Hybrid Discontinuous Finite Element/Finite Difference Method for Maxwell's Equations

L.Beilina

Department of Mathematical Sciences, Chalmers University of Technology and Gothenburg University, 41296 Gothenburg

Abstract. A fully explicit, discontinuous hybrid finite element/finite difference method is proposed for the numerical solution of Maxwell's equations in the time domain. We call the method hybrid since the different numerical methods, interior penalty discontinuous finite element method, developed in [1], and finite difference method [2], are used in different parts of the computational domain. Thus, the flexibility of finite elements is combined with the efficiency of finite differences. Our numerical experiment illustrates stability of the proposed new method.

Keywords: adaptive finite element method, discontinuous finite element method, hybrid FEM/FDM methods, Maxwell's equations
PACS: 41.20.Jb, 42.30.Va, 42.30.Wb, 45.10.Db, 46.40Cd

INTRODUCTION

In real life applications unknown material coefficients and geometric singularities occupy only a small part of the computational domain Ω. It is well known that the Finite Difference Time-Domain (FDTD) scheme [2] is simple and efficient for solution of Maxwell's equations. However, it can be applied only on structured meshes. On other hand, Finite Element Method (FEM) can handle complex geometries using unstructured mesh discretization. Thus, hybrid FEM/FDTD method for solution of Maxwell's equations combines advantages of both schemes by using flexibility of Finite Elements with efficiency of FDTD scheme.

In our hybrid method the computational domain Ω is divided into two subregions, Ω_{FDM} and Ω_{FEM}, corresponding to the FD and the FE regions, respectively, such that $\Omega = \Omega_{FDM} \cup \Omega_{FEM}$. These two regions are meshed using structured and triangular/tetrahedral meshes, respectively, with common nodes shared at the interface. Typically, the unstructured region Ω_{FEM} is much smaller than Ω_{FDM}. We assume that Ω_{FEM} lies strictly inside Ω, that is away from the physical boundary Γ. It may consist of one or more subdomains and typically covers only a small part of Ω.

While in Ω_{FDM} FDTD scheme [2] is used, for the FE discretization of Maxwell's equations in Ω_{FEM} we use interior penalty discontinuous Galerkin method (IPDGFEM) developed in [1] which lead to diagonal mass matrice and fully explicit scheme. By adding suitable bilinear forms (numerical fluxes) to the standard variational formulation continuity across element interfaces is weakly enforced and thus, implementation of IPDGFEM using piecewise-linear functions is allowed. Efficiency of the resulting hybrid scheme in Ω is obtained by using mass lumping in both space and time in Ω_{FEM}, which makes the scheme fully explicit [1].

MAXWELL'S EQUATIONS

We consider Maxwell's equations in an inhomogeneous isotropic medium in a bounded domain $\Omega \subset \mathbb{R}^d,\ d = 2,3$ with boundary Γ:

CP1281, *ICNAAM, Numerical Analysis and Applied Mathematics, International Conference 2010, Vol.I*
edited by T. E. Simos, G. Psihoyios, and Ch. Tsitouras

$$
\begin{aligned}
\frac{\partial D}{\partial t} - \nabla \times H &= -J, \qquad \text{in } \Omega \times (0,T), \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \qquad \text{in } \Omega \times (0,T), \\
D &= \varepsilon E, \\
B &= \mu H, \\
E(x,0) &= E_0(x), \\
H(x,0) &= H_0(x).
\end{aligned} \tag{1}
$$

Here $E(x,t)$ and $H(x,t)$ are the (unknown) electric and magnetic fields, whereas $D(x,t)$ and $B(x,t)$ are the electric and magnetic inductions, respectively. The dielectric permittivity, $\varepsilon(x) > 0$, and magnetic permeability, $\mu(x) > 0$, together with the current density, $J(x,t) \in \mathbb{R}^d$, are given and assumed piecewise smooth. Moreover, the electric and magnetic inductions satisfy the relations

$$\nabla \cdot D = \rho,\ \nabla \cdot B = 0 \qquad \text{in } \Omega \times (0,T), \tag{2}$$

where $\rho(x,t)$ is a given charge density. For simplicity, we restrict ourselves to perfectly conducting boundary conditions

$$
\begin{aligned}
n \times E &= 0, \qquad \text{on } \Gamma \times (0,T), \\
H \cdot n &= 0, \qquad \text{on } \Gamma \times (0,T),
\end{aligned} \tag{3}
$$

where n is the outward normal on Γ.

By eliminating the magnetic field from (1) we obtain vector wave equation with for the electric field E

$$\varepsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times E) = -j, \tag{4}$$

with appropriate initial conditions and perfectly conducting boundary condition (3). In (4) the source function j is defined as $j = \frac{\partial J}{\partial t}$. Equation for magnetic field H can be obtained similarly by eliminating the electric field from (1).

Discontinuous finite element method

For the finite element discretization of (4) we use IPDGFEM developed in [1] together with initial conditions

$$\frac{\partial E}{\partial t}(x,0) = E(x,0) = 0, \qquad \text{in } \Omega, \tag{5}$$

and perfectly conducting boundary condition

$$n \times E = 0, \qquad \text{on } \Gamma \times (0,T). \tag{6}$$

In Ω_{FEM} we associate with K_h a mesh function h, which represents the mesh size of K_h given by $h = \max_{K \in K_h} h_K$, where h_K is the diametr of the element K. Next, we denote by F_h^I the set of all interior faces of elements in K_h and by F_h^B the set of all boundary faces of elements in K_h, and denote by $F_h := F_h^I \cup F_h^B$. If μ is discontinuous then the local mesh sizes are of bounded variation such that there exists a constant $k > 0$ which depends only on the shape regularity of the mesh such that $kh_K^+ \leq h_K^- \leq k^{-1} h_K^+$ where K^+ and K^- are neighboring elements in the mesh.

For the time discretization we let $J_\tau = \{J\}$ be a partition of the time interval $I = [0,T]$, where $0 = t_0 < t_1 < ... < t_N = T$ is a sequence of discrete time steps with associated time intervals $J = (t_{k-1}, t_k]$ of constant length $\tau = t_k - t_{k-1}$.

Let w is a piecewise smooth vector-valued function and $f \in F_h^I$ be an interior face shared by two neighboring elements K^+ and K^-. Denoting by $w^\pm$ the traces of w taken from $K^\pm$, respectively, the tangential jumps and averages across f defines as follows

$$[[w]] := n^+ \times w^+ + n^- \times w^-,\ \ \{\{w\}\} := \frac{w^+ + w^-}{2}, \tag{7}$$

respectively. On boundary faces we set $[[w]] := n \times w$ and $\{\{w\}\} := w$.

For a piecewise smooth scalar function φ with $\varphi^{\pm} := \varphi|_K^{\pm}$ the tangential jumps and averages across f defines as follows

$$[[\varphi]] := n^+ \times \varphi^+ + n^- \times \varphi^-, \ \ \{\{w\}\} := \frac{\varphi^+ + \varphi^-}{2}, \tag{8}$$

respectively. On boundary faces we set $[[\varphi]] := n \times \varphi$ and $\{\{\varphi\}\} := \varphi$. We note that the jump $[[\varphi]]$ of the scalar function φ across f is a vector tangential to the normal to f, and the jump of vector function w is a scalar quantity.

To formulate a finite element method for (4), (5) and (6) we introduce the finite element trial space W_h^E, defined by

$$W_h^E := \{w \in W^E : w|_{K\times J} \in [P_1(K) \times P_1(J)]^3, \ \ \forall K \in K_h, \ \ \forall J \in J_\tau\},$$

where $P_1(K)$ and $P_1(J)$ denote the set of discontinuous linear functions on K and continuous on J, respectively, and

$$W^E := \{w \in [H^1(\Omega \times I)]^3 : w(\cdot,0) = 0, \ \ n \times w|_\Gamma = 0\}.$$

We also define the following L_2 inner products and norms

$$((p,q)) := \int_\Omega \int_0^T pq\, dx\, dt, \quad \|p\|^2 = ((p,p)),$$

$$(\alpha,\beta) := \int_\Omega \alpha\beta\, dx, \quad |\alpha|^2 = (\alpha,\alpha),$$

The discontinuous finite element method for (4) now reads: Find $E_h \in W_h^E$ such that $\forall \bar{\varphi} \in W_h^E$,

$$-((\varepsilon \frac{\partial E_h^k}{\partial t}, \frac{\partial \bar{\varphi}}{\partial t})) \quad + \quad ((\frac{1}{\mu}\nabla \times E_h^k, \nabla \times \bar{\varphi})) + ((j^k, \bar{\varphi})) - \int_0^T \int_{F_h} [[E_h^k]] \cdot \{\{\frac{1}{\mu}\nabla \times \bar{\varphi}\}\}\, ds\, dt \tag{9}$$

$$- \quad \int_0^T \int_{F_h} [[\bar{\varphi}]] \cdot \{\{\frac{1}{\mu}\nabla \times E_h^k\}\}\, ds\, dt + \int_0^T \int_{F_h} a[[E_h^k]] \cdot [[\bar{\varphi}]]\, ds\, dt = 0. \tag{10}$$

Here, the initial condition $\frac{\partial E_h}{\partial t}(x,0) = 0$ and perfectly condacting boundary conditions are imposed weakly through the variational formulation. We also used the notation $\int_{F_h} \varphi ds := \sum_{f\in F_h} \int_f \varphi ds$.

The function a is called interior penalty stabilization function and is defined as

$$a := \alpha m^{-1} h^{-1}, \tag{11}$$

where $\alpha > 0$ is a parameter independent of the mesh size and the wave number. The functions h and m are defined as in [1].

The explicit scheme for the electric field

To solve (9) we apply discontinuous finite element method of piecewise linear functions in space and continuous in time, and seek a discrete solution $\mathbf{E} \in W_h^E$ presented by functions $\mathbf{E}(x,t) = \sum_{l=1}^N \sum_{i=1}^M \mathbf{E}_i^l \varphi_i(x)\psi_l(t)$, where $\{\varphi_i(x)\}_{i=1}^M$ is basis of discontinuous piecewise linear functions in space and $\{\psi_l(t)\}_{l=1}^N$ is a basis of continuous linear functions in time. This yields the linear system of equations:

$$\begin{aligned} M(\mathbf{E}^{k+1} - 2\mathbf{E}^k + \mathbf{E}^{k-1}) \quad &= \quad -\tau^2 F^k - \tau^2 K(\frac{1}{6}\mathbf{E}^{k-1} + \frac{2}{3}\mathbf{E}^k + \frac{1}{6}\mathbf{E}^{k+1}) \\ &+ \quad \tau^2 A^k \Big(\frac{1}{6}[[\mathbf{E}]]^{k-1} + \frac{2}{3}[[\mathbf{E}]]^k + \frac{1}{6}[[\mathbf{E}]]^{k+1}\Big) \\ &+ \quad \tau^2 B^k \Big(\frac{1}{6}\{\{\mathbf{E}\}\}^{k-1} + \frac{2}{3}\{\{\mathbf{E}\}\}^k + \frac{1}{6}\{\{\mathbf{E}\}\}^{k+1}\Big) \\ &- \quad a\, \tau^2 C^k \Big(\frac{1}{6}[[\mathbf{E}]]^{k-1} + \frac{2}{3}[[\mathbf{E}]]^k + \frac{1}{6}[[\mathbf{E}]]^{k+1}\Big), \end{aligned} \tag{12}$$

with initial conditions $\mathbf{E}^0$ and $\mathbf{E}^1$ set to zero because of (5). Here, M is the block mass matrix in space, K is the block stiffness matrix corresponding to the curl term, A, B and C are the stiffness matrices corresponding to the tangential jumps, averages and penalization terms in (9), correspondingly, F^k is the load vector at time level t_k corresponding to $j(\cdot,\cdot)$, whereas $\mathbf{E}^k$ denotes the nodal values of $E(\cdot,t_k)$.

At the element level the matrix entries in (12) are explicitly given by: $M^e_{i,j} = (\varepsilon\ \varphi_i, \varphi_j)_e, K^e_{i,j} = (\mu^{-1}\ \nabla\times\varphi_i, \nabla\times\varphi_j)_e, A_{i,j}{}^f = (\varphi_i, \{\{\mu^{-1}\ \nabla\times\varphi_j\}\})_f, B^f_{i,j} = (\mu^{-1}\ \nabla\times\varphi_i, [[\varphi_j]])_f, C_{i,j}{}^f = (\varphi_i, [[\varphi_j]])_f, F^e_j = (j, \varphi_j)_e$.

Since mass matrix in discontinuous finite element method is always block-diagonal, it can be inverted and yield fully explicit time stepping method, if mass lumping is used also in time with replacing the terms $\frac{1}{6}\{\{\mathbf{E}\}\}^{k-1} + \frac{2}{3}\{\{\mathbf{E}\}\}^k + \frac{1}{6}\{\{\mathbf{E}\}\}^{k+1}$ and $\frac{1}{6}[[\mathbf{E}]]^{k-1} + \frac{2}{3}[[\mathbf{E}]]^k + \frac{1}{6}[[\mathbf{E}]]^{k+1}$ by $\{\{\mathbf{E}\}\}^k$ and $[[\mathbf{E}]]^k$, correspondingly, in (12):

$$\begin{aligned}\mathbf{E}^{k+1} = &-\tau^2 M^{-1}F^k + 2\mathbf{E}^k - \tau^2 M^{-1}K\mathbf{E}^k - \mathbf{E}^{k-1}\\ &+ \tau^2 M^{-1}A^k[[\mathbf{E}]]^k + \tau^2 M^{-1}B^k\{\{\mathbf{E}\}\}^k - a\,\tau^2 M^{-1}C^k[[\mathbf{E}]]^k.\end{aligned} \tag{13}$$

THE HYBRID METHOD

To formulate the hybrid method we note first, that the interior nodes of the computational domain Ω belong to either of the following sets:

ω_o nodes 'o' interior to Ω_{FDM} that lie on the boundary of Ω_{FEM},
$\omega_\times$ nodes '$\times$' interior to Ω_{FEM} that lie on the boundary of Ω_{FDM},
ω_* nodes '$*$' interior to Ω_{FEM} that are not contained in Ω_{FDM},
ω_D nodes 'D' interior to Ω_{FDM} that are not contained in Ω_{FEM}.

At every time step we perform the following operations:
Algorithm.

1. On the structured part of the mesh Ω_{FDM} compute $H^{n+\frac{1}{2}}$, with $H^{n-\frac{1}{2}}$ known, and then compute E^{n+1} with E^n known and $H^{n+\frac{1}{2}}$ given by Yee scheme [2].
2. On the unstructured part of the mesh Ω_{FEM} compute E^{n+1} by using the explicit finite element scheme (13).
3. Use the values of the electric field E^{n+1}_{FEM} at nodes $\omega_\times$ as a boundary condition for the finite difference method in Ω_{FDM}.
4. Use the values of the electric field E^{n+1}_{FDM} at nodes ω_o as a boundary condition for the finite element method in Ω_{FEM}.

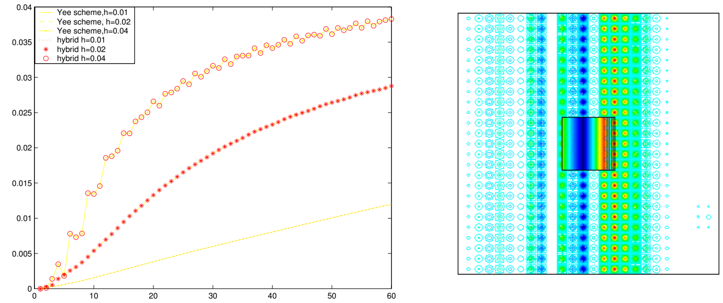
NUMERICAL EXAMPLE

We analyze the hybrid interior penalty discontinuous FEM/FDM method in the computational domain $\Omega = [0, 1.0]^2$. The domain $\Omega = [0, 1.0]^2$ separates into a finite element domain, $\Omega_{FEM} = [0.4, 0.6]^2$, and a surrounding finite difference domain, Ω_{FDM}. In all computations we choose the time step τ according to the CFL condition while the interior penalty factor in (9) is set to $a = 10$. We set also $j = 0$, $\varepsilon = \mu = 1.0$ and launch a wave by forcing the time dependent boundary condition

$$\begin{aligned}&E_1(x,y,t) = 0,\\ &E_2(x,y,t) = 0.1(\sin(50\,t - \pi/2) + 1), \quad 0 \le t \le \frac{2\pi}{50},\end{aligned} \tag{14}$$

at the left boundary of Ω_{FDM}. The initial electric and magnetic fields in the domain are zero. At the top and bottom boundaries of Ω_{FDM} we use periodic boundary conditions, and at the right boundary - absorbing boundary condition.

The simulations was run in time $T = [0, 1]$, what was enough for the wavefront to propagate to the length of the computational domain. We choose the time step $\tau = 1.0/600$ which satisfies CFL condition. Comparison of the computed electric field using Yee scheme and stability of hybrid IPDGFEM/FDTD scheme is presented on Fig. 1.



a) b)

FIGURE 1. a) L^2 norms at time interval $T = [0, 1]$ of hybrid DGFEM/FDM method and Yee scheme in Ω_{FEM}; b) Isolines of the computed solution in hybrid method in Ω at time $t = 0.8$.

REFERENCES

1. M. J. Grote, A. Schneebeli, and D. Schötzau. Interior penalty discontinuous galerkin method for maxwell's equations: optimal l^2 norm error estimates. IMA Journal of Numerical Analysis, 28:440-468, 2008.
2. K.S. Yee. Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Trans. Antennas Propag.*, 14:302-307, 1966.