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Branching-stable point processes

Contents

Introduction

A random measure on a complete separable metric space (c.s.m.s.) $\mathcal{X}$ is called strictly α-stable (StαS) if

$$t^{1/\alpha}\xi' + (1-t)^{1/\alpha}\xi'' \overset{\mathcal{D}}{=} \xi \qquad \forall t \in [0,1], \tag{1}$$

where ξ' and ξ'' are independent copies of ξ and $\overset{\mathcal{D}}{=}$ denotes the equality in distribution. Trying to extend the notion of strict stability to point processes we face the problem that the scalar multiplication doesn't preserve the integer-valued nature of point processes. We need a well-defined “multiplication" acting on point processes. The simplest way to obtain it is to use a stochastic analogous of multiplication: independent thinning, which we will denote by $\circ$. Thus we say that a point process Φ on a c.s.m.s. $\mathcal{X}$ is discrete α-stable (DαS) if

$$t^{1/\alpha} \circ \Phi + (1-t)^{1/\alpha}\Phi' \overset{\mathcal{D}}{=} \Phi \qquad \forall t \in [0,1],$$

where Φ' and Φ'' are independent copies of Φ. Davidov, Molchanov and Zuyev in [3] study DαS point processes and prove that they are Cox processes (doubly stochastic point processes) directed by StαS random measures. Therefore DαS point processes inherit properties from StαS random measures, like spectral and LePage representations. Davidov, Molchanov and Zuyev also provide a cluster representation for such processes based on Sibuya point processes. In the second chapter of the present work, after having provided basic notions of point process theory in the first chapter, we go through the main results of their article.

In the third chapter we propose a generalization of discrete stability for point processes considering a stochastic operation which is more general then thinning. We allow every point to be replaced by a random number of points rather than just deleting it or retaining it as in the thinning case. We refer to this operation as branching. Every branching operation is constructed from a subcritical Markov branching process $\big(Y(t)\big)_{t>0}$ with generator semigroup $\mathcal{F} = (F_t)_{t\geq 0}$ and satisfying $Y(0) = 1$. Following what Steutel and Van Harn did in the integer-valued random variables case (see [4]) we denote this operation by $\circ_{\mathcal{F}}$. In this setting when a point process is “multiplied" by a real number $t \in (0, 1]$ every point is replaced by a bunch of points located in the same position of their progenitor. The number of points in the bunch is stochastically distributed according to the distribution of $Y(-\ln(t))$. This operation preserves distributivity and associativity with respect to superposition and generalize thinning.

Then we characterize stable point processes with respect to branching operations $\circ_{\mathcal{F}}$, which we call $\mathcal{F}$-stable point processes. Let Y_∞ denote the limit distribution of the branching process $\big(Y(t)\big)_{t>0}$ conditional to the survival of the process. We prove that if we replace every point of a DαS point process with a stochastic number of points on the same location according to Y_∞ we obtain an $\mathcal{F}$-stable point process. Viceversa every $\mathcal{F}$-stable point process can be constructed in this way. Further we deduce some properties of $\mathcal{F}$-stable point process.

Trying to move to a broader context we asked ourselves which class of operations is the most general and appropriate one to study stability.

Given a stochastic operation $\circ$ on point processes the associative and distributive property of $\circ$ are enough to prove that Φ is stable with respect to $\circ$ if and only if

$$\forall n \in \mathbb{N} \; \exists c_n \in [0,1] \;\; : \;\; \Phi \stackrel{\mathcal{D}}{=} c_n \circ (\Phi^{(1)} + ... + \Phi^{(n)}),$$

where $(\Phi^{(1)}, ..., \Phi^{(n)})$ are independent copies of Φ. In such a context stable point processes arise inevitably in various limiting schemes similar to the central limit theorem involving superposition of point processes and therefore are worth being considered. That's why in the fourth chapter we study and characterize this class of stochastic operations. We prove that a stochastic operation on point processes satisfies associativity and distributivity if and only if it presents a branching structure: "multiplying" by t a point process is equivalent to let the process evolve for $-\ln(t)$ time according to some general Markov branching process (therefore including diffusion and general branching).

Chapter 1

Preliminaries

1.1 Definition of a Point Process

Spaces of measures

This first chapter follows Daley and Vere-Jones approach ([1] and [2]). In the whole chapter $\mathcal{X}$ will be a complete separable metric space (c.s.m.s.), $\mathcal{B}(\mathcal{X})$ its Borel σ-algebra, and μ will denote a measure on $\mathcal{B}(\mathcal{X})$.

Definition 1.

1. *$\mathcal{M}_{\mathcal{X}}$ is the space of all finite measures on $\mathcal{B}(\mathcal{X})$, i.e. measures μ such that $\mu(\mathcal{X}) < +\infty$;*
2. *$\mathcal{N}_{\mathcal{X}}$ is the space of all finite, integer-valued measures on $\mathcal{B}(\mathcal{X})$, i.e. finite measures μ such that $\mu(A) \in \mathbb{N}$ for every $A \in \mathcal{B}(\mathcal{X})$;*
3. *$\mathcal{M}^{\#}_{\mathcal{X}}$ is the space of all boundedly finite measure on $\mathcal{B}(\mathcal{X})$, i.e. measures μ such that $\mu(A) < +\infty$ for every A bounded, $A \in \mathcal{B}(\mathcal{X})$;*
4. *$\mathcal{N}^{\#}_{\mathcal{X}}$ is the space of all boundedly finite, integer-valued measure (counting measures for short) on $\mathcal{B}(\mathcal{X})$;*
5. *$\mathcal{N}^{\#*}_{\mathcal{X}}$ is the space of all simple counting measures on $\mathcal{B}(\mathcal{X})$, i.e.*

counting measure μ such that $\mu(x) = 0$ or 1 for every $x \in \mathcal{X}$.

Counting measures play a central role in this work, we therefore give the following results.

Proposition 1. *A boundedly finite measure μ on $\mathcal{B}(\mathcal{X})$ is a counting measure iff*

$$\mu = \sum_{i \in I} k_i \delta_{x_i} \tag{1.1}$$

where $\{x_i\}_{i \in I}$ is a set of countable many distinct points indexed by I, with at most finitely many in every bounded set, k_i are positive integers and δ_{x_i} represents the Dirac measure with center in x_i.

Definition 2. *Let μ be a counting measure written in the form of equation (1.1): $\mu = \sum_{i \in I} k_i \delta_{x_i}$. The support counting measure of μ is*

$$\mu^* = \sum_{i \in I} \delta_{x_i}$$

Proposition 2. *Let μ be a counting measure on $\mathcal{X}$. μ is simple (i.e. $\mu \in \mathcal{N}_{\mathcal{X}}^{\#*}$) iff $\mu = \mu^*$ a.s..*

Topologies and σ-alebras

We will need to define random elements on $\mathcal{M}_{\mathcal{X}}^{\#}$ and $\mathcal{N}_{\mathcal{X}}^{\#}$. In order to do that we need σ-algebras.

Definition 3. (w$^{\#}$-convergence) *Let $\{\mu_n\}_{n \in \mathbb{N}}, \mu \in \mathcal{M}_{\mathcal{X}}^{\#}$. Then $\mu_n \to \mu$ weakly$^{\#}$ if $\int f d\mu_n \to \int f d\mu$ for all f bounded and continuous on $\mathcal{X}$ that vanishes outside a bounded set.*

Remark 1. *The w$^{\#}$-convergence can be seen as metric convergence thanks to the Prohorov metric, which is defined as follows. Given $\mu, \nu \in \mathcal{M}_{\mathcal{X}}$*

$$\begin{aligned} d(\mu, \nu) = \inf \big\{ \epsilon > 0 : \quad & \mu(F) < \nu(F^{\epsilon}) + \epsilon \text{ and} \\ & \nu(F) < \mu(F^{\epsilon}) + \epsilon \quad \forall F \subseteq \mathcal{X} \text{ closed subset} \big\} \end{aligned}$$

where $F^\epsilon = \{x \in X : \rho(x, F) < \epsilon\}$. *The Prohorov metric* d*, whose convergence is equivalent to the weak convergence, can be extended to a metric* $d^\#$ *on* $\mathcal{M}^\#_\mathcal{X}$. *Given* $\mu, \nu \in \mathcal{M}^\#_\mathcal{X}$

$$d^\#(\mu, \nu) = \int_0^{+\infty} e^{-r} \frac{d(\mu^{(r)}, \nu^{(r)})}{1 + d(\mu^{(r)}, \nu^{(r)})} dr$$

where, having fixed a point $O \in \mathcal{X}$ *to be the origin of the space* $\mathcal{X}$, $\mu^{(r)}$ *(and analogously* $\nu^{(r)}$*) is defined as*

$$\mu^{(r)}(A) = \mu(A \cap S(O, r)) \qquad \forall A \in \mathcal{B}(\mathcal{X})$$

where $S(O, r)$ *denotes the open sphere with radius* r *and centre* O.

Proposition 3. *Let* $\{\mu_n\}_{n\in\mathbb{N}}, \mu \in \mathcal{M}^\#_\mathcal{X}$. $\mu_n \to \mu$ *weakly*$^\#$ *iff* $d^\#(\mu_n, \mu) \to 0$.

Given the w$^\#$-topology we have a Borel σ-algebra on $\mathcal{M}^\#_\mathcal{X}$, which we will call $\mathcal{B}(\mathcal{M}^\#_\mathcal{X})$. It is a very natural σ-algebra, as the next proposition shows.

Proposition 4. $\mathcal{B}(\mathcal{M}^\#_\mathcal{X})$ *is the smallest* σ*-algebra such that the mappings* $\mu \to \mu(A)$ *from* $\mathcal{M}^\#_\mathcal{X}$ *to* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *are measurable for every* $A \in \mathcal{B}(\mathcal{X})$.

Since $\mathcal{N}^\#_\mathcal{X}$ is a measurable (indeed closed) subset of $\mathcal{M}^\#_\mathcal{X}$, we have an analogous result for the Borel σ-algebra of $\mathcal{N}^\#_\mathcal{X}$: $\mathcal{B}(\mathcal{N}^\#_\mathcal{X})$.

Proposition 5.

1. $A \in \mathcal{B}(\mathcal{N}^\#_\mathcal{X})$ *iff* $A \in \mathcal{B}(\mathcal{M}^\#_\mathcal{X})$ *and* $A \subseteq \mathcal{N}^\#_\mathcal{X}$;
2. $\mathcal{B}(\mathcal{N}^\#_\mathcal{X})$ *is the smallest* σ*-algebra such that the mappings* $\mu \to \mu(A)$ *from* $\mathcal{N}^\#_\mathcal{X}$ *to* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *are measurable for every* $A \in \mathcal{B}(\mathcal{X})$.

Random measures and point processes

We can now define the main notions of this section.

Definition 4. *1. A random measure ξ with phase space $\mathcal{X}$ is a measurable mapping from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to $(\mathcal{M}^{\#}_{\mathcal{X}}, \mathcal{B}(\mathcal{M}^{\#}_{\mathcal{X}}))$;*

2. A point process (p.p.) Φ with phase space $\mathcal{X}$ is a measurable mapping from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to $(\mathcal{N}^{\#}_{\mathcal{X}}, \mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}}))$. A point process Φ is simple if $\Phi \in \mathcal{N}^{\#*}_{\mathcal{X}}$ a.s. (i.e. $\Phi = \Phi^*$ a.s.).

From this definition and Propositions 4 and 5 we obtain the following result.

Proposition 6. *A mapping ξ $\big[\Phi\big]$ from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to $\mathcal{M}^{\#}_{\mathcal{X}}$ $\big[\mathcal{N}^{\#}_{\mathcal{X}}\big]$ is a random measure $\big[$point process$\big]$ iff $\xi(A, \cdot)$ $\big[\Phi(A, \cdot)\big]$ is a random variable for every bounded $A \in \mathcal{B}(\mathcal{X})$.*

A random measure is uniquely characterized by its finite dimensional distributions.

Definition 5. *Let Φ be a point process on $\mathcal{X}$. The finite dimensional distributions (fidi distributions) of Φ are the distributions of the random variables $\big(\Phi(A_1), ..., \Phi(A_k)\big)$. For every finite family of bounded Borel sets $\{A_1, ..., A_k\}$ and nonnegative integers $\{n_1, ..., n_k\}$*

$$P_k(A_1, ..., A_k; n_1, ..., n_k) = Pr\big(\Phi(A_1) = n_1, ..., \Phi(A_k) = n_k\big).$$

Proposition 7. *The distribution of a random measure on $\mathcal{X}$ is totally determined by the finite dimensional distributions of all finite families $\{A_1, ..., A_k\}$ of bounded disjoint Borel sets.*

1.2 Intensity Measure and Covariance Measure

We firstly need to introduce the notion of moment measures.

Lemma 1. *Given a point process Φ $M : \mathcal{B}(\mathcal{X}) \to \mathbb{R}$ defined by*

$$M(A) = \mathbb{E}(\Phi(A)) \tag{1.2}$$

is a measure on $\mathcal{B}(\mathcal{X})$

Proof. M inherits the finite additivity from the finite additivity of Φ and of the expectation. Moreover M is continuous from below because if $A_n \uparrow A$ then $\Phi(A_n) \uparrow \Phi(A)$ pointwise and for the monotone convergence $M(A_n) \uparrow M(A)$. $\square$

Definition 6. *Given a point process* Φ*,* M *defined as in equation (1.2) is the first-order moment measure of* Φ*.*

There exist also higher order moment measures.

Definition 7. *Let* Φ *be a point process. We denote by* $\Phi^{(n)}$ *the n-th fold product measure of* Φ*, i.e. the (random) measure* $\Phi^{(n)}$ *on* $\mathcal{B}(\mathcal{X} \times ... \times \mathcal{X}) = \mathcal{B}(\mathcal{X}^n)$ *defined by*

$$\Phi^{(n)}(A_1 \times ... \times A_n) = \Phi(A_1) \cdot ... \cdot \Phi(A_n)$$

with $A_i \in \mathcal{B}(\mathcal{X})$ *for i=1,...,n.*

The definition is well-posed and the measure is uniquely determined because the semiring of the rectangles generates the product σ-algebra $\mathcal{B}(\mathcal{X}^n)$.

Definition 8. *Let* Φ *be a point process. The k-th order moment measure,* M_n*, is the expected value of* $\Phi^{(n)}$

$$M_n(A) = \mathbb{E}(\Phi^{(n)}(A)) \qquad \forall A \in \mathcal{B}(\mathcal{X}^n).$$

We now turn to the intensity and correlation measures. In order to introduce the notion of intensity measure we need the definition of dissecting system.

Definition 9. *A dissecting system for* $\mathcal{X}$ *is a sequence* $\{\tau_n\}_{n\geq 1}$ *of partitions of* $\mathcal{X}$*,* $\tau_n = \{A_{ni}\}_{i\in I_n}$*, that satisfies the following properties:*

- *Nesting property:* $A_{n-1,i} \cap A_{nj} = \emptyset$ *or* A_{nj};
- *Separating property: given* $x, y \in \mathcal{X}$, $x \neq y$ *there exists an* $n = n(x,y)$ *and an* $i \in I_n$ *such that* $x \in A_{ni}$ *and* $y \notin A_{ni}$.

Definition 10. *The intensity measure of a point process* Φ *is a measure* Λ *on* $\mathcal{B}(\mathcal{X})$ *defined as*

$$\Lambda(A) = \sup_{n \geq 1} \sum_{i \in I_n} P(\Phi(A_{ni}) \geq 1) \qquad \forall A \in \mathcal{B}(\mathcal{X})$$

where $\{\tau_n\}_{n \geq 1}$ *is a dissecting system for* A.

We can give also another characterization of the intensity measure, which will guarantee the intensity measure to be a well-defined measure, not depending on the dissecting system chosen.

Theorem 1. *(Khinchin's existence theorem)*
Given a point process Φ *on* $\mathcal{X}$, *and its intensity measure* Λ *it holds*

$$\Lambda(A) = M^*(A) \qquad \forall A \in \mathcal{B}(\mathcal{X})$$

where M^* *is the first-order moment measure of the support* Φ^*.

The next proposition follows as an immediate consequence of Khinchin's existence theorem and Proposition 2.

Proposition 8. *Let* Φ *be a simple point process. Then* $M(A) = \Lambda(A)$ *for every* $A \in \mathcal{B}(\mathcal{X})$.

We now define the notion of covariance measure.

Definition 11. *Given a point process* Φ, *its covariance measure* C_2 *is a measure on* $\mathcal{B}(\mathcal{X} \times \mathcal{X})$. *For every Borel sets* A *and* B

$$C_2(A \times B) = M_2(A \times B) - M(A) \cdot M(B).$$

1.3 Probability Generating Functional

Dealing with random measures a useful tool is the Laplace functional.

Definition 12. *Let ξ be a random measure. For every $f \in BM_+(\mathcal{X})$, the space of positive, bounded and measurable functions with compact support defined over $\mathcal{X}$, the Laplace functional is*

$$L_\xi[f] = \mathbb{E}[\exp\big\{\int_{\mathcal{X}} f(x)\xi(dx)\big\}].$$

The distribution of a random measure is uniquely fixed by its Laplace functional. An analogous instrument that is more appropriate for point processes is the probability generating functional.

Definition 13. *$\mathcal{V}(\mathcal{X})$ denotes the set of all measurable real-valued functions defined on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that $0 \leq h(x) \leq 1$ for every $x \in \mathcal{X}$ and $1 - h$ vanishes outside a bounded set.*

Definition 14. *Let Φ be a point process on $\mathcal{X}$. The probability generating functional (p.g.fl.) of Φ is the functional*

$$G[h] = \mathbb{E}\Big[\exp\big(\int_{\mathcal{X}} \log h(x) d\Phi(x)\big)\Big],$$

defined for every $h \in \mathcal{V}(\mathcal{X})$. Since $h \equiv 1$ outside a bounded set this expression can be seen as the expectation of a finite product

$$G[h] = \mathbb{E}\Big[\prod_i h(x_i)\Big],$$

where the product runs over the points of Φ belonging to the support of $1-h$. In case no point of Φ falls into the support of $1-h$ the product's value is one.

Theorem 2. *Let G be a real-valued functional defined on $\mathcal{V}(\mathcal{X})$. G is a p.g.fl. of a point process Φ if and only if the following three condition hold.*

1. *For every h of the form*

$$1 - h(x) = \sum_{k=1}^{n} (1 - z_k) \mathbb{1}_{A_k}(x),$$

   where $A_1, ..., A_n$ are disjoint Borel sets and $|z_k| < 1$ for every k, the p.g.fl. $G[h]$ reduces to the joint p.g.f. $P_n(A_1, ..., A_n; z_1, ..., z_n)$ of an n-dimensional integer-valued random variable;

2. *if $\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{V}(X)$ and $h_n \downarrow h \in \mathcal{V}(\mathcal{X})$ pointwise then $G[h_n] \to G[h]$;*

3. *$G[\mathbf{1}] = 1$, where **1** denotes the function identically equal to unity in $\mathcal{X}$.*

Moreover, whether these conditions are satisfied, the p.g.fl. G uniquely determines the distribution of Φ.

1.4 Some examples: Poisson, Cluster and Cox.

1.4.1 Poisson Process

Definition 15. *Let Λ be a boundedly finite measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, $\mathcal{X}$ being a complete separable metric space (c.s.m.s.). The Poisson point process Φ with parameter measure Λ is a point process on $\mathcal{X}$ such that for every finite collection of disjoint Borel sets $\{A_i\}_{i=1,\dots,k}$*

$$Pr\big(\Phi(A_i) = n_i : \quad i = 1, ..., n\big) = \prod_{i=1}^{n} \frac{e^{-\Lambda(A_i)} \Lambda(A_i)^{n_i}}{n_i!} \ .$$

We give now a first result about Poisson process characterization.

Theorem 3. *Let Φ be a point process. Φ is a Poisson process iff there exists a boundedly finite measure Λ on $\mathcal{B}(\mathcal{X})$ such that $\Phi(A)$ has a Poisson distribution with parameter $\Lambda(A)$ for every bounded Borel set A.*

Remark 2. *A Poisson process* Φ *can have fixed atoms, i.e. points* $x \in \mathcal{X}$ *such that* $Pr\big(\Phi(\{x\}) > 0\big) > 0$. x *is a fixed atom for a Poisson process* Φ *if and only if* $\Lambda(\{x_0\}) > 0$.

There is another property of p.p. which will be fundamental for the next results: the orderliness.

Definition 16. *A p.p.* Φ *is said to be orderly if for every* $x \in \mathcal{X}$

$$Pr\big(\Phi(S(x,\epsilon)) > 1\big) = o\big(Pr(\Phi(S(x,\epsilon)) > 0)\big) \quad \epsilon \to 0,$$

where $S(x,\epsilon)$ *denotes the open sphere of centre* x *and radius* ϵ.

It can be shown that for a Poisson process to be orderly is equivalent to have no fixed point. Using orderliness we can give two more results regarding Poisson process characterization.

Theorem 4. *Let* Φ *be an orderly p.p.. Then* Φ *is a Poisson process iff there exists a boundedly finite measure* Λ *with no atoms* ($\Lambda(\{x\}) = 0$ $\forall x \in \mathcal{X}$) *such that*

$$P_0(A) \doteq Pr\big(\Phi(A) = 0\big) = e^{-\Lambda(A)} \qquad \forall A \in \mathcal{B}(\mathcal{X}).$$

The Poisson process can also be identified using the complete independence property.

Theorem 5. *Let* Φ *be a p.p. with no fixed atoms.* Φ *is Poisson process iff the following conditions hold.*

(i) Φ *is orderly;*

(ii) for every finite collection $A_1, ..., A_k$ *of disjoint, bounded Borel sets the random variables* $\Phi(A_1), ..., \Phi(A_k)$ *are independent (complete independence property).*

The p.g.fl. of a Poisson process Φ with parameter measure Λ is

$$G_\Phi[h] = \exp\{-\int_{\mathcal{X}} 1 - h(x)\Lambda(dx)\}. \tag{1.3}$$

1.4.2 Cox Process

In order to define the Cox process, also called doubly stochastic Poisson process, we need some instruments.

Definition 17. *A family $\{\Phi(\cdot|y) : y \in \mathcal{Y}\}$ of p.p. on the c.s.m.s. $\mathcal{X}$, indexed by the elements of a c.s.m.s. $\mathcal{Y}$, is a measurable family of p.p. if $\mathcal{P}(A|y) \doteq Pr\big(\Phi(\cdot|y) \in A\big)$ is a $\mathcal{B}(\mathcal{Y})$-measurable function of y for every bounded set $A \in \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$.*

Proposition 9. *If we have a measurable family of point processes on $\mathcal{X}$ $\{\Phi(\cdot|y) : y \in \mathcal{Y}\}$, and a random measure ξ on the c.s.m.s. $\mathcal{Y}$ with distribution Π then*

$$\mathcal{P}(A) = \int_{\mathcal{Y}} \mathcal{P}(A|y)\Pi(dy). \tag{1.4}$$

defines a probability on $\mathcal{N}_{\mathcal{X}}^{\#}$ and therefore a point process Φ on $\mathcal{X}$.

Whether relation (1.4) holds we say that $\mathcal{P}(\cdot|y)$ is the distribution of Φ conditional to the realization y of ξ. We can now define the Cox process.

Definition 18. *Given a random measure ξ, a Cox Process directed by ξ is a point process Φ such that the distribution of Φ conditional on ξ, $\Phi(\cdot|\xi)$, is the one of a Poisson point process with parameter measure ξ.*

In order to show that such a process is well-defined we can use Proposition 9 but we need to check that the indexed family of p.p. we're using is a measurable family.

Lemma 2. *A necessary and sufficient condition for a family of p.p. on $\mathcal{X}$ indexed by the elements of $\mathcal{Y}$ to be a measurable family is that the fidi distributions $P_k(B_1, ..., B_k; n_1, ..., n_k|y)$ are $\mathcal{B}(\mathcal{Y})$-measurable functions of y for all the finite collections $\{B_1, ..., B_k\}$ of disjoint sets of $\mathcal{B}(\mathcal{X})$, and for all the choices of the nonnegative integers $n_1, ..., n_k$.*

In the definition of Cox process we have $\mathcal{Y} = \mathcal{N}_{\mathcal{X}}^{\#}$ and the fidi distributions are the ones of a Poisson process directed by ξ, which are measurable functions of $\big(\xi(B_i)\big)_{i=1,\dots,n}$, which themselves are random variables. Therefore we can apply the lemma.
Using Proposition 9 we can evaluate the fidi probabilities for a Cox Process. For example, given $B \in \mathcal{B}(\mathcal{X})$ and $k \in \mathbb{N}$

$$P(B,k) = Pr(\Phi(B) = k) = \mathbb{E}\big(\frac{\xi(B)^k e^{-\xi(B)}}{k!}\big) = \int_0^{+\infty} \frac{x^k e^{-x}}{k!} F_B(dx)$$

where F_B is the distribution function for $\xi(B)$.
A Cox point process Φ directed by ξ has p.g.fl.

$$G_\Phi[h] = \mathbb{E}\Big[\exp\{-\int_{\mathcal{X}} \big(1 - h(x)\big)\xi(dx)\}\Big] = L_\xi[1-h]. \tag{1.5}$$

1.4.3 Cluster Process

Definition 19. *A point process Φ on a c.s.m.s. $\mathcal{X}$ is a cluster process with centre process Φ_c on the c.s.m.s. $\mathcal{Y}$ and component processes (or daughter processes) the measurable family of point processes $\{\Phi(\cdot|x) : y \in \mathcal{Y}\}$ if for every bounded set $A \in \mathcal{B}(\mathcal{X})$*

$$\Phi(A) = \int_{\mathcal{Y}} \Phi(A|y)\Phi_c(dy) = \sum_{y \in \Phi_c} \Phi(A|y).$$

The component processes are often required to be independent. In that case we have an indipendent cluster process and if $\Phi(\{y_i\}) > 1$ multilpe indipendent copies of $\Phi(A|y_i)$ are taken.
We give an existence result for indipendent cluster processes.

Proposition 10. *An independent cluster process exists iff for any bounded set $A \in \mathcal{B}(\mathcal{X})$*

$$\int_{\mathcal{Y}} p_A(y)\Phi_c(dy) = \sum_{y_i \in \Phi_c} p_A(y_i) < +\infty \qquad \Pi_c\, a.s.,$$

where $p_A(y) = Pr(\Phi(A|y) > 0)$ *and* Π_c *is the distribution of the centre process* Φ_c*.*

From now on we will deal only with independent cluster processes, and we will just call them cluster processes. Using the independence property we obtain that

$$\begin{aligned} G[h] = \mathbb{E}\big(G[h|\Phi_c]\big) =& \mathbb{E}\Big(\exp\big\{ - \int_{\mathcal{Y}} (-\log G_d[h|y])\Phi_c(dy)\big\}\Big) = \\ =& \, G_c\big[G_d[h|\cdot]\big] \end{aligned} \tag{1.6}$$

1.5 Campbell Measure and Palm Distribution

Definition 20. *Given a p.p.* Φ *on a c.s.m.s.* $\mathcal{X}$ *and the associated distribution* $\mathcal{P}$ *on* $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$*, we can define the Campbell measure* $\mathcal{C}_{\mathcal{P}}$ *as a measure on* $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$ *such that*

$$\mathcal{C}_{\mathcal{P}}(A \times U) = \mathbb{E}\big(\Phi(A)\mathbb{1}_U(\Phi)\big) \qquad \forall A \in \mathcal{B}(\mathcal{X}), U \in \mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}}). \tag{1.7}$$

Remark 3. *The set function defined in equation (1.7) is clearly* σ*-additive, and it can be shown to be always* σ*-finite. Therefore, being the rectangles a semiring generating* $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$*, the set function extends to a unique* σ*-finite measure. Thus* $\mathcal{C}_{\mathcal{P}}$ *is well-defined.*

Lemma 3. *Let* $\mathcal{P}$ *be a probability measures on* $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}})$ *and* $\emptyset$ *denote the zero measure on* $\mathcal{X}$*. Then* $\mathcal{P}$ *is uniquely determined on* $\mathcal{B}(\mathcal{N}^{\#}_{\mathcal{X}}) \backslash \{\emptyset\}$ *by its Campbell measure* $\mathcal{C}_{\mathcal{P}}$*.*

Remark 4. *There is a strong relationship between Campbell measure and the first-order moment measure. In fact from the definition of Campbell measure it follows that* M *is the marginal distribution of* $\mathcal{C}_{\mathcal{P}}$*:*

$$\mathcal{C}_{\mathcal{P}}(A \times \mathcal{N}^{\#}_{\mathcal{X}}) = \mathbb{E}(\Phi(A)) = M(A) \qquad \forall A \in \mathcal{B}(\mathcal{X}).$$

From this remark it follows that given a point process Φ, its Campbell measure $\mathcal{C}_{\mathcal{P}}$ and a fixed set $U \in \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$ the measure $\mathcal{C}_{\mathcal{P}}(\cdot \times U)$ is absolutely continuous with respect to $M(\cdot)$. We therefore can define a Radon-Nikodin derivative, $\mathcal{P}_x(U) : \mathcal{X} \to \mathbb{R}$, such that

$$\mathcal{C}_{\mathcal{P}}(A \times U) = \int_A \mathcal{P}_x(U) dM(x) \qquad \forall A \in \mathcal{B}(\mathcal{X}).$$

For every $U \in \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$ $\mathcal{P}_x(U)$ is fixed up to sets which have zero measure with respect to M. We can chose a family $\{\mathcal{P}_x(U) : x \in \mathcal{X}, U \in \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})\}$ such that the following conditions hold.

1. $\forall U \in \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$, $\mathcal{P}_x(U)$ is a measurable real-valued, M-integrable function defined on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$;

2. $\forall x \in \mathcal{X}$, $\mathcal{P}_x(\cdot)$ is a probability measure on $\mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$.

Definition 21. *Given a point process Φ, a family $\{\mathcal{P}_x(U)\}_{x\in\mathcal{X}}$ defined as above and satisfing condition 1) and 2) is called Palm kernel for Φ. For each point $x \in \mathcal{X}$ the probability measure $\mathcal{P}_x(\cdot)$ is called local Palm distribution.*

Proposition 11. *Let Φ be a point process with finite first moment measure M. Then Φ admitts a Palm kernel $\{\mathcal{P}_x(U)\}_{x\in\mathcal{X}}$. Every local Palm distribution $\mathcal{P}_x(\cdot)$ is uniquely fixed up to zero measure sets with respect to M. Moreover for any function g measurable with respect to $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$, that is positive or $\mathcal{C}_{\mathcal{P}}$-integrable*

$$\mathbb{E}\Big(\int_{\mathcal{X}} g(x,\Phi)\Phi(dx)\Big) = \int_{\mathcal{X}\times\mathcal{M}_{\mathcal{X}}^{\#}} g(x,\Phi)\mathcal{C}_{\mathcal{P}}(dx \times d\Phi) = \int_{\mathcal{X}} \mathbb{E}_x\big(g(x,\Phi)\big)M(dx), \tag{1.8}$$

where for every $x \in \mathcal{X}$

$$\mathbb{E}_x\big(g(x,\Phi)\big) = \int_{\mathcal{M}_{\mathcal{X}}^{\#}} g(x,\Phi)\mathcal{P}_x(d\Phi).$$

1.6 Slivnyak Theorem

Lemma 4. *Let Φ be a poisson process with first moment measure M finite. Let $L[f]$ be the Laplace functional associated to Φ, and $L_x[f]$ the ones associated to the Palm kernel $\{\mathcal{P}_x(U)\}_{x\in\mathcal{X}}$. Then for every $f, g \in BM_+(\mathcal{X})$*

$$\lim_{\epsilon\downarrow 0} \frac{L[f] - L[f+\epsilon g]}{\epsilon} = \int_X g(x) L_x[f] M(dx). \tag{1.9}$$

Theorem 6. *(Slivnyak, 1962). Let Φ be a p.p. with finite first moment measure M. $\mathcal{P}$ denotes the distribution of Φ and $\mathcal{P}_x$ its Palm kernel. Then Φ is a Poisson process iff*

$$\mathcal{P}_x = \mathcal{P} * \delta_x \tag{1.10}$$

where $*$ denotes the convolution of distributions, which corresponds to the superposition of point processes, and δ_x denotes the random measure identically equal to the Dirac measure with centre x.

Proof. Let Φ be a Poisson process with parameter measure μ. The Laplace functional for a Poisson process has the following form

$$\log L[f] = -\int_{\mathcal{X}} (1 - e^{-f(x)})\mu(dx).$$

Then

$$\begin{aligned} -\frac{dL[f+\epsilon g]}{d\epsilon} &= L[f+\epsilon g] \frac{d}{d\epsilon}\Big(\int_{\mathcal{X}} (1 - e^{-f(x)-\epsilon g(x)})\mu(dx)\Big) \\ &= L[f+\epsilon g]\Big(\int_{\mathcal{X}} g(x) e^{-f(x)-\epsilon g(x)} \mu(dx)\Big) \\ &\to L[f] \int_{\mathcal{X}} g(x) e^{-f(x)} \mu(dx) \qquad as\ \epsilon \to 0. \end{aligned} \tag{1.11}$$

Comparing with (1.9) we notice that the left-hand terms are the same, and using that $M(\cdot) = \mu(\cdot)$ we deduce

$$L_x[f] = L[f] e^{-f(x)} = L[f] L_{\delta_x}[f] \qquad \Lambda - a.s..$$

Thanks to Laplace functional properties this relation is equivalent to (1.10).

We now prove the converse. Suppose $\mathcal{P}$ and $\mathcal{P}_x$ satisfy (1.10). Then, using equation (1.9), we obtain

$$\frac{dL[\epsilon f]}{d\epsilon} = -L[\epsilon f] \int_{\mathcal{X}} f(x) e^{-\epsilon f(x)} M(dx).$$

Since $\log\big(L[0]\big) = \log(1) = 0$

$$-\log\big(L[0]\big) = \int_{\mathcal{X}} \int_0^1 f(x) e^{-\epsilon f(x)} d\epsilon M(dx) = \int_{\mathcal{X}} (1 - e^{-f(x)}) M(dx),$$

which is the Laplace functional of a Poisson process with parameter measure equal to M. □

1.7 Infinitely Divisibile Point Processes and KLM Measures

In the proceding of the work the notion of infinite disibility will be of great importance.

Definition 22. *A point process Φ is said to be infinitely divisible if for every positive integer k, there exists k independent and identically distributed (i.i.d.) point processes $\{\Phi_i^{(k)}\}_{i=1,\ldots,k}$ such that*

$$\Phi = \Phi_1^{(k)} + \ldots + \Phi_k^{(k)}. \tag{1.12}$$

If we move to p.g.fl. condition (1.12) becomes

$$G[h] = \big(G_{1/k}[h]\big)^k,$$

where $G_{1/k}$ denotes the p.g.fl. of one of the i.i.d. point processes $\Phi_i^{(k)}$. Therefore being infinitely divisible for a point process means that for every positive integer k the positive k-th root of the p.g.fl. G, we call it $G_{1/k}$, is a p.g.fl. itself.

We give a characterization for the infinite divisible p.p. in the case of finite point processes.

Theorem 7. *Let Φ be a p.p. with p.g.fl. $G_\Phi[h]$. Then Φ is a.s. finite and infinitely divisible iff there exist a point process $\widetilde{\Phi}$, a.s. finite and nonnull, and $c > 0$ such that*

$$G_\Phi[h] = \exp\{-c(1 - G_{\widetilde{\Phi}}[h])\}, \tag{1.13}$$

where $G_{\widetilde{\Phi}}$ is the p.g.fl. of $\widetilde{\Phi}$.

Remark 5. *Representation (1.13) has a probabilistic interpretation. It means that every finite and infinitely divisible p.p. Φ can be obtained as a Poisson randomization of a finite p.p. $\widetilde{\Phi}$, and conversely that every Poisson randomization of a finite p.p. $\widetilde{\Phi}$ is a finite and infinitely divisible p.p. Φ. By Poisson randomization of a p.p. $\widetilde{\Phi}$ we mean the superposition of a Poisson distributed random number of independent copies of $\widetilde{\Phi}$. The expression was introduced by Milne in [6]. Using (1.6) and recalling that the p.g.f. of a Poisson random variable with mean $c > 0$ is*

$$F(z) = \exp\{-c(1 - z)\},$$

it is immediate to deduce that the p.g.fl. expresses in (1.13) is exactly the one of the Poisson randomization of $\widetilde{\Phi}$. In such a context the infinite divisibility of Φ follows immediately from the infinite divisibility of Poisson distributed random variables.

This result can be generalized to the case of infinite divisible p.p. (not necessarily finite) using KLM measures.

Definition 23. *A KLM measure $\boldsymbol{Q}(\cdot)$ is a boundedly finite measure on the space of nonnull counting measures $\mathcal{N}_{\mathcal{X}}^{\#} \backslash \{0\}$ (see Definition 1)* *such that*

$$\boldsymbol{Q}\big(\{\varphi \in \mathcal{N}^{\#}_{\mathcal{X}}\backslash\{0\} : \ \varphi(A) > 0\}\big) < +\infty \qquad \forall A \textit{ measurable and bounded.} \tag{1.14}$$

Theorem 8. *A p.p. Φ is infinitely divisible if and only if its p.g.fl. can be represented as*

$$G_\Phi[h] = \exp\Big\{ - \int_{\mathcal{N}^{\#}_{\mathcal{X}}\backslash\{0\}} \big[1 - e^{\langle \log(h), \varphi\rangle}\big]\, \boldsymbol{Q}(d\varphi)\Big\}, \tag{1.15}$$

where $\langle \log(h), \varphi\rangle$ is a short notation for $\int_{\mathcal{X}} \log\big(h(x)\big)\varphi(dx)$ and $\boldsymbol{Q}(\cdot)$ is a KLM measure. The KLM measure satisfying (1.15) is unique.

Example 1. *The Poisson p.p. is infinitely divisible, therefore there must exist a KLM measure $\boldsymbol{Q}(\cdot)$ such that (1.15) reduces to (1.3). If we consider counting measures consisting of one point ($\varphi = \delta_x$ with $x \in \mathcal{X}$) then*

$$1 - e^{\langle \log(h), \varphi\rangle} = 1 - h(x).$$

Let us consider a KLM measure $\boldsymbol{Q}(\cdot)$ which is concentrated only on such counting measures, which means that

$$\boldsymbol{Q}\big(\{\varphi \in \mathcal{N}^{\#}_{\mathcal{X}} : \ \varphi(A) \neq 1\}\big) = 0,$$

and such that

$$\boldsymbol{Q}\big(\{\varphi \in \mathcal{N}^{\#}_{\mathcal{X}} : \ \varphi(A) = 1\}\big) = \Lambda(A) \qquad \forall A \textit{ measurable,}$$

where Λ is a boundedly finite measure on $\mathcal{X}$. With this KLM measure $\boldsymbol{Q}(\cdot)$ (1.15) becomes

$$G_\Phi[h] = \exp\Big\{ - \int_{\mathcal{X}} \big(1 - h(x)\big)\Lambda(dx)\Big\},$$

which is exactly the p.g.fl. of a Poisson point process with intensity measure Λ.

Using the association with KLM measures it is possible to define regular and singular infinite divisible point processes.

Definition 24. *An infinitely divisible point process Φ is called regular if its KLM measure $\mathcal{Q}(\cdot)$ is concentrated on the set*

$$\mathcal{N}_f = \{\varphi \in \mathcal{N}_{\mathcal{X}}^{\#} \backslash \{0\} : \ \varphi(\mathcal{X}) < +\infty\},$$

and singular if it is concentrated on

$$\mathcal{N}_\infty \{\varphi \in \mathcal{N}_{\mathcal{X}}^{\#} \backslash \{0\} : \ \varphi(\mathcal{X}) = +\infty\}.$$

Theorem 9. *Every infinitely divisible p.p. Φ can be written as*

$$\Phi = \Phi_r + \Phi_s,$$

where Φ_r and Φ_s are independent and infinitely divisible point processes, the first one being regular and the second one singular.

Chapter 2

Stability for random measures and point processes

2.1 Strict stability

A random vector X is called strictly α-stable (StαS) if

$$t^{1/\alpha}X' + (1-t)^{1/\alpha}X'' \stackrel{\mathcal{D}}{=} X \qquad \forall t \in [0,1],$$

where X' and X'' are independent copies of X and $\stackrel{\mathcal{D}}{=}$ denotes the equality in distribution. It is well-known (Feller, 1971, An Introduction to Probability Theory and its Application, 2nd Volume, Ch 6.1) that non-trivial StαS random variables exist only for $\alpha \in (0,2]$. Moreover if X is nonnegative α must belong to $(0,1]$.

If we provide a definition of sum and multiplication for a scalar in the context of random measures on complete separable metric spaces, then we can extend the definition of stability to that context. Let

$$\begin{aligned} &(\mu_1+\mu_2)(\cdot) = \mu_1(\cdot) + \mu_2(\cdot) \qquad \forall \mu_1, \mu_2 \in \mathcal{M}^{\#}_{\mathcal{X}}, \\ &(t\mu)(\cdot) = t\mu(\cdot) \qquad \forall t \in \mathbb{R}, \forall \mu \in \mathcal{M}^{\#}_{\mathcal{X}}. \end{aligned} \tag{2.1}$$

Definition 25. *A random measure ξ on a c.s.m.s. $\mathcal{X}$ is said to be strictly α-stable (StαS) if*

$$t^{1/\alpha}\xi' + (1-t)^{1/\alpha}\xi'' \stackrel{\mathcal{D}}{=} \xi \qquad \forall t \in [0,1], \tag{2.2}$$

where ξ' and ξ'' are indipendent copies of ξ.

Remark 6. *(2.2) implies that $\xi(A)$ is a StαS random variable for every measurable set A. Since $\xi(A)$ is always nonnegative non-trivial StαS random measures exist only for $\alpha \in (0,1]$.*

Definition 26. *A Levy measure Λ is a boundedly finite measure on $\mathcal{M}^{\#}_{\mathcal{X}}\backslash\{0\}$ homogeneous of order $-\alpha$ $\big($i.e. $\Lambda(tA) = t^{-\alpha}\Lambda(A)$ for every $A \in \mathcal{B}(\mathcal{M}^{\#}_{\mathcal{X}}\backslash\{0\})$ and $t > 0\big)$, such that*

$$\int_{\mathcal{M}^{\#}_{\mathcal{X}}\backslash\{0\}} (1 - e^{\langle h,\mu\rangle})\Lambda(d\mu) < +\infty \qquad \forall h \in BM(\mathcal{X}), \tag{2.3}$$

where $\langle h, \mu\rangle$ stands for $\int_{\mathcal{X}} h(x)\mu(dx)$.

Theorem 10. *A random measure ξ is StαS if and only if there exists a Levy measure Λ such that the Laplace functional of ξ has the form*

$$L_\xi[h] = \exp\Big\{ - \int_{\mathcal{M}^{\#}_{\mathcal{X}}\backslash\{0\}} (1 - e^{\langle h,\mu\rangle})\Lambda(d\mu)\Big\} \qquad \forall h \in BM(\mathcal{X}). \tag{2.4}$$

Since Λ is homogeneous we can decompose it into radial and directional components. To do that we have to define a polar decomposition for $\mathcal{M}^{\#}_{\mathcal{X}}\backslash\{0\}$. Let $B_1, B_2, ...$ be a countable base for the topology of $\mathcal{X}$ made of bounded sets. Put $B_0 = \mathcal{X}$. Then for every $\mu \in \mathcal{M}^{\#}_{\mathcal{X}}$ the sequence $\mu(B_0), \mu(B_1), \mu(B_2), ...$ is finite apart from $\mu(B_0)$, which can be finite or infinite. Let $i(\mu)$ be the smallest integer such that $0 < \mu(B_{i(\mu)}) < +\infty$. We define now the set

$$\mathbb{S} = \{\mu \in \mathcal{M}^{\#}_{\mathcal{X}} : \ \mu(B_{i(\mu)}) = 1\},$$

which can be easily proved to be measurable. There exists a unique measurable mapping $\mu \to \hat{\mu}$ from $\mathcal{M}^{\#}_{\mathcal{X}} \backslash \{0\}$ to $\mathbb{S}$ such that $\mu = \mu(B_{i(\mu)})\hat{\mu}$. The measurable mapping $\mu \to (\hat{\mu}, \mu(B_{i(\mu)}))$ is a polar decomposition of $\mathcal{M}^{\#}_{\mathcal{X}} \backslash \{0\}$ into $\mathbb{S} \times \mathbb{R}_+$.

The Levy measure Λ of a StαS random measure ξ induces a measure $\hat{\sigma}$ on $\mathbb{S}$

$$\hat{\sigma}(A) \doteq \Lambda\big(\{t\mu : \ \mu \in A, t \geq 1\}\big),$$

for every A measurable subset of $\mathbb{S}$. It is useful to define a scaled version of this measure: $\sigma = \Gamma(1-\alpha)\hat{\sigma}$, which is called spectral measure of ξ. Because of the homogeneity of Λ it holds $\Lambda(A \times [a,b]) = \hat{\sigma}(A)(a^{-\alpha} - b^{-\alpha})$, which means that $\Lambda = \hat{\sigma} \otimes \theta_\alpha$, where θ_α is the unique measure on $\mathbb{R}_+$ such that $\theta_\alpha\big([a, +\infty)\big) = a^{-\alpha}$. Condition (2.3) becomes

$$\int_{\mathbb{S}} \mu(B)^\alpha \sigma(d\mu) < +\infty \qquad \forall B \in \mathcal{B}(\mathcal{X}) \text{ bounded.} \tag{2.5}$$

It holds the following result regarding the spectral measure σ.

Theorem 11. *Let ξ be a StαS random measure with spectral measure σ and Laplace functional L_ξ. Then*

$$L_\xi[h] = \exp\Big\{ - \int_{\mathbb{S}} \langle h, \mu \rangle^\alpha \sigma(d\mu) \Big\} \qquad \forall h \in BM(\mathcal{X}). \tag{2.6}$$

We now give a result which provides a LaPage representation of a StαS random measure.

Theorem 12. *A random measure ξ is StαS if and only if*

$$\xi \overset{\mathcal{D}}{=} \sum_{\mu_i \Psi} \mu_i,$$

where Ψ is a Poisson point process on $\mathcal{M}^{\#}_{\mathcal{X}}$ with intensity measure Λ being a Levy measure. The convergence is in the sense of the vague convergence of measures. In this context Λ is the same Levy measure of (2.4).

2.2 Discrete Stability with respect to thinning

2.2.1 Definition and characterization

In trying to extend the definition of stability to point processes (p.p.) we face the problem of the definition of multiplication: if we define multiplication of a p.p. for a scalar as the multiplication of its values (see (2.1)) it would no longer be a p.p., because it would no longer be integer-valued. We therefore define a stochastic multiplication called independent thinning.

Definition 27. *Given a p.p.* Φ *and* $t \in [0,1]$ *the result of an independent thinning operation on* Φ *is a p.p.* $t \circ \Phi$ *obtained from* Φ *by retaining every point with probability* t *and removing it with probability* $1-t$*, acting independently on every point.*

The thinned process p.g.fl. is

$$G_{t \circ \Phi}[h] = G_\Phi[th + 1 - t] = G_\Phi[1 + t(h-1)], \tag{2.7}$$

where G_Φ is the p.g.fl. of Φ (see Daley and Vere-Jones, 2008, p.155 for details). From (2.7) it is easy to deduce that the thinning operation $\circ$ is associative, commutative and distributive with respect to the superposition of point processes. Having such an operation we can give the following definition.

Definition 28. *A p.p.* Φ *is said to be discrete* α*-stable or* α*-stable with respect to thinning (D*α*S) if*

$$t^{1/\alpha} \circ \Phi' + (1-t)^{1/\alpha} \circ \Phi'' \stackrel{\mathcal{D}}{=} \Phi \qquad \forall t \in [0,1], \tag{2.8}$$

where Φ' *and* Φ'' *are indipendent copies of* Φ*.*

The next result gives a straightforward characterization of DαS point processes, showing the strong link occuring between DαS point processes and StαS random measures.

Theorem 13. *A point process Φ is DαS if and only if it is a Cox process Π_ξ directed by a StαS intensity measure ξ.*

Starting from Theorem 13 and using (1.5) and (2.6) we obtain the following result.

Corollary 1. *A point process Φ with p.g.fl. G_Φ is DαS with $\alpha \in (0,1]$ if and only if*

$$G_\Phi[h] = \exp\Big\{ -\int_S \langle 1-u, \mu\rangle^\alpha \sigma(d\mu)\Big\} \qquad \forall\, u \in \mathcal{V}(\mathcal{X}), \tag{2.9}$$

where σ is a boundedly finite spectral measure defined on $\mathbb{S}$ and satisfying (2.5).

Another important consequence of Theorem 13 is that we can use the LaPage representation for StαS random measures to obtain an analogous result for DαS point processes.

Corollary 2. *A DαS point process Φ with Levy measure Λ can be represented as*

$$\Phi = \sum_{\mu_i \in \Psi} \Pi_{\mu_i},$$

where Ψ is a Poisson process on $\mathcal{M}^{\#}_{\mathcal{X}} \backslash \{0\}$ with intensity measure Λ.

2.2.2 Cluster representation with Sibuya point processes

Since every DαS p.p. is a Cox process Π_ξ directed by a StαS random measure ξ, using (1.5) and (2.4) we obtain

$$G_{\Pi_\xi}[h] = L_\xi[1-h] = \exp\Big\{ -\int_{\mathcal{M}^{\#}_{\mathcal{X}} \backslash \{0\}} (1 - e^{-\langle 1-h, \mu\rangle})\Lambda(d\mu)\Big\},$$

where Λ is the Levy measure of ξ. Using (1.3) and (1.6) we conclude that every DαS p.p. can be represented as a cluster process with centre process being a Poisson process on $\mathcal{M}^{\#}_{\mathcal{X}}$ with intensity measure Λ

and daughter processes being Poisson processes with intensity measure $\mu \in supp(\Lambda)$.

We give now another cluster representation assuming that Λ is supported by finite measures.

Definition 29. *Let μ be a probability measure on $\mathcal{X}$. A Sibuya point process with exponent α and parameter measure μ is a point process $\Upsilon = \Upsilon(\mu)$ on $\mathcal{X}$ such that*

$$G_{\Upsilon}[h] = 1 - \langle 1 - u, \mu \rangle^{\alpha} \qquad \forall h \in \mathcal{V}(\mathcal{X}), \tag{2.10}$$

where G_{Υ} is the p.g.fl. of Υ. We will denote the distribution of such a process by $Sib(\alpha, \mu)$.

From this definition and from (2.9) it follows that given a DαS p.p. Φ such that Λ is supported by finite measure it holds

$$G_{\Phi}[h] = \exp\left\{ \int_{\mathbb{M}_1} (G_{\Upsilon(\mu)}[h] - 1)\sigma(d\mu) \right\} \qquad \forall h \in \mathcal{V}(\mathcal{X}), \tag{2.11}$$

where $G_{\Upsilon(\mu)}$ satisfies (2.10), $\mathbb{M}_1$ is the space of probability measure on $\mathcal{X}$ and σ is the spectral measure of Λ. Together with (1.3) and (1.6) it implies the following result.

Theorem 14. *A DαS point process with Levy measure supported by finite measure can be represented as a cluster process driven by the spectral measure σ on $\mathbb{M}_1$ and with daughter processes being distributed as $Sib(\alpha, \mu)$ with $\mu \in supp(\sigma)$.*

Since Sibuya processes are almost surely finite and different from the zero measure it follows that whether a DαS p.p. is finite or not depends only from the centre process.

Corollary 3. *A DαS p.p. is finite if and only if its spectral measure σ is finite and supported by finite measures.*

2.2.3 Regular and singular DαS processes

Iterating 2.8 we obtain

$$t^{-1/\alpha} \circ \Phi^{(1)} + \dots + t^{-1/\alpha} \circ \Phi^{(m)} \stackrel{\mathcal{D}}{=} \Phi,$$

where Φ is a DαS point process and $\Phi^{(1)}, \dots, \Phi^{(n)}$ are independent copies of it. Therefore DαS processes are infinitely divisible.

Remark 7. *We can obtain a KLM representation (equation (1.15)) for them. From Theorem 13 every DαS process Φ is a Cox process driven by a StαS random measure ξ. Therefore using (2.4) we have that*

$$G_\Phi[h] = L_\xi[1-h] = \exp\Big\{ - \int_{\mathcal{M}^\#_{\mathcal{X}} \setminus \{0\}} (1 - e^{\langle 1-h, \mu \rangle}) \Lambda(d\mu) \Big\}$$

which, using the expression for the p.g.fl. of a Poisson p.p. Π_μ (equation (1.3)), becomes

$$\exp\Big\{ - \int_{\mathcal{M}^\#_{\mathcal{X}} \setminus \{0\}} (1 - G_{\Pi_\mu}[h]) \Lambda(d\mu) \Big\} =$$

$$= \exp\Big\{ - \int_{\mathcal{M}^\#_{\mathcal{X}} \setminus \{0\}} \big(1 - \int_{\mathcal{N}^\#_{\mathcal{X}}} e^{\langle \log(h), \varphi \rangle} \mathcal{P}_\mu(d\varphi)\big) \Lambda(d\mu) \Big\} =$$

$$= \exp\Big\{ - \int_{\mathcal{M}^\#_{\mathcal{X}} \setminus \{0\}} \int_{\mathcal{N}^\#_{\mathcal{X}} \setminus \{0\}} 1 - e^{\langle \log(h), \varphi \rangle} \mathcal{P}_\mu(d\varphi) \Lambda(d\mu) \Big\} =$$

$$= \exp\Big\{ - \int_{\mathcal{N}^\#_{\mathcal{X}} \setminus \{0\}} \big[1 - e^{\langle \log(h), \varphi \rangle}\big] \boldsymbol{Q}(d\varphi) \Big\},$$

where $\boldsymbol{Q}(\cdot) = \int_{\mathcal{M}^\#_{\mathcal{X}} \setminus \{0\}} \mathcal{P}_\mu(\cdot) \Lambda(d\mu)$. The last expression is the KLM representation for DαS processes we were looking for.

Starting from the decomposition for infinitely divisible point processes given in Theorem 9 we can obtain the following decomposition for DαS point processes.

Definition 30. *Given a complete separable metric space (c.s.m.s.) $\mathcal{X}$ we define*

$$\mathbb{M}_f = \left\{ \mu \in \mathcal{M}^{\#}_{\mathcal{X}} \backslash \{0\} : \ \mu(\mathcal{X}) < +\infty \right\}$$

and

$$\mathbb{M}_\infty = \left\{ \mu \in \mathcal{M}^{\#}_{\mathcal{X}} : \ \mu(\mathcal{X}) = +\infty \right\}.$$

Theorem 15. *A $D\alpha S$ p.p. Φ with Levy measure Λ can be represented as the sum of two independent $D\alpha S$ processes*

$$\Phi = \Phi_r + \Phi_s,$$

the first one being regular and the second one being singular. The first one is a $D\alpha S$ p.p. with Levy measure being $\Lambda\big|_{\mathbb{M}_f} = \Lambda(\cdot \mathbb{I}_{\mathbb{M}_f})$ and the second one is a $D\alpha S$ p.p. with Levy measure $\Lambda\big|_{\mathbb{M}_\infty}$.

Remark 8. *With the decomposition given in Theorem 19 we've separated every $D\alpha S$ process into two components. The regular one which can be represented as a Sibuya cluster p.p. with p.g.fl. given by (2.11) with spectral measure being $\sigma\big|_{\mathbb{M}_1}$, and the singular one is not a Sibuya cluster p.p. and his p.g.fl. is given by (2.9) with spectral measure being $\sigma\big|_{\mathbb{S}\backslash\mathbb{M}_1}$.*

Chapter 3

$\mathcal{F}$-stability for point processes

In this chapter we extend the discrete stability of point processes to an operation more general than thinning. We will consider an operation defined through branching processes and we will characterize stable point processes with respect to this operation. This has already been done in the context of random variables, see e.g. Steutel and Van Harn [4], and random vectors, see e.g. Bouzar [5], but not for point processes. Following Steutel and Van Harn's notation we will denote the "branching" operation by $\circ_{\mathcal{F}}$ and the related class of stable point processes by $\mathcal{F}$-stable processes (the reason to use the letter $\mathcal{F}$ will become clear in the following).

3.1 Some remarks about branching processes

Before proceeding in this chapter we need to clarify which kind of branching processes we will use and recall some useful properties (complete proofs for this section can be found in the literature regarding branching processes). We will consider a continuous-time Markov branching process $Y(s)$, with $Y(0) = 1$ a.s.. Such a branching process is governed by a family of p.g.f.s $\mathcal{F} = (F_s)_{s \geq 0}$, where F_s is the p.g.f. of

$Y(s)$ for every $s \geq 0$. The transition matrix $\big(p_{ij}(s)\big)_{i,j\in\mathbb{N}}$ of the Markov process can be obtained by $\mathcal{F}$ from the following equation:

$$\sum_{j=0}^{\infty} p_{ij}(s) z^j = \{F_s(z)\}^i.$$

It is easy to prove that the family $\mathcal{F}$ is a composition semigroup, meaning that

$$F_{s+t}(\cdot) = F_s\big(F_t(\cdot)\big) \quad \forall s, t \geq 0. \tag{3.1}$$

Throughout the whole chapter we will require the branching process $\{Y(\cdot)\}_{i\in\mathbb{N}}$ to be subcritical, which in our case means $\mathbb{E}[Y(1)] < 1$. We can also suppose $F_s'(1) = e^{-s}$ without loss of generality (it can be obtain through a linear transformation of the time coordinate). Moreover we require the following conditions to hold:

$$\lim_{s\downarrow 0} F_s(z) = F_0(z) = z, \tag{3.2}$$

$$\lim_{s\to\infty} F_s(z) = 1. \tag{3.3}$$

Some reasons for these requirements will be given in Remark 10. Equations (3.1) and (3.2) implies the continuity $F_s(z)$ with respect to s. It can be also shown that $F_s(z)$ is differentiable with respect to s and thus we can define

$$U(z) \doteq \frac{\partial}{\partial s} F_s(z)\Big|_{s=0} \qquad 0 \leq z \leq 1.$$

$U(\cdot)$ is continuous and we it can be use to obtain the *A-function* relative to the branching process

$$A(z) \doteq \exp\Big[-\int_0^z \frac{1}{U(x)} dx \Big] \qquad 0 \leq z \leq 1, \tag{3.4}$$

which is a continuous strictly decreasing function such that $A(0) = 1$ and $A(1) = 0$. Since it holds that

$$U\big(F_s(z)\big) = U(z) F_s'(z) \qquad s \geq 0,\ 0 \leq z \leq 1,$$

we obtain the first property of A-functions we're interested in:

$$A\big(F_s(z)\big) = e^{-s}A(z) \qquad s \geq 0,\ 0 \leq z \leq 1. \tag{3.5}$$

Moreover it can proved that

$$B(z) \doteq 1 - A(z) = \lim_{s \to +\infty} \frac{F_s(z) - F_s(0)}{1 - F_s(0)} \qquad 0 \leq z \leq 1. \tag{3.6}$$

From the last expression we can see that $B(\cdot)$ is a p.g.f. of a $\mathbb{Z}_+$-valued distribution, which is the limit conditional distribution of the branching process $Y(\cdot)$ (we condition on the survival of $Y(s)$ and then we let the time go to infinity). We will call $B(\cdot)$ the *B-function* of the process $Y(\cdot)$, and the limit conditional distribution Y_∞. It is worth noticing that since both A and B are continuous, strictly monotone, and surjective functions from $[0,1]$ to $[0,1]$ then they are bijective and they can be inverted obtaining A^{-1} and B^{-1}, which will be continuous, strictly monotone and bijective functions from $[0,1]$ to $[0,1]$.
We give now two examples of branching process where A and B have known and explicit expressions.

Example 2. *Let $Y(\cdot)$ be a continuous-time pure-death process starting with one individual, meaning that*

$$Y(s) = \begin{cases} 1 & \textit{if } s < \tau \\ 0 & \textit{if } s \geq \tau \end{cases}, \tag{3.7}$$

where τ is an exponential random variable with parameter 1. The composition semigroup $\mathcal{F} = \big(F_s\big)_{s\geq 0}$ driving the process $Y(\cdot)$ is

$$F_s(z) = 1 - e^{-s} + e^{-s}z \qquad 0 \leq z \leq 1. \tag{3.8}$$

It is straightforward to see that $\mathcal{F} = \big(F_s\big)_{s\geq 0}$ satisfies the requirements previously listed $\big($(3.1), (3.2), (3.3) and $F_s'(1) = e^{-s}\big)$. The generator $U(z)$ and the A-function of $Y(s)$ are given by

$$U(z) = A(z) = 1 - z \qquad 0 \leq z \leq 1, \tag{3.9}$$

while the B-function equals the identity function

$$B(z) = z \qquad 0 \le z \le 1. \tag{3.10}$$

Example 3. *Let the semigroup $\mathcal{F} = \left(F_s\right)_{s\ge 0}$ be defined by*

$$F_s(z) = 1 - \frac{2e^{-s}(1-z)}{2+(1-e^{-s})(1-z)} = (1-\gamma_s) + \gamma_s \frac{z(1-p_s)}{1-p_s z}, \tag{3.11}$$

where $\gamma_s = 2e^{-s}/(3-e^{-s})$, $p_s = \frac{1}{3}(1-\gamma_s)$ and $0 \le z \le 1$. The second expression for F_s can be recognized as the composition of two p.g.f.s, $P_1(P_2(z))$. The first one is the p.g.f. of a binomial distribution with parameter γ_s

$$P_1(z) = (1-\gamma_s) + \gamma_s z,$$

and the second one the p.g.f. of a geometric distribution with parameter p_s (number of trials to get the first success)

$$P_2(z) = \frac{z(1-p_s)}{1-p_s z}.$$

This implies that F_s is a p.g.f. itself. Using the first and the second expression for F_s conditions (3.1), (3.2), (3.3) and $F_s'(1) = e^{-s}$ can be easily proved. The functions U, A and B defined on $[0,1]$ have the following expressions:

$$U(z) = \frac{1}{2}(1-z)(3-z), \quad A(z) = 3\frac{1-z}{3-z}, \quad B(z) = \frac{2z}{3-z}, \tag{3.12}$$

where we can notice that $B(\cdot)$ is the p.g.f. of a geometric$(\frac{1}{3})$ distribution on $\mathbb{N}$.

3.2 $\mathcal{F}$-stability for random variables

We can interpret a $\mathbb{Z}_+$-valued random variable X as a point process on a space $\mathcal{X}$ reduced to a single point. Given $t \in [0,1]$ the thinning operation works on X as a discrete multiplication. We can express the thinned process $t \circ X$ in the following way:

$$t \circ X \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} Z_i \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} Y_i\big(-ln(t)\big),$$

where $\{Z_i\}_{i\in\mathbb{N}}$ are independent and identically distributed (i.i.d.) random variables with Bernoulli distribution $Bin(1,t)$ and $Y_i(\cdot)$ are i.i.d. pure-death processes starting with one individual (see definition 3.7). We can now think of a more general operation which acts on X by replacing every unit with a more general branching process then the pure-death one.

Definition 31. *Let $\{Y_i(\cdot)\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. continuous-time Markov branching processes driven by a semigroup $\mathcal{F} = (F_s)_{s\geq 0}$ satisfying the conditions listed in the previous section. Given $t \in (0,1]$ and a $\mathbb{Z}_+$-valued random variable X (independent of $\{Y_i(\cdot)\}_{i\in\mathbb{N}}$) we define*

$$t \circ_{\mathcal{F}} X \doteq \sum_{i=1}^{X} Y_i\big(-\ln(t)\big). \tag{3.13}$$

Let $P(z)$ be the p.g.f. of X and $P_{t\circ_{\mathcal{F}}X}(z)$ be the p.g.f. of $t \circ_{\mathcal{F}} X$. It follows from (3.13) and from the independence of the random variables $\{Y_i(\cdot)\}_{i\in\mathbb{N}}$ that

$$P_{t\circ_{\mathcal{F}}X}(z) = P\big(F_s(z)\big) \qquad \text{with } s = -\ln(t) \text{ and } 0 \leq z \leq 1. \tag{3.14}$$

Remark 9. *The $\circ_{\mathcal{F}}$ operation for random variables includes the thinning one and is more general. In fact if we consider the branching process driven by the semigroup defined by (3.8) (i.e. the pure-death process) we obtain*

$$P_{t\circ_{\mathcal{F}}X}(z) = P\big(F_{-\ln(t)}(z)\big) = P(1 - e^{\ln(t)} + e^{\ln(t)}z) =$$

$$= P(1 - t + tz) = P_{t\circ X}(z),$$

which implies that in this case the two operations, $\circ_{\mathcal{F}}$ and $\circ$, are equivalent. Example 3 shows that the $\circ_{\mathcal{F}}$ operation involves also different situation from the thinning.

Using equation (3.14) it is easy to verify the following proposition.

Proposition 12. *The branching operation $\circ_{\mathcal{F}}$ is associative, commutative and distributive with respect to sum of random variables, i.e.*

$$t_1 \circ_{\mathcal{F}} (t_2 \circ_{\mathcal{F}} X) \stackrel{\mathcal{D}}{=} (t_1 t_2) \circ_{\mathcal{F}} X \stackrel{\mathcal{D}}{=} t_2 \circ_{\mathcal{F}} (t_1 \circ_{\mathcal{F}} X),$$

$$t \circ_{\mathcal{F}} (X + X') \stackrel{\mathcal{D}}{=} t \circ_{\mathcal{F}} X + t \circ_{\mathcal{F}} X'',$$

for $t, t_1, t_2 \in [0,1]$ and X,X' independent random variables.

Remark 10. *As shown in Steutel and Van Harn (2004), Section V.8, equations (3.2) and (3.3) turn out to be good requirements to have some "multiplication-like" properties of the operation $\circ_{\mathcal{F}}$. In particoular (3.2) implies (besides the continuity of the semigroup) that $\lim_{t\uparrow 1} t \circ_{\mathcal{F}} X = 1 \circ_{\mathcal{F}} X = X$ and (3.3) together with $F_s'(1) = e^{-s}$ implies that, in case the expectation of X is finite, $\mathbb{E}[t \circ_{\mathcal{F}} X] = t\mathbb{E}[X]$.*

Proceeding in the same way as for strict and discrete stability we can define the notion of $\mathcal{F}$-stability.

Definition 32. *A $\mathbb{Z}_+$-valued random variable X is said to be $\mathcal{F}$-stable with exponent α if*

$$t^{1/\alpha} \circ_{\mathcal{F}} X' + (1-t)^{1/\alpha} \circ_{\mathcal{F}} X'' \stackrel{\mathcal{D}}{=} X \qquad \forall t \in [0,1], \tag{3.15}$$

where X' and X'' are independent copies of X.

Let $P(z)$ be the p.g.f. of X. Then (3.15) turns into the following condition on $P(z)$:

$$P(z) = P\big(F_{-\ln(t)/\alpha}(z)\big) \cdot P\big(F_{-\ln(1-t)/\alpha}(z)\big) \qquad 0 \le z \le 1. \tag{3.16}$$

Remark 11. *Iterating (3.15) m times we obtain*

$$m^{-1/\alpha} \circ_{\mathcal{F}} X^{(1)} + ... + m^{-1/\alpha} \circ_{\mathcal{F}} X^{(m)} \stackrel{\mathcal{D}}{=} X, \tag{3.17}$$

where $\left(X^{(1)}, ..., X^{(m)}\right)$ *are independent copies of X. Thus an $\mathcal{F}$-stable random variable is infinitely divisible. Equation* (3.25) *can be written as*

$$P(z) = \left[P(F_{\ln(m)/\alpha}(z))\right]^m \qquad m \in \mathbb{N}, 0 \leq z \leq 1, \tag{3.18}$$

where $P(z)$ is the p.g.f. of x. As it is shown in Steutel and Van Harn (2004), Section V.5, a p.g.f. P(z) satisfies (3.18) if and only if it satisfies

$$P(z) = \left[P(F_{-\ln(t)}(z))\right]^{t^{-\alpha}} \qquad t \in [0, 1], 0 \leq z \leq 1. \tag{3.19}$$

Moreover equation (3.19) (or equivalently (3.18)) is not only a necessary condition for a distribution to be $\mathcal{F}$-stable but also sufficient. In fact using the associativity of the operation $\circ_{\mathcal{F}}$ it is easy to show that if a p.g.f. $P(z)$ satisfies condition (3.25) then it also satisfies condition (3.15), and thus is $\mathcal{F}$-stable. Therefore we can say that a distribution is $\mathcal{F}$-stable if and only if it satisfies (3.19).

The following theorem gives a characterization of $\mathcal{F}$-stable distribution through their probability generating functions.

Theorem 16. *Let X be a $\mathbb{Z}_+$-valued random variable and P(z) its p.g.f., then X is $\mathcal{F}$-stable with exponent α if and only if* $0 < \alpha \leq 1$ *and*

$$P(z) = \exp\left\{ - cA(z)^\alpha\right\} \qquad 0 \leq z \leq 1, \tag{3.20}$$

where A is the A-function associated to the branching process driven by the semigroup $\mathcal{F}$ and $c > 0$.

Proof. See Steutel and Van Harn (2004), Theorem V.8.6. □

3.3 $\mathcal{F}$-stability for point processes

3.3.1 Definition and characterization

Let $Y(\cdot)$ be a continuous-time Markov branching process driven by a semigroup $\mathcal{F} = (F_s)_{s\geq 0}$ satisfying conditions described in Section 3.1. We now want to extend the branching operation $\circ_{\mathcal{F}}$ to point processes. Given a point process Φ and $t \in (0,1]$, $t \circ_{\mathcal{F}} \Phi$ will be a point process obtained from Φ by replacing every point with a bunch of points located in the same position, where the number of points is given by an independent copy of $Y\big(-\ln(t)\big)$. A good way to provide a formal definition of $t \circ_{\mathcal{F}} \Phi$ is through a cluster structure. We first define the component processes.

Definition 33. *Given a continuous-time Markov branching process $Y(\cdot)$ and a point $x \in \mathcal{X}$, $Y_x(s)$ is the point process having $Y(s)$ points in x and no points in $\mathcal{X}\backslash\{x\}$, or equivalently having p.g.fl. defined by*

$$G_{Y_x(s)}[h] = \mathbb{E}[h(x)^{Y(s)}] = F_s\big(h(x)\big). \tag{3.21}$$

We can now define the operation $\circ_{\mathcal{F}}$ for point processes.

Definition 34. *Let Φ be a point process (p.p.) and $t \in (0,1]$. Then $t \circ_{\mathcal{F}} \Phi$ is the (independent) cluster point process with center process Φ and component processes $\big\{Y_x\big(-\ln(t)\big),\, x \in \mathcal{X}\big\}$.*

Equivalently $t \circ_{\mathcal{F}} \Phi$ can be defined as the p.p. having p.g.fl.

$$G_{t\circ_{\mathcal{F}}\Phi}[h] = G_\Phi[F_{-\ln(t)}(h)], \tag{3.22}$$

where G_Φ is the p.g.fl. of Φ. We are now ready to define the $\mathcal{F}$-stability for point processes.

Definition 35. *A p.p.* Φ *is* $\mathcal{F}$*-stable with exponent* α *(*α*-stable with respect to* $\circ_{\mathcal{F}}$*) if*

$$t^{1/\alpha} \circ_{\mathcal{F}} \Phi' + (1-t)^{1/\alpha} \circ_{\mathcal{F}} \Phi'' \stackrel{\mathcal{D}}{=} \Phi \qquad \forall t \in (0,1], \tag{3.23}$$

where Φ' *and* Φ'' *are independent copies of* Φ*.*

Condition (3.23) can be rewritten in the p.g.fl. form obtaining

$$G_{\Phi}[h] = G_{\Phi}\left[F_{-\ln(t)/\alpha}(h)\right] \cdot G_{\Phi}\left[F_{-\ln(1-t)/\alpha}(h)\right] \qquad \forall t \in (0,1], \forall h \in \mathcal{V}(\mathcal{X}). \tag{3.24}$$

Iterating this formula m-times as done in Remark 11 we obtain

$$m^{-1/\alpha} \circ_{\mathcal{F}} \Phi^{(1)} + ... + m^{-1/\alpha} \circ_{\mathcal{F}} \Phi^{(m)} \stackrel{\mathcal{D}}{=} \Phi, \tag{3.25}$$

where $\left(\Phi^{(1)}, ..., \Phi^{(m)}\right)$ are independent copies of Φ. Therefore an $\mathcal{F}$-stable point process is infinitely divisible.

Remark 12. *The branching operation* $\circ_{\mathcal{F}}$ *for point processes is a generalization of the thinning operation. In fact if we take as a branching process the pure-death process with semigroup* $\mathcal{F} = (F_s)_{s\geq 0}$ *defined by equation (3.8) we obtain*

$$G_{t\circ_{\mathcal{F}}\Phi}[h] = G_{\Phi}\left[F_{-\ln(t)}(h)\right] = G_{\Phi}[1 - e^{\ln(t)} + e^{\ln(t)}h] =$$

$$G_{\Phi}[1 - t + th] = G_{t\circ\Phi}[h] \qquad \forall h \in \mathcal{V}(X),$$

which implies that the process $t \circ_{\mathcal{F}} \Phi$ *has the same distribution as the thinned process* $t\circ\Phi$*, meaning that the two operations become equivalent. Therefore D*α*S point processes can be seen as a particular case of* $\mathcal{F}$*-stable point processes, obtained by choosing* $\mathcal{F} = (F_s)_{s\geq 0}$ *as in equation (3.8).*

We prove the following result which gives a characterization of $\mathcal{F}$-stable point processes.

Theorem 17. *A functional $G_\Phi[\cdot]$ is the p.g.fl. of an $\mathcal{F}$-stable point process Φ with exponent of stability α if and only if there exist a StαS random measure ξ such that*

$$G_\Phi[h] = L_\xi\big[A(h)\big] \qquad \forall h \in \mathcal{V}(\mathcal{X}), \tag{3.26}$$

where $A(z)$ is the A-function of the branching process driven by $\mathcal{F}$.

Proof. Sufficiency: We suppose (3.26). $L_\xi\big[1-h\big]$ as a functional of h is the p.g.fl. of a Cox point process and the $B(z)$ is the p.g.f. of a random variable (the limit conditional distribution of the branching process $Y(t)$. Therefore the functional $G_\Phi[h] = L_\xi\big[A(h)\big] = L_\xi\big[1 - B(h)\big]$ is the p.g.fl. of a (cluster) point process, say Φ. We need to prove that Φ is $\mathcal{F}$-stable with exponent α. Given $t \in (0,1]$ it holds

$$\begin{gathered} G_\Phi\big[F_{-\ln(t)/\alpha}(h)\big] \cdot G_\Phi\big[F_{-\ln(1-t)/\alpha}(h)\big] = \\ = L_\xi\Big[A\big(F_{-\ln(t)/\alpha}(h)\big)\Big] \cdot L_\xi\Big[A\big(F_{-\ln(1-t)/\alpha}(h)\big)\Big] \overset{(3.5)}{=} \\ = L_\xi\big[t^{1/\alpha}A(h)\big] \cdot L_\xi\big[(1-t)^{1/\alpha}A(h)\big], \end{gathered}$$

for every $h \in \mathcal{V}(\mathcal{X})$. Since ξ is α-stable we can use its spectral representation:

$$\begin{gathered} L_\xi\big[t^{1/\alpha} \cdot A(h)\big] = \exp\Big\{ - \int_S \langle t^{1/\alpha} \cdot A(h), \mu\rangle^\alpha \sigma(d\mu)\Big\} = \\ = \exp\Big\{ - t \cdot \int_S \langle A(h), \mu\rangle^\alpha \sigma(d\mu)\Big\} = \big(L_\xi\big[A(h)\big]\big)^t. \end{gathered}$$

Therefore

$$\begin{gathered} L_\xi\big[t^{1/\alpha}A(h)\big] \cdot L_\xi\big[(1-t)^{1/\alpha}A(h)\big] = L_\xi\big[A(h)\big]^t \cdot L_\xi\big[A(h)\big]^{1-t} = \\ = L_\xi\big[A(h)\big] = G_\Phi[h], \end{gathered}$$

and thus Φ is $\mathcal{F}$-stable with exponent α.

Necessity: We now suppose that Φ is $\mathcal{F}$-stable with exponent α. Firstly we need to prove that

$$L[u] \doteq G_\Phi[A^{-1}(u)] \tag{3.27}$$

is the Laplace functional of a StαS random measure. While the functional L in the left-hand side should be defined on all (bounded) functions with compact supports, the p.g.fl. G_Φ in the right-hand side of (3.27) is well defined just for functions with values on $[0,1]$ because $A^{-1} : [0,1] \to [0,1]$. To overcome this difficulty we employ (3.25) which can be written as

$$G_\Phi[h] = \big(G_\Phi[F_{\frac{\ln m}{\alpha}}(h)]\big)^m \qquad \forall h \in \mathcal{V}(\mathcal{X}),$$

and define

$$L[u] = \Big(G_\Phi[F_{\frac{\ln m}{\alpha}}\big(A^{-1}(u)\big)]\Big)^m \overset{(3.5)}{=} \Big(G_\Phi[A^{-1}(m^{-1/\alpha}u)]\Big)^m. \tag{3.28}$$

Since $u \in BM(\mathcal{X})$, for sufficiently large m the function $A^{-1}(m^{-1/\alpha}u)$ does take values in $[0,1]$ and equals 1 outside a compact set. Since (3.28) holds for all m, it is possible to pass to the limit as $m \to \infty$ to see that

$$\begin{aligned} L[u] &= \exp\Big\{ -\lim_{m\to\infty} m(1 - G_\Phi[A^{-1}(m^{-1/\alpha}u)])\Big\} \\ &= \exp\Big\{ -\lim_{t\to 0^+} t^{-\alpha}(1 - G_\Phi[A^{-1}(tu)])\Big\}. \end{aligned}$$

We need the following fact

$$\lim_{t\to 0^+} t^{-\alpha}(1 - G_\Phi[A^{-1}(tu)]) = \lim_{t\to 0^+} t^{-\alpha}(1 - G_\Phi[e^{A^{-1}(tu)-1}])\,,$$

which using the p.g.f. $B(z)$ of the limit conditional distribution can be also written as

$$\lim_{t\to 0^+} t^{-\alpha}(1 - G_\Phi[1 - B^{-1}(tu)]) = \lim_{t\to 0^+} t^{-\alpha}(1 - G_\Phi[e^{-B^{-1}(tu)}])\,. \tag{3.29}$$

Indeed, for any constant $c > 0$ and sufficiently small $t > 0$,

$$1 - t(u+g) \leq e^{-tu} \leq 1 - t(u-g)$$

for a function g that vanishes outside of the support of u and otherwise is bounded by c. From $B^{-1}(tu) = tu(B^{-1})'(0) + o(t)$ as $t \to 0$, with $(B^{-1})'(0) \neq 0$, it can be obtained, under the same hypotheses for c,t and g,

$$1 - B^{-1}(t(u+g)) \leq e^{-B^{-1}(tu)} \leq 1 - B^{-1}(t(u-g))$$

Then

$$L[u-g] \leq \lim_{t \to 0^+} t^{-\alpha}(1 - G_\Phi[e^{-B^{-1}(tu)}]) \leq L[u+g]\,,$$

and the continuity of L yields (3.29). By the Schoenberg theorem (see [7] Theorem 3.2.2), L is positive definite if $\lim m(1 - G_\Phi[1 - B^{-1}(m^{-1/\alpha}u)])$ is negative definite, i.e. in view of (3.29)

$$\sum_{i,j=1}^{n} c_i c_j \lim_{t \to 0} t^{-\alpha}(1 - G_\Phi[e^{-t(u_i+u_j)}]) \leq 0$$

for all $n \geq 2$, $u_1, \dots, u_n \in BM(\mathcal{X})$ and $c_1, \dots, c_n$ with $\sum c_i = 0$. In view of the latter condition, the required negative definiteness follows from the positve definiteness of G_Φ.

Thus, $L_\xi[\sum_{i=1}^{k} t_i h_i]$ as a function of $t_1, \dots, t_k \geq 0$ is the Laplace transform of a random vector. Arguing as in the proof of sufficiency in Theorem **??**, it is easy to check the continuity of L, so that L is indeed the Laplace functional of a random measure ξ.

To prove that ξ is StαS we consider the case of functions u with values in $[0,1]$ to simplify the calculations (the general case can be done with analogous calculations). Given $t \in (0,1]$ we have

$$L_\xi[u] = G_\Phi[A^{-1}(u)] = G_\Phi\big[F_{-\ln(t)/\alpha}(A^{-1}(h))\big] \cdot G_\Phi\big[F_{-\ln(1-t)/\alpha}(A^{-1}(h))\big] \overset{(3.5)}{=}$$

$$G_\Phi\big[A^{-1}(t^{1/\alpha}h)\big]\cdot G_\Phi\big[A^{-1}((1-t)^{1/\alpha}h)\big] = L_\xi[t^{1/\alpha}h]\cdot L_\xi[(1-t)^{1/\alpha}h],$$

which implies that ξ is StαS. $\square$

Corollary 4. *A p.p. Φ is $\mathcal{F}$-stable with exponent α if and only if it is a cluster process with a DαS centre process Ψ on $\mathcal{X}$ and component processes $\big\{\widetilde{Y}_x,\ x\in\mathcal{X}\big\}$. $\widetilde{Y}_x$ denotes the p.p. having Y_∞ points in x and no points in $\mathcal{X}\backslash\{x\}$, where Y_∞ is the conditional limit distribution of the branching process Y, with p.g.f. given by* (3.6)*.*

Proof. From Theorem 17 and (3.6) it follows that Φ is $\mathcal{F}$-stable if and only if its p.g.fl. satisfies

$$G_\Phi[h] = L_\xi\big[1-B(h)\big],$$

where $B(\cdot)$ is the p.g.f. of Y_∞, and ξ is a StαS random measure. Then from Theorem 17 and equation (1.5) we obtain

$$G_\Phi[h] = G_\Psi\big[B(h)\big],$$

where Ψ is a DαS point process. The result follows from the cluster representation for p.g.fl. (1.6). $\square$

Remark 13. *These corollary clarify the relationship between $\mathcal{F}$-stable and DαS point processes. $\mathcal{F}$-stable processes are an extension of DαS ones where every point is given an additional multiplicity according to independent copies of a $\mathbb{Z}_+$-valued random variable Y_∞ fixed by the branching process which is being considered. We notice that when the branching operation reduces to the thinning operation the random variable Y_∞ reduces to a deterministic variable equal to 1 (see Example 2). This implies that the cluster process described in Corollary 4 reduces to the DαS centre process itself.*

Corollary 5. *Let $\alpha \in (0,1]$. A p.p. Φ is $\mathcal{F}$-stable with exponent α if and only if its p.g.fl. can be written as*

$$G_\Phi[u] = \exp\big\{ - \int_S \langle 1 - B(u), \mu\rangle^\alpha \sigma(d\mu)\big\}. \tag{3.30}$$

where σ is a locally finite spectral measure on $\mathbb{S}$ satysfying (2.5)

Proof. This result is a straightforward consequence of Theorems 11 and 17. In fact if Φ is an $\mathcal{F}$-stable point process with stability exponent α thanks to Theorem 17 there exist a StαS random measure ξ such that

$$G_\Phi[h] = L_\xi\big[A(h)\big] \qquad h \in \mathcal{V}(\mathcal{X}).$$

Then (3.30) follows from spectral representation (2.6). Conversely if we have a locally finite spectral measure σ on $\mathbb{S}$ satisfying (2.5) and $\alpha \in (0,1]$ then σ is the spectral measure of a StαS random measure ξ, whose Laplace functional is given by (2.6). Therefore (3.30) can be written as

$$G_\Phi[h] = L_\xi\big[1 - B(h)\big],$$

which, by Theorem 17 implies the $\mathcal{F}$-stability of Φ. □

3.3.2 Sibuya representation for $\mathcal{F}$-stable point processes

Thanks to Theorem 17 every $\mathcal{F}$-stable point process (p.p.) is uniquely associated to a StαS random measure and thus to a Levy measure Λ and a spectral measure σ. Corollary 5 enlightens the relationship between an $\mathcal{F}$-stable p.p. Φ and the associated spectral measure σ. If we consider the case of $\mathcal{F}$-stable processes associated to Levy measures Λ supported by finite measures, representation (3.30) becomes

$$G_\Phi[h] = \exp\big\{ - \int_{\mathbb{M}_1} \langle 1 - B(h), \mu\rangle^\alpha \sigma(d\mu)\big\} \qquad \forall h \in \mathcal{V}(\mathcal{X}), \tag{3.31}$$

where $\mathbb{M}_1$ is the space of probability measures on $\mathcal{X}$. Using the definition of Sibuya point processes (see equation (2.10)) we can rewrite this equation as

$$\begin{aligned} G_\Phi[h] &= \exp\big\{ - \int_{\mathbb{M}_1} 1 - (1 - \langle 1 - B(h), \mu\rangle^\alpha)\sigma(d\mu)\big\} = \\ &= \exp\big\{ - \int_{\mathbb{M}_1} \big(1 - G_{\Upsilon(\mu)}[B(h)]\big)\sigma(d\mu)\big\} \qquad \forall h \in \mathcal{V}(\mathcal{X}), \end{aligned} \tag{3.32}$$

where $\Upsilon(\mu)$ denotes a point process following a *Sibuya* distribution with parameters (α, μ). We notice that, since $B(\cdot)$ is the p.g.f. of the distribution Y_∞ $\big($see (3.6)$\big)$, $G_{\Upsilon(\mu)}[B(h)]$ is the p.g.fl. of the point processes obtained from a $Sib(\alpha, \mu)$ by giving to every point a multiplicity according to independent copies of Y_∞. Therefore we can generalize Theorem 14 in the following way.

Theorem 18. *An $\mathcal{F}$-stable point process with Levy measure Λ supported only by finite measures can be represented as a cluster process with centre process being a Poisson process on $\mathbb{M}_1$ driven by the spectral measure σ and daughter processes having p.g.fl. $G_{\Upsilon(\mu)}[B(h)]$, where $\Upsilon(\mu)$ are $Sib(\alpha, \mu)$ distributed point processes and $B(\cdot)$ is the B-function of the branching process driven by $\mathcal{F}$.*

3.3.3 Regular and singular $\mathcal{F}$-stable processes

We can extend the decomposition in regular and singular components for DαS processes (see Theorem 19) to $\mathcal{F}$-stable processes.

Theorem 19. *An $\mathcal{F}$-stable p.p. Φ with Levy measure Λ can be represented as the sum of two independent $\mathcal{F}$-stable point processes*

$$\Phi = \Phi_r + \Phi_s,$$

where Φ_r is regular and Φ_s singular. Φ_r is an $\mathcal{F}$-stable p.p. with Levy *measure being* $\Lambda\big|_{\mathbb{M}_f} = \Lambda(\cdot\mathbb{I}_{\mathbb{M}_f})$ *and* Φ_s *is a* $D\alpha S$ *p.p. with Levy measure* $\Lambda\big|_{\mathbb{M}_\infty}$.

Remark 14. *In an analogous way to the* $St\alpha S$ *case (see Remark 8) the regular component of* Φ*, that we call* Φ_r*, can be represented as a Sibuya cluster p.p. with p.g.fl. given by (3.32)*

$$G_\Phi[h] = \exp\big\{ - \int_{\mathbb{M}_1} \big(1 - G_{\Upsilon(\mu)}[B(h)]\big)\tilde{\sigma}(d\mu)\big\} \qquad \forall h \in \mathcal{V}(\mathcal{X}),$$

with spectral measure $\tilde{\sigma} = \sigma\big|_{\mathbb{M}_1}$*, where* σ *is the spectral measure of* Φ*. On the other hand the singular component* Φ_s *is not a Sibuya cluster p.p., and his p.g.fl. can be represented by (2.9) with spectral measure being* $\sigma\big|_{\mathbb{S}\setminus\mathbb{M}_1}$.

Chapter 4

Definition of the general branching stability

4.1 Markov branching processes on $\mathcal{N}_{\mathbb{R}^n}$

In this section we follow Asmussen and Hering treatment in [8], Chapter V.

4.1.1 Definition

Let $(\Psi_t^{\varphi})_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ be a stochastic process on $\big(\mathcal{N}_{\mathbb{R}^n}, \mathcal{B}(\mathcal{N}_{\mathbb{R}^n})\big)$ where $t \geq 0$ is the time parameter and $\varphi \in \mathcal{N}_{\mathbb{R}^n}$ is the starting configuration. We require $(\Psi_t^{\varphi})_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ to be a time-homogeneous Markov branching process, meaning that, if we denote by $(P_t(\varphi,\cdot))_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ the probability distribution of Ψ_t^{φ}, given $t, s \geq 0$ we have

$$P_{t+s}(\varphi, A) = \int_{\mathcal{N}_{\mathbb{R}^n}} P_s(\psi, A) P_t(\varphi, d\psi).$$

In this framework it can be shown (see [8], Chapter V, section 1) that the following two conditions are equivalent.

Condition 1.

$$No\ immigration: \ \ P_t(\emptyset,\{\emptyset\}) = 1 \ \forall t \geq 0;$$

$$Independent\ branching: \ \ \forall \varphi_0 \in \mathcal{N}_{\mathbb{R}^n},\ \varphi_0 = \sum_{i=1}^{k} \delta_{x_i} \ \ with\ x_i \in \mathbb{R}^n$$

$$P_t(\varphi_0, \{\varphi \in \mathcal{N}_{\mathbb{R}^n} : \varphi(A_j) = n_j, j = 1, ..., m\}) =$$

$$= \sum_{\{n_{j_1}+...+n_{j_k}=n_j,\ \forall j=1,...m\}} \prod_{i=1}^{k} P_t(\delta_{x_i}, \{\varphi \in \mathcal{N}_{\mathbb{R}^n} : \varphi(A_j) = n_{j_i}, j = 1, ..., m\}).$$

Condition 2. *Let $G_{t,\varphi}[\cdot]$ be the p.g.fl. of Ψ_t^{φ}. Then*

$$G_{t,\varphi}[h] = G_{\varphi}\big[G_{t,\delta_x}[h]\big] \qquad \forall h \in BC(\mathbb{R}^n),\ \forall t \geq 0,\ \forall \varphi \in \mathcal{N}_{\mathbb{R}^n}. \quad (4.1)$$

Definition 36. *A Markov branching process on $\mathcal{N}_{\mathbb{R}^n}$ is a (time-homogeneous) Markov process on $\big(\mathcal{N}_{\mathbb{R}^n}, \mathcal{B}(\mathcal{N}_{\mathbb{R}^n})\big)$ which satisfies the two equivalent conditions above.*

4.1.2 Construction

Given the definition of Markov branching processes on $\mathcal{N}_{\mathbb{R}^n}$ (which are sometimes called branching particle systems) we ask ourselves if such processes exist and how they can be constructed. For our purposes it's enough to give the main ideas on how such processes can be obtained and then provide some references where details can be found.
We follow the construction given by [8], Chapter V. Firstly we add two points, $\{\partial, \Delta\}$, to $\mathbb{R}^n$ making a two point compactification $\mathbb{R}^{n*} := \mathbb{R}^n \cup \{\partial, \Delta\}$. The intuitive meaning of ∂ and Δ will be clear in a few lines. Let $\big(X(t)\big)_{t\geq 0}$ be a strong Markov process on $\mathbb{R}^{n*}$, right continuous with left limit. Let its transition semigroup be denoted by $Q_t(x, B)$, where $t \geq 0$, $x \in \mathbb{R}^{n*}$ and $B \in \mathcal{B}(\mathbb{R}^{n*})$. ∂ and Δ work as traps for the process $\big(X(t)\big)_{t\geq 0}$, i.e.

$$Q_t(\partial, \{\partial\}) = 1 \text{ and } Q_t(\Delta, \{\Delta\}) = 1 \ \forall t \geq 0.$$

Let us define a kernel $F(x, A)$

$$F : \mathbb{R}^n \times \mathcal{B}(\mathcal{N}_{\mathbb{R}^n}) \to [0, 1],$$

such that for every $x \in \mathbb{R}^n$ $F(x, \cdot)$ is a probability measure on $\big(\mathcal{N}_{\mathbb{R}^n}, \mathcal{B}(\mathcal{N}_{\mathbb{R}^n})\big)$ and for every $A \in \mathcal{B}(\mathcal{N}_{\mathbb{R}^n})$ $F(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable.
A Markov branching process $(\Psi_t^{\varphi})_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ can be defined in the following way:

1. every particle moves independently according to the transition semigroup of $\big(X(t)\big)_{t\geq 0}$, $Q_t(x, B)$;
2. if a particle hits ∂ it dies out;
3. if a particle hits Δ it branches: if the hitting time was T the particle is replaced by an offspring according to $F(X(T^-), \cdot)$, where $X(T^-)$ represents the left limit of $X(t)$ as $t \uparrow T$. Branching operations of different particle are independent.

Asmussen and Hering in [8] show that such processes are well defined and are indeed Markov branching processes on $\mathcal{N}_{\mathbb{R}^n}$ (they work with more general space then $\mathbb{R}^n$). They do not prove that every Markov branching processes on $\mathcal{N}_{\mathbb{R}^n}$ can be represented in this way. A result of that type is given in [9],[10] and [11]: given a compact metrizable space $\mathcal{X}$ every Markov branching process on $\mathcal{N}_{\mathcal{X}}$ which is an Hunt process with reference-measure admits a representation as the one shown above (with diffusion $\big(X(t)\big)_{t\geq 0}$ and branching given by the kernel $F(x, A)$).
Another classical way of constructing Markov branching processes on $\mathcal{N}_{\mathbb{R}^n}$ doesn't use the two-point compactification as above, and particles' life-times are distributed according to exponential distributions (see [12] section 3.2 for details).

4.2 The general branching operation for point processes

Let us consider a finite configuration of points in $\mathbb{R}^n$, which we represent as a finite counting measure on $\mathbb{R}^n$, $\varphi \in \mathcal{N}_{\mathbb{R}^n}$. In this section we want to define a stochastic "multiplication" of φ for a real number. We denote such an operation with the symbol $\circ$ and we define it for the couples $(t, \varphi) \in (0, 1] \times \mathcal{N}_{\mathbb{R}^n}$. Although φ is deterministic $t \circ \varphi$ is a stochastic point process on $\mathbb{R}^n$. This operation can be viewed as acting on the probability distributions on $\mathcal{N}_{\mathbb{R}^n}$ so that:

$$G_{t\circ\Phi}[h] = \int_{\mathcal{N}_{\mathbb{R}^n}} G_{t\circ\varphi}[h]\mathcal{P}_\Phi(d\varphi) \qquad \forall h \in BC(\mathbb{R}^n), \tag{4.2}$$

where Φ is any finite p.p. on $\mathbb{R}^n$, $\mathcal{P}_\Phi$ its probability distribution and $G_{t\circ\Phi}$ and $G_{t\circ\varphi}$ the p.g.fl.s of $t \circ \Phi$ and $t \circ \varphi$ respectively.

Definition 37. *Let $\circ$ be an operation defined on the couples (t, Φ), where $t \in (0, 1]$ and Φ is a finite p.p. on $\mathbb{R}^n$, such that the outcome $t \circ \Phi$ is a finite p.p. on $\mathbb{R}^n$. Let $\circ$ satisfy (4.2). Such an operation is a (general) branching operation if it satisfies the following three requirements:*

1. *Associativity with respect to superposition: $\forall\ \varphi \in \mathcal{N}(\mathbb{R}^n)$ and $\forall$ $t_1, t_2 \in (0, 1]$*

$$G_{t_1\circ(t_2\circ\varphi)}[h] = G_{(t_1t_2)\circ\varphi}[h] = G_{t_2\circ(t_1\circ\varphi)}[h] \qquad \forall h \in BC(\mathbb{R}^n); \tag{4.3}$$

2. *Distributivity with respect to superposition: $\forall\ \varphi_1, \varphi_2 \in \mathcal{N}(\mathbb{R}^n)$ and $\forall\ t \in (0, 1]$*

$$G_{t\circ(\varphi_1+\varphi_2)}[h] = G_{t\circ\varphi_1}[h]G_{t\circ\varphi_2}[h], \qquad \forall h \in BC(\mathbb{R}^n); \tag{4.4}$$

3. *Continuity:* $\forall\ \varphi \in \mathcal{N}(\mathbb{R}^n)$

$$t \circ \varphi \rightharpoonup \varphi \qquad t \uparrow 1, \tag{4.5}$$

where $\rightharpoonup$ *denotes the weak convergence of measure.*

The reason to call these operations "branching operation" is that there is a bijection between them and right-continuous Markov branching processes on $\mathcal{N}(\mathbb{R}^n)$, as it is proved in Theorem 13.

Remark 15. *Using* (4.2) *it is easy to prove that the three conditions that characterize (general) branching operations are equivalent to the followings:*

1'. *Associativity with respect to superposition: for every finite p.p. on* $\mathbb{R}^n$ Φ *and* $\forall\ t_1, t_2 \in (0,1]$

$$G_{t_1\circ(t_2\circ\Phi)}[h] = G_{(t_1t_2)\circ\Phi}[h] = G_{t_2\circ(t_1\circ\Phi)}[h] \qquad \forall h \in BC(\mathbb{R}^n);$$

2'. *Distributivity with respect to superposition:* $\forall\ t \in (0,1]$ *and for every couple of finite independent p.p.s on* $\mathbb{R}^n$ Φ_1 *and* Φ_2

$$G_{t\circ(\Phi_1+\Phi_2)}[h] = G_{t\circ\Phi_1}[h]G_{t\circ\Phi_2}[h], \qquad \forall h \in BC(\mathbb{R}^n).$$

3'. *Continuity: for every finite p.p. on* $\mathbb{R}^n$ Φ *and for every* $t_0 \in (0,1]$

$$t \circ \Phi \rightharpoonup \Phi \qquad t \uparrow t_0, \tag{4.6}$$

where $\rightharpoonup$ *denotes the weak convergence of measure.*

Example 4. *The simplest non trivial example of such a multiplication is thinning. Also the* $\mathcal{F}$*-operation described in chapter 3 satisfies the requirements above.*

Proposition 13. *Let $\circ$ be an operation acting on point processes and satisfying* (4.2)*. Then $\circ$ is a general branching operation if and only if there exists a right continuous Markov branching process on $\big(\mathcal{N}_{\mathbb{R}^n}, \mathcal{B}(\mathcal{N}_{\mathbb{R}^n})\big)$, $(\Psi_t^{\varphi})_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ such that*

$$\Psi_t^{\varphi} \stackrel{D}{=} e^{-t} \circ \varphi \qquad \forall t \in [0, +\infty),\ \varphi \in \mathcal{N}(\mathbb{R}^n). \tag{4.7}$$

Proof. Necessity: Give a general branching operation $\circ$ we denote the probability distribution of $e^{-t}\circ\varphi$ by $P_t(\varphi, \cdot)$. We want $\big(P_t(\varphi, \cdot)\big)_{t\geq 0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ to be the transition probability functions of a Markov branching process on $\mathcal{N}_{\mathbb{R}^n}$. Therefore we need to prove Chapman-Kolmogorov equations. Since $\circ$ is defined on $\mathcal{N}_{\mathbb{R}^n}$ and then extended to point processes (see (4.2)) for every finite point process Φ and $t \geq 0$ we have that

$$Pr\big\{t \circ \Phi \in A\big\} = \int_{\mathcal{N}_{\mathbb{R}^n}} P_t(\varphi, A) P_{\Phi}(d\varphi) \quad \forall A \in \mathcal{B}(\mathcal{N}_{\mathbb{R}^n}), \tag{4.8}$$

where $P_{\Phi}(\cdot)$ is the probability distribution of Φ. Using this equation we obtain that given $t_1, t_2 \geq 0$ and $\varphi \in \mathcal{N}_{\mathbb{R}^n}$ the distribution of $e^{-t_2} \circ (e^{-t_1} \circ \varphi)$ is given by

$$Pr\big\{e^{-t_2} \circ (e^{-t_1} \circ \varphi) \in A\big\} = \int_{\mathcal{N}_{\mathbb{R}^n}} P_{t_2}(\psi, A) P_{t_1}(\varphi, d\psi) \quad \forall A \in \mathcal{B}(\mathcal{N}_{\mathbb{R}^n}).$$

From the associativity of $\circ$ we know that

$$e^{-t_1} \circ (e^{-t_2} \circ \varphi) \stackrel{D}{=} (e^{-t_1-t_2}) \circ \varphi,$$

from which Chapman-Kolmogorov equations follow

$$\int_{\mathcal{N}_{\mathbb{R}^n}} P_{t_1}(\psi, A) P_{t_2}(\varphi, d\psi) = P_{t_1+t_2}(\varphi, A) \quad \forall A \in \mathcal{B}(\mathcal{N}_{\mathbb{R}^n}).$$

We denote the Markov process on $\mathcal{N}_{\mathbb{R}^n}$ associated to $\big(P_t(\varphi, \cdot)\big)_{t\geq 0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ by Ψ_t^{φ} and its p.g.fl. by $G_{t,\varphi}[\cdot]$. The independent branching property of Ψ_t^{φ} (see (4.1)) follows from the distributivity of $\circ$. In fact using the definition of $P_t(\varphi, \cdot)$ and the distributivity of $\circ$ we obtain

$$G_{t,\varphi}[h] = G_{e^{-t}\circ\varphi}[h] = G_{\varphi}\big[G_{e^{-t}\circ\delta_x}[h]\big] = G_{\varphi}\big[G_{e^{-t}\circ\delta_x}[h]\big] = G_{\varphi}\big[G_{t,\delta_x}[h]\big].$$

From the left continuity of $\circ$ it follows immediately that $\Psi_{t,\varphi}$ is right continuous in the weak topology.

Sufficiency: Let $(\Psi_t^{\varphi})_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$ be a Markov branching process on $\mathcal{N}_{\mathbb{R}^n}$. We consider the operation $\circ$ induced by (4.7), i.e.

$$t \circ \varphi \stackrel{D}{=} \Psi^{\varphi}_{-\ln(t)}. \tag{4.9}$$

We start proving associativity of $\circ$, which means that $\forall\ \varphi \in \mathcal{N}_{\mathbb{R}^n}$ and $\forall\ t_1, t_2 \in (0,1]$

$$t_1 \circ (t_2 \circ \varphi) \stackrel{D}{=} (t_1 t_2) \circ \varphi. \tag{4.10}$$

Using (4.9) and (4.2) we obtain that the distribution of $t_1 \circ (t_2 \circ \varphi)$ is

$$Pr\big(t_1 \circ (t_2 \circ \varphi) \in A\big) = \int_{\mathcal{N}_{\mathbb{R}^n}} P_{-\ln t_1}(\psi, A) P_{-\ln t_2}(\varphi, d\psi) \quad \forall A \in \mathcal{B}(\mathcal{N}_{\mathbb{R}^n}),$$

where $P_t(\varphi,\cdot)$ is the distribution of Ψ_t^{φ}. Using Chapman-Kolmogorov equations the right hand side of the equation becomes $P_{\ln(t_1t_2)}(\varphi, A)$ and therefore associativity (i.e. (4.10)) holds. We prove distributivity. Using the definition of $\circ$ and the independent branching property of Ψ_t^{φ} it follows

$$G_{t\circ(\varphi_1+\varphi_2)}[h] \stackrel{(4.9)}{=} G_{-\ln t, \varphi_1+\varphi_2}[h] \stackrel{(4.1)}{=} G_{\varphi_1+\varphi_2}\big[G_{-\ln t,\delta_x}[h]\big] \quad \forall h \in BC(\mathbb{R}^n).$$

Since φ_1 and φ_2 are deterministic measure they're independent and so

$$G_{\varphi_1+\varphi_2}\big[G_{t\circ\delta_x}[h]\big] = G_{\varphi_1}\big[G_{t\circ\delta_x}[h]\big] G_{\varphi_2}\big[G_{t\circ\delta_x}[h]\big] = G_{t\circ\varphi_1}[h] G_{t\circ\varphi_2}[h].$$

From the last two equations distributivity of $\circ$ follows. Finally the continuity of $\circ$ follows immediately from the definition of $\circ$ (see (4.9)) and the right continuity of $\Psi_t^{\varphi} \in \mathcal{N}_{\mathbb{R}^n}$. $\square$

4.3 Two simple examples of general branching operations

As shown before every general branching operation for point processes corresponds to a general Markov branching process in $\mathcal{N}_{\mathbb{R}^n}$. Such processes are basically made of two components: a diffusion one and a branching one (see 4.1.2). We present here two examples of these processes and of the induced branching operation on point processes.

4.3.1 Simple diffusion

The first case we consider is the one in which there is only diffusion and no branching. Let $X(t)$ be a strong Markov process on $\mathbb{R}^n$, right continuous with left limits. We can associate to X a diffusion process $(\Psi_{t,\varphi})_{t>0,\varphi\in\mathcal{N}_{\mathbb{R}^n}}$: starting from a point configuration φ every particle moves according to an independent copy of $X(t)$. We denote by $\circ_d$ the branching operation associated through (4.7). $\circ_d$ acts on a finite point process Φ by shifting every point x_i by $X_i\big(-\ln(t)\big)$, where $(X_i)_{i\in\mathbb{N}}$ are independent copies of X. We denote by f_t the density function of the distribution of $X\big(-\ln(t)\big)$. Then, given a p.p. Φ with p.g.fl. $G_\Phi[h]$, the p.g.fl. of $t\circ_d\Phi$ is

$$G_{t\circ_d\Phi}[h]=\mathbb{E}[\prod_{x_i\in t\circ_d\Phi}h(x_i)]=\mathbb{E}[\prod_{x_i\in\Phi}h\big(x_i+X_i(-\ln(t))\big)]=$$

$$=\mathbb{E}[\mathbb{E}[\prod_{x_i\in\Phi}h(x_i+X_i(-\ln(t)))|\Phi]]=\mathbb{E}[\prod_{x_i\in\Phi}\mathbb{E}[h(x_i+X_i(-\ln(t)))]]=$$

$$=\mathbb{E}[\prod_{x_i\in\Phi}f_t*h(x_i)]=G_\Phi[f_t*h].$$

4.3.2 Thinning with diffusion

The second case of general Markov branching process that we consider is the following: every particle moves independently according to $X(t)$, a strong Markov process on $\mathbb{R}^n$ right continuous with left limits, and after exponential time it dies. We call this operation *thinning with diffusion* and denote it by $\circ_{td}$. This operation acts on a point process Φ as the composition of the thinning and the diffusion operation (the order in which the operations are applied is not relevant, see Remark 16). We give the following definition.

Definition 38. *Let $X(t)$ be a strong Markov process on $\mathbb{R}^n$ right continuous with left limits. Let f_t denotes the density function of the distribution of $X\big(-\ln(t)\big)$. We denote the thinning with diffusion operation associated to $X(t)$ by $\circ_{td}$. Given a finite p.p. on $\mathbb{R}^n$ Φ, the process $t \circ_{td} \Phi$ is defined through its p.g.fl.:*

$$G_{t\circ_{td}\Phi}[h] = G_\Phi[1-t+t(f_t * h)] \qquad \forall h \in BC(\mathbb{R}^n), \tag{4.11}$$

where G_Φ is the p.g.fl. of Φ.

Remark 16. *The density function f_t has mass 1, therefore $1-t+t(f_t * h) = f_t * (1-t+th)$. This means that for every finite point process Φ on $\mathbb{R}^n$*

$$t \circ_{td} \Phi \stackrel{D}{=} t \circ_d (t \circ \Phi) \stackrel{D}{=} t \circ (t \circ_d \Phi),$$

where $\circ$ denotes thinning and $\circ_d$ the diffusion operation described in subsection 4.3.1. This means that thinning with diffusion is the composition of the thinning and the diffusion operation where the order with which these two operations are applied is not relevant.

4.4 Notion of stability for subcritical general branching operations

Let $\circ$ be a general branching operation for point processes associated to a Markov branching process on $\mathcal{N}_{\mathbb{R}^n}$ Ψ_t^φ. Ψ_t^φ is obtained from $\circ$ as shown in Theorem 13. We say that the operation $\circ$ is subcritical in the case it is associated to a subcritical branching process Ψ_t^{φ} (meaning that the mean number of particle is decreasing, i.e. $\mathbb{E}[\Psi_t^{\varphi}(\mathbb{R}^n)] < \varphi(\mathbb{R}^n)$.

Proposition 14. *Let $\circ$ be a subcritical branching operation for point processes. Let Φ be a finite point process on a c.s.m.s. $\mathcal{X}$ and $(\Phi^{(1)}, ..., \Phi^{(n)})$ independent copies of it. Φ is called (strictly) stable with respect to $\circ$ if it holds one of the following equivalent conditions:*

1. $\forall\ n \in \mathbb{N}\ \exists\ c_n \in (0,1]$ *such that*

$$\Phi \stackrel{\mathcal{D}}{=} c_n \circ (\Phi^{(1)} + ... + \Phi^{(n)});$$

2. $\forall\ \lambda > 0\ \exists\ t \in [0,1]$ *such that*

$$G_{\Phi}[h] = \big(G_{t\circ\Phi}[h]\big)^{\lambda};$$

3. $\exists\ \alpha > 0$ *such that* $\forall\ n \in \mathbb{N}$

$$\Phi \stackrel{\mathcal{D}}{=} (n^{-\frac{1}{\alpha}}) \circ (\Phi^{(1)} + ... + \Phi^{(n)}); \tag{4.12}$$

4. $\exists\ \alpha > 0$ *such that* $\forall\ t \in [0,1]$

$$G_{\Phi}[h] = \big(G_{t\circ\Phi}[h]\big)^{t^{-\alpha}};$$

5. $\exists\ \alpha > 0$ *such that* $\forall\ t \in [0,1]$

$$t^{1/\alpha} \circ \Phi^{(1)} + (1-t)^{1/\alpha} \circ \Phi^{(2)} \stackrel{\mathcal{D}}{=} \Phi. \tag{4.13}$$

Proof. 4) $\Rightarrow$ 2) $\Rightarrow$ 1) are obvious implications. If we prove 1)$\Rightarrow$4) then 1),2) and 4) are equivalent.
1)$\Rightarrow$4) : $\forall m, n \in \mathbb{N}$ using distributivity and associativity we get

$$\Phi \stackrel{\mathcal{D}}{=} c_n \circ (\Phi^{(1)} + ... + \Phi^{(n)}) \stackrel{\mathcal{D}}{=}$$

$$\stackrel{\mathcal{D}}{=} c_n \circ \Big(c_m \circ (\Phi^{(1)} + ... + \Phi^{(m)}) + ... + c_m \circ (\Phi^{(n-1)m+1} + ... + \Phi^{(nm)}) \Big) \stackrel{\mathcal{D}}{=}$$

$$\stackrel{\mathcal{D}}{=} (c_n c_m) \circ (\Phi^{(1)} + ... + \Phi^{(nm)}),$$

which implies that

$$c_{nm} = c_n c_m. \tag{4.14}$$

Given $n, m \in \mathbb{N}$ since we are considering the subcritical case we have

$$n > m \Rightarrow c_n < c_m. \tag{4.15}$$

We then define a function $c : [1, +\infty) \cap \mathbb{Q} \to (0, 1]$. For every $1 \leq m \leq n < +\infty$, $m, n \in \mathbb{N}$

$$c\Big(\frac{n}{m}\Big) := \frac{c_n}{c_m}. \tag{4.16}$$

The function c is well defined because of (4.14) and has value in $(0, 1]$ because of (4.15).

Using associativity, distributivity and hypothesis *1)*

$$\Big(G_{\frac{c_n}{c_m} \circ \Phi}[h]\Big)^{\frac{n}{m}} = \Big(G_{\frac{c_n}{c_m} \circ \big(c_m \circ (\Phi^{(1)} + ... + \Phi^{(m)})\big)}[h]\Big)^{\frac{n}{m}} =$$

$$= \Big(\Big(G_{c_n \circ \Phi}[h]\Big)^m\Big)^{\frac{n}{m}} = \Big(G_{c_n \circ \Phi}[h]\Big)^n = G_\Phi[h].$$

Therefore

$$G_\Phi[h] = \Big(G_{c(x) \circ \Phi}[h]\Big)^x \quad \forall x \in [1, +\infty) \cap \mathbb{Q}. \tag{4.17}$$

We want to extend this relationship for $x \in [1, +\infty) \cap \mathbb{R}$. Firstly we notice that from (4.15) and (4.16) we obtain that c is a strictly decreasing function. Therefore we can define a function $\tilde{c} : [1, +\infty) \to (0, 1]$ in the following way

$$\tilde{c}(x) := \inf\{c(y) |\ y \in [1, x) \cap \mathbb{Q}\}.$$

Since $\tilde{c}(x) = c(x)$ for every $x \in [1, +\infty) \cap \mathbb{Q}$ we will call both functions c. It is easy to see from (4.14) and (4.15), taking limits over rational numbers, that $c(xy) = c(x)c(y)$ for every $x, y \in [1, +\infty)$. The only monotone functions c from $[1, +\infty)$ to $(0, 1]$ such that $c(0) = 1$ and $c(xy) = c(x)c(y)$ for every $x, y \in [1, +\infty)$ have the following form $c(x) = x^r$ with $r \in \mathbb{R}$. Since our function is decreasing then $r < 0$. We fix $r := -\frac{1}{\alpha}$ with $\alpha > 0$ exponent of stability.

Let $\{x_n\}_{n\in\mathbb{N}} \subset [1, +\infty) \cap \mathbb{Q}$ be such that $x_n \downarrow x$ as $n \to +\infty$, and therefore $x_n^{-\frac{1}{\alpha}} \uparrow x^{-\frac{1}{\alpha}}$ as $n \to +\infty$. Since $\circ$ is left-continuous in the weak topology it holds

$$x_n^{-\frac{1}{\alpha}} \circ \Phi \rightharpoonup x^{-\frac{1}{\alpha}} \circ \Phi \qquad n \to +\infty,$$

where $\rightharpoonup$ denotes the weak convergence. From (4.17) we have

$$\big(G_\Phi[h]\big)^{\frac{1}{x}} = \lim_{n\to+\infty} G_{c(x_n)\circ\Phi}[h] = \lim_{n\to+\infty} G_{x_n^{-\frac{1}{\alpha}}\circ\Phi}[h].$$

If we have a sequence of point processes $\{\mu_n\}_{n\in\mathbb{N}}$ such that their p.g.fl., $G_n[h]$, converge pointwise to a functional $G[h]$ such that $G[h] \to 1$ for every $h \uparrow 1$, then there exist a random measure μ such that $\mu_n \rightharpoonup \mu$ and $G[h]$ is the p.g.fl. of μ (see Exercise 5.1 in Kallenberg (1983)). Since $G_\Phi[h] \uparrow 1$ as $h \uparrow 1$ then also $\big(G_\Phi[h]\big)^{\frac{1}{x}} \uparrow 1$ as $h \uparrow 1$, and thus $\big(G_\Phi[h]\big)^{\frac{1}{x}}$ is the p.g.fl. of $x^{-\frac{1}{\alpha}} \circ \Phi$.

4) $\Rightarrow$ 3) $\Rightarrow$ 1) are obvious implications and so also 3) is equivalent to 1),2) and 4).

4)$\Rightarrow$5): Let $x, y \in [1, +\infty)$. Then because of 4)

$$G_\Phi[h] = G_{(x+y)^{-\frac{1}{\alpha}}\circ\Phi}[h]^{x+y} = G_{x^{-\frac{1}{\alpha}}\left(\frac{x+y}{x}\right)^{-\frac{1}{\alpha}}\circ\Phi}[h]^x \cdot G_{y^{-\frac{1}{\alpha}}\left(\frac{x+y}{y}\right)^{-\frac{1}{\alpha}}\circ\Phi}[h]^y =$$

$$= G_{\left(\frac{x+y}{x}\right)^{-\frac{1}{\alpha}}\circ\Phi}[h] \cdot G_{\left(\frac{x+y}{y}\right)^{-\frac{1}{\alpha}}\circ\Phi}[h] = G_{\left(\frac{x+y}{x}\right)^{-\frac{1}{\alpha}}\circ\Phi + \left(\frac{x+y}{y}\right)^{-\frac{1}{\alpha}}\circ\Phi'}[h],$$

where Φ' is an independent copy of Φ. From the arbitrariness of $x, y \in [1, +\infty)$ follows the thesis.

5)$\Rightarrow$3): (4.12) is obviously true for $n = 1$. We suppose (4.12) true for

$n-1$ and we prove it for n. Putting $t = \frac{1}{n}$ in (4.13) we obtain

$$\Phi \stackrel{\mathcal{D}}{=} n^{-\frac{1}{\alpha}} \circ \Phi' + (1 - \frac{1}{n})^{\frac{1}{\alpha}} \circ \Phi'' \stackrel{\mathcal{D}}{=} n^{-\frac{1}{\alpha}} \circ \Phi' + (\frac{n-1}{n})^{\frac{1}{\alpha}} \circ \Phi'',$$

and using (4.12) for $n-1$

$$n^{-\frac{1}{\alpha}} \circ \Phi' + (\frac{n-1}{n})^{\frac{1}{\alpha}} \circ \Phi'' \stackrel{\mathcal{D}}{=} n^{-\frac{1}{\alpha}} \circ \Phi' + (\frac{n-1}{n})^{\frac{1}{\alpha}} \circ \Big((n-1)^{-\frac{1}{\alpha}} \circ (\Phi^{(1)} + \ldots + \Phi^{(n-1)})\Big),$$

which is exactly (4.12) for n. $\square$

Future perspectives

The natural continuation of this work is to study and try to characterize stable point processes with respect to the general branching operation described in the fourth chapter. We are working on this problem and we are hopeful to succeed. We already obtained some results in the case branching operations made by a diffusion and a thinning components. In this case stable point processes admit a Cox representation similar to the one given for DαS p.p. in Chapter 2 (Theorem 13). We are now trying to understand how to deal with the case of a general branching (i.e. when the particle branches it is replaced by particles on different locations). The first aspect that could be worth exploring is the role of the limit conditional distribution of the branching process (Y_∞ in the notation of Chapter 3) in this general case.

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