A combined discontinuous Galerkin and finite volume scheme for multi-dimensional VPFP system

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Abstract. We construct a numerical scheme for the multi-dimensional Vlasov–Poisson–Fokker-Planck system based on a combined finite volume (FV) method for the Poisson equation in spatial domain and the streamline diffusion (SD) and discontinuous Galerkin (DG) finite element in time, phase-space variables for the Vlasov-Fokker-Planck equation.

Keywords: Vlasov-Poisson-Fokker-Planck system, finite volume method, streamline diffusion method, discontinuous Galerkin method.

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INTRODUCTION

We study the approximate solution for the deterministic multi-dimensional Vlasov–Poisson–Fokker-Planck (VPFP) system: given the parameters $\beta$ and $\sigma$ and the initial data $f_0(x,v), (x,v) \in \Omega := \Omega_x \times \Omega_v \subset \mathbb{R}^d \times \mathbb{R}^d$, $d = 1, 2, 3$, find the density function $f(x,v,t)$ in the Dirichlet initial-boundary value problem for the Vlasov-Fokker-Planck equation

\[
(P1) \begin{cases}
    f_t + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f - \text{div}_v (\beta v f) - \sigma \Delta_v f = S, & \text{in } \Omega \times [0,T], \\
    f(x,v,0) = f_0(x,v), & \text{in } \Omega_x \times \Omega_v, \\
    f(x,v,t) = 0, & \text{on } \Gamma_G^{-} \times [0,T],
\end{cases}
\]

where $S$ is source, $G := (v, -\nabla_x \phi)$, $\Gamma_G^{-} := \{(x,v) \in \Gamma := \partial \Omega | \nabla \cdot n < 0\}$, $n$ is the outward unit normal, and $\phi$ satisfies

\[
(P2) \begin{cases}
    -\Delta_x \phi = \int_{\Omega_v} f(x,v,t) dv, & \text{in } \Omega_x \times [0,T], \\
    |\nabla_v \phi(x,t)| = 0, & (\phi \text{ p.w. constant, or } \phi = 0) \text{ on } \partial \Omega_x \times [0,T].
\end{cases}
\]

We solve (P2) replacing $f$ by a given function $g$, and insert the corresponding solution $\phi_g$ in (P1) to obtain an equation for $f$. Thus, we link $f$ to the given data $g$ as, say, $f = \Lambda[g]$. Hence, the solution for the VPFP system is a fixed point of the operator $\Lambda$, i.e. $f = \Lambda[f]$. We may study existence and uniqueness using a Schauder fixed point theorem. For the discrete version this step can be repeated using a Brouwer type fixed point argument, see, e.g. [1] and [3].

Conventional numerical methods for the Vlasov-Poisson and related equations have been dominated by the particle methods see, e.g. [5] and [11]. A 1-dimensional finite volume scheme for the Vlasov-Poisson is studied in [6].

To approximate (P2) we use a finite volume approach in $3d$: $\Omega_x \subset \mathbb{R}^3$. As for (P1) we employ streamline-diffusion and discontinuous Galerkin methods based on the studies in [8] and [4]. We shall only give sketch of the proofs. Detailed proofs are obtained following the techniques in [10] for finite volume, and [1]-[2] and [8] for finite elements.

THE FINITE VOLUME METHOD FOR POISSON EQUATION IN 3D

The cell-center finite volume (FV) scheme for problem (P2), in standard domain $\Omega_x = (0,1)^3$, is given by

\[
-\nabla_x^2 \phi = \rho, \quad \text{in } \Omega_x = (0,1) \times (0,1) \times (0,1) \quad |\nabla_x \phi| = 0, \quad \text{on } \partial \Omega_x,
\]

where $\rho = \int_{\Omega_v} f dv$. Existence uniqueness and regularity studies for (3) are extensions of two-dimensional results in [7]: $\rho \in H^{-1}(\Omega_x)$ implies that $\exists \! \phi \in H^1_0(\Omega_x)$, and for $\rho \in H^s(\Omega_x)$, with $-1 \leq s < 1$, $s \neq \pm 1/2$, $\phi \in H^{s+2}(\Omega_x)$.

Theorem 1. The, respective, optimal FV error estimates for general non-uniform and quasi-uniform meshes are

\[
\|\phi - \phi_h\|_{1,h} \leq Ch^s |\phi|_{H^{s+1}}, \quad \text{and} \quad \|\phi - \phi_h\|_{1} \leq Ch^s |\phi|_{H^{s+1}}, \quad 1/2 < s \leq 2; \quad (\text{for } \|\cdot\|_{1,h}, \text{ see (7)}).
\]
The corresponding finite element estimates can be read from the theorem:

**Theorem 2.** a) For the finite element solution of the problem (P2), with a quasiuniform triangulation, we have that:

\[ \| \varphi - \varphi_h \|_{1,\infty} \leq C h^r |\log h| \times \| \varphi \|_{r+1,\infty}, \quad r \leq 2 \]

b) \( \forall \varepsilon \in (0,1) \) small, \( \exists C_\varepsilon \) such that \( | \| \varphi - \varphi_h \|_{1,\infty} \geq C_\varepsilon h^{-r-\varepsilon} \| \varphi \|_r, \) cf [9].

To derive the finite volume formula we consider the Cartesian mesh:

\[
I_x^h : = \{ x_i : i = 0, 1, \ldots, J \}; \quad x_0 = 0, \quad x_i - x_{i-1} = h_i; \quad x_J = 1, \]

\[
I_y^h : = \{ y_j : j = 0, 1, \ldots, J \}; \quad y_0 = 0, \quad y_j - y_{j-1} = k_j; \quad y_J = 1, \]

\[
I_z^h : = \{ z_n : n = 0, 1, \ldots, N \}; \quad z_0 = 0, \quad z_n - z_{n-1} = \ell_n; \quad z_N = 1. \]

With each \((x_i, y_j, z_n)\) we associate a finite volume box:

\[
\omega_{ijn} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}) \times (z_{n-1/2}, z_{n+1/2}),
\]

and choose central finite volume boxes inside each 27-points stencil element with the characteristic functions as:

\[
\chi_{ijn} = \text{Char} \left[ \left( -\frac{h_{i+1/2}}{2}, \frac{h_i}{2} \right) \times \left( -\frac{k_{j+1}}{2}, \frac{k_j}{2} \right) \times \left( -\frac{\ell_{n+1}}{2}, \frac{\ell_n}{2} \right) \right] \in H^\tau (\mathbb{R}^3), \quad \forall \tau < 1/2.
\]

Let now \( \rho \in H^\tau (\Omega_h), s > -1/2, \) and extend \( \rho \) to \( \mathbb{R}^3 \) preserving its Sobolev class. Thus, we may define

\[
\frac{1}{|\omega_{ijn}|} \int_{\partial \omega_{ijn}} \frac{\partial \varphi}{\partial n} ds = \frac{1}{|\omega_{ijn}|} \left( \chi_{ijn} * \rho \right)(x_i, y_j, z_n) \quad (5)
\]

using three dimensional convolutions \( \chi_{ijn} * \rho, \) which is continuous in \( \mathbb{R}^3. \) Recalling that \( \rho \in L^1_{\text{loc}}(\Omega_h) \) we may write

\[
\frac{1}{|\omega_{ijn}|} \int_{\partial \omega_{ijn}} \frac{\partial \varphi}{\partial n} ds = \frac{1}{h_i k_j \ell_n} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{z_{n-1/2}}^{z_{n+1/2}} \rho(x, y, z) dx dy dz. \quad (6)
\]

Let \( \mathcal{V}_h \) be the set of piecewise bilinear functions on the box \( \Omega_h \) induced by \( \hat{\Omega}_h \), i.e. \( \mathcal{V}_h = \{ F \in \mathcal{V}_h \mid F = 0 \text{ on } \partial \Omega_h \}. \)

**Definition 1.** The finite volume approximation of the solution \( \varphi \) for the Poisson equation: \( \varphi \in \mathcal{V}_h \) is defined (implicitly) through the following algorithm:

\[
-\frac{1}{h_i k_j \ell_n} \int_{\partial \omega_{ijn}} \frac{\partial \varphi_h}{\partial n} ds = \frac{1}{h_i k_j \ell_n} \left( \chi_{ijn} * \rho \right)(x_i, y_j, z_n), \quad (x_i, y_j, z_n) \in \hat{\Omega}_h.
\]

Stability and convergence of this method are generalizations of Süli’s [10] results in two dimensions for the Dirichlet problem. For \( |\nabla \psi| = 0 \) on \( \partial \Omega_h \) with extended \( \varphi(\infty) = 0 \) yield \( \varphi = 0 \) on \( \partial \Omega_h. \) The first assertion in Theorem 1, may be proved repeating the arguments in [10] (we skip) for the 3d case in discrete \( H^1(\Omega_h^d) \) and \( L_2(\Omega_h^d) \) norms:

\[
\| \psi \|_{1,h} = \left( \| \psi \|^2 + \| \psi \|_{1,h} \right)^{1/2}, \quad \text{and} \quad \| \psi \| = \left( \| \psi \| + \| \psi \|_{1,h} \right)^{1/2}, \quad (7)
\]

where \( \langle \psi, \varphi \rangle = \sum_{i=1}^{J} \sum_{j=1}^{J} \sum_{n=1}^{N} h_i k_j \ell_n \varphi_{ijn} \psi_{ijn}, \) and \( |\varphi|_{1,h} = \left( |\Delta_x \varphi|^2 + |\Delta_y \varphi|^2 + |\Delta_z \varphi|^2 \right)^{1/2}, \) with divided differences \( \Delta_x \psi_{ijn} = (\varphi_{ijn} - \psi_{i-1,j,n})/h_i, \Delta_y \psi_{ijn} = (\psi_{i+1,j,n} - \psi_{i,j,n})/k_j \) and \( \Delta_z \psi_{ijn} = (\varphi_{i+1,j,n} - \psi_{i,j,n})/\ell_n \), and the, one-sided discrete \( L_2 \)-norms:

\[
|\Delta_x \psi |_{x}^2 = \langle \psi, \psi \rangle_x, \quad (\phi, \psi)_x = \sum_{i=1}^{J} \sum_{j=1}^{J} \sum_{n=1}^{N} h_i k_j \ell_n \phi_{ijn} \psi_{ijn},
\]

with similar notations for \( y \) and \( z \) variables. Our numerical results justify convergence of FV scheme.
STREAMLINE DIFFUSION AND DISCONTINUOUS GALERKIN APPROACHES

For a finite element scheme in $\Omega_T := [0,T] \times \Omega$ we use a subdivision of $\Omega$ into the product of tetrahedral elements $\tau$, and $\bar{\tau}$, as $\mathcal{T}_h := \{ \tau = \bar{\tau} \times \bar{\tau} \}$ combined with a partition of the time interval $[0,T)$: $0 = t_0 < t_1 < \ldots < t_M = T$, and let $I_m := (t_m,t_{m+1})$; $m = 0, 1, \ldots, M - 1$. Then the corresponding partition of $\Omega_T$ is given by the prism-type triangulation

$$\mathcal{G}_h := \{ K[K := \tau \times I_m, \tau \in \mathcal{T}_h, \quad h = \max(\text{diam} \tau) \}.$$ 

We seek piecewise polynomial approximations for the solution of problem (1) in a finite dimensional space

$$V_h := \{ f \in \mathcal{H} : f|_K \in \mathcal{P}_k(\tau) \times \mathcal{P}_k(I_m) \}; \forall K = \tau \times I_m \in \mathcal{G}_h,$$

with $\mathcal{V}_h$ being continuous in $x$ and $v$, possibly discontinuous in $t$ across time levels $t_m$ and $\mathcal{H} := \prod_{m=0}^{M-1} H^1(\Omega_m)$; $\Omega_m = \Omega \times I_m$. We shall also use the jumps $|g| = g^+ - g^-$ with $g^+ = \lim_{s \to 0^+} g(x,v,t+s)$, and the standard notation

$$(f,g)_m = (f,g)_{\Omega_m} = \int_{\Omega_m} f g \, dx \, dv \, dt,$$

$$(f,g) = (g,g)^{1/2}, \quad (f,g)_m = \int_{\Omega_m} f g^\sigma \, dv,$$

$$(f^+,g^+)_{\Gamma^+} = \int_{\Gamma^+} f^+ g^\sigma \, d\Gamma,$$

Using notation $\nabla f := (\nabla_x f, \nabla_v f)$ and $G := (v_1, \ldots, v_d, -\partial \phi/\partial x_1, \ldots, -\partial \phi/\partial x_d)$, we get $\div G(f) = 0$. For finite element procedure (both in the SD and the DG cases) we let $\mathcal{F}$ to be a certain (linear) function space, $\tilde{f} \in \mathcal{F}$ an approximation of $f$ and $\Pi f \in \mathcal{F}$ a projection of $f$ into $\mathcal{F}$, then to estimate the approximation error

$$f - \tilde{f} = (f - \Pi f) + (\Pi f - \tilde{f}) = \eta + \xi; \quad \xi \in \mathcal{F}.$$

We use interpolation to estimate $||| \eta |||$, and establish $||| \xi ||| \leq C ||| \eta |||$, $\forall \xi \in \mathcal{F}$. SD or DG, below.

Now we consider the streamline diffusion (SD) method (for (P1)) with test functions of the form $u + \delta (u + G(\tilde{f}) \cdot \nabla u)$ with $\delta \sim h$, the mesh size. For convenience we use the notation $\delta u := w_0 + G(f_h) \cdot \nabla w$ and formulate the SD method for problem (P1) as follows: given $f_h$ and $\Pi f$, find $f_h$ such that for $m = 0, \ldots, M - 1$,

$$(P_m) \quad B^\delta_m(G(f_h);f_h,u) - J^\delta_m(f_h,u) = L^\delta_m(u), \quad \forall u \in V_h.$$ (8)

$$B^\delta_m := (\mathcal{D}f_h,u + \delta \mathcal{D}u)_m + \sigma(\nabla_v f_h, \nabla_v u)_m + (f_h,u)_m - \delta \mathcal{D}(\mathcal{D}f_h,f_H)_m,$$

$$J^\delta_m := (\nabla \cdot (\beta v f_h), u + \delta \mathcal{D}u)_m, \quad \text{and} \quad L^\delta_m := (S,u + \delta \mathcal{D}u)_m + (f^+,u^+)_\lambda_m + (f^-,u^-)_\lambda_m.$$ (10)

$P_m$ is a linear system of equations leading to an implicit scheme and to solve $P_m$ is equivalent to find $f_h$ such that

$$B^\delta(G(f_h);f_h,u) - J^\delta(f_h,u) = L^\delta(u), \quad \forall u \in V_h,$$

$$B^\delta := \sum_{m=0}^{M-1} B^\delta_m, \quad J^\delta := \sum_{m=0}^{M-1} J^\delta_m, \quad L^\delta := \sum_{m=0}^{M-1} L^\delta_m.$$ (11)

Stability and error estimates

**Lemma 1.** For the SD method we have the coercivity and stability estimates $B^\delta(G(f^0);g,g) \geq \frac{1}{2} ||| g |||_{SD}^2, \forall g \in \mathcal{H},$

$$||| g |||_{\mathcal{H}^1(\Omega_T,SD)}^2 = \frac{1}{2} \left[ 2\sigma ||| \nabla_v g |||_{\mathcal{H}^1}^2 + ||| g |||_{\mathcal{H}^1}^2 + \sum_{m=1}^{M-1} ||| g |||_{\Omega_m}^2 + \int_{\Gamma^+} g^2 |G^h \cdot n| \right] + \int_{\Gamma^-} g^2 |\mathcal{D}f^\delta|,$$

$$||| g |||_{L^2(\Omega_T,SD)}^2 \leq \left[ \frac{1}{C_1} ||| \mathcal{D}f^\delta ||^2 + \sum_{m=1}^{M-1} ||| g |||_{\Omega_m}^2 + \int_{\partial \Omega_T} g^2 |G^h \cdot n| \right] \delta e^{C_1}, \quad \forall \delta \geq 0.$$

**Remark 1.** In the discontinuous Galerkin case, $||| g |||_{DG}$ and $||| g |||_{L^2(\Omega_T,DG)}$ are defined by replacing the $\int_\cdot$-term, in the SD case, by $\int_{\partial K^1(G')} |g|_D^2 |G^h \cdot n| \, ds$ where $\partial K^1(G') = \{ (x,v,t) \in \partial K^1(G') : n(x,v,t) = 0 \}$. 59
Theorem 3. Assume that there is a constant $C$ such that $\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla \eta\|_\infty \leq C$. Then, we have the following error estimate for the streamline diffusion method for (P1):

$$\|f - f_{SD}\|_{SD} \leq C h^{k+1/2} \|f\|_{H^{k+1}(\Omega_T)}.$$  

Proof. (Sketchy) Let $\tilde{f}^h$ be an interpolant of $f$, split the error as $e = f - f_{SD} = f - \tilde{f}^h + \tilde{f}^h - f_{SD} := \eta - \xi$. Then,

$$\frac{1}{2} \|\xi\|_{SD}^2 \leq B(G(f^h); \xi, \xi) = B(G(f); f, \xi) - B(G(f^h); f, \xi) + J(f^h, \xi) - J(f, \xi)$$

$$:= \Delta B + \Delta J \leq \frac{1}{8} \|\xi\|_{SD}^2 + C_B \|\eta\|_{SD}^2 + \frac{1}{8} \|\xi\|_{SD}^2 + C_T \|\eta\|_{SD}^2,$$

where we have used the inverse estimate. The interpolation error following error estimate for the streamline diffusion method for (P1):

Theorem 3. Under the assumptions of Theorem 3 and for the exact solution $f$ Gaussian function:

$$u_K \text{ responds to the element } \tau \text{ element } \xi \text{ where we have used the inverse estimate.}$$

In the DG case, we assume also discontinuities in $x$ and $v$ over the interelement boundaries and use discrete spaces

$$W_h = \{ g \in L_2(Q_T) : g|_K \in P_k(K) \quad \forall K \in \mathcal{C}_h \}, \quad W_h^d = \{ w \in [L_2(Q_T)]^d : w|_K \in [P_k(K)]^d \quad \forall K \in \mathcal{C}_h \}.$$

Theorem 4. Under the assumptions 3 and for the exact solution $f \in H^{k+1}(\Omega_T) \cap W^{k+1,\infty}(\Omega_T)$, we have that the discontinuous Galerkin approximation $f_{DG} \in W_h^d$ for $f$ in (P1) satisfies the error estimate

$$\|f - f_{DG}\|_{DG} \leq C h^{k+1/2} \left( \|f\|_{H^{k+1}(\Omega_T)} + \|f\|_{W^{k+1,\infty}(\Omega_T)} \right).$$

Proof. (Sketchy) Here we demonstrate only the terms that are involved in estimations of the interelement jump terms, which are additional to those in the SD-case. To this end, we introduce $R$: $W_h \rightarrow W^d$, see [4], defined by

$$R(g)w = - \sum_{\tau \times \lambda \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_{\tau \times \lambda}} \int_e \|g\| n_v \cdot (w) \, dv, \quad \forall w \in W_h^d,$$

$E_v$ is the set of all interior edges of $T_h^v$. Define $(\chi)^0 = \frac{x + x_{ext}}{2}$ and $|\chi| = \chi - \chi_{ext}$, where $\chi_{ext}$ is the value of $\chi$ in the element $\tau_{ext}$ having $e \in E_v$ common edge with $\tau$. Let $r_e$ be the restriction of $R$ to the elements sharing the edge $e \in E_v$:

$$r_e(g)w = - \sum_{\tau \times \lambda \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_{\tau \times \lambda}} \int_e \|g\| n_v \cdot (w) \, dv, \quad \forall w \in W_h^d.$$

Hence, we may easily verify that $\sum_{e \in \partial \tau \times \lambda \in \mathcal{E}_v} r_e = R$ on $\tau \rightarrow \|R(g)\|_K^2 \leq \gamma \sum_{e \in \partial \tau \times \lambda \in \mathcal{E}_v} \|r_e(g)\|^2_K$, where $\gamma$ corresponds to the element $K$ and $\gamma = \gamma(d) > 0$ is a constant. Furthermore, since the support of each $r_e$ is the union of elements sharing the edge $e$, $\sum_{e \in \mathcal{E}_v} \|r_e(g)\|^2 = \sum_{K \in \mathcal{C}_h} \sum_{e \in \partial \tau \times \lambda \in \mathcal{E}_v} \|r_e(g)\|^2_K$. The DG method reads as: find $f_h \in W_h$ such that $B_{DG}(G(f_h); f_h, g) - K(f_h, g) = L(g)$, $\forall g \in W_h$, $(Kf, g) = \left( \nabla_v (\bar{\beta} v f), g + h \partial g \right)$. Proving the coercivity: $B_{DG}(f_h; f_h, g) \geq \alpha \|g\|^2$, $\forall g \in W_h$, (compared to $B_{SD}$, contains also interelement jumps) yields the DG estimate.

\[\square\]

NUMERICAL EXAMPLES FOR FVM

Our implementations are done, in 2d, in the R statistical package for:

**Gaussian function:** $u_1(x, y) = \exp \left( -((\cos(\pi x))^2 + (\cos(\pi y))^2) \right) \cdot \chi_{(0,1) \times (0,1)}(x, y)$, with

$$\Delta u_1(x, y) = \frac{1}{2} \pi^2 \exp \left( -((\cos(\pi x))^2 + (\cos(\pi y))^2) \right) \cdot \left( \sum_{\tau \times \lambda \in \mathcal{T}_h} \frac{1 + 6 \cos(2 \pi \tau) + \cos(4 \pi \tau)}{\sin^6(\pi \tau)} \right) \cdot \chi_{(0,1) \times (0,1)}(x, y). \quad (14)$$

**Mollifier:** $u_2(x, y) = \exp \left( -\frac{1}{1 - (4(y - 0.5)^2 + 4(x - 0.5)^2)} \right) \cdot \chi_{(0,1) \times (0,1)}(x, y)$, with

$$\Delta u_2(x, y) = \exp \left( -\frac{1}{1 - (4(y - 0.5)^2 + 4(x - 0.5)^2)} \right) \cdot \left\{ \begin{array}{l}
\frac{128 \left( (y - 0.5)^2 + (x - 0.5)^2 \right)^2}{(1 - (4(y - 0.5)^2 + 4(x - 0.5)^2))^4} + \\
+ \frac{64 \left( (y - 0.5)^2 + (x - 0.5)^2 \right)^2}{(1 - (4(y - 0.5)^2 + 4(x - 0.5)^2))^3} - \frac{16}{(1 - (4(y - 0.5)^2 + 4(x - 0.5)^2))^2} \end{array} \right\} \cdot \chi_{(0,1) \times (0,1)}(x, y). \quad (15)$$
FIGURE 1. Left: error for Gaussian $u_1$ function. Center: error for mollifier $u_2$ function. Right: potential $u_3$ function. Top row: $L^2$ errors, bottom row relative errors (the relative errors for small meshes are not shown, as they were extremely high and also the relative error was not evaluated if the value of the true function was less than $10^{-5}$).

The FVM performs reasonably well for $u_1$ and $u_2$. Errors, for different mesh sizes, and graphs are in Figs. 1 and 2.

Mollifier difference — potential: Let $u_2(x,y)$ denote the mollifier function defined in section Mollifier. To create a potential we take a difference of two mollifiers, one shrunk to a smaller region.

$$u_3(x,y) = 3 \cdot u_2(x,y) - 10 \cdot u_2\left(\frac{x}{2} - \frac{1}{8}, \frac{y}{2} - \frac{1}{8}\right) \cdot \chi\left(\frac{1}{4}, \frac{3}{4}\right)(x,y)$$

$$\Delta u_3(x,y) = 3 \cdot \Delta u_2(x,y) - 10 \cdot \Delta u_2\left(\frac{x}{2} - \frac{1}{8}, \frac{y}{2} - \frac{1}{8}\right) \cdot \chi\left(\frac{1}{4}, \frac{3}{4}\right)(x,y)$$

(16)

Note that the characteristic function in the definition of $u_3$ does not cause any problems for the derivatives as the mollifier goes smoothly to 0.
FIGURE 2. Top left: true Gaussian function, top right: FVM approximation of Gaussian function, center left: true mollifier function, center right: FVM approximation of mollifier function, bottom left: true potential function, bottom right: FVM approximation of potential function. The FVM was calculated on a random grid of 50 internal nodes in each dimension. The points were the same in both dimensions. The graphs are done (by wireframe() function) on a random grid (different from the FVM one) of 100 internal nodes in each dimension. The points are the same in both dimensions.

REFERENCES