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A Table of Upper Bounds for Binary Codes

Erik Agrell, Alexander Vardy, and Kenneth Zeger

Abstract— **Let $A(n,d)$ denote the maximum possible number of codewords in an (n,d) binary code. We establish 4 new bounds on $A(n,d)$, namely $A(21,4) \leqslant 43689$, $A(22,4) \leqslant 87378$, $A(22,6) \leqslant 6941$, and $A(23,4) \leqslant 173491$. Furthermore, using recent upper bounds on the size of constant-weight binary codes, we reapply known methods to generate a table of bounds on $A(n,d)$ for all $n \leqslant 28$. This table extends the range of parameters compared with previously known tables.**

Keywords— **Binary codes, constant-weight codes, Delsarte inequalities, linear programming, upper bounds.**

I. Introduction

An (n,d) *binary code* is a set of binary vectors (or *codewords*) of length n such that the Hamming distance between any two of them is at least d. An (n,d,w) *constant-weight* binary code is an (n,d) binary code in which all codewords have the same number w of ones. The *size* of a code is its cardinality. The maximum possible sizes of binary codes and constant-weight binary codes are denoted $A(n,d)$ and $A(n,d,w)$, respectively. Known methods to bound $A(n,d)$ often assume that bounds on $A(n,d,w)$ are known. Motivated by the recently published [1] tables of upper bounds on $A(n,d,w)$, we compute bounds on $A(n,d)$ for all lengths $n \leqslant 28$. This generates Table I, which is the main result of this correspondence. The table gives upper bounds for longer codes than existing tables; it also includes several updates to bounds in these tables.

The latest published table of upper bounds on $A(n,d)$ is [5, p. 248], for the range $n \leqslant 24$ and $d \leqslant 10$. A wider range of parameters is included in [4, Table II]. Updates to the combination of the upper bounds in [4] and [5] are given in boldface in Table I. Specifically, we establish four new bounds on $A(n,d)$ for $n \leqslant 24$, namely $A(21,4) \leqslant 43689$, $A(22,4) \leqslant 87378$, $A(22,6) \leqslant 6941$, and $A(23,4) \leqslant 173491$. Superscripts in Table I indicate the method used to obtain each upper bound, where integers refer to theorem numbers in this correspondence while S refers to bounds for specific parameters (discussed in the last paragraph of the next section). The best known lower bounds are included for completeness; these are taken from [9].

Online versions of the tables of bounds on $A(n,d)$ and $A(n,d,w)$ are available at [2]. We welcome reports of any updates, which will be recorded at [2] upon verification.

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II. A table of bounds on $A(n,d)$

We start with a brief review of known upper bounds on $A(n,d)$ that are referenced in Table I. The following bounds are due to Plotkin [12].

Theorem 1.

$$\begin{aligned} A(n,d) &\leqslant 2A(n-1,d) \\ A(n,d) &\leqslant 2\left\lfloor \frac{d}{2d-n} \right\rfloor, && \text{if } n < 2d \\ A(n,d) &\leqslant 2n, && \text{if } n = 2d. \end{aligned}$$

Johnson [10, p. 532] showed that the sphere-packing bound can be improved as follows.

Theorem 2. *For every positive integer δ,*

$$A(n,2\delta) \leqslant 2^{n-1}\left(\binom{n-1}{0} + \cdots + \binom{n-1}{\delta-1} + \frac{\binom{n-1}{\delta} - \binom{2\delta-1}{\delta-1} A(n-1,2\delta,2\delta-1)}{\left\lfloor \binom{n-1}{\delta} \right\rfloor}\right)^{-1}.$$

The best known bounds on $A(n,d,w)$ are tabulated in [1,2]. One useful result of Theorem 2 is $A(24,4) \leqslant 344308$. This was known to Johnson [8, Table I] in 1971, but has been overlooked in later tables [4, 5].

The *distance distribution* of a binary code $\mathscr{C}$ is defined as the sequence $A_i = |\{(\boldsymbol{c}_1, \boldsymbol{c}_2) \in \mathscr{C} \times \mathscr{C} : d(\boldsymbol{c}_1, \boldsymbol{c}_2) = i\}|/|\mathscr{C}|$ for $i = 0, 1, \ldots, n$, where $d(\cdot,\cdot)$ is the Hamming distance. It is known that $A(n,d) = A(n+1,d+1)$ if d is odd. Furthermore, for any (n,d) binary code with even d, there exists another (n,d) binary code with the same number of codewords, in which all codewords have even weight. Hence, the search for $A(n,d)$ can be limited to those codes for which d is even and $A_i = 0$ for all odd i. The linear programming bound was introduced by Delsarte [6], who showed that the distance distribution of any code satisfies

$$\sum_{i=0}^{n} A_i P_k(i) \geqslant 0$$

for $k = 0, 1, \ldots, n$, where $P_k(x)$ is the *Krawtchouk polynomial* of degree k, given by:

$$P_k(x) = \sum_{j=0}^{k} (-1)^j \binom{x}{j}\binom{n-x}{k-j}.$$

As discussed above, it would suffice to consider only even values of d, while assuming that $A_i = 0$ except for A_0 and $A_d, A_{d+2}, \ldots, A_{2\lfloor n/2 \rfloor}$. This leads to the following theorem.

Erratum: In Theorem 2, the denominator "\lfloor {n-1 \choose \delta} \rfloor" should be "\lfloor (n-1)/\delta \rfloor"

TABLE I
A table of bounds on $A(n,d)$. Boldface denotes updates to [4, 5].

n	d = 4	6	8	10	12	14
6	4^1	2^1				
7	8^1	2^1				
8	16^1	2^1	2^1			
9	20^4	4^1	2^1			
10	40^1	6^1	2^1	2^1		
11	72^{S}	12^1	2^1	2^1		
12	144^{S}	24^1	4^1	2^1	2^1	
13	256^3	32^{S}	4^1	2^1	2^1	
14	512^3	64^3	8^1	2^1	2^1	2^1
15	1024^2	128^3	16^1	4^1	2^1	2^1
16	2048^2	256^2	32^1	4^1	2^1	2^1
17	$2720-3276^3$	$256-340^{\mathrm{S}}$	$36-37^{\mathrm{S}}$	6^1	2^1	2^1
18	$5312-6552^1$	$512-680^1$	$64-72^{\mathrm{S}}$	10^1	4^1	2^1
19	$10496-13104^1$	$1024-1288^4$	$128-144^4$	20^1	4^1	2^1
20	$20480-26208^1$	$2048-2372^4$	$256-279^3$	40^1	6^1	2^1
21	$36864-\mathbf{43689}^4$	$2560-4096^{\mathrm{S}}$	512^{S}	$42-48^{\mathrm{S}}$	8^1	4^1
22	$73728-\mathbf{87378}^1$	$4096-\mathbf{6941}^4$	1024^3	$50-88^{\mathrm{S}}$	12^1	4^1
23	$147456-\mathbf{173491}^3$	$8192-13774^4$	2048^3	$76-150^4$	24^1	4^1
24	$294912-\mathbf{344308}^2$	$16384-24106^4$	4096^2	$128-280^3$	48^1	6^1
25	$524288-\mathbf{599185}^4$	$16384-\mathbf{48148}^3$	$4096-\mathbf{6425}^4$	$176-\mathbf{549}^4$	$52-\mathbf{56}^{\mathrm{S}}$	8^1
26	$1048576-\mathbf{1198370}^1$	$32768-\mathbf{86132}^4$	$4096-\mathbf{10336}^4$	$270-\mathbf{1029}^4$	$64-\mathbf{98}^{\mathrm{S}}$	14^1
27	$2097152-\mathbf{2396740}^1$	$65536-\mathbf{162400}^4$	$8192-\mathbf{17804}^3$	$512-\mathbf{1764}^3$	$128-\mathbf{169}^4$	28^1
28	$4194304-\mathbf{4793480}^1$	$131072-\mathbf{291269}^4$	$16384-\mathbf{32205}^4$	$1024-\mathbf{3200}^3$	$178-\mathbf{288}^3$	56^1

Theorem 3. *For every positive even integer d,*

$$A(n,d) \leqslant 1 + \lfloor \max(A_d + A_{d+2} + \cdots + A_{2\lfloor n/2 \rfloor}) \rfloor \quad (1)$$

subject to the constraints

$$0 \leqslant A_i \leqslant A(n,d,i), \qquad i = d, d+2, \ldots, 2\lfloor n/2 \rfloor$$
$$\sum_{j=d/2}^{\lfloor n/2 \rfloor} A_{2j} P_k(2j) \geqslant -\binom{n}{k}, \qquad k = 1, 2, \ldots, \lfloor n/2 \rfloor. \quad (2)$$

In some cases, the right-hand side of (2) can be slightly increased, as in the following theorem [3, Theorems 5, 8].

Theorem 4. *The distance distribution of an (n,d) binary code of odd size M satisfies*

$$\sum_{j=d/2}^{\lfloor n/2 \rfloor} A_{2j} P_k(2j) \geqslant \frac{1-M}{M}\binom{n}{k}, \qquad k = 1, 2, \ldots, \lfloor n/2 \rfloor$$

while if $M \equiv 2 \pmod 4$, then for at least one $l \in \{0, \ldots, n\}$

$$\sum_{j=d/2}^{\lfloor n/2 \rfloor} A_{2j} P_k(2j) \geqslant \frac{(2-M)\binom{n}{k} + 2P_k(l)}{M}, \quad k = 1, \ldots, \lfloor n/2 \rfloor.$$

Finally, some bounds hold only for specific values of n and d. The following bounds, that do not follow from Theorems 1–4, are included in Table I. $A(13,6) \leqslant 32$ was proved by linear programming in [10, pp. 538–540], using constraints specifically derived for these parameters. In a similar manner, van Pul [13, pp. 32–39] proved $A(18,8) \leqslant 72$, $A(21,10) \leqslant 48$, and $A(22,10) \leqslant 88$, while Honkala [7, pp. 25–27] obtained $A(25,12) \leqslant 56$ and $A(26,12) \leqslant 98$. The bounds $A(17,6) \leqslant 340$, $A(21,6) \leqslant 4096$, $A(17,8) \leqslant 37$, and $A(21,8) \leqslant 512$ have been derived in [3], apparently by linear programming, although the specific inequalities used in the optimization are not disclosed in [3]. The bounds $A(11,4) \leqslant 72$ and $A(12,4) \leqslant 144$ have been established in [11] with the help of a computer-assisted search method (thereby proving a long-standing conjecture).

References

[1] E. Agrell, A. Vardy, and K. Zeger, "Upper bounds for constant-weight codes," *IEEE Trans. Inform. Theory*, vol. 46, pp. 2373–2395, Nov. 2000.

[2] E. Agrell, A. Vardy, and K. Zeger, "Tables of binary block codes," available online at `www.s2.chalmers.se/~agrell`.

[3] M. R. Best, A. E. Brouwer, F. J. MacWilliams, A. M. Odlyzko, and N. J. A. Sloane, "Bounds for binary codes of length less than 25," *IEEE Trans. Inform. Theory*, vol. 24, pp. 81–93, Jan. 1978.

[4] A. E. Brouwer, J. B. Shearer, N. J. A. Sloane, and W. D. Smith, "A new table of constant weight codes," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1334–1380, Nov. 1990.

[5] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. New York, NY: Springer, 3rd ed., 1999.

[6] Ph. Delsarte, "Bounds for unrestricted codes, by linear programming," *Philips Res. Reports*, vol. 27, pp. 272–289, June 1972.

[7] I. Honkala, "Bounds for binary constant weight and covering codes," *Licentiate thesis*, Dept. of Mathematics, Univ. of Turku, Turku, Finland, Mar. 1987.

[8] S. M. Johnson, "On upper bounds for unrestricted binary error-correcting codes," *IEEE Trans. Inform. Theory*, vol. 17, pp. 466–478, July 1971.

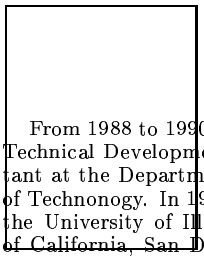
[9] S. Litsyn, "An updated table of the best binary codes known," in *Handbook of Coding Theory* (V. S. Pless and W. C. Huffman, eds.), vol. 1, pp. 463–498, Amsterdam: Elsevier, 1998.

[10] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam: North-Holland, 1977.

[11] P. R. J. Östergård, T. Baicheva, and E. Kolev, "Optimal binary one-error-correcting codes of length 10 have 72 codewords," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1229–1231, May 1999.

[12] M. Plotkin, "Binary codes with specified minimum distance," *IRE Trans. Inform. Theory*, vol. 6, pp. 445–450, Sept. 1960.

[13] C. L. M. van Pul, "On bounds on codes," *Master's thesis*, Dept. of Mathematics and Computing Science, Eindhoven Univ. of Technology, Eindhoven, The Netherlands, Aug. 1982.



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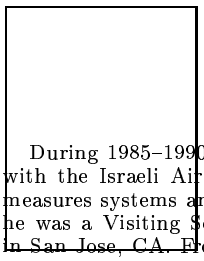

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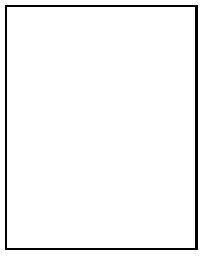

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