact by [**E**], Theorem 20.9 (cf. Section 2.6, where $Z_k = Z(I_k)$).

In particular, since $\mathcal{O}/\mathcal{J}$ has a minimal free resolution of length 2, with $\operatorname{rank} E_2 = 2$, Z is Cohen-Macaulay but not a complete intersection. However, Z is in fact a set-theoretic complete intersection. Let $f = (z^2 - x^2y, x^4 + y^3 - 2xyz)$, and $\mathcal{I} = \mathcal{J}(f)$. One can verify that $Z(\mathcal{I}) = Z$, and since $\operatorname{codim} Z = 2$, Z is indeed a set-theoretic complete intersection.

Now, let (E, φ) be the free resolution of $\mathcal{O}/\mathcal{J}$, and (F, ψ) be the Koszul complex of f, which is a free resolution of $\mathcal{O}/\mathcal{I}$ since f is a complete intersection. Since $\mathcal{O}/\mathcal{J}$ is Cohen-Macaulay and $Z(\mathcal{I}) = Z(\mathcal{J})$, we can apply Theorem 1.3 to (F, ψ) and (E, φ). One verifies that $a : (F, \psi) \to (E, \varphi)$,

$$a_2 = \begin{bmatrix} x^3 - yz \\ y^2 - xz \end{bmatrix}, a_1 = \begin{bmatrix} 0 & y \\ 0 & x \\ -1 & 0 \end{bmatrix} \text{ and } a_0 = \begin{bmatrix} 1 \end{bmatrix},$$

is a morphism of complexes extending the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$. See Appendix A for an example of how to compute these things with the help of a computer algebra system. Since the current associated to the Koszul complex of a complete intersection f is the Coleff-Herrera product of f, we get by Theorem 1.3 that

$$R^E = \bar\partial \frac{1}{x^4 + y^3 - 2xyz} \wedge \bar\partial \frac{1}{z^2 - x^2y} \wedge \begin{bmatrix} x^3 - yz \\ y^2 - xz \end{bmatrix}.$$

The fact that we can express the residue current corresponding to the ideal above in terms of a Coleff-Herrera product can be done more generally, as the following example shows.

Example 2. Let $\mathcal{J} \subseteq \mathcal{O}$ be a Cohen-Macaulay ideal of codimension p, and let $Z = Z(\mathcal{J})$. Then, there exists a complete intersection $(f_1,\dots,f_p)$ such that $Z \subseteq Z(f)$, see for example [**L2**], Lemma 19. By the Nullstellensatz, there exist N_i such that $f_i^{N_i} \in \mathcal{J}$. Thus, by replacing f_i by $f_i^{N_i}$, we can assume that $(f_1,\dots,f_p)$ is a complete intersection such that $\mathcal{J}(f_1,\dots,f_p) \subseteq \mathcal{J}$. Let (F,ψ) be the Koszul complex of f and let (E,φ) be a free resolution of $\mathcal{O}/\mathcal{J}$ of length p. By Theorem 1.3, we then have that

$$R_p^{\mathcal{J}} = \bar\partial \frac{1}{f_p} \wedge \cdots \wedge \bar\partial \frac{1}{f_1} \wedge a_p(e_1 \wedge \cdots \wedge e_p),$$

where a_p is the morphism in Theorem 1.3, since the current associated with the Koszul complex of f is the Coleff-Herrera product of f.

Remark 3. The transformation law for Coleff-Herrera products is a corollary of Theorem 1.3 in the following way. Let f and g be two complete intersections of codimension p, and assume that there exists a matrix A of holomorphic functions such that $f = gA$.

Since f and g are complete intersections, the Koszul complexes $(\bigwedge \mathcal{O}^{\oplus p}, \delta_f)$ and $(\bigwedge \mathcal{O}^{\oplus p}, \delta_g)$ are free resolutions of $\mathcal{O}/\mathcal{J}(f)$ and $\mathcal{O}/\mathcal{J}(g)$. Since $\mathcal{J}(f) \subseteq \mathcal{J}(g)$, we get a morphism a of the Koszul complexes of f and g induced by the inclusion $\pi : \mathcal{O}/\mathcal{J}(f) \to \mathcal{O}/\mathcal{J}(g)$ by Proposition 3.1. In fact, the morphism $a_k : \bigwedge^k \mathcal{O}^{\oplus p} \to \bigwedge^k \mathcal{O}^{\oplus p}$ is readily verified to be $\bigwedge^k A : \bigwedge^k \mathcal{O}^{\oplus p} \to \bigwedge^k \mathcal{O}^{\oplus p}$, see [**L1**], Lemma 7.2. In particular, $a_p = \bigwedge^p A = \det A$, so since the Andersson-Wulcan currents associated to the Koszul complexes of f and g are the Coleff-Herrera products of f and g, the transformation law $\mu^g = (\det A)\mu^f$ follows directly from Theorem 1.3.

In fact, the proof of Theorem 1.3 in this particular situation becomes exactly the proof of the transformation law for Coleff-Herrera products given in [**L1**], Theorem 7.1.

As mentioned above, the transformation law for Coleff-Herrera products is a special case of Theorem 1.3. In [**DS2**], two proofs of the transformation law are given, and in fact, we can essentially use the same argument as the second proof of the transformation law in [**DS2**], pages 54–55, to prove Theorem 1.3.

Alternative proof of Theorem 1.3. Consider $\mathcal{E}^p_{\mathcal{J}} := \mathcal{E}xt^p_{\mathcal{O}}(\mathcal{O}/\mathcal{J},\mathcal{O})$. One way of computing $\mathcal{E}^p_{\mathcal{J}}$ is by taking a free resolution (E,φ) of $\mathcal{O}/\mathcal{J}$, applying $\operatorname{Hom}(\bullet,\mathcal{O})$ and taking cohomology, i.e., $\mathcal{E}^p_{\mathcal{J}} \cong \mathcal{H}^p(\operatorname{Hom}(E_\bullet,\mathcal{O}))$. On the other hand, it can also be computed by taking an injective resolution of

$\mathcal{O}$, which can be taken as the complex of $(0,*)$-currents, $(\mathcal{C}^{0,\bullet},\bar\partial)$, and applying $\mathrm{Hom}(\mathcal{O}/\mathcal{J},\bullet)$ to this complex and taking cohomology, i.e., $\mathcal{E}^p_{\mathcal{J}} \cong \mathcal{H}^p(\mathrm{Hom}(\mathcal{O}/\mathcal{J},\mathcal{C}^{0,\bullet}))$.

Since these are different realizations of $\mathcal{E}xt$, they are naturally isomorphic, and by [**A5**], Theorem 1.5, this isomorphism is given by

$$\phi : [\xi]_{\mathcal{H}^p(\mathrm{Hom}(E_\bullet,\mathcal{O}))} \mapsto [\xi R^E_p]_{\mathcal{H}^p(\mathrm{Hom}(\mathcal{O}/\mathcal{J},\mathcal{C}^{0,\bullet}))}. \tag{4.1}$$

We now consider the map $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$, which induces a map $\pi^* : \mathcal{E}^p_{\mathcal{J}} \to \mathcal{E}^p_{\mathcal{I}}$. In the first realization of $\mathcal{E}xt$, π^* becomes the map $a^*_p : \mathcal{H}^p(\mathrm{Hom}(E_\bullet,\mathcal{O})) \to \mathcal{H}^p(\mathrm{Hom}(F_\bullet,\mathcal{O}))$ induced by $a : (F,\psi) \to (E,\varphi)$. In the second realizations of $\mathcal{E}xt$, the map becomes just the identity map on the currents (due to the fact that currents annihilated by $\mathcal{J}$ are also annihilated by $\mathcal{I}$). Thus, using the naturality of π^*, and the isomorphism (4.1), we get from the commutative diagram

$$\begin{tikzcd} \mathcal{H}^p(\mathrm{Hom}(E_\bullet,\mathcal{O})) \arrow[r, "\pi^*"] \arrow[d, "\phi"] & \mathcal{H}^p(\mathrm{Hom}(F_\bullet,\mathcal{O})) \arrow[d, "\phi"] \\ \mathcal{H}^p(\mathrm{Hom}(\mathcal{O}/\mathcal{J},\mathcal{C}^{0,\bullet})) \arrow[r, "\pi^*"] & \mathcal{H}^p(\mathrm{Hom}(\mathcal{O}/\mathcal{I},\mathcal{C}^{0,\bullet})) \end{tikzcd} \tag{4.2}$$

that $[(a^*_p)\xi R^F_p]_{\bar\partial} = [\xi R^E_p]_{\bar\partial}$, where ξ is a holomorphic section of $\ker\varphi^*_{p+1}$, so $\xi a_p R^F_p = \xi R^E_p + \bar\partial\eta_\xi$, where η_ξ is annihilated by $\mathcal{I}$. Since $\mathcal{O}/\mathcal{J}$ is Cohen-Macaulay, $\varphi_{p+1} = 0$, so the equality holds for all holomorphic sections ξ of E_p, i.e., $a_p R^F_p = R^E_p + \bar\partial\eta$ for some (vector-valued) current η annihilated by $\mathcal{I}$. Since a_p is holomorphic, and R^F_p and R^E_p are in CH_Z, see Section 2.5, where $Z = Z(\mathcal{I})$, we get from the decomposition $\ker(\mathcal{C}^{0,p}_Z \stackrel{\bar\partial}{\to} \mathcal{C}^{0,p+1}_Z) = CH_Z \oplus \bar\partial\mathcal{C}^{0,p-1}_Z$, see [**DS2**], Theorem 5.1, that $\bar\partial\eta = 0$, where $\mathcal{C}^{0,p}_Z$ is the sheaf of $(0,p)$-currents supported on Z.

The only difference of the proof here, to the proof in [**DS2**] is that we have the isomorphism (4.1) from [**A5**], while in [**DS2**], this isomorphism was only available if $\mathcal{J}$ was a complete intersection ideal, see the proof of Proposition 3.5 in [**DS1**]. $\square$

We end this section with an example of how we can express Andersson-Wulcan currents associated to Cohen-Macaulay ideals in terms of Bochner-Martinelli currents.

Example 3. Let $f = (f_1,\cdots,f_k)$ be a tuple of holomorphic functions such that $Z = Z(f)$, and assume that $\operatorname{codim} Z = p$. Note that we do not assume that f is a complete intersection, i.e., that $k = p$. Let $\mathcal{O}^{\oplus k}$ be the trivial vector bundle with frame $e_1,\cdots,e_k$, and consider f as a section of $(\mathcal{O}^{\oplus k})^*$, $f = \sum f_i e^*_i$. Let R^f be the Bochner-Martinelli current associated with f, and write $R^f_p = \sum R_I \wedge e_I$, i.e., $R_I \wedge e_I$ is the component of R^f_p with values in

$e_I := e_{i_1} \wedge \cdots \wedge e_{i_p} \in \bigwedge^p \mathcal{O}^{\oplus k}$. In [**A4**], Andersson proves that if $\mu \in CH_Z$, then there exist holomorphic $(*,0)$-forms α_I such that $\mu = \sum \alpha_I R_I$ (first, replacing f_i by $f_i^{N_i}$ such that $f_i^{N_i}\mu = 0$). In particular, this applies in our case to R^Z, see Section 2.5. In [**A4**], the α_I are not explicitly given, but when $\mu = R^Z$, we can obtain them from Theorem 4.1. We let (F,ψ) be the Koszul complex of f, and (E,φ) a minimal free resolution of $\mathcal{O}/\mathcal{I}_Z$. Since the current associated with the Koszul complex of f is the Bochner-Martinelli current of f, Theorem 4.1 gives the factorization

$$R^Z = \sum \alpha_I R_I,$$

where $\alpha_I = a_p(e_I)$.

5. A non Cohen-Macaulay example

When the ideals involved in the comparison formula are not Cohen-Macaulay, the comparison formula does not have as simple form as in the Cohen-Macaulay case in Section 4. In this section, we illustrate with an example how one could still use the comparison formula also to compute the residue current associated to a non Cohen-Macaulay ideal.

Example 4. Let $Z \subseteq \mathbb{C}^4$ be the variety $Z = \{x = y = 0\} \cup \{z = w = 0\}$. The ideal $\mathcal{I}_Z$ of holomorphic functions on $\mathbb{C}^4$ vanishing on Z equals $\mathcal{I}_Z = \mathcal{J}(xz, xw, yz, yw)$. It can be verified that $\mathcal{I}_Z$ has a minimal free resolution (E,φ) of the form

$$0 \to \mathcal{O} \xrightarrow{\varphi_3} \mathcal{O}^{\oplus 4} \xrightarrow{\varphi_2} \mathcal{O}^{\oplus 4} \xrightarrow{\varphi_1} \mathcal{O} \to \mathcal{O}/\mathcal{I}_Z,$$

where

$$\varphi_3 = \begin{bmatrix} w \\ -z \\ -y \\ x \end{bmatrix}, \varphi_2 = \begin{bmatrix} -y & 0 & -w & 0 \\ 0 & -y & z & 0 \\ x & 0 & 0 & -w \\ 0 & x & 0 & z \end{bmatrix} \text{ and}$$

$$\varphi_1 = \begin{bmatrix} xz & xw & yz & yw \end{bmatrix}.$$

Note that Z has codimension 2, while the free resolution above, which is minimal, has length 3, so Z is not Cohen-Macaulay.

We compare this resolution with the Koszul complex (F,ψ) of the complete intersection ideal $\mathcal{I} = \mathcal{J}(xz, yw)$. One can verify that the morphism $a : (F,\psi) \to (E,\varphi)$

$$a_2 = \frac{1}{2}\begin{bmatrix} w \\ z \\ y \\ x \end{bmatrix}, a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } a_0 = [1]$$

is a morphism of complexes extending the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{I}_Z$ as in Proposition 3.1.

We compute the current R_2^E by using the comparison formula, Theorem 3.2, $R_2^E = a_2 R_2^F + \varphi_3 M_3 - \bar\partial M_2$, where M_k is part of M in (3.1) with values in $\mathrm{Hom}(F_0, E_k)$. Since M_2 is a pseudomeromorphic $(0,1)$-current with support on $Z(\mathcal{I})$, which has codimension 2, M_2 is zero by Proposition 2.3. We now consider the terms of M_3,

$$M_3 = \bar\partial |G|^{2\lambda} \wedge \left(\bar\partial \sigma_3^E \sigma_2^E a_1 \sigma_1^F + \sigma_3^E a_2 \sigma_2^F \bar\partial \sigma_1^F \right)\Big|_{\lambda=0}$$

(where we have used that $(\bar\partial\sigma_3^E)\sigma_2^E = \sigma_3^E \bar\partial \sigma_2^E$). We claim that in fact, the first term of the right-hand side is 0. Note that if we choose the trivial metrics on E, then

$$\sigma_3^E = 1/(|x|^2+|y|^2+|z|^2+|w|^2)\begin{bmatrix} \overline{w} & -\bar z & -\bar y & \bar x \end{bmatrix}.$$

Outside of $\{0\}$, $\bar\partial \sigma_3^E$ is smooth, and the first term of M_3 is $\bar\partial\sigma_3^E$ times $\bar\partial |G|^{2\lambda} \wedge \sigma_2^E a_1 \sigma_1^F\big|_{\lambda=0}$, where the last current is a pseudomeromorphic $(0,1)$-current with support on $Z(\mathcal{I})$ of codimension 2, so it is 0 by the dimension principle. Thus, the first term of M_3 has support at 0, and being a pseudomeromorphic $(0,2)$-current supported at 0, it is zero everywhere, again by the dimension principle. Thus, outside of $\{0\}$,

$$R_2^E = (I_{E_2} - \varphi_3\sigma_3) a_2 R_2^F$$

(the minus sign in front of φ_3 is due to $\bar\partial|G|^{2\lambda}$ and σ_3 anti-commuting). Then, R_2^E is the standard extension in the sense of [**B2**], Section 6.2, of $(I_{E_2} - \varphi_3\sigma_3)a_2 R_2^F$. One way to interpret the standard extension here is that since R_2^E is a pseudomeromorphic $(0,2)$-current defined on all of $\mathbb{C}^4$, its extension from $\mathbb{C}^4 \setminus \{0\}$ is uniquely defined by the dimension principle.

We have that

$$(I_{E_2} - \varphi_3\sigma_3)a_2 = \frac{1}{|x|^2+|y|^2+|z|^2+|w|^2}\begin{bmatrix} w(|y|^2+|z|^2) \\ z(|x|^2+|w|^2) \\ y(|x|^2+|w|^2) \\ x(|y|^2+|z|^2) \end{bmatrix}.$$

Since $R_2^F = \bar\partial(1/yw)\wedge\bar\partial(1/xz)$, see Theorem 2.1, we get from the transformation law and Proposition 2.3 that R_2^E is the standard extension of

$$\frac{1}{|x|^2+|y|^2+|z|^2+|w|^2}\begin{bmatrix} |z|^2 \bar\partial\frac{1}{y}\wedge\bar\partial\frac{1}{xz} \\ |w|^2\bar\partial\frac{1}{yw}\wedge\bar\partial\frac{1}{x} \\ |x|^2\bar\partial\frac{1}{w}\wedge\bar\partial\frac{1}{xz} \\ |y|^2\bar\partial\frac{1}{yw}\wedge\bar\partial\frac{1}{z} \end{bmatrix}.$$

Using again the transformation law and Proposition 2.3, one gets that R_2^E is the standard extension of

$$R_2^E = \frac{1}{|z|^2+|w|^2}\begin{bmatrix} \bar{z} \\ \overline{w} \\ 0 \\ 0 \end{bmatrix} \wedge \bar\partial\frac{1}{y} \wedge \bar\partial\frac{1}{x} + \frac{1}{|x|^2+|y|^2}\begin{bmatrix} 0 \\ 0 \\ \overline{x} \\ \overline{y} \end{bmatrix} \wedge \bar\partial\frac{1}{w} \wedge \bar\partial\frac{1}{z}.$$

6. Andersson-Wulcan currents on an analytic variety

In this section, we prove Theorem 1.4. The special case of Theorem 1.4 when $\mathcal{J} = \{0\}$ will be the basis of the proof in the general case.

Proposition 6.1. *Let $(Z,0)$ be an analytic subvariety of $(\mathbb{C}^n,0)$, and let $\mathcal{I}_Z$ be the ideal of germs of holomorphic functions vanishing at Z. Then $R^{\mathcal{I}_Z} \wedge dz$ defines a current on $(Z,0)$.*

In case Z has pure dimension, this will follow from [**AS1**], Proposition 3.3, where $R^{\mathcal{I}_Z} \wedge dz$ is expressed as the push-forward of a current on Z. In [**L3**], Proposition 5.2, we generalize this construction to the case when Z does not have pure dimension.

Now, we let (F,ψ) and (E,φ) be free resolutions of the ideals $\mathcal{I}_Z$ and $\mathcal{J}+\mathcal{I}_Z$, and let $a : (F,\psi) \to (E,\varphi)$ be the map induced from the natural surjection $\pi : \mathcal{O}/\mathcal{I}_Z \to \mathcal{O}/(\mathcal{J}+\mathcal{I}_Z)$, as in Proposition 3.1. Let σ^E and σ^F be the forms associated to (E,φ) and (F,ψ) as in Section 2.2, and let G be a tuple of holomorphic functions such that $Z(G) \supseteq Z$. Define

$$M_k^l = \bar\partial|G|^{2\lambda} \wedge \bar\partial\sigma_k^E \bar\partial\sigma_{k-1}^E \dots \sigma_{l+1}^E a_l \sigma_l^F \bar\partial\sigma_{l-1}^F \dots \bar\partial\sigma_1^F|_{\lambda=0}. \tag{6.1}$$

Note that by using that $\bar\partial\sigma_{j+1}\sigma_j = \sigma_{j+1}\bar\partial\sigma_j$, it follows that the current M in (3.1) is exactly $\sum_{l<k} M_k^l$ (and in particular, the existence of the analytic continuation in the definition of M_k^l follows in the same way). However, in the definition of M_k^l, we also allow $k = l$, which we interpret as containing no σ^E's at all. The reason we allow $k = l$ is to be able to start the induction in the next lemma.

The proof of Theorem 1.4 will be a simple consequence of Proposition 6.1 and this lemma.

Lemma 6.2. *Let M be the current defined by* (3.1)*. Then $M \wedge dz$ defines a current on Z.*

Proof. Note that as we remarked above, $M = \sum_{l<k} M_k^l$, where M_k^l is defined by (6.1). We show by induction over $k-l$, that if ϕ is a test form such that $\phi|_{Z_{\text{reg}}} = 0$, then $\phi \wedge M_k^l \wedge dz = 0$. Since $M_l^l = a_l R_l^{\mathcal{I}_Z}$, the case $k = l$ follows from Proposition 6.1.

We let $Z_k = Z_k^{\mathcal{J}+\mathcal{I}_Z}$. For $k = l+1$, we have $M_k^l = -\sigma_k^E M_{k-1}^l$ (the minus sign in front of σ_k^E comes from σ_k^E and $\bar\partial|G|^{2\lambda}$ anti-commuting), where σ_k^E is smooth outside of Z_k, so by induction over l, ϕ annihilates $M_k^l \wedge dz$ outside of Z_k. Thus,

$$\phi \wedge M_k^l \wedge dz = \phi \wedge \mathbf{1}_{Z_k} M_k^l \wedge dz = 0,$$

since $\mathbf{1}_{Z_k} M_k^l$ is a $(0,k-1)$-current with support on Z_k of codimension $\geq k$.

For $k > l+1$, $M_k^l = (\bar\partial \sigma_k^E) \wedge M_{k-1}^l$ outside of Z_k, and as above, $\phi \wedge M_k^l = \phi \wedge \mathbf{1}_{Z_k} M_k^l = 0$ since $\mathbf{1}_{Z_k} M_k^l$ is a $(0,k-1)$-current with support on Z_k of codimension $\geq k$. □

Proof of Theorem 1.4. The fact that $R^{\mathcal{J}+\mathcal{I}_Z} \wedge dz$ defines a current on Z will follow from the formula

$$R^{\mathcal{J}+\mathcal{I}_Z} \wedge dz = aR^{\mathcal{I}_Z} \wedge dz + \nabla_\varphi M \wedge dz$$

in Theorem 3.2. The term $aR^{\mathcal{I}_Z} \wedge dz$ defines a current on Z by Proposition 6.1, and since $M \wedge dz$ defines a current on Z by Lemma 6.2, so does $\nabla_\varphi M \wedge dz$, since if ϕ vanishes on Z, then $\bar\partial\phi$ also vanishes on Z, so

$$\phi \wedge \nabla_\varphi M \wedge dz = \nabla_\varphi(\phi \wedge M) \wedge dz + \bar\partial\phi \wedge M \wedge dz = 0.$$

□

7. The Jacobian determinant of a holomorphic mapping

Proof of Theorem 1.5. As explained before the statement of Theorem 1.5, the 'only if' direction follows from the Poincaré-Lelong formula and the duality theorem. Thus, it remains to prove that if $\operatorname{codim} Z(f) < n$, then $J_f \in \mathcal{J}(f_1,\dots,f_n)$. We consider a free resolution (E,φ) of $\mathcal{O}/\mathcal{J}(f)$ of length $\leq n$, which exists by Hilbert's syzygy theorem, and the Koszul complex $(\bigwedge \mathcal{O}^{\oplus n}, \delta_f)$ of f. By Proposition 3.1, there exists a morphism $a : (\bigwedge \mathcal{O}^{\oplus n}, \delta_f) \to (E,\varphi)$ extending the identity morphism $\operatorname{coker}(\delta_f)_1 \cong \operatorname{coker}\varphi_1$. Thus, we get from Theorem 3.2 that

$$R^E = aR^f + \nabla M, \tag{7.1}$$

where R^f is the Bochner-Martinelli current of f, i.e., the currents associated to the Koszul complex of f. Since $\operatorname{ann} R^E = \mathcal{J}(f)$, we are done if we can prove that J_f annihilates both the currents of the right-hand side of (7.1), or equivalently that $df := df_1 \wedge \cdots \wedge df_n$ annihilates these currents.

We consider first the terms $R_k^f \wedge df$. From the proof of Lemma 8.3 in [**A1**], it follows that there exists a modification $\pi : \tilde X \to (\mathbb{C}^n, 0)$, such that $R_k^{\pi^* f} \wedge \pi^* df$ is of the form

$$\bar\partial \frac{1}{f_0^k} \wedge (f_0^{n-1} df_0 \wedge \eta_1 + f_0^n \eta_2),$$

where f_0 is a single holomorphic function, such that $\{f_0 = 0\} = \{\pi^* f = 0\}$ and η_1 and η_2 are smooth forms. By the Poincaré-Lelong formula and the duality theorem, this equals $2\pi i [f_0 = 0] f_0^{n-k} \eta_1$. In particular, if $k < n$,

$R^f_k \wedge df = 0$ since $R^f_k \wedge df = \pi_*(R^{\pi^* f}_k \wedge \pi^* df)$. If $k = n$, it is thus sufficient that a_n vanishes on $Z(f)$ to prove that df annihilates $a_n R^f_n$. Since we assume that $\dim Z(f) > 0$, by continuity, it is enough to prove that a_n vanishes generically at $Z(f)$. An ideal is generically Cohen-Macaulay since if $p = \operatorname{codim} Z(\mathcal{J})$, then the set where $\mathcal{J}$ is not Cohen-Macaulay is Z_{p+1}, which has codimension $\geq p+1$, see Section 2.6. Thus, we can assume that we are at a point $z_0 \in Z(f)$ such that $\mathcal{J}(f)_{z_0}$ is Cohen-Macaulay, and of codimension $p < n$ and we want to prove that $a_n(z_0) = 0$.

We consider a minimal free resolution (F, ψ) of $\mathcal{O}_{z_0}/\mathcal{J}(f)_{z_0}$, and let $b : (\bigwedge \mathcal{O}^{\oplus n}_{z_0}, \delta_f) \to (F, \psi)$ be the morphisms induced by the identity morphism by Proposition 3.1. Since a minimal free resolution is a direct summand of any free resolution, we get an inclusion $i : (F, \psi) \to (E, \varphi)$. Thus, one choice of $a' : (\bigwedge \mathcal{O}^{\oplus n}_{z_0}, \delta_f) \to (E, \varphi)$ would be $a' = ib$. Since b_n is 0 (because we assume that $\mathcal{J}(f)_{z_0}$ is Cohen-Macaulay of codimension $p < n$, so $F_n = 0$), $a'_n = 0$. Thus, there exists one choice of $a : (\bigwedge \mathcal{O}^{\oplus n}_{z_0}, \delta_f) \to (E, \varphi)$ such that $a_n(z_0) = 0$. We need to prove that this holds for any choice of a. By Proposition 3.1, we have that there exists $s : (\bigwedge \mathcal{O}^{\oplus n}_{z_0}, \delta_f) \to (E, \varphi)$ of degree -1 such that $a_k - a'_k = \varphi_{k+1} s_k - s_{k-1}(\delta_f)_k$, and in particular, if $k = n$, then $\varphi_{n+1} = 0$, so $a_n = a'_n + s_{n-1}(\delta_f)_n$. Thus, $a_n(z_0) = 0$ since $a'_n(z_0) = 0$ and $(\delta_f)_n = 0$ on $Z(f)$.

Finally, we want to prove that df annihilates M (note that df is holomorphic, so df commutes or anti-commutes with ∇, depending on the degree of df). Let

$$M^l_k = \bar\partial |f|^{2\lambda} \wedge \bar\partial \sigma_k \cdots \bar\partial \sigma_{l+2} \sigma_{l+1} a_l u^f_l \Big|_{\lambda=0}, \tag{7.2}$$

so that $M = \sum_{l<k} M^l_k$, cf. (6.1). Note that if $k = l$ (which we interpret as M^l_k containing no σ's at all), then $M^l_l = a_l R^f_l$. Thus, if $k = l < n$, then by the first part, $df \wedge M^l_l = 0$. Then, one finishes the proof of showing that $df \wedge M^l_k = 0$ for $l < k$ by induction over $k - l$, the argument is exactly the same as in the proof of Lemma 6.2. □

A. Using Macaulay2 to compute induced morphisms

The computation of the induced morphisms as in Proposition 3.1 can be performed with the help of the computer algebra program Macaulay2[1]. The following code computes the induced morphisms in Example 1.

```
-- Create the ambient ring (R), generators (h),
-- and corresponding ideal (I), and its radical
-- ideal (J)
R = QQ[x,y,z]
h = matrix{{z^2-x^2*y,x^4+y^3-2*x*y*z}}
I = ideal(h)
```

[1] http://www.math.uiuc.edu/Macaulay2/

```
J = radical(I)

-- Create free resolutions of I and J
-- (h is a complete intersection, so the
-- Koszul complex is a free resolution)
E = res J
F = koszul(h)

-- Create the map induced from the natural
-- surjection O/J -> O/I
a_0 = inducedMap(F_0,E_0)
a = extend(E,F,a_0)

-- Print a, E and F
a
E.dd
F.dd
```

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PAPER IV

Residue currents with prescribed annihilator ideals on singular varieties

Richard Lärkäng

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Abstract. Given an ideal $\mathcal{J}$ on a complex manifold, Andersson and Wulcan constructed a current $R^{\mathcal{J}}$ such that $\operatorname{ann} R^{\mathcal{J}} = \mathcal{J}$, generalizing the duality theorem for Coleff-Herrera products. We describe a way to generalize this construction to ideals on singular varieties.

1. Introduction

Let $f \in \mathcal{O}$ be a germ of a holomorphic function, where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$ is the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$. Consider the problem of finding a current U such that $fU = 1$. Such currents were proven to exist abstractly by Schwartz in [**S1**]. A canonical and explicit choice of such a current, as constructed in [**HL**], is the *principal value current* $1/f$, which can be defined by

$$\frac{1}{f} := \lim_{\epsilon \to 0^+} \frac{\bar{f}}{|f|^2 + \epsilon},$$

where the limit is taken in the sense of currents. The existence of this limit over $Z(f)$ as a current is non-trivial if $n > 1$, relying on Hironaka's theorem on resolution of singularities. Nevertheless, $1/f$ exists as a explicit limit of smooth functions. In addition, it is canonical in the sense that any "reasonable" way of cutting off the singularities followed by a limiting procedure will result in the same current.

Since we have defined the principal value current $1/f$, one can also give meaning to meromorphic currents g/f and residue currents $\bar{\partial}(1/f)$. The residue current $\bar{\partial}(1/f)$ is closely related to the ideal $\mathcal{J}(f)$ generated by f in the following way: Let $\operatorname{ann}_{\mathcal{O}} \bar{\partial}(1/f)$ be the *annihilator* of $\bar{\partial}(1/f)$, i.e., the ideal of holomorphic functions g such that $g\bar{\partial}(1/f) = 0$. Then $g \in \operatorname{ann}_{\mathcal{O}} \bar{\partial}(1/f) = 0$ if and only if $\bar{\partial}(g/f) = 0$ and, by regularity of the $\bar{\partial}$-operator on $(0,0)$-currents, this holds if and only if $g/f \in \mathcal{O}$, i.e., $g \in \mathcal{J}(f)$. Hence, $\operatorname{ann}_{\mathcal{O}} \bar{\partial}(1/f) = \mathcal{J}(f)$.

Let $f = (f_1, \dots, f_p) \in \mathcal{O}^{\oplus p}$ be a tuple of holomorphic functions. In [**CH**], Coleff and Herrera showed that one can give a meaning to products

$$\bar{\partial}\frac{1}{f_p} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1},$$

what is nowadays called the *Coleff-Herrera product* of f, and which we will also denote by μ^f.

Such products are "nicely" behaved if f defines a *complete intersection*, i.e., if $\operatorname{codim} Z(f) = p$. Maybe the most important property is the following *duality theorem* for Coleff-Herrera products.

Theorem 1.1. *Let $f = (f_1, \dots, f_p)$ be a holomorphic mapping on a complex manifold defining a complete intersection. Then locally,*

$$\operatorname{ann} \mu^f = \mathcal{J}(f_1, \dots, f_p).$$

This result thus extends the description of the annihilator for one single holomorphic function described above. It was proven independently by Dickenstein and Sessa in [**DS**] and Passare in [**P**].

Another way in which the Coleff-Herrera product is nicely behaved in the case of complete intersection is the following. Let $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ be two tuples of holomorphic functions defining complete intersections. If there exists a matrix A of holomorphic functions such that $f = gA$, then the *transformation law* for Coleff-Herrera products states that $\mu^g = (\det A)\mu^f$. In particular, if f and g define the same ideal, then A is invertible, so $\det A$ is a non-vanishing holomorphic function. Thus, we can view the Coleff-Herrera product as an essentially canonical current associated to a complete intersection ideal.

Coleff-Herrera products have had various applications, for example to explicit versions of the Ehrenpreis-Palamodov fundamental principle by Berndtsson and Passare, [**BP**], the $\bar{\partial}$-equation on singular varieties by Henkin and Polyakov, [**HePo**], and effectivity questions in division problems by Berenstein and Yger, [**BY**].

In [**AW1**], Andersson and Wulcan generalized the construction of the Coleff-Herrera product from complete intersection ideals to arbitrary ideals. From a Hermitian resolution (E, φ) (i.e., a locally free resolution equipped with Hermitian metrics) of an ideal $\mathcal{J}$, they constructed explicitly a vector-valued current $R^{\mathcal{J}}$ with values in E such that $\operatorname{ann}_{\mathcal{O}} R^{\mathcal{J}} = \mathcal{J}$. In case $\mathcal{J} = \mathcal{J}(f_1, \dots, f_p)$ is a complete intersection ideal, the current they constructed coincides with the Coleff-Herrera product of f.

In case the ideal is Cohen-Macaulay, i.e., if $\mathcal{O}/\mathcal{J}$ has a free resolution of length equal to $\operatorname{codim} Z(\mathcal{J})$, the current $R^{\mathcal{J}}$ is essentially canonically associated to $\mathcal{J}$, in the sense that it does not depend on the Hermitian metrics chosen, and choosing different minimal free resolutions only changes the current by an invertible holomorphic matrix (just like the Coleff-Herrera product changes by an invertible holomorphic function by changing the generators). In addition, the construction "globalizes" in the same way as free resolutions in the sense that if we construct the current $R^{\mathcal{J}}$ globally, and restrict it to a neighbourhood of a point z, we can express $R^{\mathcal{J}}$ there as a smooth matrix times the current constructed locally around z (just as considering a global (locally) free resolution will in general not restrict to a minimal free resolution locally, but only that the local minimal free resolution is a direct summand of the restriction of the global one).

The construction is explicit both in the sense that it is explicitly described in terms of a free resolution of the ideal, and also in the sense that it not only describes ideal membership in terms of its annihilator, but also explicitly realizes this ideal membership, by appearing in integral representation formulas, see [**AW1**], Section 5.

The applications described for Coleff-Herrera products have been generalized in various ways to Andersson-Wulcan currents, thereby being able to remove assumptions about complete intersection, see for example [**AS1**, **AS2**, **ASS**, **AW1**, **AW3**, **S2**].

The aim of this article, is to generalize the construction in [**AW1**], to currents with prescribed annihilator ideals on singular varieties. Describing this construction more precisely, and how the construction generalizes the one of Andersson and Wulcan requires more knowledge about their construction, which we leave for later parts of the article, see in particular Theorem 3.2 and Theorem 5.3. In the introduction, we instead describe a special case where many of the technicalities of the construction disappears, while it still illustrates much of the ideas behind the construction.

1.1. Principal ideals on hypersurfaces. Let $Z \subseteq \Omega$ be a *reduced hypersurface* of an open set $\Omega \subseteq \mathbb{C}^n$, i.e., $Z = Z(h)$, where h is a holomorphic function on Ω such that dh is non-vanishing generically on Z. In particular, $\mathcal{O}_Z = \mathcal{O}/\mathcal{J}(h)$.

One of the simplest examples of an ideal in $\mathcal{O}_Z$ would be a principal ideal $\mathcal{J} = \mathcal{J}(f) \subseteq \mathcal{O}_Z$, where we also assume that f is a non-zero-divisor in $\mathcal{O}_Z$, i.e., f does not vanish identically on any irreducible component of Z. We then want to find an intrinsic current R on Z such that $\operatorname{ann}_{\mathcal{O}_Z} R = \mathcal{J}$. Currents on analytic varieties can either be defined in a similar manner as on manifolds, or in terms of currents in the embedding, see Section 2.1. Of particular importance here will be that the construction of principal-value currents works just as well on singular varieties. Since the residue current $\bar\partial(1/f)$ of f exists on Z, it would be a natural candidate for the current R. However, in [**Lä2**], we show that if $\operatorname{codim} Z_{\text{sing}} = 1$ (as would be the case for example for any singular planar curve), then one can always find a holomorphic function f such that $\operatorname{ann}_{\mathcal{O}_Z} \bar\partial(1/f) \neq \mathcal{J}(f)$.

We instead start by considering currents in the ambient space. Let $\tilde f$ be a representative of f in the ambient space Ω. The current

$$T := \bar\partial\frac{1}{\tilde f} \wedge \bar\partial\frac{1}{h} \wedge dz,$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$, has the same annihilator as $\bar\partial(1/\tilde f) \wedge \bar\partial(1/h)$, i.e., $\mathcal{J}(\tilde f, h)$ by the duality theorem. Since the annihilator contains h, we get a well-defined multiplication with elements of $\mathcal{O}_Z = \mathcal{O}/\mathcal{J}(h)$, and the annihilator of T over $\mathcal{O}_Z$ equals $\mathcal{J}(f)$. Thus, we have found a current in the ambient space with the correct annihilator, and then if we can find a current R on Z such that $i_* R = T$, where $i : Z \to \Omega$ is the inclusion, then R will be a current with the correct annihilator.

We consider the current $(1/f)\omega$ on Z, where ω is the Poincaré residue of dz/h, see Example 2 below. One way of characterizing the Poincaré residue ω is that $i_*\omega = \bar\partial(1/h) \wedge dz$, so

$$i_*\left(\frac{1}{f}\omega\right) = \frac{1}{\tilde f}\bar\partial\frac{1}{h} \wedge dz.$$

Thus, by Leibniz' rule, see (2.1),

$$i_*\left(\bar\partial\left(\frac{1}{f}\omega\right)\right) = \bar\partial\left(\frac{1}{\tilde f}\bar\partial\frac{1}{h} \wedge dz\right) = \bar\partial\frac{1}{\tilde f} \wedge \bar\partial\frac{1}{h} \wedge dz = T,$$

and we have proved the following.

Proposition 1.2. *Let Z be a reduced hypersurface defined by a holomorphic function h, and let ω be the Poincaré residue of dz/h on Z. If $f \in \mathcal{O}_Z$ and R^f_Z is the current $\bar\partial((1/f)\omega)$ on Z, then*

$$\operatorname{ann}_{\mathcal{O}_Z} R^f_Z = \mathcal{J}(f).$$

Note that, since $\bar\partial\omega = 0$, we have formally that $R^f_Z = \bar\partial(1/f) \wedge \omega$. However, it might very well happen that ω has its poles (which are contained in Z_{sing}) on $Z(f) = \operatorname{supp}\bar\partial(1/f)$. In that case, the product $\bar\partial(1/f) \wedge \omega$ can not be defined in a "robust" way. For example, it is natural to regularize the factors one at a time, and in that case, the product will in general depend on in which order one regularizes, so we refrain from giving such products any meaning. However, in case $\operatorname{codim} Z_{\text{sing}} \cap Z(f) \geq 2$ in Z, then $\bar\partial(1/f) \wedge \omega$ can be defined in a "robust" way, and it coincides with R^f_Z.

If we let $U = (1/f)$, then we have by Leibniz' rule, and a natural cancellation property for residue currents, see (2.1), that

$$R^f_Z = \omega - \nabla(U\omega), \tag{1.1}$$

where $\nabla = f - \bar\partial$, and in addition,

$$i_* R^f_Z = \bar\partial\frac{1}{\tilde f} \wedge \bar\partial\frac{1}{h} \wedge dz. \tag{1.2}$$

In this article, we generalize this construction to arbitrary ideals on arbitrary varieties. The starting point of generalizing this construction is to replace the right-hand side of (1.2) with the Andersson-Wulcan current $R^{\tilde{\mathcal{J}}}$

associated to a maximal lifting $\tilde{\mathcal{J}}$ of the ideal $\mathcal{J}$, which will give a current in the ambient space with the correct annihilator. In Section 2, we describe the construction of residue currents from [**AW1**] and other necessary background on residue currents. In order to prove that this current corresponds to a current on Z, we show that $R^{\tilde{\mathcal{J}}} \wedge dz$ is the push-forward of a current on Z of a similar form as the right-hand side of (1.1). We treat the case when Z is of pure dimension in Section 3. The main ingredients are a comparison formula for Andersson-Wulcan currents from [**Lä3**], relating such currents associated to two different ideals, and a generalization of the Poincaré residue to arbitrary varieties of pure dimension, as introduced in [**AS1**], called the structure form associated to Z. In Section 4, we describe how this construction coincides with the construction in [**AW1**] in case Z is non-singular. In Section 5, we prove the general case of our construction, i.e., when Z is not necessarily of pure dimension. A key part is to prove the existence of a structure form also associated to such varieties. We finish in Section 6 by discussing why a more straightforward generalization of the construction in [**AW1**], by considering free resolutions on the variety itself, does not work in general.

2. Preliminaries

In this section we recall several tools which will be useful during the rest of the article, like currents on singular varieties, almost semi-meromorphic and pseudomeromorphic currents, the construction of Andersson-Wulcan of currents with prescribed annihilator ideals and a comparison formula for such currents.

2.1. Currents on analytic varieties. Since a key part in this article is that we construct intrinsic currents on the varieties, we begin by recalling what currents on analytic varieties are. The usual way to define currents on an analytic variety is to first define test forms on analytic varieties, and then define currents as continuous linear functionals on the test forms. However, it can also be described more concretely in terms of embeddings. If Z is a subvariety of pure codimension k of some complex manifold X, and i is the inclusion $i : Z \to X$, then T is a (p,q)-current on Z if $i_* T$ is a $(p+k, q+k)$-current on X which vanishes when acting on test forms ϕ on X such that $\phi|_{Z_{\text{reg}}} = 0$. Conversely, if T' is any such current on X, then T' defines a unique current T on Z such that $i_* T = T'$. Note that considered as a current in the ambient space, it is not sufficient that $\operatorname{supp} T \subseteq Z$ for it to correspond to a current on Z. For example, if $Z = \{0\} \subseteq \mathbb{C}$, then $[0]$, the integration current at $\{0\}$, corresponds to a current on Z, while $\partial/\partial_z[0]$ does not, although both have support on Z.

Example 1. The most basic example of a current on a singular variety is given by the integration current constructed by Lelong, [**Le**]. Given a subvariety Z of a complex manifold X, the integration current $[Z]$ of Z on X is defined by

$$[Z].\phi := \int_{Z_{\text{reg}}} \phi,$$

where ϕ is a test form. It is thus immediate from the description above, that $[Z]$ corresponds to a current on Z, and it is reasonable to denote it by 1, i.e., $i_*1 = [Z]$.

Multiplying the equation $i_*1 = [Z]$ by a smooth form, any smooth (p,q)-form on Z can be considered as a current on Z, and in fact, the construction of Herrera and Lieberman of principal value and residue currents works also on a singular variety, so for any meromorphic (p,q)-form η on Z, we can define its corresponding meromorphic current, which we for simplicity will also denote by η.

By a *holomorphic form* on a singular variety Z, we mean the restriction of a holomorphic form in the ambient space, and by a meromorphic form, we mean the restriction of a *meromorphic form* in the ambient space such that its polar set has positive codimension in Z. See [**HePa**] for a rather detailed discussion about different definitions of meromorphic forms, and various definitions of holomorphic forms. In order to distinguish between a meromorphic form η on Z, and a representative of it in the ambient space, we will denote the representative by $\tilde{\eta}$. In particular, we write $i_*\eta = \tilde{\eta} \wedge [Z]$.

In case we have two holomorphic functions f and g on Z such that $\operatorname{codim} Z(f) \cap Z(g) = 2$, then we can form products of residue currents and principal value currents of f and g satisfying the following natural properties.

$$f\frac{1}{f}\bar\partial\frac{1}{g} = \bar\partial\frac{1}{g}, \quad g\frac{1}{f}\bar\partial\frac{1}{g} = 0 \text{ and } \bar\partial\left(\frac{1}{f}\bar\partial\frac{1}{g}\right) = \bar\partial\frac{1}{f} \wedge \bar\partial\frac{1}{g}. \tag{2.1}$$

Example 2. Let $Z \subseteq \Omega \subseteq \mathbb{C}^n$ be a reduced hypersurface defined by a holomorphic function h. On such a hypersurface, the *Poincaré residue* ω of dz/h is a meromorphic form, which can be defined by

$$i_*\omega = \bar\partial\frac{1}{h} \wedge dz. \tag{2.2}$$

If we let $\tilde{\omega}$ be a meromorphic form on Ω such that $(dh/2\pi i) \wedge \tilde{\omega} = dz_1 \wedge \cdots \wedge dz_n =: dz$, then ω can alternatively be defined by $\omega := \tilde{\omega}|_Z$. This definition of ω does not depend on the choice of $\tilde{\omega}$. Considered as a meromorphic current, ω is $\bar\partial$-closed, see [**HePa**]. If $\partial h/\partial z_n$ does not vanish identically on any irreducible component of Z, then one can take $\omega = (-1)^{n-1}/(2\pi i \partial h/\partial z_n) dz_1 \wedge \cdots \wedge dz_{n-1}|_Z$. The Poincaré residue is a classical construction in mathematics, see for example [**Y**]. In this form it appears for example in [**HePa**], and in similar forms in for example [**B**] and [**H**].

2.2. Almost semi-meromorphic and pseudomeromorphic currents. In $\mathbb{C}_z$ the principal value current $1/z^m$ can be defined as the analytic continuation $|z|^{2\lambda}/z^m|_{\lambda=0}$, where by $|_{\lambda=0}$ we mean that it is a current-valued analytic function for $\operatorname{Re}\lambda \gg 0$, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$. We can thus also define $\bar\partial(1/z^m)$ in the sense of currents, which thus equals $\bar\partial|z^m|^{2\lambda}/z^m|_{\lambda=0}$. Hence, we can consider tensor products of such one variable currents

$$\tau = \bar\partial\frac{1}{z_1^{m_1}} \wedge \cdots \wedge \bar\partial\frac{1}{z_k^{m_k}}\frac{\alpha}{z_{k+1}^{m_{k+1}}\dots z_N^{m_N}},$$

on $\mathbb{C}^N$, where $m_1,\dots,m_N$ are non-negative integers and α is a smooth form with compact support. We call such a current an *elementary current*. Andersson and Wulcan introduced the following class of currents in [**AW2**].

Definition 1. Let Z be an analytic variety. A current μ on Z is *pseudomeromorphic*, denoted $\mu \in \mathcal{PM}(Z)$ if it can be written as a locally finite sum of push-forwards $\pi_*\tau$ of elementary currents, where π is a composition of modifications and open inclusions.

The definition in [**AW2**] was for Z a complex manifold, but allowing Z to be singular makes no difference. In [**AS1**], a slightly wider definition was used, allowing more general push-forwards, but Definition 1 will be sufficient for our purposes.

For pseudomeromorphic currents one can define natural restrictions to analytic subvarieties. If $T \in \mathcal{PM}(Z)$, $V \subseteq Z$ is a subvariety of Z, and h is a tuple of holomorphic functions such that $V = Z(h)$, one defines

$$\mathbf{1}_{Z\setminus V}T := |h|^{2\lambda}T|_{\lambda=0} \text{ and } \mathbf{1}_V T := T - \mathbf{1}_{Z\setminus V}T.$$

This definition is independent of the choice of tuple h, and $\mathbf{1}_V T$ is a pseudomeromorphic current with support on V, see [**AW2**], Proposition 2.2.

A pseudomeromorphic current $\mu \in \mathcal{PM}(Z)$ is said to have the *standard extension property*, SEP, if $\mathbf{1}_V\mu = 0$ for any subvariety $V \subseteq Z$ of positive codimension. If Z does not have pure dimension, we mean that V has positive codimension on each irreducible component of Z.

If α is a smooth form, and T is a pseudomeromorphic current, then $\mathbf{1}_V(\alpha\wedge T) = \alpha\wedge\mathbf{1}_V T$, and in particular, if T has the SEP, then $\alpha\wedge T$ also has the SEP.

An important property of pseudomeromorphic currents is that they satisfy the following *dimension principle*, Corollary 2.4 in [**AW2**].

Proposition 2.1. *If $T \in \mathcal{PM}(Z)$ is a (p,q)-current with support on a variety V, and* codim $V > q$*, then $T = 0$.*

Given f holomorphic on an analytic variety Z, as described in the introduction, Herrera and Lieberman defined the principal value current $1/f$ on Z. One way to define this is by

$$\frac{1}{f}.\phi := \int_{Z_{\mathrm{reg}}}\frac{|f|^{2\lambda}}{f}\phi\bigg|_{\lambda=0},$$

where by $|_{\lambda=0}$, we mean that right-hand side for $\operatorname{Re}\lambda \gg 0$ is analytic in λ, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$. This way of defining the principal value current by analytic continuation goes back to Atiyah, [**At**], and Bernstein-Gel′fand, [**BG**]. The proof of the existence of this analytic continuation relies on Hironaka's theorem of resolution of singularities in order to write it as a locally finite sum of push-forwards of elementary currents, and hence, principal value currents are pseudomeromorphic.

The product of a principal value current and a smooth form (i.e., the restriction of a smooth form in the ambient space) is called a *semi-meromorphic current*. In [**AS1**], the authors introduce a generalization of this called *almost semi-meromorphic* currents.

Definition 2. A current μ on an analytic variety Z is said to be *almost semi-meromorphic* if $\mu = \pi_*\tilde{\mu}$, where $\tilde{\mu}$ is semi-meromorphic and $\pi : \tilde{Z} \to Z$ is a smooth modification of Z.

Since the class of pseudomeromorphic currents is closed under multiplication with smooth functions and under push-forwards under modifications, almost semi-meromorphic currents are pseudomeromorphic. By the dimension principle, principal value currents have the SEP, and thus any semi-meromorphic current will also have the SEP.

Definition 3. The sheaf $\mathcal{W}_Z$ is the subsheaf of $\mathcal{PM}_Z$ of pseudomeromorphic currents on Z with the SEP on Z.

In particular, almost semi-meromorphic currents are in $\mathcal{W}_Z$. The fact that $\mathcal{W}_Z$ allows a natural multiplication with semi-meromorphic currents will be crucial for the description of the currents we construct, Proposition 2.7 in [**AS1**].

Proposition 2.2. *Let α be an almost semi-meromorphic current on Z. If $\mu \in \mathcal{W}(Z)$, then the current $\alpha \wedge \mu$, a priori defined where α is smooth has a unique extension as a current in $\mathcal{W}(Z)$, which we also denote by $\alpha \wedge \mu$.*

2.3. Andersson-Wulcan currents. Here we recall the construction in [**AW1**] of residue currents with prescribed annihilator ideals on complex manifolds. Let $\mathcal{J} \subseteq \mathcal{O}$ be an ideal of holomorphic functions, and let (E,φ) be a Hermitian resolution of $\mathcal{O}/\mathcal{J}$, i.e., (E,φ) is a free resolution

$$0 \to E_N \xrightarrow{\varphi_N} E_{N-1} \xrightarrow{\varphi_{N-1}} \cdots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 \to \mathcal{O}/\mathcal{J},$$

where the free modules $E_k \cong \mathcal{O}^{r_k}$ are equipped with Hermitian metrics.

To construct the current associated to E, one first defines, outside of $Z = Z(\mathcal{J})$, right inverses $\sigma_k : E_{k-1} \to E_k$ to φ_k which are minimal with respect to some metric on E, i.e., $\varphi_k \sigma_k|_{\operatorname{Im}\varphi_k} = \operatorname{Id}_{\operatorname{Im}\varphi_k}$, $\sigma_k = 0$ on $(\operatorname{Im}\varphi_k)^\perp$, and $\operatorname{Im}\sigma_k \perp \ker\varphi_k$. One lets $\sigma = \sigma_1 + \cdots + \sigma_N$, and

$$u^E = \sigma + \sigma\bar{\partial}\sigma + \cdots + \sigma(\bar{\partial}\sigma)^N. \tag{2.3}$$

Letting ∇_{End} be the morphism on $\mathcal{D}(\mathrm{End}E)$ induced by $\nabla = \varphi - \bar{\partial}$ by $\nabla_{\mathrm{End}}(\alpha) = \nabla \circ \alpha - \alpha \circ \nabla$, one has that $\nabla_{\mathrm{End}} u^E = I_E$ outside of Z. The form u^E, which is smooth outside of Z, has a current extension $U^E := |F|^{2\lambda} u^E|_{\lambda=0}$ over Z, where $F \not\equiv 0$ is a holomorphic function vanishing at Z and for

$\operatorname{Re}\lambda \gg 0$, the right-hand side is is a (current-valued) analytic function in λ, and $|_{\lambda=0}$ denotes the analytic continuation to $\lambda = 0$. The residue current R^E associated to E is defined as

$$R^E := I_E - \nabla_{\mathrm{End}} U^E. \tag{2.4}$$

Alternatively, one could define R^E by

$$R^E = \bar{\partial}|F|^{2\lambda} \wedge u^E|_{\lambda=0}. \tag{2.5}$$

See [**AW1**] for more details. From the proof of the existence of U^E and R^E, it follows that they are pseudomeromorphic.

Let R^E_k denote the part of R^E with values in E_k, i.e., R^E_k is a E_k-valued $(0,k)$-current. If $Z = Z(\mathcal{J})$, and $\operatorname{codim} Z = p$, then we will in fact have that

$$R^E = R^E_p + \cdots + R^E_N, \tag{2.6}$$

where N is the length of the free resolution (E,φ).

The fundamental property of the current R^E is the following, [**AW1**], Theorem 1.1.

Theorem 2.3. *Let R^E be the current associated to a free resolution (E,φ) of an ideal $\mathcal{J}$. Then* $\operatorname{ann} R^E = \mathcal{J}$.

In particular, if $\mathcal{J}$ is a complete intersection ideal, $\mathcal{J} = \mathcal{J}(h_1,\dots,h_p)$, then the Koszul complex of h is a free resolution of $\mathcal{O}/\mathcal{J}_Z$. In that case, both the Coleff-Herrera product of h and the current associated to the Koszul complex are currents with annihilator equal to $\mathcal{J}$, and in fact they turn out to coincide. Here, we identify the tuple f with a section of G^*, where $G \cong \mathcal{O}^{\oplus p}$ with a frame $e_1,\dots,e_p$, so that $f = \sum f_i e_i^*$.

Theorem 2.4. *Let $f = (f_1,\dots,f_p)$ be a tuple of holomorphic functions defining a complete intersection. Let R^f be the current associated to the Koszul complex of f, $R^f = \mu \wedge e_1 \wedge \cdots \wedge e_p$, and let μ^f be the Coleff-Herrera product of f. Then $\mu = \mu^f$.*

The current R^f was originally introduced by Passare, Tsikh and Yger in [**PTY**] (defined more directly), referred to as a Bochner-Martinelli type residue current. The equality of the Coleff-Herrera product and the Bochner-Martinelli type residue current was originally proved in [**PTY**], Theorem 4.1, see also [**An1**], Corollary 3.2 for an alternative proof.

The definition of the Coleff-Herrera product and Bochner-Martinelli type current works also in the singular case, and the equality of those in the case of complete intersection, Theorem 2.4 also holds; the proof in [**An1**] works also in the singular case, see [**Lä1**], Theorem 6.4.

Note that from Theorem 2.3 and Theorem 2.4, the construction by Andersson and Wulcan of a current with a prescribed annihilator ideal can be seen as a generalization of the Coleff-Herrera product and the duality theorem for Coleff-Herrera products.

We introduce the notation

$$R_X^{\mathcal{J}} := R^E \wedge \omega_X = \omega_X - \nabla(U^E \wedge \omega_X), \tag{2.7}$$

where R^E is the current associated to a minimal free resolution (E,φ) of $\mathcal{O}/\mathcal{J}$, and ω_X is a global holomorphic non-vanishing $(n,0)$-form on X (for example if X is an open subset of $\mathbb{C}^n_z$, we can take $\omega_X = dz := dz_1 \wedge \dots dz_n$). Note that since ω_X is assumed to be holomorphic and non-vanishing, we will have that $\operatorname{ann} R_X^{\mathcal{J}} = \operatorname{ann} R^E = \mathcal{J}$, so in this setting, the advantage of multiplying with the factor ω_X will not be very apparent, but it will be important when we generalize this to singular varieties.

2.4. A comparison formula for residue currents. An important tool in this article will be a comparison formula for Andersson-Wulcan currents, [**Lä3**], which can be seen as a generalization of the transformation law for Coleff-Herrera products.

Let $\mathcal{I} \subseteq \mathcal{J}$ be two ideals of holomorphic functions, and let (F,ψ) and (E,φ) be free resolutions of $\mathcal{O}/\mathcal{I}$ and $\mathcal{O}/\mathcal{J}$ respectively. Since $\mathcal{I} \subseteq \mathcal{J}$, we have the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$. By a rather straightforward diagram chase, one can show that there exists a morphism of complexes $a : (F,\psi) \to (E,\varphi)$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & E_N \arrow[r, "\varphi_N"] & E_{N-1} \arrow[r, "\varphi_{N-1}"] & \cdots \arrow[r, "\varphi_1"] & E_0 \arrow[r] & \mathcal{O}/\mathcal{J} \arrow[r] & 0 \\ 0 \arrow[r] & F_N \arrow[r, "\psi_N"] \arrow[u, "a_N"] & F_{N-1} \arrow[r, "\psi_{N-1}"] \arrow[u, "a_{N-1}"] & \cdots \arrow[r, "\psi_1"] & F_0 \arrow[r] \arrow[u, "a_0"] & \mathcal{O}/\mathcal{I} \arrow[r] \arrow[u, "\pi"] & 0. \end{tikzcd} \tag{2.8}$$

The comparison formula, Theorem 1.2 in [**Lä3**], is expressed in terms of this morphism a.

Theorem 2.5. *Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{O}$ be two ideals of germs of holomorphic functions such that $\mathcal{I} \subseteq \mathcal{J}$, and let (E,φ) and (F,ψ) be minimal free resolutions of $\mathcal{O}/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}$ respectively. Let $a : (F,\psi) \to (E,\varphi)$ be the morphism in* (2.8) *induced by the natural surjection $\pi : \mathcal{O}/\mathcal{I} \to \mathcal{O}/\mathcal{J}$. Then,*

$$R^E a_0 = aR^F + \nabla_\varphi M, \tag{2.9}$$

where $\nabla_\varphi = \sum \varphi_k - \bar\partial$,

$$M = \bar\partial |G|^{2\lambda} \wedge u^E \wedge a u^F \Big|_{\lambda=0}, \tag{2.10}$$

and G is a tuple of holomorphic functions such that $\{G = 0\}$ contains the set where (E,φ) and (F,ψ) are not pointwise exact.

2.5. Singularity subvarieties of free resolutions. In the study of residue currents associated to free resolutions of ideals, an important ingredient is certain singularity subvarieties associated to the ideal. Given a free resolution (E,φ) of an ideal $\mathcal{J}$, the variety $Z_k = Z_k^E$ is defined as the set where φ_k does not have optimal rank. These sets are independent of the choice of free resolution. If $\operatorname{codim} Z(\mathcal{J}) = p$, then $Z_k = Z$ for $k \le p$, Corollary 20.12 in

[**E**]. In addition, Corollary 20.12 says that $Z_{k+1} \subseteq Z_k$, and $\operatorname{codim} Z_k \geq k$ by Theorem 20.9 in [**E**]. In fact, Theorem 20.9 in [**E**] is a characterization of exactness, the Buchsbaum-Eisenbud criterion, which says that a generically exact complex of free modules is exact if and only if $\operatorname{codim} Z_k \geq k$.

However, more precise information is obtained about which irreducible components Z_k that are of maximal dimension. By Corollary 20.14, if $\operatorname{codim} V = k$, then $V \subseteq Z_k$ if and only if $\mathcal{I}_V \in \operatorname{Ass} \mathcal{J}$, i.e., if the ideal of holomorphic functions vanishing on V is an associated prime of $\mathcal{O}/\mathcal{J}$. In particular, if $\mathcal{J}$ is reduced, $\operatorname{Ass} \mathcal{J}$ correspond exactly the irreducible components of $Z = Z(\mathcal{J})$. In that case, if we let W^d be the union of the irreducible components of Z of codimension $p = n - d$, then $Z_p = W^d \cup Z_p'$, where $\operatorname{codim} Z_p' \geq p + 1$. If we consider $e > d$, then $\operatorname{codim} W^d \cap W^e \geq p + 1$, so we get that

$$\operatorname{codim} W^e \cap Z_p \geq p + 1. \tag{2.11}$$

2.6. Tensor products of free resolutions. In this section, we describe how under suitable conditions on "proper" intersection, one can construct a free resolution of a sum of ideals from free resolutions of the individual ideals. To begin with, let (E, φ) and (F, ψ) be two complexes. The tensor product complex $(E \otimes F, \varphi \otimes \psi)$ is defined by $(E \otimes F)_k = \oplus_{p+q=k} E_p \otimes F_q$ and $(\varphi \otimes \psi)(\xi \otimes \eta) = \varphi(\xi) \otimes \eta + (-1)^i \xi \otimes \psi(\eta)$ if $\xi \in E_i$ and $\eta \in F_j$. Note in particular that if (E, φ) and (F, ψ) are minimal free resolutions of ideals $\mathcal{J}$ and $\mathcal{I}$, then $E_0 \cong \mathcal{O} \cong F_0$, and $(\varphi \otimes \psi)_1 : E_1 \oplus F_1 \to \mathcal{O}$, $(\varphi \otimes \psi)_1 = \varphi_1 \oplus \psi_1$, so if the tensor product complex is exact, it is a free resolution of $\mathcal{J} + \mathcal{I}$. The tensor product complex will be exact if the corresponding singularity subvarieties intersect properly in the following sense.

Proposition 2.6. *Let (E, φ) and (F, ψ) be free resolutions of ideal sheaves $\mathcal{J}$ and $\mathcal{I}$, and let Z_k^E and Z_l^F be the associated sets where φ_k and ψ_l do not have optimal rank. Then $(E \otimes F, \varphi \otimes \psi)$ is a free resolution of $\mathcal{I} + \mathcal{J}$ if and only if $\operatorname{codim}(Z_k^E \cap Z_l^F) \geq k + l$ for all $k \geq \operatorname{codim} Z(\mathcal{J})$, $l \geq \operatorname{codim} Z(\mathcal{I})$.*

In addition, if $\mathcal{I}$ and $\mathcal{J}$ are Cohen-Macaulay ideals, and (E, φ) and (F, ψ) are free resolutions of minimal length, then

$$R^{E \otimes F} = (I_E - \nabla_\varphi(|G|^{2\lambda} u^E)) \wedge R^F \big|_{\lambda=0}, \tag{2.12}$$

where G is a tuple of holomorphic functions vanishing on $Z(\mathcal{J})$ but not identically on any irreducible component of $Z(\mathcal{I})$.

A proof of the first part can be found in [**An2**], Remark 4.6, which we have reformulated slightly, by only requiring the condition to hold for $k \geq \operatorname{codim} Z(\mathcal{J})$, $l \geq \operatorname{codim} Z(\mathcal{I})$ instead of $k, l \geq 1$. However, this reformulation follows from the fact that $Z_k^E = Z_p^E$ for $k \leq \operatorname{codim} Z(\mathcal{J})$ (and similarly for Z_l^F). The second part is part of Theorem 4.2 in [**An2**].

When E and F are equipped with Hermitian metrics, we will assume that $E \otimes F$ is equipped with the product metric induced from the metrics of E and F.

3. Currents with prescribed annihilator ideals on singular varieties of pure dimension

Let Z be an analytic subvariety of pure dimension d of $\Omega \subseteq \mathbb{C}^n_z$. We first consider the current $R^{\mathcal{I}_Z}$ associated to $\mathcal{I}_Z$, the ideal of holomorphic functions on Ω vanishing on Z. In [**AS1**], Andersson and Samuelsson showed that there exists what they call a *structure form* ω_Z associated to Z, generalizing the Poincaré residue in Section 2.1. The following part of Proposition 3.3 in [**AS1**] will be sufficient for our purposes.

Proposition 3.1. *Let (F, ψ) be a Hermitian resolution of $\mathcal{O}_\Omega/\mathcal{I}_Z$, and let $R^{\mathcal{I}_Z}$ be the associated residue current. Then there exists an almost semi-meromorphic current*

$$\omega_Z = \omega_0 + \cdots + \omega_{d-1}$$

on Z, where $\dim Z = d$, $\operatorname{codim} Z = p$, *and ω_r has bidegree (d, r) and takes values in F_{p+r}, such that*

$$i_* \omega_Z = R^{\mathcal{I}_Z} \wedge dz, \tag{3.1}$$

where $i : Z \to \Omega$ is the inclusion and $dz := dz_1 \wedge \cdots \wedge dz_n$.

The structure form ω_Z plays an important role in [**AS1**] and [**AS2**] related to the $\bar{\partial}$-equation on singular varieties. It also appears (more implicitly) in [**ASS**], related to the Briançon-Skoda theorem on a singular variety.

Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be an ideal. We will use the comparison formula from Section 2.4 in order to construct intrinsically on Z the current with the prescribed annihilator ideal in terms of almost semi-meromorphic currents. Let $\tilde{\mathcal{J}} \subseteq \mathcal{O}_\Omega$ be the largest lifting of the ideal $\mathcal{J}$, i.e., the largest ideal $\tilde{\mathcal{J}}$ such that $i^* \tilde{\mathcal{J}} = \mathcal{J}$, where $i^* : \mathcal{O}_\Omega \to \mathcal{O}_Z$ is induced by the inclusion $i : Z \to \Omega$. Note that $\mathcal{I}_Z \subseteq \tilde{\mathcal{J}}$ (since $i^* \mathcal{I}_Z = 0$), so if (E, φ) and (F, ψ) are free resolutions of $\tilde{\mathcal{J}}$ and $\mathcal{I}_Z$ respectively, we get a morphism of complexes

$$a : (F, \psi) \to (E, \varphi) \tag{3.2}$$

extending the natural surjection $\pi_* : \mathcal{O}_\Omega/\mathcal{I}_Z \to \mathcal{O}_\Omega/\tilde{\mathcal{J}}$ as in (2.8).

On $Z \setminus Z^E_{p+1}$, let

$$\nu := \sum_{m \geq k \geq p+1} \sigma^E_m \bar{\partial} \sigma^E_{m-1} \ldots \bar{\partial} \sigma^E_k. \tag{3.3}$$

Note that since σ^E_l and $\bar{\partial} \sigma^E_l$ are smooth outside Z_l, we get that ν is smooth outside Z_{p+1}. Since $\operatorname{codim} Z^E_{p+1} \geq p + 1 > p = \operatorname{codim} Z$, ν is defined and smooth generically on Z.

Now we are ready to state our main theorem.

Theorem 3.2. *Let $Z \subseteq \Omega \subseteq \mathbb{C}^n$ be an analytic subvariety of Ω of pure dimension, where Ω is an open set in $\mathbb{C}^n$. Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be an ideal. Then ν defined by* (3.3) *has an extension as an almost semi-meromorphic current to Z, which we denote by V^E. If we let*

$$R_Z^{\mathcal{J}} := a\omega_Z - \nabla(V^E \wedge a\omega_Z), \tag{3.4}$$

where $a : (F,\psi) \to (E,\varphi)$ is the morphism in (3.2)*, then*

$$\operatorname{ann}_{\mathcal{O}_Z} R_Z^{\mathcal{J}} = \mathcal{J}. \tag{3.5}$$

Moreover,

$$i_* R_Z^{\mathcal{J}} = R_\Omega^{\tilde{\mathcal{J}}}, \tag{3.6}$$

where $\tilde{\mathcal{J}} \subseteq \mathcal{O}_\Omega$ is the maximal lifting of $\mathcal{J}$, and $R_\Omega^{\tilde{\mathcal{J}}}$ is the current associated to $\tilde{\mathcal{J}}$ as in (2.7).

Proof. By applying the comparison formula (2.9) to $a : (F,\psi) \to (E,\varphi)$, and taking the wedge product with $\omega_\Omega = dz$, we get that

$$R_\Omega^{\tilde{\mathcal{J}}} = aR_\Omega^{\mathcal{I}_Z} + \nabla(M \wedge \omega_\Omega). \tag{3.7}$$

If we show that $M \wedge \omega_\Omega$ in (3.7) is the push-forward of $-V^E \wedge a\omega_Z$, then (3.6) will follow from (3.7) together with Proposition 3.1, and (3.5) follows from the fact that $\operatorname{ann}_{\mathcal{O}} R_\Omega^{\tilde{\mathcal{J}}} = \tilde{\mathcal{J}}$.

The proof that $M \wedge \omega_\Omega$ is the push-forward of $-V^E \wedge a\omega_Z$ will be rather similar to the proof of Lemma 6.2 in [**Lä3**] (which says that $M \wedge \omega_\Omega$ corresponds to a current on Z). We let

$$M_k^l = \bar\partial |G|^{2\lambda} \wedge \bar\partial\sigma_k^E \bar\partial\sigma_{k-1}^E \dots \sigma_{l+1}^E a_l \sigma_l^F \bar\partial\sigma_{l-1}^F \dots \bar\partial\sigma_1^F|_{\lambda=0}.$$

Note that by using that $\bar\partial\sigma_{j+1}\sigma_j = \sigma_{j+1}\bar\partial\sigma_j$, it follows that the current M in (2.10) is exactly $\sum_{l<k} M_k^l$. However, in the definition of M_k^l we also allow $k = l$, which we interpret as containing no σ^E's at all. The reason we allow $k = l$ is that we use it as a starting point for an inductive argument.

If $j \geq p+1$, then σ_j^E and $\bar\partial\sigma_j^E$ are smooth outside $Z_j \subseteq Z_{p+1}$, which has codimension $\geq p+1$, and since $\operatorname{codim} Z = p$, Z_{p+1} has codimension ≥ 1 in Z. As in the proof of Proposition 3.3 in [**AS1**], one sees that the restrictions of σ_j^E and $\bar\partial\sigma_j^E$ to Z are almost semi-meromorphic on Z. Hence, when $l \geq p$, we can define

$$V_k^l := i^*\bar\partial\sigma_k^E \dots i^*\bar\partial\sigma_{l+2}^E i^*\sigma_{l+1}^E \tag{3.8}$$

as a product of almost semi-meromorphic currents on Z by Proposition 2.2.

Note that if $l \geq p$, we have outside of Z_{p+1} that

$$M_k^l \wedge \omega_\Omega = -\bar\partial\sigma_k^E \dots \bar\partial\sigma_{l+2}^E \sigma_{l+1}^E a_l R_l^F \wedge \omega_\Omega = i_*(V_k^l \wedge a_l \omega_{l-p}), \tag{3.9}$$

where the minus sign in the first equality is due to $\bar\partial\sigma_k^E \dots \bar\partial\sigma_{l+2}^E\sigma_{l+1}^E$ being of odd degree and hence anti-commuting with $\bar\partial|G|^{2\lambda}$, and the second equality is due to (3.1) and (3.8).

The right-hand side of (3.9) has a unique extension as a product of almost semi-meromorphic currents by Proposition 2.2 and this extension has the SEP with respect to Z. Hence, this extension will coincide with $M_k^l \wedge \omega_\Omega$ if we can prove that $M_k^l \wedge \omega_\Omega$ also has the SEP with respect to Z. When $l < p$, we interpret the right-hand side of (3.9) as 0, and we thus also want to prove that $M_k^l = 0$ if $l < p$. We will prove both these statements, i.e., that $M_k^l = 0$ if $l < p$, and that $M_k^l \wedge \omega_\Omega$ has the SEP with respect to Z if $l \geq p$, by induction over $k - l$.

For $k = l$, M_l^l is a pseudomeromorphic $(0,l)$-current (note that M_k^l is a $(0,k-1)$-current when $k > l$, but an $(0,k)$-current when $k = l$) with support on Z, which has codimension p, so if $l < p$, then $M_l^l = 0$ by the dimension principle. For $l \geq p$, note that $M_l^l = a_l R_l^{\mathcal{I}_Z}$, so $M_l^l \wedge \omega_\Omega = i_*(a_l \omega_{l-p})$, and since ω_{l-p} is almost semi-meromorphic on Z, it has the SEP with respect to Z.

We thus now assume that $M_k^l = 0$ for $l < p$, and $M_k^l \wedge \omega_\Omega$ has the SEP with respect to Z for $l \geq p$, and we want to prove the same for M_{k+1}^l. We first consider the case $k = l+1$. Then $M_{l+1}^l = \sigma_{l+1} M_l^l$ outside of Z_{l+1}. If $l < p$, we thus get that $\operatorname{supp} M_{l+1}^l \subseteq Z_{l+1}$, and since M_{l+1}^l is a pseudomeromorphic $(0,l)$-current, we get by the dimension principle that $M_{l+1}^l = 0$. If $l \geq p$, then since $M_{l+1}^l = \sigma_{l+1} M_l^l$ outside of Z_{l+1}, and $M_l^l \wedge \omega_\Omega$ has the SEP with respect to Z, we get that $\operatorname{supp} \mathbf{1}_V M_{l+1}^l \wedge \omega_\Omega \subseteq Z_{l+1}$ if V is a subvariety of Z of codimension ≥ 1. Since $\mathbf{1}_V M_{l+1}^l \wedge \omega_\Omega$ is a pseudomeromorphic (n,l)-current, it is 0 by the dimension principle, i.e., $M_{l+1}^l \wedge \omega_\Omega$ has the SEP with respect to Z. The argument for $k \geq l+1$ follows in exactly the same way as for $k = l$, with the only change that $M_{k+1}^l = \bar\partial\sigma_{k+1} M_k^l$ instead.

Thus, we see from (3.9) that $i_*(-V^E \wedge a\omega_Z) = M \wedge \omega_\Omega$ since $M = \sum_{k>l} M_k^l$, $V^E = \sum_{k>l\geq p} V_k^l$ and $M_k^l = 0$ if $l < p$. □

We now consider some examples of this construction.

Example 3. Let $Z \subseteq \Omega$ be a Cohen-Macaulay variety, i.e., if $\operatorname{codim} Z = p$, then $\mathcal{O}_\Omega/\mathcal{I}_Z$ has a free resolution of length p. Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be an ideal with a lifting $\widehat{\mathcal{J}}$ of $\mathcal{J}$ to $\mathcal{O}_\Omega$ such that if $\operatorname{codim} Z(\mathcal{J}) = q$ in Z, then $\widehat{\mathcal{J}}$ is a Cohen-Macaulay ideal of codimension q in Ω. Note that we want to take the lifting $\widehat{\mathcal{J}}$ to be as small as possible, in contrast to Theorem 3.2, where we take the largest lifting.

One example is when $\mathcal{J} = \mathcal{J}(f_1,\dots,f_q) \subseteq \mathcal{O}_Z$ is a complete intersection ideal.

With these conditions, we can apply Proposition 2.6 to the ideals $\mathcal{I}_Z$ and $\widehat{\mathcal{J}}$, so the tensor product complex $(E \otimes F, \varphi \otimes \psi)$ is a free resolution of

$\mathcal{O}_\Omega/(\mathcal{I}_Z + \widehat{\mathcal{J}}) = \mathcal{O}_\Omega/\tilde{\mathcal{J}}$, where (E,φ) and (F,ψ) are free resolutions of $\mathcal{O}_\Omega/\widehat{\mathcal{J}}$ and $\mathcal{O}_\Omega/\mathcal{I}_Z$ respectively.

Since $i_*\omega_Z = R^F \wedge dz$ and $i_* R_Z^{\mathcal{J}} = R^{E\otimes F} \wedge dz$, we thus get by (2.12) that

$$i_* R_Z^{\mathcal{J}} = i_*((I_E - \nabla_\varphi(|G|^{2\lambda}u^E)) \wedge \omega_Z)\big|_{\lambda=0}.$$

Since Z is Cohen-Macaulay, $\omega_Z = \omega_0$, so $\nabla_\varphi(\omega_Z) = -\bar\partial\omega_Z = 0$ (note the φ, not ψ), since $\bar\partial\omega_0 = \psi_{p+1}\omega_1 = 0$. In addition, i_* is injective on currents on Z, so

$$R_Z^{\mathcal{J}} = \omega_Z - \nabla_\varphi(|G|^{2\lambda}u^E \wedge \omega_Z\big|_{\lambda=0}). \tag{3.10}$$

From (3.10), we can see that the current R_Z^f we defined in Proposition 1.2 in the introduction is the current given by Theorem 3.2. When Z is a reduced hypersurface defined by h, then $R^Z = \bar\partial(1/h)$, so the structure form ω_Z becomes just the Poincaré residue of dz/h on Z. In addition, the free resolution (E,φ) of $\mathcal{O}/\mathcal{J}(f)$ becomes just the complex $\mathcal{O} \xrightarrow{(f)} \mathcal{O}$. Hence,

$$R_Z^{\mathcal{J}(f)} = \omega_Z - (f - \bar\partial)\left(\frac{1}{f}\omega_Z\right) = \bar\partial\left(\frac{1}{f}\omega_Z\right).$$

The structure form ω_Z here plays a bit similar role as in [**AS1**]. In [**AS1**], for example $\bar\partial$-closedness for a current $T \in \mathcal{W}_Z$ is expressed as $\bar\partial(T\wedge\omega_Z) = 0$, not just $\bar\partial T = 0$. In the case of $(0,0)$-currents, $\bar\partial$-closed currents in this sense become just holomorphic functions, i.e., as expected from the smooth case, while there can exist $\bar\partial$-closed $(0,0)$-currents in the usual sense which are not holomorphic functions when Z is singular. Here, we get that the annihilator of $\bar\partial(1/f)$ might be larger than the ideal generated by f, while adding ω_Z, the annihilator of $\bar\partial((1/f)\omega_Z)$ becomes exactly f.

We finish this section with an example not covered by Example 3.

Example 4. Consider the cusp $Z = \{z^3 - w^2 = 0\} \subseteq \mathbb{C}^2$, and the maximal ideal at 0, $\mathfrak{m} = \mathcal{J}(z,w) \subseteq \mathcal{O}_Z$. Note that since $z^3 - w^2 \in \mathcal{J}(z,w)$, the maximal lifting of $\mathfrak{m}$ to $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2}$ equals $\tilde{\mathfrak{m}} = \mathcal{J}(z,w) \subseteq \mathcal{O}$. It is easily verified that the morphism $a : (F,\psi) \to (E,\varphi)$ from (3.2), where (F,ψ) and (E,φ) are free resolutions of $\mathcal{O}/\mathcal{I}_Z$ and $\mathcal{O}/\tilde{\mathfrak{m}}$, becomes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r, "\varphi_2"] & \mathcal{O}^{\oplus 2} \arrow[r, "\varphi_1"] & \mathcal{O} \arrow[r] & \mathcal{O}/\tilde{\mathfrak{m}} \arrow[r] & 0 \\ & 0 \arrow[r] & \mathcal{O} \arrow[r, "\psi_1"] \arrow[u, "a_1"] & \mathcal{O} \arrow[r] \arrow[u, "a_0"] & \mathcal{O}/\mathcal{I}_Z \arrow[r] \arrow[u, "\pi"] & 0, \end{tikzcd} \tag{3.11}$$

where

$$\varphi_2 = \begin{pmatrix} -w \\ z \end{pmatrix}, \varphi_1 = \begin{pmatrix} z & w \end{pmatrix}, \psi_1 = \begin{pmatrix} z^3 - w^2 \end{pmatrix} \text{ and } a_1 = \begin{pmatrix} z^2 \\ -w \end{pmatrix}.$$

Choosing the trivial metric on E, the minimal right-inverse σ_2 of φ_2 is $\begin{pmatrix} -\bar{w} & \bar{z} \end{pmatrix}/\left(|z|^2+|w|^2\right)$. Since Z is a reduced hypersurface defined by z^3-w^2, the structure form ω becomes $2\pi i dz/(2w)|_Z$ as in Example 2.

We let $\tau : \mathbb{C} \to Z$, $\tau(t) = (t^2, t^3)$, which is a smooth modification of Z (in fact, it is the normalization of Z). Then, one can verify that $\tau^*(\sigma_2 a_1) = -t$, and since $\tau^*(dz/(2w)) = dt^2/(2t^3) = dt/t^2$, we get that $\tau^*(V^E a\omega) = -2\pi i dt/t$. Thus,

$$V^E a\omega = -2\pi i \tau_*(dt/t)$$

(since $\tau_*\tau^* = \mathrm{Id}$ for currents with the SEP on Z, where τ is a modification). Since $\operatorname{supp} R^{\mathfrak{m}}_Z \subseteq Z(\mathfrak{m}) = \{0\}$, we get by the dimension principle that $R^{\mathfrak{m}}_Z = -\bar\partial(V^E a_1\omega)$, since the right-hand side here is the only part of $R^{\mathfrak{m}}_Z$ as defined by (3.4) of bidegree $(*,1)$ on Z. Thus,

$$R^{\mathfrak{m}}_Z = 2\pi i\bar\partial\tau_*\left(\frac{dt}{t}\right) = 2\pi i\tau_*\left(\bar\partial\left(\frac{dt}{t}\right)\right) = \tau_*((2\pi i)^2[0]) = (2\pi i)^2[0].$$

This could also have been seen directly in this case from (3.6), since

$$R^{\tilde{\mathfrak{m}}}_{\mathbb{C}^2} = \bar\partial(1/w)\wedge\bar\partial(1/z)\wedge dz\wedge dw = (2\pi i)^2[0].$$

Note that since $\tau^*(dz/(2z)) = dt/t$, we can also express this as

$$R^{\mathfrak{m}}_Z = \bar\partial\left(2\pi i \left.\frac{dz}{2z}\right|_Z\right).$$

4. The construction in the case that Z is smooth

Note the similarity of the definition of $R^{\mathcal{J}}_Z$ in (2.7) and (3.4). In fact, it is easy to see that if $Z = \Omega \subseteq \mathbb{C}^n$, then the definitions of R from (2.4) and (3.4) coincide, since then, (F,ψ) becomes just $F_0 \cong \mathcal{O}$, and $a = a_0$ is the isomorphism $a_0 : F_0 \cong \mathcal{O} \cong E_0$, $V^E = U^E$ and $\omega_Z = dz$. In fact, even more holds.

Proposition 4.1. *Let Z be a smooth subvariety of Ω. Let $\mathcal{J}$ be a Cohen-Macaulay ideal on Z. Then $R^{\mathcal{J}}_Z$ for an ideal $\mathcal{J} \subseteq \mathcal{O}_Z$ defined intrinsically on Z as in* (2.7) *as the current associated to a free resolution on Z coincides with the current defined in* (3.4).

In particular, it is motivated to use the same notation $R^{\mathcal{J}}_Z$ for both the currents defined by (2.7) and (3.4).

Proof. We assume that locally, $Z = \{w_1 = \cdots = w_m = 0\} \subseteq \mathbb{C}^n_z \times \mathbb{C}^m_w$, i.e., z are local coordinates on Z and $\mathcal{I}_Z = \mathcal{J}(w_1,\dots,w_m)$. We let R be the current $R^{\mathcal{J}}_Z$ defined by (3.4), and let R' be the current $R^{\mathcal{J}}_Z$ defined by (2.7). We let $\widehat{\mathcal{J}}$ be the ideal $\mathcal{J}$ considered as an $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n_z\times\mathbb{C}^m_w}$-module. We also let $(\widehat{E},\widehat{\varphi}) := (\pi^*E, \pi^*\varphi)$, where (E,φ) is a free resolution of $\mathcal{O}_{\mathbb{C}^n_z}/\mathcal{J}$ and $\pi : \mathbb{C}^n_z\times\mathbb{C}^m_w \to \mathbb{C}^n_z$ is the projection. Then $\widehat{\mathcal{J}}$ is also Cohen-Macaulay since $Z(\widehat{\mathcal{J}}) = Z(\mathcal{J})\times\mathbb{C}^m_w$, i.e., $\operatorname{codim}_{\mathbb{C}^n_z} Z(\mathcal{J}) = \operatorname{codim}_{\mathbb{C}^n_z\times\mathbb{C}^m_w} Z(\widehat{\mathcal{J}})$ and $(\widehat{E},\widehat{\varphi})$ is a free resolution of $\mathcal{O}_{\mathbb{C}^n_z\times\mathbb{C}^m_w}/\widehat{\mathcal{J}}$ since $\mathcal{O}_{\mathbb{C}^n_z\times\mathbb{C}^m_w}$ is a flat $\mathcal{O}_{\mathbb{C}^n_z}$-module, see [**F**], Proposition 3.17.

Let (F,ψ) be the Koszul complex of $(w_1,\dots,w_m)$, which is a free resolution of $\mathcal{O}/\mathcal{I}_Z$. The maximal lifting $\tilde{\mathcal{J}}$ of $\mathcal{J}$ equals $\widehat{\mathcal{J}}+\mathcal{I}_Z$, so by Proposition 2.6, $(\widehat{E}\otimes F,\widehat{\varphi}\otimes\psi)$ is a free resolution of $\mathcal{O}/\tilde{\mathcal{J}}$. Thus, by (3.6),

$$i_*R = R^{\widehat{E}\oplus F}\wedge dz\wedge dw,$$

and by (2.12),

$$i_*R = R^{\widehat{E}}\wedge R^F\wedge dz\wedge dw.$$

Since $\widehat{\varphi}$ only depends on z, $R^{\widehat{E}} = R^E$, and by Theorem 2.4, $R^F = \mu^w\wedge e_1\wedge\cdots\wedge e_m$, and by the Poincaré-Lelong formula, $\mu^w\wedge dw = (2\pi i)^m[w=0]$, so

$$i_*R = cR^E\wedge[w=0]\wedge e_1\wedge\cdots\wedge e_m,$$

for some non-zero constant c. Note also that $i_*R' = R^E\wedge[w=0]$, so $i_*R' = i_*R$, (up to $ce_1\wedge\cdots\wedge e_m$), i.e., $R' = R$ (up to isomorphism). □

5. Currents with prescribed annihilator ideals on arbitrary varieties

We will here consider the construction of a current $R_Z^{\mathcal{J}}$ with annihilator $\mathcal{J}$ on a variety Z as in Section 3, but without the assumption of pure dimension, i.e., Z may consist of irreducible components of different dimensions.

The construction will be essentially the same, when seen from the right viewpoint. However, treating the case of pure dimension separately should hopefully illustrate the main ideas better, without needing to delve in to certain technicalities in the general case.

To begin with, we note that on a variety which is not of pure dimension, talking about the bidegree of a current does not have any meaning, while the bidimension (i.e., the bidegree of the test forms it is acting on) still does. For example, considering the union Z of a complex line and a complex plane in $\mathbb{C}^3$, intersecting at the origin, then the integration current $[0]$ is a current on Z of bidimension $(0,0)$. However, if we consider $[0]$ as a current on the line, it would have bidegree $(1,1)$, while on the plane, it would have bidegree $(2,2)$. Note also that the bidimension of a current is preserved under push-forwards under inclusions (in contrast to the bidegree in the case of pure dimension, which increases by the codimension under push-forwards). We will thus in this section need to reformulate statements in terms of bidimension instead of bidegree of currents. For example, the dimension principle needs to be formulated in the following natural form.

Proposition 5.1. *If $T\in\mathcal{PM}(Z)$ is a current of bidimension (c,d) with support on a variety V, and $\dim V<d$, then $T=0$.*

The proof works the same as in the smooth case, by first proving that $\overline{h}T=0$ and $d\overline{h}\wedge T=0$ if h is a holomorphic function vanishing on $\operatorname{supp}T$. Then, if $i: Z\to\Omega\subseteq\mathbb{C}^n$ is a local embedding, one proves that $i_*T=0$ by induction over $\dim V$, by proving that $i_*T=0$ on V_{reg} (considered as a subvariety of Ω).

The rest from Section 2.2 about restrictions of pseudomeromorphic currents, the SEP and almost semi-meromorphic currents works the same as in the case of pure dimension, as is assumed in [**AS1**]. However, one must make sure to interpret the SEP in the right way. A pseudomeromorphic current T has the SEP with respect to Z if $\mathbf{1}_V T = 0$ for all subvarieties V of Z of positive codimension. By positive codimension, we mean that $\operatorname{codim} V \cap Z_i > \operatorname{codim} Z_i$ for all irreducible components Z_i of Z. Note however, that this is not the same as saying that $\operatorname{codim} V > \operatorname{codim} Z$, which for example any irreducible component not of maximal dimension would satisfy.

The existence of the structure form ω_Z takes the following form.

Proposition 5.2. *Let (F, ψ) be a Hermitian resolution of $\mathcal{O}_\Omega/\mathcal{I}_Z$, where Z is a subvariety of Ω of not necessarily pure dimension. Let $R^{\mathcal{I}_Z}$ be the associated residue current of (F, ψ), and let W^e be the union of the irreducible components of Z of dimension e. Then there exists an almost semi-meromorphic current*

$$\omega_Z = \omega^d + \cdots + \omega^0$$

on Z, where $\dim Z = d$, ω^e has bidimension $(0, e)$, support on $\cup_{f \geq e} W^f$ and takes values in F_{n-e}, such that

$$i_* \omega_Z = R^{\mathcal{I}_Z} \wedge dz, \tag{5.1}$$

where $i : Z \to \Omega$ is the inclusion and $dz := dz_1 \wedge \cdots \wedge dz_n$.

We can now state the main theorem also in the case when the dimension is not pure. The setting will be the same as in Section 3, with $\mathcal{J}$ an ideal in $\mathcal{O}_Z$, $\tilde{\mathcal{J}}$ a lifting of the ideal, the morphism $a : (F, \psi) \to (E, \varphi)$ between the free resolutions (E, φ) and (F, ψ) of $\mathcal{O}_\Omega/\tilde{\mathcal{J}}$ and $\mathcal{O}_\Omega/\mathcal{I}_Z$ respectively. We also let as above, W^e be the union of the irreducible components of Z of dimension e. On $W^e \setminus Z_{\text{sing}}$, define

$$\nu^e := \sum_{m \geq k \geq p+1} \sigma^E_m \bar\partial \sigma^E_{m-1} \ldots \bar\partial \sigma^E_k,$$

where $p = n - e = \operatorname{codim} W^e$. We then let

$$\nu = \nu^d + \cdots + \nu^0, \tag{5.2}$$

defined on Z_{reg} (where we extend ν^e from $W^e \setminus Z_{\text{sing}}$ to $Z \setminus Z_{\text{sing}}$ by 0).

Theorem 5.3. *Let $Z \subseteq \Omega \subseteq \mathbb{C}^n$ be an analytic subvariety of Ω of not necessarily pure dimension, where Ω is an open set in $\mathbb{C}^n$. Let $\mathcal{J} \subseteq \mathcal{O}_Z$ be an ideal. Then ν defined by* (5.2) *has an extension as an almost semi-meromorphic current to Z, which we denote by V^E. If we let*

$$R^{\mathcal{J}}_Z := a\omega_Z - \nabla(V^E \wedge a\omega_Z), \tag{5.3}$$

where $a : (F, \psi) \to (E, \varphi)$ is the morphism in (3.2), *then*

$$\operatorname{ann}_{\mathcal{O}_Z} R^{\mathcal{J}}_Z = \mathcal{J} \tag{5.4}$$

and

$$i_* R_Z^{\mathcal{J}} = R_\Omega^{\tilde{\mathcal{J}}}, \tag{5.5}$$

where $\tilde{\mathcal{J}} \subseteq \mathcal{O}_\Omega$ is the maximal lifting of $\mathcal{J}$, and $R_\Omega^{\tilde{\mathcal{J}}}$ is the current associated to $\tilde{\mathcal{J}}$ as in (2.7).

Proof. Only minor changes need to be done to the proof of Theorem 3.2 in order for it to work in this situation as well.

First of all, one defines V_k^l in (3.8) as almost semi-meromorphic currents on irreducible components of codimension $\geq l$ (which works in the same way as in (3.8) since in a smooth modification, the components of different dimension will split into disjoint manifolds), and then extend it by 0 to the irreducible components of smaller codimension.

Then, in (3.8) and the rest of the proof, ω_{l-p} is replaced by ω^{n-l}, and the equality in (3.8) will now follow from (5.1) instead of (3.1), together with the fact that $\operatorname{supp} \omega^{n-l} \subseteq \cup_{e \geq n-l} W^e$ (where we as before assume that W^e consists of the irreducible components of Z of dimension e).

Finally, the induction argument that $M_k^l = 0$ if $l < \operatorname{codim} Z$ is replaced by $\operatorname{supp} M_k^l \subseteq \cup_{e \geq n-l} W^e$. The base case for this follows from that $\operatorname{supp} \omega^{n-l} \subseteq \cup_{e \geq n-l} W^e$ by Proposition 5.2, and the induction step follows as in the proof of Theorem 3.2 by the dimension principle. □

We now turn to the proof of Proposition 5.2. Only the case of pure dimension is treated in [**AS1**]. We will essentially go through the proof of Proposition 3.3 in [**AS1**], and explain how to adapt the proof to cover also the case when the dimension is not pure. In order to keep this proof to a bit more manageable length, we split out the following lemma, which corresponds to the first step in the proof of Proposition 3.3 in [**AS1**].

Lemma 5.4. *Using the notation of Proposition 5.2, let $R' := \mathbf{1}_{W^e} R_{n-e}^{\mathcal{I}_Z} \wedge dz$. Then, there exists an almost semi-meromorphic current $\tilde{\omega}^e$ on W^e such that $j_* \tilde{\omega}^e = R'$, where $j : W^e \to \Omega$ is the inclusion.*

Proof. In Proposition 3.3 in [**AS1**], Z is assumed to have pure codimension p. A vector bundle G and a morphism $g : G \to F_p$ is defined such that $\psi_{p+1} g = 0$, and g has a minimal right-inverse σ_G, defined and smooth outside of Z_{p+1} (in the notation of [**AS1**], $g : F \to E_p$, and σ_G is denoted σ_F). We do the same construction for $p = n-e$; it is not essential for this construction that $p = \operatorname{codim} Z$ or that Z is of pure dimension.

The first step in the proof in [**AS1**] is to define ω_0 on Z_{reg}. On $W^e \setminus Z_{\mathrm{sing}}$, we define $\tilde{\omega}^e$ in the same way as ω_0 is defined in [**AS1**]; this definition on the regular part does not rely on Z being of pure dimension. By construction, $i_* \tilde{\omega}^e = R_p^{\mathcal{I}_Z} \wedge dz = R'$ on $W^e \setminus Z_{\mathrm{sing}}$. We have that R' corresponds to a current on $W^e \setminus Z_{\mathrm{sing}}$ since it is the push-forward of $\tilde{\omega}^e$ there. In fact, R' will correspond to a current on all of W^e, since if $\phi|_{Z_{\text{reg}}} = 0$, then $\operatorname{supp}\phi \wedge R' \subseteq Z_{\text{sing}} \cap W^e$, so

$$\phi \wedge R' = \mathbf{1}_{Z_{\text{sing}} \cap W^e}(\phi \wedge R') = \phi \wedge \mathbf{1}_{Z_{\text{sing}} \cap W^e} R' = 0,$$

where the last equality holds since $\operatorname{codim} Z_{\text{sing}} \cap W^e > \operatorname{codim} W^e = p$, and R' is a pseudomeromorphic (n,p)-current, so $\mathbf{1}_{Z_{\text{sing}} \cap W^e} R' = 0$ by the dimension principle. Thus, $\tilde{\omega}^e$ has an extension as a current to W^e. If we let $\vartheta = g\tilde{\omega}^e$ on $W^e \setminus Z_{\text{sing}}$, then, as in the equation following (3.19) in [**AS1**], $\bar{\partial}\vartheta = 0$ on $W^e \setminus Z_{\text{sing}}$ and $\tilde{\omega}^e = \sigma_G \vartheta$. In addition, since $\tilde{\omega}^e$ has an extension as a current to W^e, so does $\vartheta = g\tilde{\omega}^e$, since g is holomorphic (and in particular, smooth). By Example 2.8 in [**AS1**], ϑ then has a meromorphic extension to W^e.

As in the proof of Proposition 3.3 in [**AS1**], by principalization of the Fitting ideal of g, followed by a resolution of singularities, one gets a smooth modification $\tau : \tilde{Z} \to Z$ of Z such that the Fitting ideal of $\tau^* g$ is locally principal on $\tilde{Z}$. Thus, there exists a line bundle on $\tilde{Z}$ with section s_G generating this Fitting ideal. Then, $\tau^*\sigma_G = \beta_G / s_G$, where β_G is smooth. We thus get that $j^*\sigma_G$, is almost semi-meromorphic on W^e since it is smooth outside of Z_{p+1}, which has codimension $\geq p+1$. Hence, $\tilde{\omega}^e = \sigma_G \vartheta$ has an extension to W^e as a product of almost semi-meromorphic currents and this extension has the SEP with respect to W^e by Proposition 2.2. Since $i_*\tilde{\omega}^e = R'$ on $W^e \setminus Z_{\text{sing}}$, and both sides have extensions over Z_{sing}, this equality will hold on all of W^e if we show that also R' has the SEP with respect to W^e. That R' has the SEP with respect to Z follows from the dimension principle, since R' is a pseudomeromorphic (n,p)-current with support on W^e of codimension p (so $\mathbf{1}_V R'$ will be a pseudomeromorphic $(0,p)$-current with support on V of codimension ≥ 1 in W^e).

□

Proof of Proposition 5.2. For $d = \dim Z$ (i.e., the dimension of the irreducible components of maximal dimension), we define $\omega^d := \tilde{\omega}^d$, where $\tilde{\omega}^d$ is from Lemma 5.4. Since by the dimension principle, $R_p \wedge dz$ has support on W^d, $R' = \mathbf{1}_{W^d} R_p \wedge dz = R_p \wedge dz$. Thus, since $i_*\tilde{\omega}^d = R'$, we get that $i_*\omega^d = R_p \wedge dz$, ω^d is almost semi-meromorphic, and $\operatorname{supp}\omega^d \subseteq W^d$.

As in the proof of Lemma 5.4, by principalization of the Fitting ideals of φ_k for $k \geq \operatorname{codim} Z$, followed by a resolution of singularities, one gets a smooth modification $\tau : \tilde{Z} \to Z$ of Z such that all the Fitting ideals are locally principal on $\tilde{Z}$. Thus, there exists line bundles on $\tilde{Z}$ with sections s_k generating the Fitting ideals of $\tau^*\varphi_k$. Then, as in the proof of Proposition 3.3 in [**AS1**], $\tau^*\sigma_k = \beta_k / s_k$, where β_k are smooth, and $\tau^*\bar{\partial}\sigma_k = \bar{\partial}\beta_k / s_k$. We thus get that $i^*\sigma_k$ and $i^*\bar{\partial}\sigma_k$ are almost semi-meromorphic on the irreducible components of Z where they are generically defined.

We will now by backwards induction over e define ω^e, such that $i_*\omega^e = R_{n-e} \wedge dz$, ω^e is almost semi-meromorphic and $\operatorname{supp}\omega^e \subseteq \bigcup_{f \geq e} W^f$. Assume hence that this holds for ω^{e+1}, and let $p = n - e$. On $\operatorname{supp}\omega^{e+1} \subseteq$

$\bigcup_{f\geq e+1} W^f =: V$, we have that $j^*\bar\partial\sigma_p$ is almost semi-meromorphic, where $j : V \to \Omega$ is the inclusion, since it is generically defined outside of Z_p, which has dimension $\leq e$. Then, we let

$$\omega^e := \tilde\omega^e + j^*(\bar\partial\sigma_p)\omega^{e+1}.$$

Since $i_*\omega^{e+1} = R_{p-1}\wedge dz$, and $R_p = \bar\partial\sigma_p R_{p-1}$ outside of Z_p, we get that $i_*\omega^e = R^p\wedge dz$ outside of Z_p. In addition, we have that $i_*\omega^e = R^p\wedge dz$ on $W^e \setminus Z_{\text{sing}}$ by construction of $\tilde\omega^e$ and the fact that ω^{e+1} has no support there. In conclusion, $i_*\omega^e = R_p\wedge dz$ outside of $(W^e\cap Z_{\text{sing}})\cup(V\cap Z_p)$. Both sides have current extensions over this set, and ω^e being almost semi-meromorphic thus has the SEP on $W^e\cup V$. It thus remains to see that also $R_p\wedge dz$ has the SEP in order to finish the induction step. This will hold by the dimension principle, since $R^p\wedge dz$ is of bidegree (n,p), and $\dim((W^e\cap Z_{\text{sing}})\cup(V\cap Z_p)) < e$. To see this last part, we note first that $W^e\cap Z_{\text{sing}} = W^e_{\text{sing}}\cup(W^e\cap(\cup_{f\neq e}W^f))$, of which both of the sets in this union have codimension ≥ 1 in W^e. In addition, by (2.11), $\dim V\cap Z_p < e$. □

We consider an example of such a structure form. The calculation becomes rather involved, even though this is probably the simplest case of a variety which is not of pure dimension.

Example 5. Let $Z = \{x = y = 0\}\cup\{z = 0\} = Z(xz, yz)\subseteq\mathbb{C}^3$. Then $\mathcal{O}/\mathcal{I}_Z$ has a free resolution

$$0\to\mathcal{O}\xrightarrow{\varphi_2}\mathcal{O}^{\oplus 2}\xrightarrow{\varphi_1}\mathcal{O}\to\mathcal{O}/\mathcal{I}_Z,$$

where

$$\varphi_2 = \begin{pmatrix} -y \\ x\end{pmatrix} \text{ and } \varphi_1 = \begin{pmatrix} xz & yz\end{pmatrix},$$

i.e., it is like the Koszul complex of (x,y), except for the factors z of the entries in φ_1. We first compute the current R^E associated to this free resolution. Since R^E has support on Z, by the dimension principle, we get that R^E_1 has support on $\{z = 0\}$. Looking first on $\{z = 0\}\setminus\{x = 0\}$, $\mathcal{I}_Z$ is generated by z. Applying the comparison formula to (E,φ), and the free resolution (F,ψ) of $\mathcal{O}/\mathcal{J}(z)$, where $F_1\cong F_0\cong\mathcal{O}$ and $\psi_1 = z$, we get that the morphism $a : (F,\psi)\to(E,\varphi)$ becomes $a_1 = \begin{pmatrix} 1/x & 0\end{pmatrix}^t$. Since the current associated to F equals $\bar\partial(1/z)$, we get by (2.9) and (2.10) that $R^E_1 = (I_{E_1} - \varphi_2\sigma^E_2)\begin{pmatrix} 1/x & 0\end{pmatrix}^t\bar\partial(1/x)$. Using that $\sigma^E_2 = \begin{pmatrix} -\bar y & x\end{pmatrix}/(|x|^2+|y|^2)$, we get that outside of $\{x = z = 0\}$,

$$R^E_1 = \frac{1}{|x|^2+|y|^2}\begin{pmatrix}\bar x \\ \bar y\end{pmatrix}\bar\partial\frac{1}{z}. \tag{5.6}$$

By the dimension principle, this holds everywhere, since R^E_1 is a pseudomeromorphic $(0,1)$-current, and $\operatorname{codim}\{x = z = 0\} = 2$. Regarding what this means at $\{0\}$, cf., the discussion of standard extensions in Example 5 in [**Lä3**].

Outside $\{z=0\}$, then $\mathcal{I}_Z=(x,y)$, and the free resolution (E,φ) of $\mathcal{O}/\mathcal{I}_Z$ will differ from the Koszul complex of (x,y) only by the factor z in the entries φ_1. This will cause an extra factor $1/z$ in σ_1^E compared to the σ_1 associated to the Koszul complex. Since the current associated to the Koszul complex of (x,y) is $\bar\partial(1/y)\wedge\bar\partial(1/x)$, we get that

$$R_2^E=\frac{1}{z}\bar\partial\frac{1}{y}\wedge\bar\partial\frac{1}{x} \tag{5.7}$$

outside of $\{z=0\}$. On the other hand, since $Z_2^E=\{x=y=0\}$, we have outside of Z_2^E that $R_2^E=\bar\partial\sigma_2^E R_1^E$, and combining this with (5.6) and (5.7), we get that

$$\begin{aligned}R^E=&\frac{1}{|x|^2+|y|^2}\begin{pmatrix}\bar x\\ \bar y\end{pmatrix}\bar\partial\frac{1}{z}+\frac{1}{z}\bar\partial\frac{1}{y}\wedge\bar\partial\frac{1}{x}+\\ &\bar\partial\left(\frac{\begin{pmatrix}-\bar y & \bar x\end{pmatrix}}{|x|^2+|y|^2}\right)\frac{1}{|x|^2+|y|^2}\begin{pmatrix}\bar x\\ \bar y\end{pmatrix}\wedge\bar\partial\frac{1}{z}\end{aligned}$$

outside of $\{x=y=0\}\cap\{z=0\}=\{0\}$. By the dimension principle, this thus holds everywhere, since the components of R^E are of either bidegree $(0,1)$ or $(0,2)$ and $\operatorname{codim}\{0\}=3$. Taking the wedge product with $\omega_{\mathbb{C}^3}=dx\wedge dy\wedge dz$, and using that $\bar\partial(1/y)\wedge\bar\partial(1/x)\wedge dx\wedge dy=(2\pi i)^2[x=y=0]$ and $\bar\partial(1/z)\wedge dz=2\pi i[z=0]$, we get by (5.1) that

$$\begin{aligned}\omega_Z=(2\pi i)^2\chi_{\{x=y=0\}}\frac{dz}{z}+2\pi i\chi_{\{z=0\}}&\left(\frac{dx\wedge dy}{|x|^2+|y|^2}\begin{pmatrix}\bar x\\ \bar y\end{pmatrix}+\right.\\ &\left.\bar\partial\left(\frac{\begin{pmatrix}-\bar y & \bar x\end{pmatrix}}{|x|^2+|y|^2}\right)\frac{dx\wedge dy}{|x|^2+|y|^2}\begin{pmatrix}\bar x\\ \bar y\end{pmatrix}\right),\end{aligned}$$

where $\chi_{\{x=y=0\}}$ and $\chi_{\{z=0\}}$ are the characteristic functions for the respective zero sets.

6. Free resolutions on singular varieties

Given an ideal $\mathcal{J}\subseteq\mathcal{O}_Z$, where $Z\subseteq\Omega$, the construction of the current $R_Z^{\mathcal{J}}$ relied on free resolutions over $\mathcal{O}_\Omega$ of the maximal lifting $\tilde{\mathcal{J}}$ of $\mathcal{J}$. A more natural generalization of the construction in [**AW1**] would be to consider free resolutions intrinsically on Z, i.e., a free resolution of $\mathcal{O}_Z/\mathcal{J}$ over $\mathcal{O}_Z$, which (at least locally) exists also on a singular variety. We discuss in this section why this approach does not work.

One of the differences between free resolutions of ideals in the smooth and singular case is that the free resolutions need not be of finite length in the latter case, see Example 7 below for an example of this. In fact, a famous result by Auslander-Buchsbaum-Serre states that a Noetherian local ring R is regular if and only if all finitely generated R-modules have free resolutions of finite length. If $R=\mathcal{O}_{Z,z}$, then R is regular if and only if

z is a regular point of Z. However, even when the ideals do have finite free resolutions, the construction of Andersson and Wulcan will in general not have the correct annihilator. This is essentially treated in [**Lä2**], but we will elaborate a bit here how this applies to our situation. We consider first an example, where one can get an indication of what can go wrong.

Example 6. Let, as in Section 1.1, Z be a reduced hypersurface defined by a holomorphic function h, and let f be a non-zero-divisor in $\mathcal{O}_Z$. Note that f being a non-zero-divisor means precisely that the complex $\mathcal{O}_Z \stackrel{(f)}{\to} \mathcal{O}_Z$ is a free resolution of $\mathcal{O}_Z/\mathcal{J}(f)$. Hence, the current associated to this free resolution is the residue current $\bar\partial(1/f)$. Consider the push-forward of $\bar\partial(1/f)$ to the ambient space, $i_*\bar\partial(1/f) = \bar\partial(1/\tilde f) \wedge [Z]$, where $\tilde f$ is a representative of f in Ω. By the Poincaré-Lelong formula, see [**CH**], Section 3.6,

$$\bar\partial\frac{1}{\tilde f} \wedge [Z] = \frac{1}{2\pi i}\bar\partial\frac{1}{\tilde f} \wedge \bar\partial\frac{1}{h} \wedge dh.$$

Now, if $\phi\bar\partial(1/\tilde f) \wedge [Z] = 0$, then the coefficients of ϕdh lie in $\mathcal{J}(\tilde f, h)$ by the duality theorem. However, since dh vanishes on Z_{sing}, this does not necessarily imply that $\phi \in \mathcal{J}(\tilde f, h)$. Indeed, we show in [**Lä2**] that if $\operatorname{codim} Z_{\text{sing}} = 1$, then one can find ϕ and f such that $\phi dh \in \mathcal{J}(\tilde f, h)$ but $\phi \notin \mathcal{J}(\tilde f, h)$. In that case, we thus get that $\operatorname{ann}\bar\partial(1/f) \neq \mathcal{J}(f)$.

We now turn to the general case. Consider a singular subvariety $Z \subseteq \Omega$ of codimension p. Let $Z^0 := Z_{\text{sing}}$ and $Z^k := Z_{k+p}$ for $k \geq 1$, where Z_{k+p} are the singularity subvarieties associated to a free resolution of $\mathcal{O}_Z$. Let q be the largest integer such that $\operatorname{codim} Z^k \geq k + q$ (since Z is assumed to be singular, $Z^0 = Z_{\text{sing}} \neq \emptyset$, and hence, $q \leq \dim Z$). By Corollary 1.6 in [**Lä2**] there exists a complete intersection $f = (f_1, \dots, f_q)$ on Z such that $\operatorname{ann}\mu^f \neq \mathcal{J}(f)$. By Theorem 2.4, μ^f equals the Bochner-Martinelli current of f, i.e., the current associated to the Koszul complex of f. We claim that in this case, the Koszul complex of f is a free resolution of $\mathcal{J}(f)$, and hence what we described above show that the naive generalization of the construction by Andersson and Wulcan does not work in this case. To see that the Koszul complex of f is exact, we note first that by Theorem 1.3 in [**Lä2**], if $f' = (f_1, \dots, f_{q'})$, where $q' < q$, then $\operatorname{ann}\mu^{f'} = \mathcal{J}(f')$, and by Lemma 7.5 in [**Lä2**], $(f_1, \dots, f_q)$ is then a regular sequence. By [**E**], Corollary 17.5, the Koszul complex of f is then a free resolution of $\mathcal{O}_Z/\mathcal{J}(f)$.

We saw however in Example 3 that if Z is Cohen-Macaulay, there was an easy remedy for this, we should consider $R = \omega - \nabla(U^f\omega)$ instead of $I - \nabla(U^f)$. If Z is not Cohen-Macaulay, or if we have an ideal which does not lift to a Cohen-Macaulay ideal, it is not as clear how to remedy this.

We consider also another issue arising when the free resolutions on the variety are not of finite length.

Example 7. Let $Z = \{x = 0\} \cup \{y = 0\} = \{xy = 0\} \subseteq \mathbb{C}^2$. Consider the ideal $\mathcal{J} = \mathcal{J}(x) \subseteq \mathcal{O}_Z$. It is easily verified that if $E_k \cong \mathcal{O}_Z$, $\varphi_{2k+1} = (x)$, $\varphi_{2k+2} = (y)$, $k = 0, 1, \cdots$, then (E, φ) is a free resolution of $\mathcal{O}_Z/\mathcal{J}(x)$ over $\mathcal{O}_Z$. In addition, since $x \in \mathfrak{m}$ and $y \in \mathfrak{m}$, where $\mathfrak{m} := \mathcal{J}(x,y)$ is the maximal ideal in $\mathcal{O}_{Z,0}$, we have that (E, φ) is a minimal free resolution over the local ring $\mathcal{O}_{Z,0}$, see [**E**], Theorem 20.2. This theorem about uniqueness of minimal free resolutions holds over any Noetherian local ring, without any requirements about regularity of the ring, so since (E, φ) is one minimal free resolution of $\mathcal{O}_Z/\mathcal{J}$ over $\mathcal{O}_Z$ of infinite length, any other free resolution must also be of infinite length.

We now consider the sets Z_k^E, where φ_k does not have minimal rank. They are $Z_{2k+1}^E = \{x = 0\}$ and $Z_{2k+2}^E = \{y = 0\}$, $k = 0, 1, \ldots$. Note that $\operatorname{codim} Z_k^E = 0$ and $Z_{2k+2}^E \not\subseteq Z(\mathcal{J})$. This shows that the Buchsbaum-Eisenbud criterion and its corollaries, as described in Section 2.5, fail. The reason for this is not directly that the ring we consider is not regular, the Buchsbaum-Eisenbud criterion holds on any Noetherian local ring. However, the criterion does not apply here, since one requirement is that the complex is of finite length. Since much of the construction of Andersson-Wulcan currents relies on the Buchsbaum-Eisenbud criterion and its corollaries, this would be an obstacle to overcome in order to construct such currents directly from free resolutions on the variety, without going to a lifting of the ideal as we do in this article.

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