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Analytical methods of solution to eddy current interaction problems

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Department of Applied Mechanics
CHALMERS UNIVERSITY OF TECHNOLOGY
Göteborg, Sweden 2012

THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING IN SOLID AND STRUCTURAL MECHANICS

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Abstract

The eddy current method is used for nondestructive evaluation of conducting materials. To achieve a greater knowledge and insure safe and reliable evaluation methods, the use of mathematical models is needed. In this thesis analytical methods of solutions are applied to solve the eddy current interaction problem which essentially is a scattering problem. This involves a Green's function technique to generate integral relations between the surface fields and the fields everywhere else. Then the key is to use suitable basis functions to describe the surface fields. In the end numerical integration is used to get the solution, the change of impedance due to the scatterer. The scatterer in this case is a model of a defect and the source is a single conductor or a single coil. The solutions are compared to Finite Element solutions.

This thesis includes two papers where two different methods of solution have been used. In the first paper, the T matrix method is applied on a 2D problem with a subsurface defect. The second paper presents a boundary integral equation method solution to a problem with a surface-breaking flat crack.

Keywords: Nondestructive evaluation, Scattering, Eddy current, T matrix, Boundary integral equation

Preface

The work in this thesis is a part of the European project ”PICASSO” and the funding from the European Commission within the FP7 programme is gratefully acknowledged.

The completion of this thesis is credited the following persons: my supervisor Anders Boström, my co-supervisors Peter Bövik and Håkan Wirdelius and my roommate Anders Rosell who also has been a member of the ”PICASSO” project, all of them also being co-authors.

I would like to thank my supervisor Anders Boström for his excellent guidance. I also would like to thank all of my colleges in the Advanced NDT group for making the duration of this work a pleasant time.

Göteborg, July 2012
Lars Larsson

Thesis

This thesis consists of an extended summary and the following appended papers:

Paper A	L. Larsson and A. Rosell. "The T matrix method for a 2D eddy current interaction problem". *AIP Conf. proc. Reveiw of progress in quantitative nondestructive evaluation.* Vol. 1430. Burlington, Vermont, 2012, pp. 316–323
Paper B	L. Larsson, A. Boström, P. Bövik, and H. Wirdelius. Integral equation method for eddy current nondestructive evaluation of a tilted, surface-breaking crack. *To be submited for international publication.* (2012)

Both papers are written with co-authors. In the first paper the author of this thesis has preformed all of the work except the finite element calculations. In the second paper the author of this thesis has carried out a large part of the derivations, all the numerical implementations and written parts of the paper.

Contents

Extended Summary

1 Introduction

Nondestructive evaluation (NDE) methods are used to insure the quality of safety-critical components, e.g. turbine blades in a jet-engine. One of these methods is the eddy current method which is used for nondestructive evaluation of conductive materials. Eddy currents are generated by applying an altering current to a coil positioned near the surface of the material. A crack will interfere with the currents and in that way affect the impedance of the coil. By measuring this impedance, cracks can be found. The current is exponentially decreasing into the depth of the material, therefore the method is mostly used to detect surface breaking or near surface defects. How fast the fields will decay is dependent of the frequency of the input current which is the same as the frequency of the involved fields. A lower frequency field penetrates deeper into the material, but it also contains less energy compared to a field of higher frequency and will therefore generate a weaker signal.

In the field of nondestructive testing and evaluation, mathematical models are important tools to secure the safety and reliability of the testing methods. With these models simulations of the testing processes can be done to achieve a greater knowledge of the process. Of special interest is the use of statistical models together with the mathematical model to estimate the probability of detection (POD). The POD is a tool that is used to ensure a methods capability in an NDE application. The POD curve shows the estimated probability to detect a defect as a function of defect size. These curves is most often produced by an expensive experimental procedure. To obtain the POD curves by use of simulations is therefore very desirable.

2 The mathematical model

The eddy current method is an electromagnetic method and the electromagnetic fields are obeying Maxwell's equations (see [9])

$$\nabla \cdot \left(\varepsilon^{-1} \boldsymbol{E}\right) = \rho \tag{2.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.3}$$

$$\nabla \times \left(\mu^{-1} \boldsymbol{B}\right) = \boldsymbol{J} + \frac{\partial (\varepsilon \boldsymbol{E})}{\partial t} \tag{2.4}$$

The presentation in this elegant form simplifies the interpretation of the equations. The electric field $\boldsymbol{E}$ can be created by charges (ρ) and the magnetic field $\boldsymbol{B}$ by moving charges, the current $\boldsymbol{J}$. Equations (2.3) and (2.4) also imply that a change of the magnetic field produces an electric field and vice versa. The material is assumed to be homogeneous, isotropic and linear and then the electric permittivity ε and the magnetic permeability μ are constants. When seeking a solution, it is preferable to reformulate Eqs. (2.1) – (2.4) in order to get fewer equations to work with. This can be done by replacing the fields with potentials. Equation (2.2) implicates that the magnetic field $\boldsymbol{B}$ can be expressed as the curl of a magnetic vector potential $\boldsymbol{A}$, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Inserting this in Eq. (2.3) gives

$$\nabla \times (\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}) = 0 \tag{2.5}$$

and the expression of $\boldsymbol{E}$ in terms of potentials become

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{2.6}$$

where V is a scalar function. The potentials are not unique by these definitions and can be changed without affecting $\boldsymbol{E}$ or $\boldsymbol{B}$. By choosing a gauge such that ∇V is eliminated Eq. (2.4) can now be written as

$$\nabla \times \nabla \times \boldsymbol{A} = \mu_0 \, \boldsymbol{J} - \mu_0 \, \varepsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2}. \tag{2.7}$$

The currents appear in form of source currents and conductive currents, $\boldsymbol{J} = \boldsymbol{J}_s + \sigma \boldsymbol{E}$, where σ is the conductivity. Further on, $\boldsymbol{E}$ is assumed to have zero divergence ($\rho = 0$) meaning that there only exists closed currents. By use of this, together with the vector identity $\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ in Eq. (2.7), gives (the harmonic time factor $e^{-i\omega t}$ is assumed throughout this thesis)

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu_0 \, \boldsymbol{J}_s, \tag{2.8}$$

where $k^2 = i\mu\omega\sigma + \omega^2 \mu\varepsilon$. Equation (2.8) together with different boundary conditions is the mathematical model that is used in this thesis to solve the eddy current interaction problem. The equation is called Helmholtz equation and it does not only describe eddy current interaction problems but also other scattering problems. Two different methods of solutions which successfully have been used to solve other scattering problems (see for example refs. [4], [5], [6], [2]) are used to solve the eddy current interaction problem in this thesis. In paper A the Transition matrix method is used to solve a 2D problem and in paper B an integral equation method is used to solve a problem with 2D geometry but with a 3D electromagnetic field.

3 The methods of solution

3.1 The integral representation

Both methods make use of a Green's function technique to get an integral representation of the problem. This integral representation expresses the fields everywhere in terms of the fields on the boundaries. In this way the dimensionality of the problem is reduced to the determination of surface fields. The Green's function $G(\boldsymbol{r}, \boldsymbol{r}', k)$ to equation (2.8) satisfies

$$\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}', k) + k^2 G(\boldsymbol{r}, \boldsymbol{r}', k) = -\delta(|\boldsymbol{r} - \boldsymbol{r}'|). \tag{3.1}$$

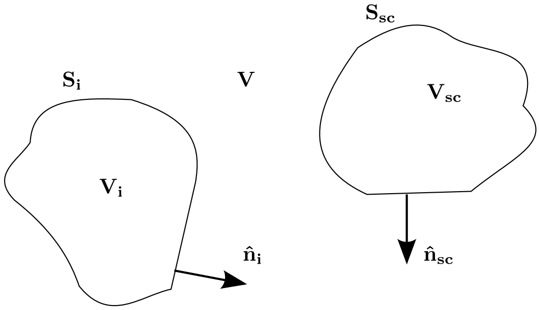

Figure 3.1: *Scattering geometry*

Or in other words the Green's function is the solution to the differential equation for a point source. To illustrate how the Green's function technique is used to solve Helmholtz equation the procedure for deriving the integral representation for a general scattering case is now given. Beginning with multiplication of equation (2.8) with $G(\boldsymbol{r}, \boldsymbol{r}', k)$ and subtracting equation (3.1) multiplied with $\boldsymbol{A}(\boldsymbol{r})$ yields

$$G(\boldsymbol{r}, \boldsymbol{r}', k)\nabla^2 \boldsymbol{A}(\boldsymbol{r}) - \boldsymbol{A}(\boldsymbol{r})\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}', k) + \mu_0 \, \boldsymbol{J}_s G(\boldsymbol{r}, \boldsymbol{r}', k) = \boldsymbol{A}(\boldsymbol{r})\delta(|\boldsymbol{r} - \boldsymbol{r}'|). \quad (3.2)$$

Now the scattering geometry described by figure 3.1 is considered. The source is positioned inside the surface S_i and the scatterer is bounded by the surface S_{sc}. Continuing with integration of equation (3.2) over the volume V (all space outside S_i and S_{sc}) and using Green's theorem gives

$$\begin{aligned} &-\int_{S_{sc}} \Big(((\nabla \cdot \boldsymbol{A}(\boldsymbol{r}))\hat{\boldsymbol{n}}_{sc} - \hat{\boldsymbol{n}}_{sc} \times (\nabla \times \boldsymbol{A}(\boldsymbol{r})))G(\boldsymbol{r}, \boldsymbol{r}', k) \\ &\qquad -(\hat{\boldsymbol{n}}_{sc} \cdot \boldsymbol{A}(\boldsymbol{r}))\nabla G(\boldsymbol{r}, \boldsymbol{r}', k) - (\hat{\boldsymbol{n}}_{sc} \times \boldsymbol{A}(\boldsymbol{r}))\nabla G(\boldsymbol{r}, \boldsymbol{r}', k)\Big) \, \mathrm{d}S_{sc} \\ &\qquad\qquad + \boldsymbol{A}(\boldsymbol{r'})_{inc} = \begin{cases} \boldsymbol{A}(\boldsymbol{r'}) & \boldsymbol{r'} \, \text{outside} \, S_{sc}, \\ 0 & \boldsymbol{r'} \, \text{inside} \, S_{sc}, \end{cases} \end{aligned} \quad (3.3)$$

where $\boldsymbol{n}_{sc}$ is the outward normal to the surface of the scatterer and $\boldsymbol{A}(\boldsymbol{r'})_{inc}$ is the incoming field. The first row in the integral representation (3.3) is used in integral equation methods while the T matrix method also makes use of the second row.

In eddy current interaction there only exist closed currents and $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$. In paper A where the 2D case is considered the third term in the surface integral in Eq. (3.3) will also be zero. The other terms will remain, but in a much simpler form. In paper B the scatterer is a flat crack where the boundary condition across the crack are taken as

$$\nabla \times \boldsymbol{A}(\boldsymbol{r})_- = \nabla \times \boldsymbol{A}(\boldsymbol{r})_+, \quad (3.4)$$

$$\hat{\boldsymbol{n}}_{sc} \cdot \boldsymbol{A}(\boldsymbol{r})_- = \hat{\boldsymbol{n}}_{sc} \cdot \boldsymbol{A}(\boldsymbol{r})_+ = 0. \quad (3.5)$$

Here the indices plus and minus denote the limit from the two sides. By introducing these boundary conditions the surface integral is reduced to only contain the last term. The resulting integral representation seems to be neat, but an even more convenient integral representation for this case can be found by using another form of Green's theorem and the Green's tensor (see Bowler [3])

$$\hat{\boldsymbol{n}}_{sc} \cdot \boldsymbol{A}(\boldsymbol{r'})_{inc} + i\omega \lim_{x \to 0^+} \int_{-\infty}^{\infty} \int_{0}^{a} G_{11}(\boldsymbol{r}, \boldsymbol{r}') V(\boldsymbol{r}') \, \mathrm{d}z' \mathrm{d}y' = 0. \tag{3.6}$$

Here $\nabla' V(\boldsymbol{r'}) = \boldsymbol{E}^-(\boldsymbol{r'}) - \boldsymbol{E}^+(\boldsymbol{r'})$ is the jump in the electric field across the crack. The crack lies in the yz-plane and has the height a. This integral representation is used in paper B.

In addition to the formulation of the integral representations (3.3) and (3.6) Green's theorem is also of great importance in the calculation of the impedance. The change in coil impedance due to a crack can be calculated by the following integration over a surface enclosing the coil (see [1])

$$\Delta Z = \frac{1}{I^2} \int_{S_i} (\boldsymbol{E}_b \times \boldsymbol{H}_a - \boldsymbol{E}_a \times \boldsymbol{H}_b) \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{3.7}$$

With the index b denoting the fields in the presence of a defect and a the fields in the absence of a defect. Again the calculation only involves surface fields. Now by use of Green's theorem and Lorentz reciprocity relation for a source-free region

$$\nabla \cdot (\boldsymbol{E}_a \times \boldsymbol{H}_b - \boldsymbol{E}_b \times \boldsymbol{H}_a) = 0, \tag{3.8}$$

Eq. (3.7) can be rewritten in an even more convenient form

$$\Delta Z = \frac{1}{I^2} \int_{S_{sc}} (\boldsymbol{E}_a \times \boldsymbol{H}_b - \boldsymbol{E}_b \times \boldsymbol{H}_a) \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{3.9}$$

Here the integration is over a surface enclosing the defect. The information about the coil geometry is now needed only to determine the incoming field. Thus it is possible to derive an expression for the change in impedance due to a specific scatterer for an arbitrary source.

3.2 Basis functions

A set of functions is called a basis if every solution can be expressed as a sum of these functions. The functions in such a set are called basis functions. It is a great advantage to expand the unknown solution in these known functions and the technique is used in many methods of solution. Then the problem consists of determining the expansion coefficients. For example, the solution to Helmholtz (2.8) in 2D on a cylindrical surface can be expanded as

$$A(r, \theta) = \sum_{m\varsigma} \zeta_{m\varsigma} Re\chi_{m\varsigma}(\boldsymbol{r}, k), \tag{3.10}$$

where $\zeta_{m\varsigma}$ are the unknown expansion coefficients and the basis are defined as

$$Re\chi_{m\varsigma}(\boldsymbol{r},k) = \frac{\sqrt{\epsilon_m}}{2} J_m(kr) \begin{cases} \cos(m\phi), & \text{if} \, \varsigma = e, \\ \sin(m\phi), & \text{if} \, \varsigma = o. \end{cases} \tag{3.11}$$

Here $J_n(kr)$ is a bessel function, $m \in N$ and ς denotes odd or even.

The solutions can also be expressed in terms of plane waves which are convenient when the scattering surface is a plane, e.g. the surface of the conductive material. By studying the plane wave solution one can get a hint of how the frequency of the fields and the electrical and magnetic properties of the material (σ, μ and ε) affect the solution. The plane wave basis functions are defined as

$$\varphi(\boldsymbol{k},\boldsymbol{r}) = \frac{1}{\sqrt{8\pi}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{3.12}$$

where $\boldsymbol{k} = k\hat{\boldsymbol{k}}$. Here $\hat{\boldsymbol{k}}$ is the unit vector in the direction of propagation. The imaginary part of k will affect the amplitude whereas the real part will affect the phase. In the eddy current interaction problem the frequency is assumed to be low such that $\sigma \gg \omega\,\varepsilon$. Because of this the imaginary part is equal to the real part, i.e. $k = (1+i)\sqrt{\mu\,\omega\,\sigma/2}$. This states that the important parameter is this product $\mu\,\omega\,\sigma$. This justifies the introduction of the variable $\delta = \sqrt{2/\mu\,\omega\,\sigma}$ which is called the skin depth. At one skin depth the amplitude of the field has been reduced to 37% (e^{-1}) of the amplitude at the surface.

The main difference of the analytical methods compared to the finite element method (FEM) or the boundary element method (BEM) is the choice of basis functions. The basis functions above are preferably used to describe the field on a cylindrical or plane surface, respectively. Other basis functions can be used for other kind of surfaces. With the BEM this is avoided by discritizing the surfaces into a set of elements. Then quite simple basis functions can be used e.g. a polynomial of low order. In the FEM the Green's functions technique is not used at all and the entire volume of interest is discritized in a set of elements.

4 Summary of appended papers

4.1 Paper A

A 2D model of the eddy current interaction problem that consists of an inhomogeneity in a conductive half space is presented. The applied analytical method of solution is the transition (T) matrix method. This involves use of the free space Green's function to generate a system of boundary integral relations. In this way, it is easy to identify the contributions to the total solution from each different scattering surface. The different parts are separated also in the computation of the impedance. This leads to low cost simulations in terms of computation time and qualify the method to be used to obtain probability of detection (POD) curves. The model is compared with a Finite Element (FE) model and numerical examples for the case with a cylindrical inhomogeneity are given.

4.2 Paper B

An integral equation method to the nondestructive evaluation problem for a flat, tilted, surface-breaking crack in a conducting half-space is presented. The method involves use of the half-space Green's function and the Bowler potential. This potential describes the jump in the electric field over the crack and is expanded in basis functions related to the Chebyshev polynomials, being a more analytical approach than the commonly used boundary element method (BEM). In the method the scatterer defines a transformation operator to be applied on the incoming field. This is practical in simulations of the eddy current inspection where this operator just has to be generated once and not for every position of the probe. The numerical calculations of the change in impedance due to the crack are compared with a Finite Element (FE) model of the problem and good agreement is found.

5 Concluding remarks

Both the T matrix method described in paper A and the integral equation method described in paper B have shown to be applicable methods of solution for the eddy current interaction problem. The T matrix method has the great advantage of being a building block method, meaning that several T matrices describing different scatterers can be easily put together into a total T matrix which generates the total scattered field. To model eddy current inspection on the inside of a pipe, the T matrix could be used to describe the inner pipe wall and with the method from paper B a surface breaking crack could be introduced. Another possible combination is to use the T matrix method together with a finite element method solution approach to manage complex scattering geometries. On the other hand, it is preferable to use other analytical methods to change the scattering geometries when possible, e.g. the method described in [10]. The integral equation method may also be used for finite flat cracks.

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Appended Papers

Paper A

The T matrix method for a 2D eddy current interaction problem

Paper B

Integral equation method for eddy current nondestructive evaluation of a tilted, surface-breaking crack