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Holographic Superconductivity

Effective Field Theoretic Approach to Layered Superconductors

Thesis for the degree Master of Science in Fundamental Physics

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Abstract

In this thesis we apply the AdS/CFT conjecture to condensed matter physics and more specifically we consider the application to layered superconductors. We start by analysing the "ordinary" 2+1 dimensional holographic superconductor where we only have a scalar field coupled to an Einstein-Maxwell theory in the bulk. We then proceed to add first order corrections to the theory by higher derivative terms in the action. This is initially done by adding a Weyl correction which allows us to interpolate between vortex- and quasi particle excitations in the superconductor. We then generalise this to adding all possible first order correction terms to the theory, this amounts to adding non-linear Maxwell terms to the bulk Lagrangian. The stability of the theory is considered and we also explore the parameter space and in particular we find that we are able to tune the energy gap $\frac{2\Delta}{T_c}$, which in the weakly coupled BCS-case is ≈ 3.5. The range of the values for the energy gap we find matches nicely with the experimentally obtained range of energy gaps for high-T_c cuprates. By tuning the coupling strength of the non-linear Maxwell terms we find that we can produce Drude behaviour at low frequencies.

Keywords: AdS/CFT, AdS/CMP, Holography, Superconductivity, Black holes, Weyl corrections, Energy gap

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Contents

1 Introduction

1.1 AdS/CFT Correspondence

The concept of strongly coupled quantum field theories have since the beginning of the century been a thorn in the side of physicists. Despite the fact that Feynman diagrams and other perturbative techniques manages to do extraordinary well in some areas such as the electroweak part of the Standard Model and Fermi Liquids in condensed matter there has always been the nagging question: what if the coupling is not small enough to give accurate results with perturbative methods? The fact that one manages to get as far as we have in the area of quantum field theory reflects perhaps the "exoticness" of the strongly coupled phenomena but there is also no doubt that strongly coupled phenomena do play a crucial role in nature. Examples are the confinement in the $SU(3)$ part of the Standard Model determining the behaviour of fundamental constituents of matter and also more recently the discovery of high-T_c superconductors, some of which are hypothesised to be strongly coupled. In this thesis we explore a new approach to strongly coupled quantum field theories known as the AdS/CFT (Anti-de Sitter[1]/Conformal Field Theory) conjecture, first discovered in [1], which tells us that strongly coupled quantum field theories in $D-1$ dimensions have a weakly coupled gravitational dual theory in D dimensions. The AdS/CFT correspondence is often called a gauge-gravity duality, a duality in the sense that it describes the same physics with different theories which we can relate to each other. The specific case of AdS/CFT states that a D-dimensional theory with gravity in AdS-space describes the same physics as a $(D-1)$-dimensional quantum field theory that exists on the boundary of the D-dimensional space. At a first glance this statement seems very strange because translated into everyday physics it seems to mean that everything that goes on *inside* a box can be described by what happens *on the boundary* of the box. This notion has been termed *holography* because it is closely related to a hologram in

[1]Anti-de Sitter geometry is a maximally symmetric solution to Einsteins vacuum equations and have constant negative curvature, for a more precise definition see section 2.3.

which 3-dimensional images are encoded on 2-dimensional surfaces. The extra dimension in the gravity theory can be understood as the energy scale, in close analogy with the RG, of the field theory on the boundary. That this idea of holography is not as far fetched as one may initially think can be seen by observing the Bekenstein-Hawking formula for the entropy of a black hole:

$$S_{BH} = \frac{k_B A}{4l_p^2},$$

where A is the area of the event horizon, k_B is the Boltzmann constant and l_P is the Planck length. This is the highest amount of entropy any object of a given size can have and the fact that it is proportional to the *area* of the black hole and not the volume is striking. An interpretation of this is that the horizon of the black hole is divided into pixels, each of size of the Planck area, and that each such pixel has a certain number of degrees of freedom. Now, if the most entropic objects can be described by d.o.f. on its boundary why not also less entropic objects? All objects might thus be describable by d.o.f. on their boundary and we have arrived at something which is very similar in nature to the AdS/CFT correspondence. The box analogy above was not strictly correct from an AdS/CFT point of view since our universe is believed to be of de Sitter geometry and not anti-de Sitter as required by this correspondence. However, the Bekenstein-Hawking formula for the entropy of a black hole seems to hint at a more general duality for general geometries of space.

Alltough all the implications of the AdS/CFT correspondence are hard to fully grasp, the basic notion of a duality between theories of different dimensionality is not as complicated. In figure 1, a 2 dimensional bulk with gravity and some interacting fields is dual to a 1 dimensional conformal field theory on the boundary. This is actually how we will always draw our pictures of the duality, no matter how many dimensions are involved. The radial direction is the only one which we depict geometrically[2], the other dimensions have more subtle effects.

[2]There is of course one more coordinate than r needed to make up the area in figure 1 but we do not specify which one it is since it does not matter.

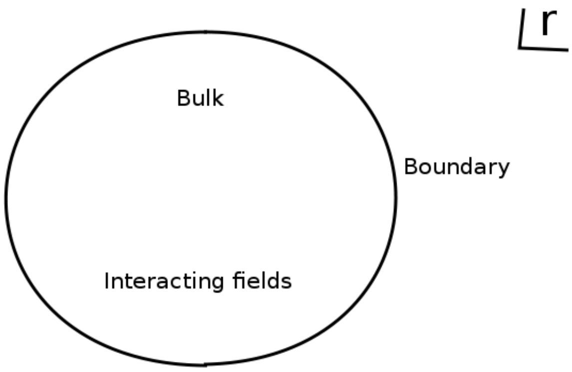

Figure 1: A visualization of the content of an AdS/CFT theory. The distance from the center of the circle depicts the radial coordinate. The CFT can be thought of as existing on the boundary, the fields in the bulk include the metric so it is clear that it is a theory with gravity.

The real statement of the AdS/CFT correspondence is that the bulk geometry only needs to be *asymptotically AdS*. As long as the geometry near the CFT on the boundary is "nearly of AdS type" the correspondence holds. This means that we can perturb the metric in the bulk far away from the boundary without breaking the assumed validity of our conjecture. We say assumed validity as it is worth stressing that the *AdS/CFT correspondence is not a proven theorem*. It is merely a conjecture which is plausible seen from the right viewpoint. It should also be pointed out that even if correct, *the AdS/CFT correspondence is in its original form only valid within a certain approximation called the Large-N limit*. This constrains the class of CFT's which can be used within the correspondence but we often choose to make an (uncontrolled) approximation that a CFT with low or zero N can be described by some bulk dual. We shall point out that it is not always clear what the interpretation of N is in the field theory, thus naturally making it difficult to know its value. The idea is to see if it is possible to get some interesting physics from this underlying bulk dual which could lead to "universal" behaviour in the field theory such as superconductivity, Fermi surfaces etc. We shall see that it is.

Two notes on the work done in this thesis.

1. The work consisted to a large extent of solving differential equations using Mathematica. The created notebooks are available upon request[1].

2. This thesis work resulted in two papers that are currently in preparation. The first paper [2] deals with the energy gap in holographic theories and is an extension of section 4.3. In the second paper [3] we take hold on the fact that we seem to be able to model Drude behaviour in the conductivity using a non linear correction term to our theory. This is somewhat surprising since Drude behaviour arises from scattering and that the non linear term encodes that is far from obvious.

1.2 Superconductivity

Superconductivity was first discovered 1911 and is characterized by zero DC-resistivity and the expulsion of magnetic fields from the superconducting bulk, the so called Meissner effect. The fact that it took until the beginning of the 20th century to discover this state of matter is due to the fact that superconductivity only sets in at very low temperatures. The temperature at which superconductivity sets in is called the critical temperature, denoted T_c, and these early experiments managed to find critical temperatures for different elements roughly in the range $0 - 20$ K. Alltough there appeared some purely phenomenological models, such as the London equations, describing different aspects of superconductivity, a microscopic explanation was not developed until 1957. This year Bardeen, Cooper and Schrieffer published what has later been named BCS theory after the three authors. The basis of BCS theory is that at low enough temperatures (below T_c) the electrons which normally repel each other suddenly start to feel a net attractive force towards each other due to their interactions with the lattice ions in the material bulk. This will result in bound quasi particle states known as Cooper pairs, which are

[1] *wenger at student.chalmers.se* or *wenger at chalmers.se*

nothing more than electrons bound together by phonons propagating through the lattice. Using this theory one can explain the properties of these early discovered superconductors, they are therefore known as BCS superconductors. These superconductors are characterized by the fact that the mechanism behind the superconductivity is the formation of Cooper pairs via phonon interaction.

BCS theory of Cooper pairs describes all the phenomena of the BCS-superconductors but in 1986 the high-T_c superconductors were discovered and as of now one has reached critical temperatures of about 130K. These high-T_c materials have some properties that cannot be explained by BCS theory and the question arises; what is the mechanism behind high-T_c superconductivity? There is a range of proposed possibilities but they can be classified into two main categories. One category is very similar to BCS theory but the difference is that the attractive force between the electrons is mediated by something other than phonons; spin attraction is one example of such a theory. The other main class is that the attraction is not weak as it is in BCS theory but that this is a strongly-coupled phenomena. It is important to remember that for strongly coupled systems the ordinary method of perturbation theory is not applicable and we really have no good methods of doing calculations for such a theory. It is here that AdS/CFT conjecture comes into play; the hopes are that using this duality one can get new insights into this state of matter because we can calculate the behaviour of a strongly coupled theory. This is the real strength of the AdS/CFT conjecture in general, if correct it gives us a tool capable of doing calculations that are impossible to do with other methods. To be able to compare the behaviour of our AdS/CFT theories for superconductors with something we need to find some general prediction(s) of BCS theory and then look for similarities and differences with our new model.

A very general result obtained from BCS theory is the energy required to break the binding between the Cooper pairs as a function of temperature. This result for $T \approx T_c$ is

$$\frac{\Delta}{T_c} = 3.5\sqrt{1 - \frac{T}{T_c}}, \tag{1}$$

where Δ is the minimum breaking energy called the *energy gap* and $k_B = 1$, i.e. energy and temperature have the same units. This has a very characteristic behaviour as it drops off near T_c. The full behaviour of the energy gap for $T < T_c$ is plotted in figure 2 taken from [4].

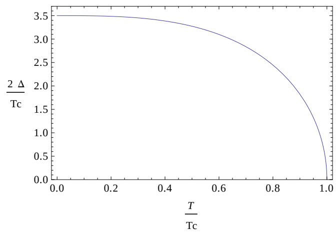

Figure 2: The full behaviour for the BCS energy gap.

1.3 AdS/CMP and Superconductors

The field of science that applies the AdS/CFT correspondence to condensed matter physics is usually called AdS/CMP for brevity. As we mentioned above one of the things that we may wish to model in AdS/CMP is superconductors and more specifically high-T_c superconductors. The main reason being that since their discovery in 1986, high-T_c superconductors have resisted a theoretical microscopic description. BCS theory that describes the "usual" superconductors seems to be inadequate to describe these systems and the hope is that AdS/CMP will somehow provide a new theoretical approach on this problem. There are indications that this is the case but the accuracy of the AdS/CFT correspondence at present makes it hard to make predictions and even to make comparisons with experiment beyond a qualitative level.

What we have to do is to take a "top-down" approach led by simplicity and calculability. We will make some approximations and the end goal is to find a simple bulk dual that allows us to calculate the behaviour of the boundary field theory and show that it behaves as a system with a phase transition like a superconductor. The main idea behind all this is not to try to fit precisely experimental data by putting our parameters at precise values but rather to get results that show us that AdS/CFT could be the right track to pursue if one wants to further investigate systems currently out of theoretical reach. Universality, the observation that many different systems are described by the same underlying theory, is a central piece of the larger picture. The hope is that high-T_c superconductors are in the same universality class, i.e. have the same underlying dynamics, as our theoretical model. Layered high-T_c superconductors are but one example of a system one may focus on describing and the one we will focus on here.

2 Theory

2.1 Graphene

In recent years since the creation of graphene, in [5], it has become intensely studied. This is in part due to its many possible applications in technology but it is also interesting for purely scientific reasons. Graphene exhibits many strange properties because it is $2+1$-dimensional, this constrains the electron movement and also makes it hard to make predictions using ordinary methods of condensed matter physics. The reason we are interested is closely tied to universality and dimensionality; since we are hoping to capture the behaviour of effectively $2+1$ dimensional materials we might hope to capture some of the physics of truly $2+1$ dimensional materials as well.

It can be shown, e.g. [6] pp. 57, that due to its hexagonal structure graphene has a cone-like band structure as depicted in figure 3. The transitions that are possible within the graphene band structure are shown in figure 3. Here we assume that the transitions happen with $\mathbf{k} = 0$, i.e. there is no change in momentum between the initial and final states. This will be a very convenient approximation when we later do our calculations but it is also a reasonable assumption to make since we want to work in the experimentally accessible regime.

The line labeled E_F in figure 3 is called a *Fermi surface* which is the highest occupied "energy-surface" at $T = 0$. The reason a collection of fermions at $T = 0$ have a Fermi surface is that they can not all be in lowest lying state since they have to obey the Pauli exclusion principle. A collection of bosons do not exhibit a Fermi surface.

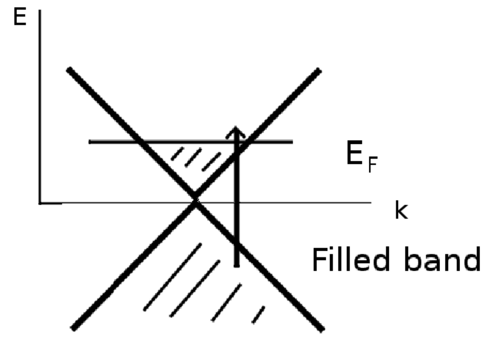

Figure 3: A transition in graphene from one filled band at the bottom of the figure to a more energetic lowly populated band. In experiments the transitions that can occur are those with very small $|\mathbf{k}|$ as depicted above by the arrow.

2.2 Coleman-Mermin-Wagner Theorem

Since we will work in i field theory that has $2 + 1$ dimensions we are forced to somehow discuss the Coleman-Mermin-Wagner (CMW) theorem. The CMW theorem states (among other things) that in $2 + 1$ dimensions it is not possible to have a spontaneous symmetry breaking since this would lead to divergent behaviour of certain d.o.f. that arise (the Goldstone bosons). This is important to us because we are constructing an effective field theory for layered materials that we claim can be thought of as effectively $2 + 1$ dimensional. We will later see that we actually get a spontaneous symmetry breaking in our calculations, how to understand this? In our model we are working in the large N limit which in some sense suppress quantum effects and the divergent behaviour forbidding the symmetry breaking is a quantum effect. It is thus not so strange that we get the breaking in our calculations but how can we then hope to describe a system in which this symmetry breaking is forbidden? In the case of the layered materials it is not so difficult to think of a way out; we know for a fact that the materials are not really $2+1$ dimensional but we are only treating them within an effective description as such, thus there is no problem interpreting this as layered high-T_c superconductors.

If we wish to describe graphene also this is a bit more tricky, true sheets of graphene are in fact "as $2+1$ dimensional" you can get in a $3+1$ dimensional world. What we can hope for is that if we manage to see that our theory with spontaneous symmetry breaking captures some physics of graphene then in this case universality is in some sense stronger than the CMW theorem. Even systems with different symmetry breakings can be described by the same theory as long as we restrict ourselves to some other observables and not the symmetry breaking itself. What we expect, since a spontaneous symmetry breaking is forbidden in $2+1$ dimensions, is that when we turn on finite N effects the symmetry breaking will change from spontaneous to infinite order breaking, which is not forbidden by the CMW theorem.

Concluding, the CMW theorem should not affect our ability to model layered superconductors but in describing graphene we could run into some problems which makes the graphene case hard to interpret. In this case we will have to let the results alone guide us and appeal to universality.

2.3 Anti-de Sitter Space and Conformal Boundary

Anti de Sitter space in D dimensions (AdS$_D$, meaning $D-1$ spatial and one time direction) is a space time with constant negative curvature and its metric can be written as

$$ds^2 = -r^2 dt^2 + \frac{1}{r^2} dr^2 + r^2 dx_i^2 \tag{2}$$

where we have put the (fixed) AdS background radius equal to one (we can think of this as the inverse of the curvature of the space) and

$r \geq 1$ is the radius. We directly see that if we let $r \to \infty$ the metric effectively "pinches off" the radial direction of the space and we are left with

$$ds^2 = r^2\left(-dt^2 + dx_i^2\right). \tag{3}$$

If we now introduce a so called conformal scaling of the metric so that $ds^2 \to \Omega^2 ds^2$ where we choose $\Omega = \frac{1}{r}$ we get

$$ds^2 = -dt^2 + dx_i^2 \tag{4}$$

which is the Minkowski metric in one dimension less than we started out with since the radial direction vanished. Since the Minkowski metric thus in some sense was obtained by going to the boundary, i.e. r very large, and then doing a conformal scaling we call it the conformal boundary of AdS space. *The conformal boundary of AdS_D is $Mink_{D-1}$.*

Empty AdS space is not very interesting but the AdS/CFT correspondence states that we need only restrict ourselves to geometries that are *asymptotically AdS*, i.e. near the boundary the space is AdS but inside the bulk we can have deviations from this. If we introduce a black hole the theory acquires an entropy and a temperature due to the horizon of the black hole. The metric of a Schwarzschild AdS black hole is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2) \tag{5}$$

where

$$f(r) = r^2 - \frac{M}{r} \tag{6}$$

where M in the metric is related to the mass of the black hole, and at $f(r) = 0$ we see the Schwarzschild radius (for AdS space) $\rho = M^{1/3}$ which is the event horizon for the black hole, we will simply call it the horizon. We know that near the boundary our space is asymptotically AdS$_D$, but how about near the horizon? We state here without proof the non-obvious fact that in a $3+1$ dimensional bulk *as long as there is an event horizon the near horizon geometry is always $AdS_2 \times \mathbf{R^2}$.* That the near horizon geometry always contains an AdS$_2$ (one time-and one spatial direction) is indeed a strange effect and it will have important consequences later on.

Since our theory only contains one energy scale, the temperature of the black hole, the only temperatures we can distinguish between are zero and non-zero because we can always do a rescaling of finite temperatures. To be able to talk about different finite temperatures we need to introduce yet another energy scale, this can be done by introducing scalar fields coupled to a Maxwell field in the bulk and this is the focus of section 3.1. Another way to introduce a different energy scale is to allow the black hole to have a charge i.e. a Reissner-Nordstrom black hole but this is more complicated and is not covered here. We do note however, that when working in the limit $G_N \to 0$, i.e. when the gravitational dynamics are neglected, the AdS Reissner-Nordstrom black hole metric coincide with the AdS Schwarzschild black hole metric. We can thus state that in this limit the Reissner-Nordstrom black hole and the Schwarzschild black hole are degenerate i.e. indistinguishable.

To get a better notion of what we are doing we want to visualise what our theory contains. In order to visualize an infinite space of several dimension we make a projection of our space onto a Poincarè disc as shown in figure 4. We will sometimes find it convenient to work in a coordinate different from r, this new coordinate we define as $z = \frac{L}{r}$ and the picture then becomes as in figure 5 instead. The z coordinate is very convenient for numerical purposes since we then work in a finite coordinate interval $z \in [0, 1]$.

2.4 Conformal Field Theories

Conformal field theories (CFT's) are quantum field theories that are invariant under the action of conformal transformations. The conformal transformations consist of rotations, translations, time translations, special conformal transformations and dilatations. We focus now on the dilatations which are scale transformations where we simultaneously scale both time (energy) and space (momentum). In

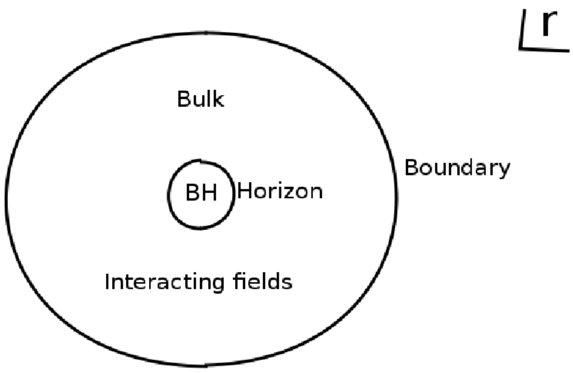

Figure 4: A visualization of the content of an AdS/CFT theory with a black hole.The distance from the center of the circles depict the radial direction.

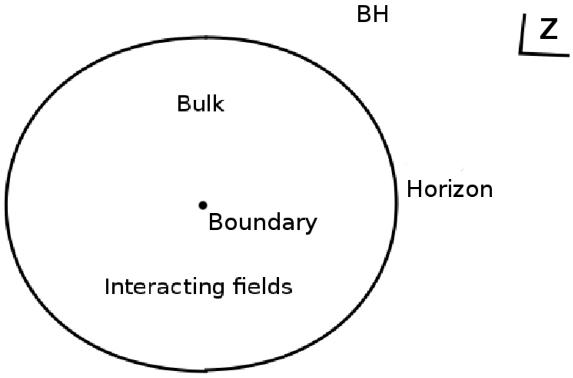

Figure 5: A visualization of the content of an AdS/CFT theory with a black hole shown in the z-coordinate. The distance from the center now depicts the z coordinate. Notice that the boundary and the horizon have "swapped places" compared to figure 4 and that the boundary is now a single point at the origin. The inside of the black hole is "outside the horizon", indicated by the label BH.

general this scaling can be of the form

$$t \to \lambda^z t, \quad \mathbf{x} \to \lambda \mathbf{x} \tag{7}$$

but in this thesis we will only consider cases where the dynamical critical exponent $z = 1$.[1] We note here that alltough there exist QFT's that are scale invariant and not conformally invariant we will use these terms as equivalent.

[1]This dynamical critical exponent z has nothing to do with the coordinate $z = \frac{L}{r}$ defined above, this is just the ordinary naming convention.

A CFT contains operators and we say that they have a scaling dimension Δ if they satisfy

$$\mathcal{O}(\mathbf{x}) \to \lambda^{\Delta}\mathcal{O}(\mathbf{x}) \tag{8}$$

when we apply the transformations (7). Note that the theory *does not look the same at all energy scales i.e. it is not self-similar* as one might be led to conclude by the name scale invariant, but in general only scales in a very specific way. The reason for the name will become clear in a moment. A given scaling dimension Δ for an operator determines how the operator in question behaves when we go between different energy scales of the theory. We see from equation (7) that by applying a $\lambda > 1$ we go to shorter distances i.e. higher energies and if $\lambda < 1$ we go to longer distances i.e. lower energies. This may seem confusing but here the scaling is done such that the argument of the operators is not changed i.e. $\lambda\mathbf{x}$ is kept fixed while scaling λ.[1] A very convenient way to study the behaviour of a CFT is to study it in an effective formulation at low energies that constitutes a "laboratory like" environment. E.g. in the theory of solids we could use full QED but if we are only interested in the low energy phonons of the theory we could scale QED down to the interesting energy scale and get an effective theory for the energies in question. We note that we can not hope to gain full insight into higher energy d.o.f. by scaling to higher and higher energies. One can only go down in energies to obtain a full theory and we say that a CFT flows to a conformal or infrared (IR) fixed point at low energies. These fixed points are points where subsequent scaling does not further change the theory at all. This means that *at these fixed points the theory is self-similar i.e. looks the same at all scales*[2] and it is these conformal fixed points that we refer to when we say that CFT's are scale invariant. The theory may also contain an ultra-violet (UV)-fix point at high energies where going to higher energies does not change the theory.

One important piece of nomenclature will be important later on. We see in equation (8) that going to lower energies (remember $\lambda < 1$)

[1] It might be a good idea to consider the difference between passive and active transformations.

[2] This is only true if the scaling is done "in the right direction" i.e. down in energy at the IR fix point and up in energy at the UV fix point.

can either: i) make the operator more important in the case $\Delta < 0$, ii) make the operator less important in the case $\Delta > 0$ or iii) the importance of the operator stays the same in the case $\Delta = 0$. These cases are called relevant, irrelevant and marginal operators respectively. For us where we are interested in the low energy behaviour of the theory the most important operators are thus the relevant operators because they grow in importance when we go down in energy.

2.5 Motivation of the AdS/CFT Correspondence

The AdS/CFT correspondence is a rather non-intuitive duality since it involves theories of different dimensionality. The following will be a brief heuristic explanation of how the duality arises from a string theory setting.

We start off by making a very important remark about the nature of the duality. We should not expect to be able to map the theories into each other by mapping the microscopic degrees of freedom onto one another. This stems from the fact that the d.o.f. in the bulk are vibrating strings and on the field theory side we expect the theory to arise from some Hamiltonian of a lattice where the d.o.f. might be (quasi) particles and the lattice vibrations. From the very construction of the duality we can not hope to obtain the underlying microscopic theory that describes the field theory but only the resulting observables.

We now consider type IIB superstring theory which contains as fundamental objects D_p-branes together with closed and open strings where the open strings must end on the D_p-branes. In the present setting the important parameters in the theory are the string coupling g and the number of D_3-branes N; we will consider a stack of N D_3-branes on top of each other in flat 10-dimensional Minkowski space. The important product in this setting is gN, i.e. the string coupling times the number of D_3-branes. The main idea in the rest of this section is to examine how this setting behaves at low energies in the limits of $gN << 1$ and $gN >> 1$ and then relating the two cases to each other which will give rise to the duality.

$gN << 1$ case

In this case the gravitational effects in the system are negligible since the string coupling is proportional to Newtons constant which in turn sets the gravitational strength. In this limit also the fields on the branes decouple from the closed strings not on the branes. In a setting of N D_3 branes the symmetry group of the fields is $U(N)$ but it is also a fact that the center of mass motion of the branes, which is not important, corresponds to $U(1)$ and $U(N)/U(1) \simeq SU(N)$ so we are left with an $SU(N)$ Super Yang-Mills (SYM) theory in the $3+1$ dimensions for the fields on the branes.

$SU(N)$ *SYM + closed strings in* 10 *dimensions*

$gN >> 1$ case

In this limit gravity is the dominant force and the low energy solution turns out to be $\mathrm{AdS}_5 \times S^5$ i.e. five dimensional Anti-de Sitter space times a five dimensional sphere. This solution can be thought of as a throat opening up in the flat space and in the end of the throat we have $\mathrm{AdS}_5 \times S^5$ space time, this is illustrated nicely in [7] pp. 539. Far away from this throat we still have closed strings that, if we still consider only low energies, are decoupled from the physics in the throat region. This is because at low energies the physical states in the throat can not overcome the gravitational potential and are thus confined to the throat region.

General relativity on $AdS_5 \times S^5$ *+ closed strings in* 10 *dimensions*

Conclusion

We have now taken our original brane setting to two different extremes and seen how to approximate the system there. It is possible to show that the decoupling between the free closed strings and the rest of the system is actually valid for all values of gN, the interpretation then is that the closed strings are mapped to the closed strings and the SYM is mapped to the AdS system when we dial gN from very small to very large. Since we nowhere altered the theory both descriptions should be valid and describe the same physics at low enough energies. If we tune the gN parameter we can go from one of the limiting cases to the other thus tracing a curve in moduli space that connects the two. Thus the same physics could be described by both theories but at different coupling strengths gN. Finally, it is possible to relate the parameters of the two theories and one can see that if we have a strongly coupled quantum field theory then we have a weakly coupled string theory description in AdS space which makes Einstein gravity a good approximation. A natural question to ask is; how general is this duality? Is it possible to relate other geometries, such as for instance AdS_4, to other sorts of gauge theories and can they be of a non-supersymmetric sort? We will proceed in this manner; having a bulk of AdS_4 and relate it to a guage theory with gauge group $U(1)$, i.e. the electromagnetic gauge group.

2.6 Applying the AdS/CFT Correspondence

We now start to make explicit what we really mean by the AdS/CFT duality. The duality states that the theory in the bulk should be type IIB superstring theory and the dual boundary theory should be a supersymmetric SU(N) gauge theory. In the spirit of ”top-down” physics we will break both of these requirements to be able to do calculations. In a low energy limit we know that type IIB superstring theory reduces to ordinary Einsteinian gravity (general relativity) and it is reasonable to assume that the duality exists even at low energies. As for the supersymmetry of the boundary theory we can assume that the supersymmetric part of the theory somehow decouple or is broken and affects the theory very little. A fact we do know is that if we hope to describe condensed matter (or a quark gluon plasma at "low" energies) the boundary theory should not have supersymmetry, at least not unbroken since this would have been experimentally detected. Thus we conclude that at least to some approximation we should be able to cope without SUSY in the boundary theory. As stated in section 2.5 we will assume that the validity of the correspondence extends beyond an $SU(N)$ SYM gauge theory and beyond $\mathrm{AdS}_5 \times S^5$.

A natural way to relate these two dual theories is to use the partition function of each theory and relate the fields in the bulk as sources to the operators e.g.

$$Z_{\text{bulk}}[\psi \to \delta\psi_{(0)}] = \langle \exp\left(i \int d^d x \sqrt{-g_{(0)}} \delta\psi_{(0)} \mathcal{O}\right)\rangle_{F.T.} \tag{9}$$

and

$$Z_{\text{bulk}}[A_\mu \to \delta A_{(0)\mu}] = \langle \exp\left(i \int d^d x \sqrt{-g_{(0)}} \delta A_{(0)\mu} J^\mu\right)\rangle_{F.T.} \tag{10}$$

and so forth for tensor fields with more indices. *We make an important note that d denotes the total number of dimensions on the boundary i.e. $d = D - 1$, also ψ is a scalar field.* A subscript with parentheses e.g. (0) means that the corresponding field lives on the boundary. Equations (9) and (10) tell us that when the boundary values of the bulk fields (e.g. $\psi_{(0)}$ and $A_{(0)\mu}$) are perturbed they source operators, $\mathcal{O}$ and J^μ, in the field theory on the boundary. It can be shown [8] that near the boundary (z=0) of AdS space a scalar field goes as

$$\psi(z) = \left(\frac{z}{L}\right)^{d-\Delta} \psi_{(0)} + \left(\frac{z}{L}\right)^{\Delta} \psi_{(1)} \tag{11}$$

and this is now where we make contact with the CFT from section 2.4. As we mentioned earlier, operators will be sourced by the scalar field and if we let $\psi_{(0)}$ be the source of the operator then by using a semiclassical limit one can show that the expectation value of the operator is given by

$$\langle \mathcal{O} \rangle = \frac{2\Delta - d}{L} \psi_{(1)} \tag{12}$$

and the scaling (conformal) dimension of the operator we see from the $\psi_{(1)}$ term in equation (11),

$$\dim[\mathcal{O}] = \Delta. \tag{13}$$

In the same way one can show that by choosing $\psi_{(1)}$ as the source then $\psi_{(0)}$ would give the expectation value and the conformal dimension of the operator would be $d - \Delta$. In the usual way when introducing a source we set the source term to zero after we have done the identifications, this means we need to find solutions to the equations of motion with either $\psi_{(0)}$ or $\psi_{(1)}$ zero in order to interpret the other as an expectation value.

There is an important difference here from section 2.4 regarding relevant, irrelevant and marginal operators. We say that our operator has scaling dimension Δ but we must remember that the source is what is important for the scaling behaviour and not only the dimension of the operator. In fact we see from equation (11) that the prefactor to the $\psi_{(0)}$ term that we have chosen as our source scales as $\lambda^{d-\Delta}$. Since near the boundary we know that we must restrict ourselves to scale-invariant AdS space we cannot introduce something that scales as this would ruin the AdS/CFT correspondence. This means that the term $\psi_{(0)}$ itself must scale as $\lambda^{\Delta-d}$ which then defines the relevance of the operator in question. Going to lower energies as in section 2.4 (again $\lambda < 1$) we see that a relevant operator now has

$$\Delta < d \tag{14}$$

and an irrelevant operator has

$$\Delta > d \tag{15}$$

and finally a marginal operator has

$$\Delta = d. \tag{16}$$

Equations (12) and (13) are the main results of this section as they provide us the information we need in order to translate between the bulk solution and the field theory solution in the scalar case.

As might be anticipated from equation (10) the mapping of fields in the bulk to sources and operators in the field theory can be generalized to fields other than scalar ones. E.g. for a vector field A_μ with near boundary behaviour $A_x^{(0)} + \frac{A_x^{(1)}}{r} + ...$ we would say that $A_x^{(0)}$ is the source and $A_x^{(1)}$ is the expectation value. This scheme is often called the "AdS/CFT dictionary" in the literature and to use it one is only required to obtain the field behaviour near the boundary, regardless of what kind of field it is (i.e. scalar or different rank of tensors).

Thus we need only to define an action for the fields in the bulk that source operators that we are interested in and solve the general relativity equations of motion in an AdS black hole background to get the field theory behavior on the boundary. This is the topic of section 3 but before that we need to consider one more aspect of the duality, namely the way phase transitions occur. Since we expect that the bulk shall be dual to a field theory of a superconductor and we know that the field theory will have at least one phase transition which is at the critical temperature. This must somehow also manifest itself in the bulk theory, but what kind of sharp transition can occur in our bulk gravity theory? We know that general relativity is a theory in which the background metric is not a fixed arena in which the physics play out but it is a dynamical field which affects and is affected by other fields. We are led to consider the fields and their interaction with the background metric and in particular the stability properties of the background. We remember that a tachyon is a particle that has $m^2 < 0$ and often a tachyon signals that a theory is unstable. In AdS/CFT we do not really know what instabilities we have but we know that we require real scaling dimension of our operators in the field theory and one can obtain an expression for the scaling dimension of the scalar operators as a function of the scalar mass and the dimensionality of AdS space as

$$\Delta = \frac{d}{2} + \sqrt{(Lm)^2 + \frac{d^2}{4}}. \tag{17}$$

The constraint that the scaling dimension is real is

$$(Lm)^2 \geq -\frac{d^2}{4} \tag{18}$$

which is known as the Breitenlohner-Friedmann (BF) bound and it states that we actually can allow negative mass-squared (i.e. not all tachyons induce instability) as long as it satisfies (18). There is one thing in particular that we should note about equation (18); the stability of the theory depends on the dimensionality of the space time. This is most interesting since we remember that in section 2.3 we saw that when we introduce a black hole into the space time, the near horizon geometry contains an AdS_2 regardless of what dimensionality we have in the bulk far from the horizon. The deep infrared of our theory thus contains instabilities that are not present in the UV sector and this will get translated into a phase transition at low temperatures (energies) in the boundary CFT. In this specific case the instability of the gravity theory will be the formation of scalar hair around the black hole, hair with negative mass squared but which is nevertheless able to break the $U(1)$ symmetry and give rise to a phase transition.

3 Applied AdS/CMP: A Simple Theory with Scalars

3.1 Schwarzschild Black Hole Background/Holographic Superconductor

This section follows closely the work presented in [9]. We will show that we get what is called a holographic superconductor out of a really simple top-down approach for our bulk theory. We will take our bulk metric to be that of a Schwarzschild AdS black hole with the metric we recall from section 2.3

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2) \tag{19}$$

where

$$f(r) = \frac{r^2}{L^2} - \frac{M}{r}. \tag{20}$$

The goal here is to describe (effectively) 2+1 dimensional systems so we need 4 dimensions in the bulk and thus choose to work with AdS_4. Since we want to describe a superconductor we obviously expect our boundary theory to at least have some Maxwell vector potential that describes the electric and magnetic fields.

In a superconductor there exists a condensate, in the BCS case this is the Cooper pairs, that breaks the $U(1)$ symmetry and in effect gives the photon a mass in the superconducting bulk. The easiest way to do this in our setting is to introduce a charged scalar field that can acquire an expectation value and thereby giving rise to a Higgs effect for the photon. Alternatively this scalar field can be seen somewhat analogously to the dynamical order parameter of Ginzburg-Landau theory but without the $|\psi|^4$ term. This means that our bulk must have a scalar field and a Maxwell field which are governed by the Klein-Gordon and Maxwell equation respectively. We remember from the end of section 2.6 that we wish to construct an action for the theory in the bulk so we find the action for the scalar field and the Maxwell field in [10] Appendix E.1:

$$S_{KG} = -\frac{1}{2}\int d^d x \sqrt{-g}\left(|\nabla\psi|^2 + m^2|\psi|^2\right) \tag{21}$$

$$S_{EM} = -\frac{1}{4}\int d^d x \sqrt{-g} F^{\mu\nu}F_{\mu\nu} \tag{22}$$

where KG stands for Klein-Gordon action and EM for electromagnetic action. To properly include gravity in our action principle we need to add a so called Einstein-Hilbert term to the action which is of the form

$$S_{EH} = \frac{1}{2\kappa^2}\int d^d x \sqrt{-g}\left(R + \frac{d(d-1)}{L^2}\right) \tag{23}$$

where R is the Ricci scalar, $\kappa = \sqrt{8\pi G}$, and the second term is a negative cosmological constant inherent in the AdS geometry. We must be careful to remember that in curved space time the derivative operator acting on a tensor is $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\rho} A^\rho$, i.e. when acting on tensor fields the Christoffel symbol Γ appears and mixes the coordinates in the derivative. Now, if we add S_{EH}, S_{KG} and S_{EM} together and scale them separately to get the correct eqm's, we obtain a theory of free scalar particles and a free EM-field interacting (rather weakly, hence the term "free") only with gravity. If the scalar field is charged, which in general it is, we know that there are interactions between the charged scalar field and the EM-field. In any book on QFT, e.g. [11], one may find the minimal gauge coupling (interaction) scheme which is promoting the derivative ∇_μ to a gauge covariant derivative $D_\mu = \nabla_\mu - iA_\mu$ [1] which yields an interacting theory

$$S_{\text{bulk}} = \int d^d x \sqrt{-g} \Bigg[\frac{1}{2\kappa^2} \left(R + \frac{d(d-1)}{L^2} \right) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - m^2 |\psi|^2 \\ - |\nabla_\mu \psi - iA_\mu \psi|^2 \Bigg]. \tag{24}$$

The full eqm's arising from varying S_{bulk} are non-linear coupled differential equations involving the scalar field ψ, the vector potential A_μ and the background metric $g_{\mu\nu}$ and are difficult to solve. Here we will work in what is called the *probe limit*, $G_N \to 0$, where G_N is Newtons constant. This means that the fields in the bulk do not backreact on the metric, i.e. the metric is fixed to be the Schwarzschild AdS metric in equation (19). This yields the action in static background as the sum of S_{KG} and S_{EM}

$$S_{\text{bulk}} = \int d^d x \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - m^2 |\psi|^2 - |\nabla_\mu \psi - iA_\mu \psi|^2 \right). \tag{25}$$

If we further make the assumption that our fields only depend on the radial coordinate we get the equation of motion

$$\psi'' + \left(\frac{f'}{f} + \frac{2}{r} \right) \psi' + \frac{\phi^2}{f^2} \psi - \frac{m^2}{2L^2 f} \psi = 0 \tag{26}$$

[1] The sign of the iA_μ term differs from [11] but follows that of [9] and is only a matter of convention.

for the scalar field. We now assume that we have only an electric field in the bulk, i.e. $A_t = \phi$ is the electric potential and $A_r = A_x = A_y = 0$ and we also impose the Lorentz gauge condition $\nabla_\mu A^\mu = 0$, which in our case yields $\nabla_t A^t = 0$. Also we observe from the Maxwell equations that the phase of ψ is constant so without loss of generality we can take ψ to be a real scalar field. This allows us the obtain the equation of motion for the electric potential as (also from [9])

$$\phi'' + \frac{2}{r}\phi' - \frac{2\psi^2}{f}\phi = 0. \tag{27}$$

Equations (26) and (27) are two coupled non-linear differential equations and we have to solve them numerically. An early part of this thesis work treated this and a Mathematica notebook was created following [9] to obtain the results. Below is a brief description of the method by which the equations are solved, in Appendix A we describe some of the subtleties in more detail, the results are then displayed and compared to experimental data from graphene in section 3.2.

In order to create a phase transition in the boundary theory we need to have a bulk theory which also undergoes a transition of some kind. We remember the BF bound and the fact that it depended on the dimensionality of space. We also remember that as soon as we have a horizon in the AdS bulk the near horizon geometry is AdS_2 and this means different BF bounds near and far away from the horizon. The idea here is to choose a mass that is above the BF bound for AdS_4 but below the bound for AdS_2, thus inducing an instability at low energies which is precisely what we are after. This argument was proposed in [12].

From the equations of motion it follows directly that the scalar field and the scalar (electric) potential near the boundary behave as

$$\psi(r) = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + ... \tag{28}$$

and

$$\phi(r) = \mu - \frac{\rho}{r} + ... \tag{29}$$

respectively. Here, μ denotes the chemical potential, ρ denotes the charge density and we remember from section 2.6 that $\psi^{(i)}$ denotes the terms that gives rise to an expectation value for a scalar operator of scaling dimension i. We are thus interested in finding solutions to the eqm's that have one $\psi^{(i)}$ finite and the other zero representing the expectation value and the source respectively. The basic idea behind the solution technique is to construct a series expansion of the fields at the horizon to some order. We then substitute these expansions into the eqm's and use *NDSolve* to integrate out to the boundary where we match the solutions of the horizon expansion with an expansion at the boundary like the ones in equations (28) and (29)[1]. We can then determine $\psi^{(i)}$, μ and ρ and in general both of $\psi^{(i)}$ will be finite so we must explicitly look for solutions with one of them zero. This gives us expectation values for operators of conformal dimension 1 an 2 and what we are interested in really is dimensionless quantities, i.e. quantities that *do not have a scaling dimension.* This means that we must "normalize" our operators because by definition they possess a specific scaling dimension and we will divide by the temperature to achieve this. The temperature can be calculated as [8]

$$T = \frac{3}{4\pi\sqrt{\rho}} \tag{30}$$

and has scaling dimension 1. The critical temperature, T_c, is the temperature where the phase transition occurs and sets a natural temperature scale we can use to normalize our quantities. Thus the dimensionless quantities we wish to compute are

$$\frac{\langle \mathcal{O}_1 \rangle}{T_c}$$

and

$$\frac{\sqrt{\langle \mathcal{O}_2 \rangle}}{T_c}$$

versus "normalized" temperature $\frac{T}{T_c}$. The results of these calculations are shown in figures 6 and 7 and they show that using this theory

[1] This is treated in more depth in Appendix A

we indeed have operators that acquire expectation values below a certain critical temperature, the operators have condensed, and this is exactly what happens in superconductors where below T_c an energy gap forms.

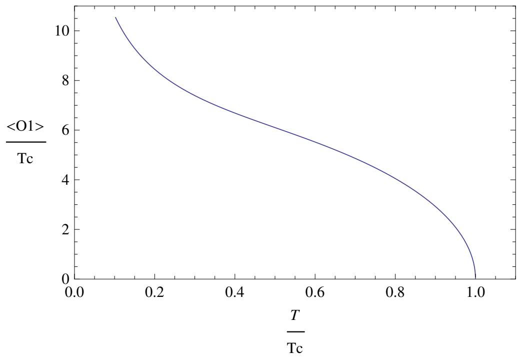

Figure 6: Numerical results of the expectation value for the dim 1 scalar operator.

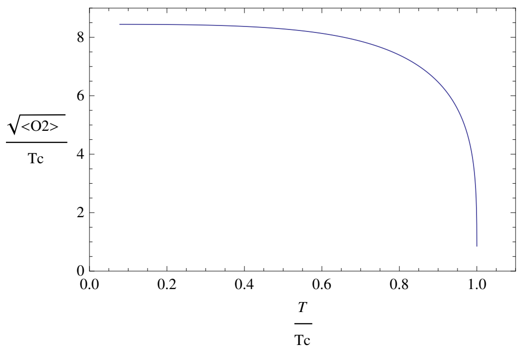

Figure 7: Numerical results of the expectation value for the dim 2 scalar operator. This has the exact same behaviour as the energy gap in the BCS case, see figure 2. The only difference is that this curve approaches 8 in the low T limit, the theoretical BCS curve approaches 3.5.

Figure 7 deserves special notice as it is very similar to the energy gap curve predicted within BCS theory which is shown in figure 2. There is one big difference however, in our case the curve tends towards 8 and the BCS curve goes to 3.5, our numerical value is typical for the gap of high-T_c cuprates which takes values $\sim 6 - 10$. It thus seems as we have actually managed to construct a theory of strong coupling which exhibits the same behaviour as high-T_c superconductors just using a straightforward top-down approach.

To really show that we have actually constructed a superconductor it is not enough to show that our theory develops a condensing operator however, we need to show that we have infinite DC-conductivity and ideally we would like to match some of our results to experimental results. We shall see that we can do both of these things. To be able to talk about conductivity we must perturb the vector potential in a spatial direction, i.e. we allow it to have some freedom in forming magnetic fields so we may talk about changing electric fields. Without loss of generality (due to rotational invariance) we assume perturbation in the x-direction and we further assume a time dependence of the form $e^{-i\omega t}$. We also assume zero momentum transfer, which means that the wavelength of the perturbation is much larger than the sample size and this is the experimentally realisable situation, and this makes the eqm for the A_x component look like

$$A_x'' + \frac{f'}{f}A_x' + \left(\frac{\omega^2}{f^2} - \frac{2\psi^2}{f}\right) A_x = 0. \tag{31}$$

The idea behind this solution is exactly the same as before: we create an expansion at the horizon and integrate out to the boundary and evaluate the conductivity at the boundary. Close to the boundary the Maxwell field behaves as

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + ... \tag{32}$$

which according to the AdS/CFT dictionary allows us to compute the electric field component E_x and the associated current $\langle J_x \rangle$. We remember that Ohm's law reads [1]

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = -\frac{\langle J_x \rangle}{\dot{A}_x} = -\frac{iA_x^{(1)}}{\omega A_x^{(0)}} \tag{33}$$

where in the last step we have used that the time dependence is of the form $e^{-i\omega t}$. We may now compute the conductivity σ as a function of the dimensionless quantity $\frac{\omega}{T}$. These results are presented and compared to real experiments in section 3.2 and there we see that we actually do have infinite conductivity in the DC case.

[1] A more familiar form of Ohm's law might be $U = RI \Leftrightarrow \frac{1}{R} = \frac{I}{U} \Leftrightarrow \sigma = \frac{I}{U}$ which in our case is exactly $\frac{\langle J_x \rangle}{E_x}$

3.2 Comparison with Experiment on Graphene

It is now time to do two things, first to really show that we do have a superconductor i.e. show that we have infinite DC-conductivity and secondly to compare our numerical results with experimental results from graphene. The numerical result for the real part of the conductivities can be seen in figures 8 and 10 and we see some interesting behaviour but for now we will actually focus on what we do not see. What we do not see in these plots is a divergent peak that goes to infinity as $\omega \to 0$ but the conductivity looks flat for small ω and it is far from obvious at this point that we have a superconductor. We will now employ the *Kramers-Kronig relation*, which relates the real and imaginary parts of functions analytical in the upper half plane, and in our case it relates the real part of the conductivity to the imaginary part (and vice versa of course). So if we instead of looking at the real parts of the conductivities look at the imaginary parts, shown in figures 9 and 11, we see that there is a divergent behaviour in the low frequency regime. By using the Kramers-Kronig relation we can show that there must be a delta function at $\omega = 0$ which our numerics fail to detect. This means that our plots of the real parts of the conductivities, figures 8 and 10, are to be modified by adding a delta function at the origin, and thus we have an infinite DC conductivity, i.e. *the action we defined in section 3.1 is actually the bulk dual of a superconductor.*

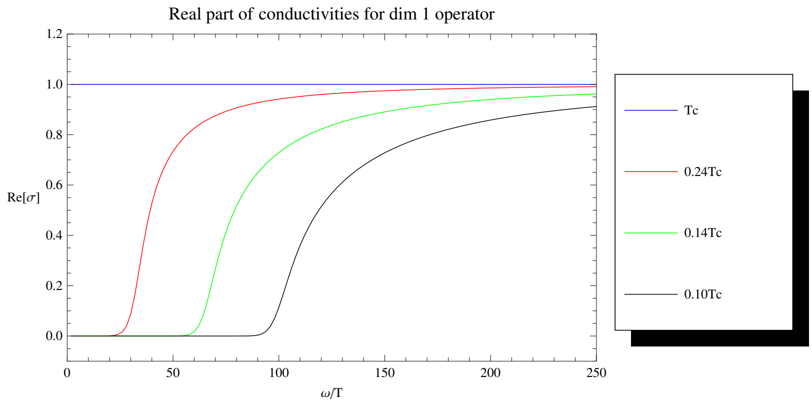

Figure 8: Numerical results for the real part of the conductivity of mono-layer graphene, remember that there is an extra delta peak at the origin. This result is for the operator of conformal dimension one.

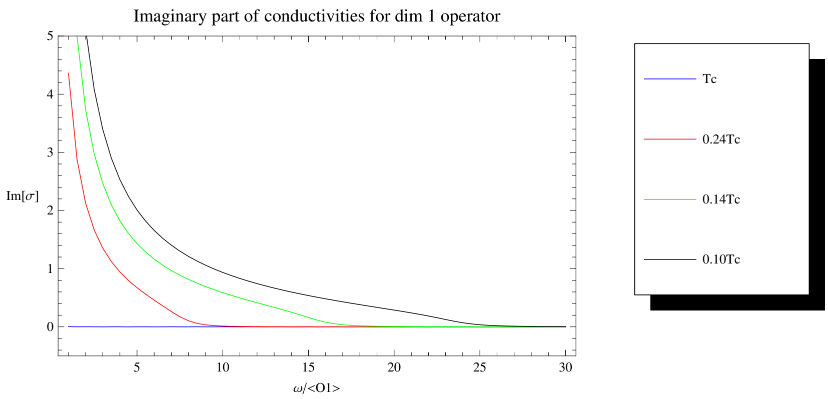

Figure 9: Numerical results for the imaginary part of the conductivity of mono-layer graphene. This result is for the operator of conformal dimension one.

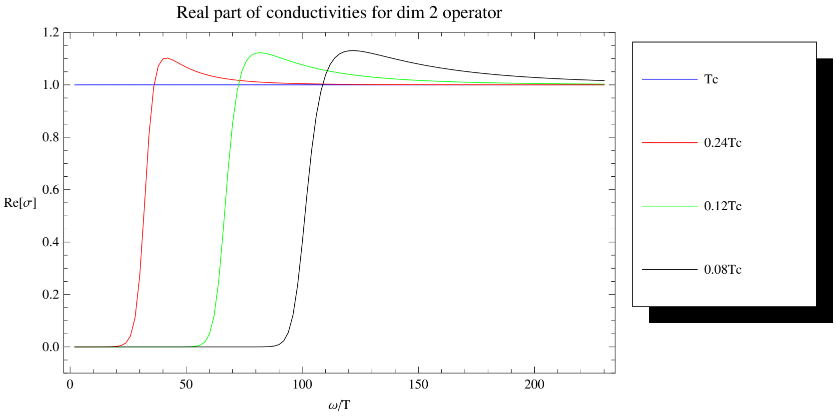

Figure 10: Numerical results for the real part of the conductivity of mono-layer graphene, remember that there is an extra delta peak at the origin. This result is for the operator of conformal dimension two.

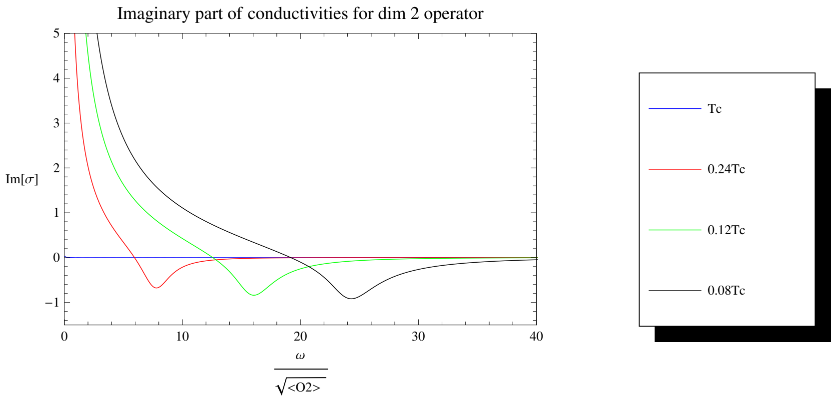

Figure 11: Numerical results for the imaginary part of the conductivity of mono-layer graphene. This result is for the operator of conformal dimension two.

Figure 2 in [13] shows experimental data on mono-layered graphene, the upper plot shows the real part and the bottom plot shows the imaginary part of the conductivity. If we compare these two plots to our two-dimensional operator and the conductivities it gives rise to, figures 10 and 11, we see some striking similarities and some differences. We must remember that we have a delta function at the origin of figure 10 and then our model seems to capture the qualitative behaviour of the experimental data. The reason the real data in [13] turns upwards much earlier than ours is due to impurities in the experimental sample whereas we have considered completely pure graphene. The main point here is that by using purely a top-down approach, considering only a scalar field and a scalar potential, we can construct something that behaves as a complicated macroscopic object, namely a 2-dimensional graphene layer. This gives some viability to the claim that the AdS/CFT conjecture can actually be a very potent tool, once more fully understood.

4 Extension of the Simple Scalar Theory

Encouraged by the fact that the very simple theory with a condensing scalar in the previous section seems to capture some of the behaviour of real 2+1 dimensional systems we want to somehow extend our theory to see if we can improve upon it. In this section we introduce and explain a Weyl correction term and two non-linear Maxwell correction terms to the original theory and we also explore, to some extent, the parameter space of the theory.

An observation made early on in the study of AdS/CMP was that an ordinary Maxwell term

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{34}$$

in the bulk gave rise to a trivial electrical response, the conductivity was constant as a function of ω. We know from the previous section that by introducing a scalar field we get a non constant conductivity but we would like to be able to also obtain this behaviour without introducing a superconducting condensate. It is then natural to ask what kind of extensions we might add to the simple theory defined by (34) in order to accomplish this. In particular it was shown in [14] how to add higher derivative corrections. The higher derivative corrections are to be viewed in an effective field theoretic sense, the possible terms may arise in different ways microscopically but their effective contribution might be the same. The Lagrangian (34) contains terms with two derivatives so we need to add corrections with higher degree of derivatives, e.g. three or four. The paper [15] states that *without* *scalar fields in the bulk* the general lowest derivative terms that respect parity, covariance and conserve the UV-conformality[1] of the theory are

$$\mathcal{L}_{\text{corr}} = \gamma C^{abcd} F_{ab} F_{cd} + \alpha_1 \left(F_{\mu\nu} F^{\mu\nu}\right)^2 + \alpha_2 F^a_b F^b_c F^c_d F^d_a. \tag{35}$$

Here C^{abcd} is the Weyl tensor, a quantity determined by the space time metric, which introduces an explicit interaction between the geometry and the field strength. The F^4 terms we will refer to as non linear terms since they give rise to non linearities in the equations of motion, see Appendix B. These terms are introduced to alter the behaviour of the theory by introducing corrections and we emphasise that according to [15] these are the most general terms we might add consistently.

4.1 Motivation of the Weyl Extension and its Connection to the Hubbard Hamiltonian

Following [16] we will now consider the γ term in absence of scalar fields, i.e. no superconductivity. We can think of this as us explicitly putting ourselves above T_c in the phase diagram where the scalar field has yet to acquire an expectation value or we can think of it as a system without a superconducting transition. The Lagrangian we will be considering first is thus a Weyl corrected Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \gamma C^{abcd} F_{ab} F_{cd}. \tag{36}$$

We also note here that in this case there is a causality bound on γ seen in an AdS/CFT setting. The resulting boundary field theory will be non-causal if we break this bound which applies to AdS_4/CFT_3, the bound is

$$|\gamma| \leq \frac{1}{12}. \tag{37}$$

We do not go into details here but this is covered in detail in [14] and [15]. In order to acquire some physical insight into the Lagrangian (36)

[1]By respecting UV-conformality we mean that near the boundary (the UV sector) we still want our theory to be conformal so we can only add terms which goes to zero as the space time asymptotes to AdS, this is true for the Weyl tensor (which is sometimes also known as the conformal tensor).

we will now try to connect it to a microscopic Hamiltonian. In condensed matter physics a Hamiltonian that captures conformal physics is the 2+1 dimensional boson Hubbard Hamiltonian

$$H = -w \sum_{<ij>} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + \frac{U}{2} \sum_i n_i(n_i - 1), \tag{38}$$

where $b_i^\dagger$ and b_i are boson creation- and annihilation operators, $n_i = b_i^\dagger b_i$ is the number operator for site i, and $\langle ij \rangle$ denotes nearest neighbour summation. We will consider the case where there are equally many bosons as lattice sites. w denotes the hopping strength between nearest neighbour sites and U is the repulsive energy between two bosons occupying the same site. We now define a coupling $g = \frac{U}{w}$, we see that very large g means that the bosons repel each other strongly and so are localized one by one on the lattice cells. We also see that small g means that many bosons can occupy the same site, since the hopping term dominates the repulsion term, and the number of bosons on each site will fluctuate. The case of large g with single bosons on each site is an insulating state and the case with large fluctuations is a superfluid state. The fact that the ground states of the system are intrinsically different implies that at some intermediate coupling, neither very large nor very small $g = g_c$, there is a phase transition between a superconducting and an insulating phase. These two phases have different basic degrees of freedom, in the insulating phase the excitations are ordinary quasi-particle excitations where a boson has left its original site and we have double occupancy somewhere else, and in the superfluid case the excitations are vortices. The question is now which one, if any, of these fundamental excitations best describe the system at and near critical coupling g_c i.e. near the phase transition. This is a rather basic question that is very general for condensed matter systems: how can we describe systems at critical points if we do not know what the fundamental excitations are? It is not even clear that there always are well defined fundamental excitations. We now move back into the holographic approach to see if we can shed some light on this.

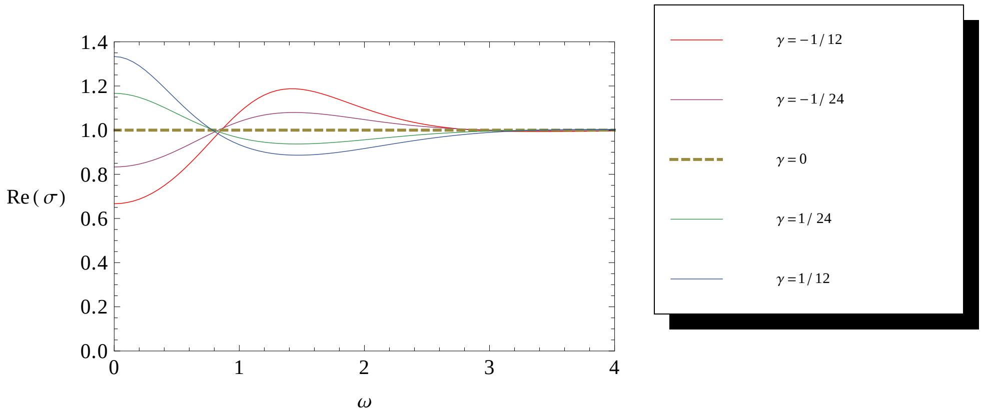

Figure 12: Holographically calculated conductivities for different values of γ.

Solving the bulk equations for the Lagrangian (36) and applying the AdS/CFT duality in much the same way as in section 3.1[2] one obtains the conductivities as shown in figure 12. The Drude peak behaviour we observe at small ω for $\gamma = 1/12$ is just what we expect from a system with quasi particle like excitations and the dip at small ω for $\gamma = -1/12$ is what we expect from vortex like excitations, [15]. It is now obvious why we spent time on the Hubbard Hamiltonian, the interpretation we make is that in that when $\gamma < 0$ and $\gamma > 0$ the effective fundamental excitations of the field theory dual to the Lagrangian (36) are vortices and quasi-particles respectively. We have thus managed to find an example of a system which in some limits is amenable to treat both with ordinary condensed matter techniques and within an AdS/CMP setting. We noted above that a problem when writing down and solving a Hamiltonian is to know the degrees of freedom and solving the Hamiltonian was a problem when we are near critical coupling g_c. Within our holographic treatment we need not know the basic degrees of freedom and we can easily tune γ in between the two extremes (i.e. extreme quasi particle- and extreme vortex behaviour) and get the curves shown in figure 12. The work presented in [15] within a holographic framework thus managed to solve a condensed matter problem that was not previously solved, i.e.

[2]Actually the present case is much simpler since we do not have to worry about temperature or a condensate.

managed to interpolate between g large and g small in the conformal system.

We finally note the fact that the simple Weyl correction term seems to capture some deep physics once we learn to interpret it via ordinary condensed matter methods, i.e. connect it to a microscopic Hamiltonian. This seems a promising approach for the non linear terms as well.

4.2 F^4 terms

Analogous to the treatment above, where we followed [15] to connect the γ parameter to some microscopic picture, we would like to do the same for the parameters α_1 and α_2. An observation we make when we try to solve the theory for negative α parameters is that the solutions all seem to diverge. A natural limitation is then to only consider positive values for the α parameters. It is conceivable that for some combination of the parameters it is possible to obtain finite field solutions for one or both of the parameters negative but we do not investigate this at great length here.

The α_2 term

We start out by considering what turns out to be the easier of the two parameters. This is also the subject of our paper [3] where this is treated more extensively. We set $\gamma = \alpha_1 = 0$ and by sweeping the parameter α_2 we may look at the conductivity, still just as in the γ case, and the results are presented in figure 12. We see that for non zero α_2 we develop a peak around $\omega = 0$ and this is very similar to what one might expect from impurity scattering. In that case we would expect a peak around $\omega = 0$ that has a width of $\sim 1/\tau_0$, i.e. the inverse of the scattering time, this knowledge allows us to a bit non-rigorously associate $\alpha_2 \sim 1/\tau_0$. If we want to capture the physics of real layered superconductors we need to have a non-zero α_2 in the system, this since most materials have some degree of impurity in them.

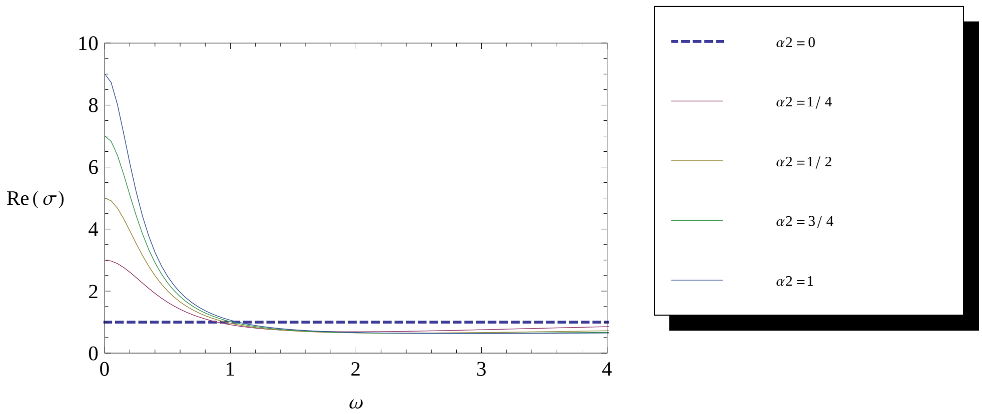

Figure 13: The conductivities for different values of α_2.

In order to make some connection with real systems we can compare the numerical results in figure 13 with figure 2 in [17] where experimental results are shown. The experiments are made on strongly correlated thin film materials and this is exactly the kind of system we might hope to describe with our model. The similarity between our holographic model and the data from [17] are quite striking, the Drude peak behaviour at low frequencies matches well. This Drude peak behaviour was also observed in [18] where this effect arises from a holographic lattice. Seen from that viewpoint the effect is not surprising since a Drude behaviour is expected when introducing scattering by the lattice. The fact that we observe the same behaviour without a lattice is somewhat surprising and this means that in some sense the lattice interaction is "hidden" inside the non-linear α_2 term.

The α_1 term

The α_1 parameter is a little harder to investigate in the same way as the γ and α_2 parameters since in doing the calculations we observe that it does not change the conductivity above T_c. However, we have not performed an exhaustive analysis of the parameter space and it could be that for some combination of γ and α_2 the α_1 parameter affects the conductivity. This leads us to conclude that if α_1 has some impact on the physics above T_c it must necessarily be on some other quantity that is not connected with the conductivity. Since we are only considering the conductivity we will take α_1 to be some parameter that is only important below T_c i.e. it is connected to the superconducting transition and the nature of the condensate. This is not in contradiction with that it could well influence some physics above T_c, it is just that we do not consider those effects here.

4.3 Extended Energy Gap

We now consider the extensions mentioned above in the superconducting case, i.e. we now also consider the dynamics of the scalar field. For a more extensive treatment, see our paper [2]. The Lagrangian for the full theory is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2|\psi|^2 - |\nabla_\mu\psi - iA_\mu\psi|^2 + \gamma C^{abcd}F_{ab}F_{cd} \\ &+ \alpha_1 \left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \alpha_2 F^a{}_b F^b{}_c F^c{}_d F^c{}_a,\end{aligned} \tag{39}$$

i.e. the superconducting theory from section 3.1 with the addition of the correction terms considered above. First we note that the causality bound on γ in the superconducting case, because of the presence of the scalar field, is changed from (37) to

$$-\frac{1}{12} \leq \gamma \leq \frac{1}{8}, \tag{40}$$

this is covered in more depth in [2]. The lower bound on γ is unchanged from the non-superconducting case but the upper bound has been pushed upwards by the presence of the scalar field. We note that when trying to create a Frobenius expansion (see Appendix A) with $\gamma = 1/8$ we get divergent behaviour which seems difficult to get away from. We take the easy way out and only solve the equations near $\gamma = 1/8$, this should not be such a large restriction since divergent solutions should have a very large backreaction. Even if we managed to solve extremely near $\gamma = 1/8$ we have not incorporated backreaction in our calculations so the solutions could not be trusted. At some point the backreaction becomes important in our calculations which means that the solutions we obtain when approaching $\gamma = 1/8$ are not on the same secure footing as the other solutions.

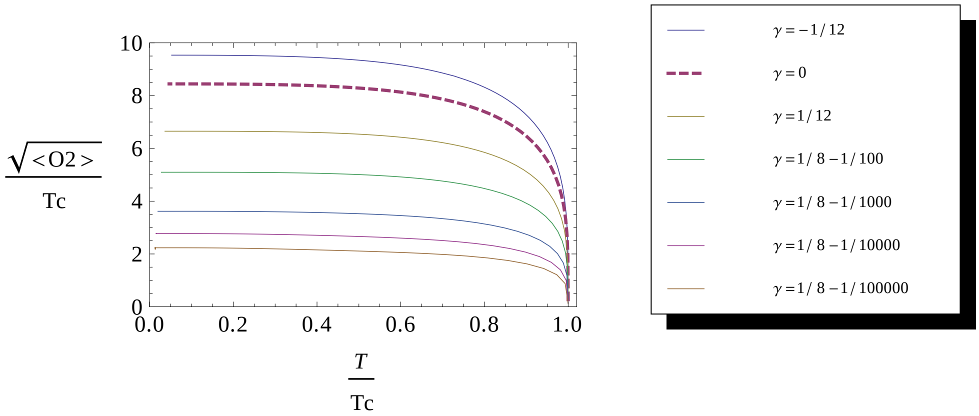

Figure 14: Calculated energy gap for the γ sweep.

We noted in section 3.1 that without corrections we get an energy gap of around 8 and that was indicative of a high-T_c superconductor but we were in that case limited to this fixed value of the gap. Observing the results presented in figure 14 we see that by tuning γ we obtain different energy gaps, we note that the region we previously said was unsure if we could trust is the one with the lowest gaps. Even ignoring these extremes we cover gaps in the range $\sim 6 - 10$ which is in accordance with experimental results shown in figure 1 in [19]. Thus we capture the energy gap range of the high-T_c cuprates in the non-underdoped regime by introducing the Weyl correction.

By also including the non linear F^4 terms we see in figures 15 and 16 that we get higher gap values than 10. According to experimental data in figure in [19] this region is what is called ”underdoped”, this is a particularly messy region where different phases are thought to affect each other. In figures 15 and 16 we obtain gap values in this region by tuning one α at the time but we could combine both of the α's and also setting $\gamma = -1/12$ (which is the γ that yields the largest gap) in order to get this effect. Perhaps this means one could adress the question of phase proximity effects in quantum critical materials within our model. In order to accomplish this a rigorous analysis of the parameter space of our model is required and has to be done in a condensed matter context, this is beyond our treatment here.

By comparing with figure 1 in [19] we see that by introducing the correction terms we essentially *cover the whole range of energy gaps for high T_c superconductors.*

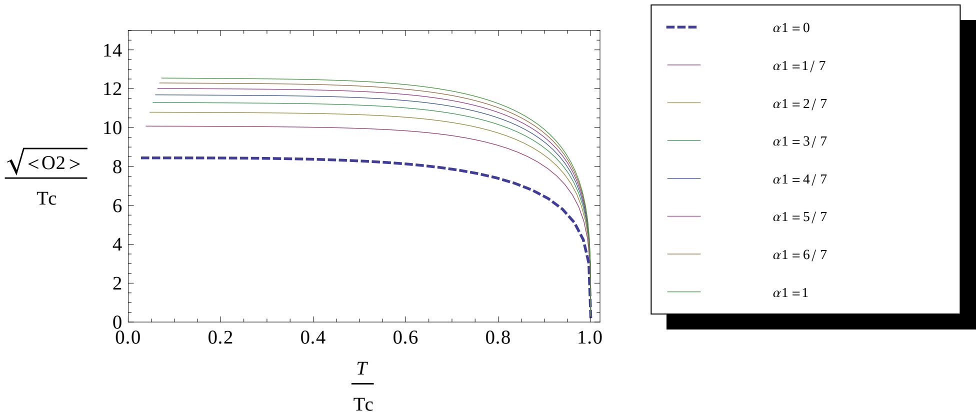

Figure 15: Calculated gaps sizes for different α_1, here $\alpha_2 = \gamma = 0$.

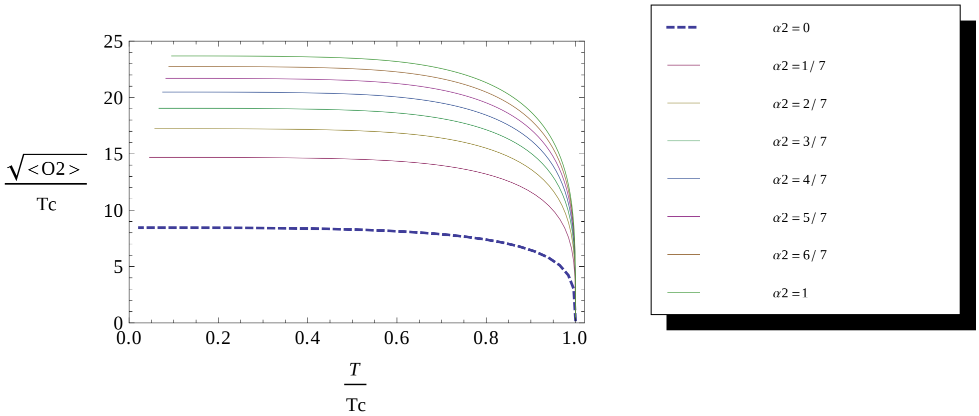

Figure 16: Calculated gaps sizes for different α_2, here $\alpha_1 = \gamma = 0$.

4.4 Conductivity Behaviour

Here we examine the behaviour of the optical conductivity in the superconducting phase. In figure 17 we see the results for different γ's with $\alpha_1 = \alpha_2 = 0$. Just below $T = T_c$ we see that the γ parameter clearly affects the conductivity behaviour of our theory at small ω. As we lower the temperature the conductivity behaviour has very similar shapes for the different parameter values and the effect of the γ parameter is just to shift the conductivity curves.

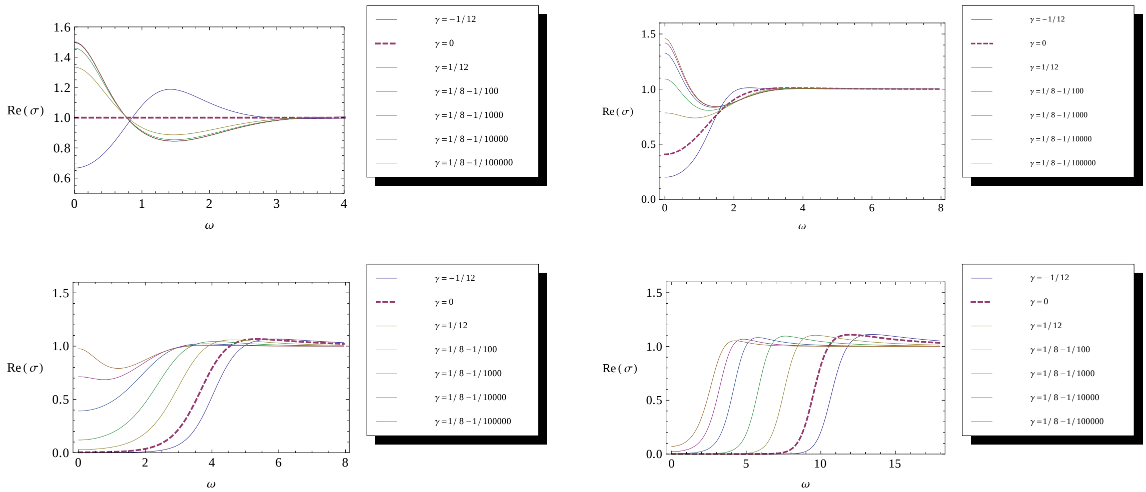

Figure 17: Calculated real part of the conductivities for different γ's at four different temperatures. From top to bottom left to right we have $T = T_c$, $T = 0.9T_c$, $T = 0.5T_c$ and $T = 0.2T_c$. In all of the figures $\alpha_1 = \alpha_2 = 0$.

5 Conclusion, Discussion and Future Work

The main conclusion one can draw about AdS/CMP is that it produces results in qualitative agreement with experiments on strongly correlated materials, one such class of materials is the high-T_c cuprates. In starting this work it was known from [9] that one could in fact construct a superconducting theory but one could only produce one fixed gap ~ 8. In later works the same authors considered backreaction in their theory in [20] where they get an extended range of the gap from 8 and higher. One of the main findings in this work was that a Weyl correction (as considered in for instance [15] in a non-superconducting setting) allows us to obtain energy gaps that roughly correspond to the full range of experimentally measured non-underdoped cuprate gaps. By also allowing non linear Maxwell terms in the theory we also show that one can obtain larger gap sizes which belongs to the region called underdoped cuprates. This region is somewhat messy and people expect the competition between different phases to have a big impact on the physics. Together these two contributions, the Weyl and non-linear terms, allow us to tune the theory to incorporate the full range of energy gaps of the cuprates. A point worth stressing is that we started with a theory that was known to be a minimal AdS/CMP superconducting theory and to it we added all possible lowest order corrections. This leads us to believe that we have constructed a very general effective field theory by help of AdS/CMP. The fact that this approach is also shown to reproduce some features of real world systems, e.g. the energy gap of high T_c superconductors, seems very promising. Another thing we have noticed in this work is that a non-linear Maxwell term in the Lagrangian introduces a Drude peak in the conductivity indicative of scattering. This had previously been considered in [18] with a more complicated set up with a holographic lattice that is allowed to "imprint" itself on the geometry. Our approach to scattering is much more simplistic, in that it does not involve any backreaction on the metric, but still seems to contain the essential Drude behaviour.

There are a number of future directions one might consider, probably the most important one is to consider backreaction. We have everywhere worked in the probe limit where we treat the background metric as fixed which we know is an approximation. It would be interesting to see how much of the behaviour gets changed and if we get some qualitatively new effects that we miss in the probe limit. In particular we expect that backreaction solves the problems at $\gamma = 1/8$, this might then mean that we could get a lower bound on the energy gap at a lowest derivative correction level in holographic theories. Another obvious thing to do is to calculate other properties of our theory, e.g. heat capacity, and compare those with experiments. If these could also be matched to experiments on the same kind of systems as the conductivities are matched to, we would have a much more solid argument for saying that we have a general theory for a wide class of layered strongly coupled materials.

A qualitatively different direction of future work would be to more thoroughly investigate the parameter space of the theory. Since we believe that our model is quite general and we also know that the γ parameter is connected to qualitatively different microscopic degrees of freedom it seems possible that one could observe some other sort of phase transition than the superconducting one. In particular it would bee very interesting to see if a pseudogap phase could be found, this is sometimes thought to accompany the high T_c superconducting phase. Finding such a phase within the phase diagram would be exciting indeed. It would also be exciting to know whether the sign of γ determines the type of the superconductor, i.e. type I or type II. The type II superconductors are associated with two different critical temperatures so a new phase has formed compared to the type I superconductors. The indicating feature of this phase is that it allows magnetic fields to penetrate in flux vortices, thus linking back to our microscopic interpretation of γ. All in all, the phase diagram and associated phenomena that occur in strongly correlated systems would be interesting too look for in our model, a good theory should perhaps incorporate them all. Since we have several times emphasised the generality of our theory it would be interesting to ”put it to the test” and actually see how much of the behaviours of strongly correlated systems (including superconducting phase transitions) it models.

Appendices

A Solving Singular Differential Equations: The Frobenius Method

An important feature of the AdS/CFT correspondence is that it allows us to compute quantities at finite temperatures by putting a black hole in the bulk which gives rise to a temperature via Hawking radiation. The presence of black hole horizons is thus closely linked to the usefulness of the AdS/CFT correspondence. Unfortunately event horizons represent singularities in space time and this manifests itself in the equations of motion one has to solve[1], for instance equations (26) and (27). To see that these equations are singular we need to insert the definition of $f(r)$ from (20), we also for numerical convenience switch to work with $z = L/r$, and simplifying we get

$$\left(2 - 2z^3 + z^2\phi[z]^2\right)\psi[z] + z\Bigg(\left(-2 + z^3 + z^6\right)\psi'[z] + z\left(-1 + z^3\right)^2\psi''[z]\Bigg) = 0 \qquad (41)$$

and

$$-2\phi[z]\psi[z]^2 - z^2\left(-1 + z^3\right)\phi''[z] = 0 \qquad (42)$$

These equations have diverging terms as z approaches 1 and 0 which is the horizon and the boundary respectively. In order to solve the equations we need to put boundary conditions on the horizon and integrate out to the boundary where we read off the solution so the problem is now twofold. Since the equations diverge on the horizon it is impossible to implement the boundary conditions there and even if we could somehow manage that problem we still would have the problem of diverging terms at the boundary where we want to look

[1] Usually one says that nothing special happens at the horizon because this is only coordinate problem, we may always choose to "follow" an object through the event horizon and thus discover that it is not a "true" singularity. We could choose to work in such coordinates called Kruskal coordinates in which the horizon is not singular, this brings with it some other issues though and since we manage to solve our problem without the Kruskal coordinates we will not consider them further.

at the solutions. A very easy way to get around this is to introduce a small number ϵ and implement the boundary conditions a distance ϵ away from the horizon and also read off the solution ϵ away from the boundary. This however is not a very robust way of solving the problem since it is hard to know how small the ϵ must be in order to really solve the equations.

The Frobenius method is tailor made for this situation, it allows us to construct a series expansion around the singular points and order by order determine the coefficients in the expansion. This can be done very fast and to high accuracy, in practice we do not need many terms in the expansion for good accuracy. What we will do is to use the expansion around the horizon and implement the boundary condition on the expansion and then use the expansion a distance ϵ away from the horizon as "effective" boundary conditions for the integration. The same applies at the boundary; we stop the integration a distance ϵ away and then use the Frobenius expansion the "propagate" the solution all the way to the boundary and read off the result. One might argue that we still have an arbitrary parameter ϵ so that we have not gained anything but this is not true. The ϵ cut-off is of crucial importance because the numerical routines break down close to the singularities but the difference now is that we have an expression for the solution even in the regime where the numerics fail. This solution is in the form of a Frobenius expansion which has been matched with the numerical solution. It is then possible to examine the error in that regime which is important in order to be certain that one has truly solved the problem.

We will now briefly explain how the Frobenius method works for ordinary singular differential equations and this treatment will follow closely to that of [21]. We will then move on to show the differences that appears for non-linear differential equations, which is what we are primarily interested in.

Consider the differential equation

$$(z-c)^2\frac{d^2u(z)}{dz^2}+(z-c)P(z)\frac{du(z)}{dz}+Q(z)u(z)=0 \tag{43}$$

where, $z\in\mathbf{R}$, $P(z)$ and $Q(z)$ are analytic, they can then be Taylor expanded and we call the Taylor coefficients p_i and q_i respectively. Let us now assume a solution to this differential equation that has the form

$$u_{sol}(z)=(z-c)^\alpha\left(1+\sum_{n=1}^{\infty}a_n(z-c)^n\right) \tag{44}$$

so that the problem now has been reduced to determine the coefficients α, $a_1....a_n....$ This we can do by plugging our solution ansatz (44) into the equation (43) and then Taylor expanding the entire expression around the singular point c and collecting the terms of equal powers of $(z-c)$, let us call these coefficients A_i so that the entire equation now looks like

$$(z-c)^\alpha\left(A_\alpha+A_1(z-c)+.......+A_n(z-c)^n+......\right) \tag{45}$$

and the coefficients A_i contains the coefficients of the Taylor expansions of the $P(z)$ and the $Q(z)$ and of course the ansatz coefficients a_i which we wish to find. In order for the resulting expression to fulfill equation (43) each coefficient A_i must be zero. The expression for the A_α is called the *indicial equation* and its solution sets the lowest power of the solution ansatz u_{sol}, the expression looks like

$$A_\alpha=\alpha^2+(p_0-1)\alpha+q_0=0. \tag{46}$$

The term A_1 has the expression

$$A_1=a_1\left((\alpha+1)^2+(p_0-1)(\alpha+1)+q_0\right)+\alpha p_1+q1=0 \tag{47}$$

and in general the expression for the term A_n looks like

$$\begin{aligned}A_n=a_n&\left((\alpha+n)^2+(p_0-1)(\alpha+n)+q_0\right)+\\+\sum_{m=1}^{n-1}a_{n-m}&\Big((\alpha+n-m)\,p_m+q_m\Big)+\alpha p_n+q_n=0.\end{aligned} \tag{48}$$

The indicial equation is of particular importance since it determines the leading power of the expansion but it also determines whether the solution has oscillatory behaviour or not, if α has a non-zero imaginary part then the corresponding solution oscillates. The solutions to the indicial equation also determines and how many solutions there are. Let us denote the solutions to the indicial equation by α_1 and α_2, there are then a number of possible scenarios:

a) $\alpha_1 = \alpha_2$

b) $\alpha_1 - \alpha_2 = p$, where p is an integer

c) none of the above.

The case c) is uncomplicated and one only solves the indicial equation and can then obtain two different solutions depending on if one chooses α_1 or α_2. Cases a) and b) are in general more complicated and two solutions might be obtained by some extra manipulations, we will not cover that here but refer to [21] where this is covered. The case we have to deal with in this thesis is c) although we have to work a little more to get there. We also note that once one has made a choice of solution from the indicial equation of either α_1 or α_2, then equation (48) defines a recursion relation by which we can determine the a_n term by term, this allows us to very efficiently create a power series that is valid in some interval around the singular point c.

We now move on to treat non-linear singular differential equations but we will not make an extensive treatment but just give a flavor of the differences with the linear case and what differs in the solution method. The reader should be warned that this text is solely based on experience since it is hard to find good reference material about non-linear differential equations, in particular singular ones. Because of this the present part should be considered more as a guide than a rigorous treatment. The problem we will look at is the coupled non-linear singular system that is comprised of equations (41) and (42). As already mentioned these equations are singular at both the horizon

$z = 1$ and the boundary $z = 0$, since we need to start by imposing boundary conditions at the horizon we will start with an expansion there. We assume a solution for both fields of the form

$$\psi_{Horizon}(z) = (1-z)^{\alpha} \left(\psi_p + \sum_{n=1}^{\infty} a_n (1-z)^n \right) \tag{49}$$

$$\phi_{Horizon}(z) = (1-z)^{\beta} \left(\phi_p + \sum_{n=1}^{\infty} b_n (1-z)^n \right) \tag{50}$$

where everything works as in the linear case except the two new parameters ψ_p and ϕ_p. These two parameters are free parameters of the solution and we will vary these in order to get different solutions on the boundary. Now, the big difference from the linear case is that if we plug both of these expansions in to equations (41) and (42) we get very complicated expressions due to the non-linear terms, these multiplies two sums so that in general we have a lot of terms of the form $a_n a_m$, $b_n b_m$ and $a_n b_m$ for each power of $(1-z)$. The same thing happens for the α and the β, it is no longer possible to create indicial equations and determine α and β separately because they get "tangled up" with the other coefficients. To simplify things a bit we note that one boundary condition we know is that the ϕ field has to be zero on the horizon to avoid infinities[1]. This gives $\beta = 1$ and plugging this into the equation for the coefficients we can determine α to be zero and now we have managed the equivalent of the indicial equation step in the linear case. Now we need to find the coefficients a_n and b_n but we are obviously prevented from creating a recursion scheme because of the mixing of the terms that arises due to the non-linearities. What we have to do is to solve a system of equations to obtain the coefficients a_n and b_n which will be functions of ψ_p and ϕ_p, this means that plugging these coefficients into the expansions (49) and (50) we have a solution near the horizon with two free parameters. This will allow us to propagate the solutions out a distance ϵ and let a numerical

[1] To see this consider $A^{\mu} = g^{\mu\nu} A_{\nu} = g^{tt} A_t = -\phi(z)/f(z)$ and on the horizon $f(1) = 0$ so $\phi(1)$ also has to be zero.

algorithm take over. However, as discussed above numerical routines break down close to the boundary at $z = 0$ because the equations are singular there as well. Thus we have to create two new Frobenius expansions but this time expand around the boundary in order to extract the true boundary behaviour of the fields, these expansions take the form

$$\psi_{Boundary}(z) = z^{\lambda}\left(\psi_1 + \psi_2 z + \sum_{n=3}^{\infty} c_n z^n\right) \tag{51}$$

$$\phi_{Boundary}(z) = z^{\kappa}\left(\mu + \rho z + \sum_{n=1}^{\infty} d_n z^n\right). \tag{52}$$

In this case it is actually possible to solve two separate indicial equations for λ and κ and one obtains $\lambda = 1$ and $\kappa = 0$. The idea here is quite similar to the where we expanded around the horizon but now we want to determine the coefficients c_n and d_n at some distance ϵ from the horizon and then read off the solutions at the boundary. The numbers we want to read off are the ones we call ψ_1, ψ_2, μ, and ρ in the expansion so it suffices to find these coefficients. This we do as following, we use the numerically calculated solution (which is obtained by first using the Frobenius expansion around the horizon) and match the boundary expansion with the numerical solution at $z = \epsilon$. In order to find solutions with the correct boundary behaviour, i.e. with either ψ_1 or ψ_2 zero, we use the free parameters ψ_p and ϕ_p from the solution ansatz in equations (49) and (50). Once we have obtained solutions with correct behaviour we have managed to solve the problem of accurately solving non-linear singular differential equations and also we have found a solution so that we can make the proper identifications to use the AdS/CFT dictionary (as described in section 2.6).

There is one more thing concerning singular differential equations of this kind worth mentioning. The case when the leading power α has a non-zero imaginary part as we briefly mentioned above is actually present in this work. If we look at equation (31) (and transform to the variable z) and do the same type of analysis as above we will see that we get oscillatory behaviour for the function $A_x(z)$. A very convenient way to deal with the oscillation is to make a second ansatz for the solution, after we have made the Frobenius ansatz and solved for α, $A_x(z) = (1-z)^\alpha S_x(z)$ and insert this into the equation of motion. We can then, since equation (31) is linear, divide out the oscillatory behaviour which is good for the numerical stability. Since we are only interested in the linear response of the theory we will always, even in the case where we have added the F^4 terms in section 4, get linear equations for the conductivity so this can always by employed.

As stressed above this scheme does not stand on any solid theoretical ground and it is not certain that this is a general solution scheme, it is only noted that it works in this case. It is however difficult to imagine a system where the expansion have no radius of convergence so it certainly feels like this could be applied in general with some care taken with the choice of ϵ. It can easily be seen by inserting the expansions into the equations of motion that the radius of convergence is quite small in the case treated here but an ϵ of the order of 10^{-5} is enough to have a very small error.

B Equations of Motion

Here we present the full equations of motion arising from the Lagrangian

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2|\psi|^2 - |\nabla_\mu\psi - iA_\mu\psi|^2 + \gamma C^{abcd}F_{ab}F_{cd} \\ &+ \alpha_1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \alpha_2 F^a{}_b F^b{}_c F^c{}_d F^c{}_a. \end{aligned} \tag{53}$$

Equation of motion for the scalar field ψ:

$$\begin{aligned}&\left(2 - 2z^3 + z^2\phi[z]^2\right)\psi[z] \\ &+ z\left(\left(-2 + z^3 + z^6\right)\psi'[z] + z\left(-1 + z^3\right)^2\psi''[z]\right) = 0.\end{aligned} \tag{54}$$

Equation of motion for the electric potential ϕ:

$$
\begin{aligned}
&-2\phi[z]\psi[z]^2+z^2\left(-1+z^3\right)\Bigg(-\phi''[z]+8z^2\Big(3\gamma\phi'[z]-4z\alpha_2\phi'[z]^3\\
&+z\left(\gamma-z(2\alpha_1+3\alpha_2)\phi'[z]^2\right)\phi''[z]\Big)\Bigg)=0. \qquad (55)
\end{aligned}
$$

Equation of motion for the electromagnetic perturbation δA_x:

$$
\begin{aligned}
&-\frac{A_x[z]\left(2\left(-1+z^3\right)\psi[z]^2+z^2\omega^2\left(1+4z^3\gamma+8z^4(2\alpha_1+\alpha_2)\phi'[z]^2\right)\right)}{-1+z^3}\\
&-z^2\left(-1+z^3\right)\left(1+4z^3\gamma+8z^4(2\alpha_1+\alpha_2)\phi'[z]^2\right)A_x''[z]\\
&-z^4A_x'[z]\Bigg[3\left(1+\left(-4+8z^3\right)\gamma\right)+8z\phi'[z]\\
&\times\Big(\left(-4\alpha_2+z^3(6\alpha_1+7\alpha_2)\right)\phi'[z]+2z\left(-1+z^3\right)\alpha_2\phi''[z]\Big)\Bigg]=0.
\end{aligned}
\qquad (56)
$$

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