SOME ANALYTIC GENERALIZATIONS OF THE BRIANÇON-SKODA THEOREM

JACOB SZNAJDMAN

Abstract. The Briançon-Skoda theorem appears in many variations in recent literature. The common denominator is that the theorem gives a sufficient condition that implies a membership $\phi \in \mathfrak{a}^l$, where $\mathfrak{a}$ is an ideal of some ring R. In the analytic interpretation R is the local ring of an analytic space Z, and the condition is that $|\phi| \leq C|\mathfrak{a}|^{N+l}$ holds on the space Z. The theorem thus relates the rate of vanishing of ϕ along the locus of $\mathfrak{a}$ to actual membership of (powers of) the ideal. The smallest integer N that works for all $\mathfrak{a} \subset R$ and all $l \geq 1$ simultaneously will be called the Briançon-Skoda number of the ring R.

The thesis contains three papers. The first one gives an elementary proof of the original Briançon-Skoda theorem. This case is simply $Z = \mathbb{C}^n$.

The second paper contains an analytic proof of a generalization by Huneke. The result is also sharper when $\mathfrak{a}$ has few generators if the geometry is not to complicated in a certain sense. Moreover, the method can give upper bounds for the Briançon-Skoda number for some varieties such as for example the cusp $z^p = w^q$.

In the third paper non-reduced analytic spaces are considered. In this setting Huneke's generalization must be modified to remain valid. More precisely, ϕ belongs to $\mathfrak{a}^l$ if one requires that $|L\phi| \leq C|\mathfrak{a}|^{N+l}$ holds on Z for a given family of holomorphic differential operators on Z. We impose the assumption that the local ring $\mathcal{O}_Z$ is Cohen-Macaulay for technical reasons.
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Contents

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Introduction

We denote by $\mathcal{O}_{\mathbb{C}^n,0}$ the local ring at $0 \in \mathbb{C}^n$. The integral closure of an ideal $\mathfrak{a}$ is defined as $\overline{\mathfrak{a}} = \{\phi : \phi^N + s_1\phi^{N-1} + \cdots + s_N = 0,\ s_i \in \mathfrak{a}^i\}$. It turns out that $\phi \in \overline{\mathfrak{a}}$ if and only if $|\phi| \leq C|\mathfrak{a}| := C\sum_1^m |a_i|$, where the a_i generate $\mathfrak{a}$. This also holds for the local ring $\mathcal{O}_{Z,x}$ of any analytic variety Z, see [9]. The Briançon-Skoda theorem, [4], states that for any ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^n,0}$ generated by m germs, we have the inclusion $\overline{\mathfrak{a}^{\min(m,n)+l-1}} \subset \mathfrak{a}^l$, where $\overline{\mathfrak{a}}$ is the integral closure of $\mathfrak{a}$. Note that, up to a multiplicative constant, $|\mathfrak{a}|$ does not depend on the choice of generators; i.e. for norms obtained from different choices we have $|\mathfrak{a}|_1 \leq K|\mathfrak{a}|_2$. Thus the Briançon-Skoda theorem is equivalent to the implication $|\phi| \leq C|\mathfrak{a}|^{\min(m,n)+l-1} \implies \phi \in \mathfrak{a}^l$.

The first proof of the theorem was given in 1974 by Joël Briançon and Henri Skoda, [4]. They gave a very short proof based upon an L^2 division theorem by Skoda, [14].

From the algebraic viewpoint it is natural to ask what happens in other rings, and in particular regular local Noetherian rings. It is easy to see that for any ring R, an ideal $\mathfrak{a}$ and its integral closure $\overline{\mathfrak{a}}$ have the same radical. Therefore the Nullstellensatz immediately gives, assuming that R is Noetherian, that there is an integer N such that $\overline{\mathfrak{a}}^N \subset \mathfrak{a}$. Since the Briançon-Skoda theorem gives an upper bound for the number N that is needed, one may think of it as some sort of effective Nullstellensatz. These theorems are in fact related; Lazarsfeld explains, [8], Section 10.5, how the L^2 theorem of Skoda can be used to obtain an effective Nullstellensatz. It is well-known that $\overline{\mathfrak{a}}^l \subset \overline{\mathfrak{a}^l}$ for any positive integer l. By possibly increasing N, and allowing it to depend on $\mathfrak{a}$ and l, we get furthermore that $\overline{\mathfrak{a}}^{N+l-1} \subset \mathfrak{a}^l$. This inclusion is slightly weaker than the one in the Briançon-Skoda theorem because of the order of taking integral closure and powers of ideals, but more importantly, one is usually only interested in *uniform* Briançon-Skoda results, i.e. to find an integer N that does not depend on $\mathfrak{a}$ nor on l. Now, if N exists and is the smallest integer such that

$$\overline{\mathfrak{a}^{N+l-1}} \subset \mathfrak{a}^l \tag{1}$$

holds for all ideals $\mathfrak{a}$ and integers $l \geq 1$ simultaneously, we call it the Briançon-Skoda number of R and denote it by $\mathrm{bs}(R)$. The original theorem of 1974 thus states that $\mathrm{bs}(\mathcal{O}_{\mathbb{C}^n,0}) \leq n$, and for ideals with few generators one can do better; if $\mathfrak{a}$ is generated by $m < n$ generators, (1) holds also with $N = m$.

In fact the original theorem is optimal. This is seen by considering $\phi = z_1^{n-1} \cdot z_2^{n-1} \cdot \cdots \cdot z_n^{n-1}$ and $\mathfrak{a} = (z_1^n, z_2^n, \ldots, z_n^n)$, and another example identical to this one, but with m instead of n.

Lipman and Sathaye, [11], proved algebraically that the theorem holds for any regular Noetherian ring in [11]. A little earlier, Lipman and Tessier, [12], proved the first part, i.e. $\mathrm{bs}(R) = n$, in [12], for a "reasonable" pseudo-rational ring R (reasonable means that the localization at each prime is also pseudo-rational). However, the improvement for few generators works only [1] for special ideals $\mathfrak{a}$, for example if $\mathfrak{a}$ has a reduction (i.e. a subideal with the same integral closure) generated by a regular sequence. They also proved that the class of pseudo-rational rings includes all regular rings. The following quote appears in the paper [12]:

> The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite trancendental deep result by Skoda in [20]. The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

The challenge was actually made explicity by Hochster at a CBMS conference held at Georege Mason University in 1979, where he was the principal speaker and concentrated on the Briançon-Skoda theorem.

In the late eighties, Craig Huneke and Melvin Hochster introduced the notion of tight closure, which is a closure operation on ideals, such that the tight closure is always contained in the integral closure. Tight closure works naturally in rings with characteristic p, but it can be used to prove statements in characteristic 0 by reducing to characteristic p. This method has been quite successful, and proofs of various statements are often remarkably short in positive characteristic. In the book [15], chapter 13, by Huneke and Swanson, it says

> The Briançon-Skoda theorem has played an important role in the development of many techniques in commutative algebra. These developments range from the theorem of Lipman and Sathaye, Theorem 13.3.3, to contributing to the development of tight closure, as well as Lipman's development of adjoint ideals.

The first tight closure proof of the Briançon-Skoda theorem for a regular ring appears in [6]. Schoutens gave an elementary proof, based on the tight closure approach, by using ultrafilters to simplify the procedure of reduction to characteristic p.

Lazarsfeld, [8], gives a geometric, rather short proof of the original Briançon-Skoda theorem. It uses the main idea of Lipman and Tessier, and multiplier ideals and vanishing theorems are the main tools. He also proves some related theorems, such as a global version of Skoda's L^2 theorem, and discusses the relation between the theorems of Skoda and of Briançon-Skoda. In Remark 9.6.29. he writes that experience

[1]The generalization was fully proven two decades later in [1], by Aberbach and Huneke.

shows that algebraic statements established by L^2 methods or multiplier ideals can also be understood via tight closure.

Lipman's notion of adjoint ideals, introduced in [10], actually coincides with multiplier ideals, but is more general. One may therefore speculate that the theorem of Briançon-Skoda has also contributed to the development of multiplier ideals, or its algebraic formulation. An argument that supports this view is that Skoda's theorem is actually a statement about multiplier ideals, although there is no explicit mention of these ideals in his work. Nevertheless, quite many theories, some of them vastly differing from the others, can be used to prove the Briançon-Skoda, which is rather fascinating itself.

A different approach to proving the Briançon-Skoda theorem and its generalizations is to use division formulas and residue currents. This approach was first taken by Berenstein, Gay, Vidras, and Yger in [3] to prove the original version of the theorem. The authors used a division formula by Berndtsson. The present author simplified this argument in the first paper of the present thesis. The proof is more elementary in the respect that it avoids the use of Hironaka's theorem of resolution of singularities. Instead it uses basic integration theory and a simple instance of the Chern-Levine-Nirenberg inequalities.

In 1992 Huneke, [7], showed that for a quite general Noetherian reduced local ring R, $\mathrm{bs}(R)$ exists; in other words, there is an integer N such that $\overline{\mathfrak{a}^{N+l-1}} \subset \mathfrak{a}^l$, for all ideals $\mathfrak{a} \subset R$ and $l \geq 1$. The case $R = \mathcal{O}_{Z,x}$ of Huneke's theorem was recently reproven using residue currents in the second paper of this thesis, [2]. The author is unaware of any proofs using L^2 theory on singular varieties.

In contrast to the case of regular rings (and of rational singularities), it is possible that $bs(R)$ is arbitrarily large for fixed dimension, even when $R = \mathcal{O}_{Z,x}$ and Z is a complex curve, as Example 0.1 shows.

Example 0.1. Let Z be the curve in $\mathbb{C}^2$ given by $z^p = w^q$, where $p > q$ are relatively prime. We will show in the appendix that the Briançon-Skoda number of this curve is $a = \lceil \frac{(p-1)(q-1)}{q} \rceil + 1$. A lower bound for $\mathrm{bs}(R)$ is $b = \lfloor \frac{p(q-1)}{q} \rfloor + 1$, since w^{q-1} is not in the ideal generated by z, but $|w^{q-1}| = |z|^{\frac{p(q-1)}{q}}$ holds on the curve. It is easy to see that $a = b$; both $\frac{(p-1)(q-1)}{q}$ and $\frac{p(q-1)}{q}$ are in $\frac{1}{q}\mathbb{Z}$ and their difference is $1 - \frac{1}{q}$.

The third paper deals with the case when $R = \mathcal{O}_{Z,x} = \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ is not reduced, i.e. there is $x \in R$ such that $x \neq 0$ but $x^k = 0$ for some k. A famous result of Ehrenpreis, [5], and Palamodov, [13] states that there is a finite set of holomorphic differential operators $L_1, \dots L_M$, such that $\phi \in \mathcal{J}$ if and only if $L_j \phi \in \sqrt{\mathcal{J}}$ for $1 \leq j \leq M$.

We present a theorem that gives a sufficient condition for $\phi \in \mathfrak{a}^l$ to hold, whenever R is Cohen-Macaulay. It is similar to the statement of the second paper, but instead of just having a single condition $|\phi| \leq C|\mathfrak{a}|^{N+l-1}$ on $Z = Z(\mathcal{J})$, we need a number of conditions $|L_j\phi| \leq C|\mathfrak{a}|^{N+l-1}$ on Z, for a certian set of holomorphic differential operators with the property above. Considering the history of the Briançon-Skoda theorem, it would be highly interesting to reformulate the theorem of the third paper in an algebraic setting and to see if it too can contribute to new developments.

Finally, the appendix contains two calculations of $\mathrm{bs}(\mathcal{O}_{Z,x})$ when Z is a cusp. The reader may wish to start by glancing at the appendix before reading the second and third papers. The notation of the appendix is however taken from these papers.

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AN ELEMENTARY PROOF OF THE BRIANÇON-SKODA THEOREM

JACOB SZNAJDMAN

ABSTRACT. We give a new elementary proof of the Briançon-Skoda theorem, which states that for an m-generated ideal $\mathfrak{a}$ in the ring of germs of analytic functions at $0 \in \mathbb{C}^n$, the ν:th power of its integral closure is contained in $\mathfrak{a}$, where $\nu = \min(m, n)$.

1. Introduction

Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. The integral closure $\overline{I}$ of an ideal I is the set of all $\phi \in \mathcal{O}_n$ such that

$$\phi^N + a_1\phi^{N-1} + \cdots + a_N = 0, \tag{1.1}$$

and $a_k \in I^k,\ k = 1, \ldots, N$.

Theorem 1.1 (Briançon-Skoda)**.** *Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_n$ generated by m germs $f_1, \ldots, f_m$. Then $\overline{\mathfrak{a}^{\min(m,n)+l}} \subset \mathfrak{a}^{l+1}$.*

In 1974 Briançon and Skoda, [Sko74], proved this theorem as a quite immediate consequence of Skoda's L^2-theorem in [Sko72].

An algebraic proof was given by Lipman and Tessier in [Tes81]. Their paper also contains a historical summary. An account of the further development and an elementary algebraic proof of the result is found in Schoutens [Sch03].

Berenstein, Gay, Vidras and Yger [Yge93] proved the theorem by finding an integral representation formula $\phi = \sum u_i f_i$ with explicit u_i. However, some estimates rely on Hironaka's theorem.

In this note, we provide a completely elementary proof along these lines. The key point is an L^1 estimate (Proposition 2.1), to which the proof is reduced in Section 3.

2. The Main Estimate

Proposition 2.1. *Let $f_1, f_2, \ldots, f_m$ be holomorphic functions defined in a neighborhood of $0 \in \mathbb{C}^n$. Then*

$$\frac{|\partial f_1 \wedge \cdots \wedge \partial f_m|}{\prod_1^m |f_i|}$$

is integrable on some ball centered at the origin.

Remark 2.2. If f_i are monomials, the assertion is trivial. By Hironaka's theorem, one can reduce to this case, since integrability is preserved under push-forward.

Lemma 2.3. *For any $F \in \mathcal{O}_n$, $F \not\equiv 0$, there is $\delta > 0$, such that $1/|F|^\delta$ is integrable on some ball centered at the origin.*

Proof. We can assume that F is a Weierstrass polynomial and that the domain is a product of a disk and a polydisk. Then we partition each disk (slice) into sets, one for each root, consisting of those points which are the closest to that root. Thus the integrand is no worse than $z_1^{-\delta s}$. □

Proof of Proposition 2.1. By Lemma 2.3 and the inequality $2ab \leq a^2 + b^2$ it suffices to show that for any positive δ the function

$$F = \frac{|\partial f_1 \wedge \cdots \wedge \partial f_m|^2}{\prod_1^m |f_j|^{2-\delta}}$$

is integrable on some ball centered at the origin. This follows from the Chern-Levine-Nirenberg inequalities (e.g. [Dem07] (3.3), Ch. III) and (2.3). We proceed however without explicitly relying on facts about positive forms or plurisubharmonic functions.

Let us first set

$$\beta = \frac{i}{2}\partial\overline{\partial}|\zeta|^2 = \frac{i}{2}\sum d\zeta_j \wedge d\overline{\zeta_j}, \quad \text{and} \quad \beta_k = \frac{\beta^k}{k!}.$$

A simple calculation gives that for any $(1,0)$-forms α_j,

$$i\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha_p} \wedge \beta_{n-p} = |\alpha_1 \wedge \cdots \wedge \alpha_p|^2 dV. \tag{2.1}$$

We now compute

$$\partial\overline{\partial}(|f_j|^2 + \varepsilon)^{\delta/2} = \frac{\delta}{2}\left(1 + \frac{\left(\frac{\delta}{2} - 1\right)|f_j|^2}{|f_j|^2 + \varepsilon}\right)(|f_j|^2 + \varepsilon)^{\delta/2-1}\partial f_j \wedge \overline{\partial f_j},$$

which yields that

$$\frac{i\partial f_j \wedge \overline{\partial f_j}}{(|f_j|^2 + \varepsilon)^{1-\delta/2}} = G_j i\partial\overline{\partial}(|f_j|^2 + \varepsilon)^{\delta/2}, \tag{2.2}$$

where

$$G_j = \frac{2}{\delta}\left[1 + \left(\frac{\delta}{2} - 1\right)\frac{|f_j|^2}{|f_j|^2 + \varepsilon}\right]^{-1}.$$

Observe that $\left(\frac{2}{\delta}\right) \leq G_j \leq \left(\frac{2}{\delta}\right)^2$. We introduce the regularized form

$$F_\varepsilon dV = \frac{|\partial f_1 \wedge \cdots \wedge \partial f_m|^2}{\prod_1^m (|f_j|^2 + \varepsilon)^{1-\delta/2}} dV = \frac{\prod_1^m \left(i\partial f_j \wedge \overline{\partial f_j}\right) \wedge \beta_{n-m}}{\prod_1^m (|f_j|^2 + \varepsilon)^{1-\delta/2}} =$$

$$= \prod G_j i\partial\overline{\partial}\left(|f_j|^2 + \varepsilon\right)^{\delta/2} \wedge \beta_{n-m}. \tag{2.3}$$

From the equality $|w|^2 = |w \wedge \overline{w}|$, that holds for all $(p,0)$-forms w, and (2.2), we get

$$F_\varepsilon dV = \frac{\left|\prod_1^m \left(i\partial f_j \wedge \overline{\partial f_j}\right)\right| dV}{\prod_1^m \left(|f_j|^2 + \varepsilon\right)^{1-\delta/2}} = \left|\prod G_j i\partial\overline{\partial}\left(|f_j|^2 + \varepsilon\right)^{\delta/2}\right| dV. \tag{2.4}$$

Let B be a ball about the origin and let χ_B be a smooth cut-off function supported in a ball of twice the radius. We now use the two expressions (2.3) and (2.4) for F_ε and integrate by parts to see that

$$\int_B F_\varepsilon \leq C \int \chi_B \left| i\partial\overline{\partial}\left(|f_1|^2+\varepsilon\right)^{\delta/2} \wedge \cdots \wedge i\partial\overline{\partial}\left(|f_m|^2+\varepsilon\right)^{\delta/2}\right| dV =$$

$$= C \int \chi_B i\partial\overline{\partial}\left(|f_1|^2+\varepsilon\right)^{\delta/2} \wedge \cdots \wedge i\partial\overline{\partial}\left(|f_m|^2+\varepsilon\right)^{\delta/2} \wedge \beta_{n-m} =$$

$$= C \left| \int \partial\overline{\partial}\chi_B \left(|f_1|^2+\varepsilon\right)^{\delta/2} \wedge \cdots \wedge i\partial\overline{\partial}\left(|f_m|^2+\varepsilon\right)^{\delta/2} \wedge \beta_{n-m}\right| \leq$$

$$\leq C \sup_{2B} |f_1|^\delta \int_{2B} \left| i\partial\overline{\partial}\left(|f_2|^2+\varepsilon\right)^{\delta/2} \wedge \cdots \wedge i\partial\overline{\partial}\left(|f_m|^2+\varepsilon\right)^{\delta/2}\right| dV.$$

By induction, we now have

$$|F_\varepsilon| \leq C \sup_{2^{m+1}B} |f_1 \cdot \cdots \cdot f_m|^\delta < \infty,$$

so if we let ε tend to zero, we get the desired bound. □

3. A Proof of the Briançon-Skoda Theorem

By a simple estimate, (1.1) with $I = \mathfrak{a}^{\min(m,n)+l}$ implies that

$$|\phi| \leq C|f|^{\min(m,n)+l}. \tag{3.1}$$

To prove Theorem 1.1, it thus suffices to show that $\phi \in \mathfrak{a}^l$, provided that (3.1) holds. We will use an explicit division formula introduced in [Ber83], but for convenience, we use the formalism from [And03] to obtain it.

Proof of Theorem 1.1. Define the operator $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}$, where $\delta_{\zeta-z}$ is contraction with the vector field

$$2\pi i \sum_1^n (\zeta_k - z_k) \frac{\partial}{\partial \zeta_k}.$$

A weight with respect to a point $z \in \mathbb{C}^n$ is a smooth differential form g for which $\nabla_{\zeta-z} g = 0$ and $g_{(0,0)}(z) = 1$ holds. If furthermore g has compact support and ϕ is holomorphic, then

$$\phi(z) = \int \phi(\zeta) g(\zeta). \tag{3.2}$$

Next, take an m-tuple $h = (h_i)$ of so called Hefer forms, which are holomorphic forms in $\mathbb{C}^{2n}$ such that $\delta_{\zeta-z} h_i = f_i(\zeta) - f_i(z)$, where f_i are as in Theorem 1.1. Also define $\mu = \min(m, n+1)$ and $\sigma_i =$

$\bar{f}_i/|f|^2$ and let $\chi_\varepsilon = \chi(|f|/\varepsilon)$ be a smooth cut-off function, where χ is approximatively the characteristic function for $[1, \infty)$.

For convenience, we assume that $l = 0$, although the general proof goes through verbatim by just replacing μ with $\mu + l$. We can now define the weight

$$\begin{aligned} g_B &= (1 - \nabla_{\zeta - z}(h \cdot \chi_\varepsilon \sigma))^\mu \\ &= \big(1 - \chi_\varepsilon + f(z) \cdot \chi_\varepsilon \sigma + h \cdot \overline{\partial}(\chi_\varepsilon \sigma)\big)^\mu = \\ &= f(z) \cdot A_\varepsilon + B_\varepsilon, \end{aligned} \tag{3.3}$$

where

$$A_\varepsilon = \sum_{k=0}^{\mu-1} C_k \chi_\varepsilon \sigma [f(z) \cdot \chi_\varepsilon \sigma]^k \big[1 - \chi_\varepsilon + h \cdot \overline{\partial}(\chi_\varepsilon \sigma)\big]^{\mu-k-1} \tag{3.4}$$

and

$$B_\varepsilon = \big(1 - \chi_\varepsilon + h \cdot \overline{\partial}(\chi_\varepsilon \sigma)\big)^\mu. \tag{3.5}$$

Let g be any weight with respect to z which has compact support and is holomorphic in z near 0 (see [And03] for the construction). An application of (3.2) to the weight $g_B \wedge g$ yields

$$\phi(z) = f(z) \cdot \int \phi(\zeta) A_\varepsilon \wedge g + \int \phi(\zeta) B_\varepsilon \wedge g. \tag{3.6}$$

To obtain the division, we begin by showing that the second term tends uniformly to zero for small $|z|$.

On the set $\{|f| > 2\varepsilon\}$, we have $B_\varepsilon = \big(h \cdot \overline{\partial}\sigma\big)^\mu$, which vanishes regardless of whether $\mu = n + 1$ or $\mu = m$. In the latter case apply $\overline{\partial}$ to $f \cdot \sigma = 1$ to see that $\overline{\partial}\sigma$ is linearly dependent. Hence, it remains to find an integrable bound that holds uniformly. A simple calculation gives that

$$\overline{\partial}\chi_\varepsilon = \mathcal{O}(1)|f|^{-1} \sum \overline{\partial f_j} \quad \text{and} \quad \overline{\partial}\sigma_i = \mathcal{O}(1)|f|^{-2} \sum \overline{\partial f_j}, \tag{3.7}$$

since $|f| \sim \varepsilon$ on the support of $\overline{\partial}\chi_\varepsilon$. Note also that $|\sigma| = |f|^{-1}$. By the assumption (3.1), the integrand $\phi(\zeta) B_\varepsilon \wedge g$ consists of terms of the type

$$\begin{aligned} &C\phi(\zeta)\big(\overline{\partial}\chi_\varepsilon h \cdot \sigma\big)^a \wedge \big(\chi_\varepsilon h \cdot \overline{\partial}\sigma\big)^b (1 - \chi_\varepsilon)^c \wedge g = \\ &= \mathcal{O}(1) \wedge |f|^{(\min(m,n) - 2(a+b))} \overline{\partial f_I}, \end{aligned}$$

where $\mu = a + b + c$, $I \subset \{1, 2 \ldots m\}$, $|I| = a + b \leq \mu \leq m$ and $\overline{\partial f_I} = \bigwedge_{i \in I} \overline{\partial f_i}$. It follows from Proposition 2.1 that the expression above is the required bound if $a + b \leq \min(m, n)$, but we also have $a + b \leq n$ due to bidegree reasons. We now know that $|\phi(\zeta) B_\varepsilon \wedge g|$ is bounded by $\chi_{|f|<2\varepsilon} F$ for an integrable function F. The convergence will therefore indeed be uniform in z.

According to (3.6), it remains to show that $\int \phi(\zeta) A_\varepsilon \wedge g$ converges as $\varepsilon \to 0$, since it is clearly holomorphic for each fixed ε. We consider first the case $m \leq n$, which implies $\mu = \min(m, n+1) = \min(m, n) = m$. A generic term in $\phi(\zeta) A_\varepsilon$ can be estimated by

$$\mathcal{O}(1) \wedge \phi(\zeta)\sigma(f(z) \cdot \chi_\varepsilon \sigma)^k \left(\overline{\partial}\chi_\varepsilon h \cdot \sigma\right)^a \wedge \left(h \cdot \overline{\partial}\sigma\right)^b,$$

where $a + b \leq \mu - k - 1$, $k \leq \mu - 1$. As in the first part of the proof this yields an integrable bound by Proposition 2.1 and (3.7). The same estimate automatically holds for

$$A = \lim_{\varepsilon \to 0} A_\varepsilon = \sum_{k=0}^{\mu-1} C_k \sigma[f(z) \cdot \sigma]^k \left[h \cdot \overline{\partial}\sigma\right]^{\mu-k-1}.$$

As above, one sees that $\int \phi(\zeta) A_\varepsilon \wedge g \to \int \phi(\zeta) A \wedge g$ uniformly.

The case $m > n$ presents an additional difficulty as $\phi A \wedge g$ will not be integrable. Since the first part of the argument does not depend on the assumption $m \leq n$, we have that ϕ is in the closure of $\mathfrak{a}$ with respect to uniform convergence. All ideals are however closed in this topology, so ϕ belongs to $\mathfrak{a}$. The second course of action, which is followed in [Sko74], is to consider a reduction of the ideal $\mathfrak{a}$. That is, an ideal $\mathfrak{b} \subset \mathfrak{a}$ generated by n germs such that $\overline{\mathfrak{b}} = \overline{\mathfrak{a}}$, e.g. Lemma 10.3, Ch. VIII in [Dem07]. □

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ON THE BRIANÇON-SKODA THEOREM ON A SINGULAR VARIETY

MATS ANDERSSON & HÅKAN SAMUELSSON & JACOB SZNAJDMAN

Abstract. Let Z be a germ of a reduced analytic space of pure dimension. We provide an analytic proof of the uniform Briançon-Skoda theorem for the local ring $\mathcal{O}_Z$; a result which was previously proved by Huneke by algebraic methods. For ideals with few generators we also get much sharper results.

1. Introduction

Let $\mathfrak{a} = (a) = (a_1, \ldots, a_m)$ be an ideal in the local ring $\mathcal{O} = \mathcal{O}_0$ of holomorphic functions at $0 \in \mathbb{C}^d$ and let $|\mathfrak{a}| = \sum_j |a_j|$. Up to constants, this function is independent of the choice of generators of $\mathfrak{a}$. In [13], Briançon and Skoda proved:
If $\phi \in \mathcal{O}$ and

$$|\phi| \le C|\mathfrak{a}|^{\min(m,d)+\ell-1}, \quad \ell = 1, 2, 3, \ldots, \tag{1.1}$$

then $\phi \in \mathfrak{a}^\ell$.

If $m \le d$, then the statement follows directly from Skoda's L^2-estimate in [26]; if $m > d$ one uses that there is an ideal $\mathfrak{b} \subset \mathfrak{a}$ such that $|\mathfrak{a}| \sim |\mathfrak{b}|$, a so-called reduction of $\mathfrak{a}$, with n generators.

If $\mathfrak{b}$ is any ideal in $\mathcal{O}$ then $|\phi| \le C|\mathfrak{b}|$ if (and in fact only if) ϕ is in theintegral closure $\overline{\mathfrak{b}}$. Therefore, the statement implies (is equivalent to) the inclusion

$$\overline{\mathfrak{a}^{\min(m,d)+\ell-1}} \subset \mathfrak{a}^\ell. \tag{1.2}$$

This is a notable example of a purely algebraic theorem that was first proved by transcendental methods. It took several years before algebraic proofs appeared, [22] and [21]. In [11] there is a proof by integral formulas and residue theory.

Assume now that Z is a germ of an analytic space of pure dimension d and let $\mathcal{O}_Z$ be its structure ring of germs of (strongly) holomorphic functions. It is non-regular if (and only if) Z is non-regular. It is easy to see that the usual Briançon-Skoda theorem cannot hold in general in the non-regular case, not even for $m = 1$, see Example 1

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below. However, Huneke proved in [17] that there is a number μ only depending on Z such that for any ideal $\mathfrak{a} \subset \mathcal{O}_Z$, and integer $\ell \geq 1$,

$$\overline{\mathfrak{a}^{\mu+\ell-1}} \subset \mathfrak{a}^{\ell}. \tag{1.3}$$

Huneke's proof is completely algebraic (and holds for some more general rings as well), so it is natural to look for an analytic proof. In this paper we give a proof by means of residue calculus, and the membership can be realized by an integral formula on Z. A problem of general interest, see, e.g., p. 657 in [18] and Remark 4.14 in [17], is to estimate the Briançon-Skoda number, μ, in Huneke's theorem in terms of invariants of the ring. Our proof relates μ to the complexity of a free resolution of $\mathcal{O}_Z$. We have also a sharper statement in case $\mathfrak{a}$ has "few" generators, and the zero set, $Z^{\mathfrak{a}}$, of the ideal does not overlap the singular set of Z "too much". To formulate this we first have to introduce certain (germs of) subvarieties, Z^r, associated with Z:

To begin with we choose an embedding of Z and consider it as a subvariety at, say, the origin of $\mathbb{C}^n$ for some n. If $\mathcal{I}$ is the corresponding radical ideal in $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n,0}$, then $\mathcal{O}_Z = \mathcal{O}/\mathcal{I}$. Let

$$0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \dots \xrightarrow{f_3} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \tag{1.4}$$

be a free resolution of $\mathcal{O}/\mathcal{I}$. Here E_k are trivial vector bundles and E_0 is a trivial line bundle. Thus f_k are just holomorphic matrices in a neighborhood of 0. We let Z_k be the set of points x such that $f_k(x)$ does not have optimal rank. These varieties are, see, [15] Ch. 20, independent of the choice of resolution, and we have the inclusions

$$\dots \subset Z_{p+2} \subset Z_{p+1} \subset Z_{sing} \subset Z_p = \dots = Z_1 = Z,$$

where $p = n - d$. Now let

$$Z^0 = Z_{sing}, \quad Z^r = Z_{p+r}, \ r > 0. \tag{1.5}$$

Since any two minimal embeddings are equivalent, and any embedding factors in a simple way over a minimal embedding, one can verify that these subsets Z^r are intrinsic subvarieties of the analytic space Z, that reflect the degree of complexity of Z. To begin with, since Z has pure dimension (Corollary 20.14 in [15]),

$$\operatorname{codim} Z^r \geq r + 1, \quad r > 0.$$

Moreover, $Z^r = \emptyset$ for $r > d - \nu$ if and only if the depth of the ring $\mathcal{O}_Z$ is at least ν. In particular, $Z^r = \emptyset$ for $r > 0$ if and only if Z (i.e., $\mathcal{O}_Z$) is Cohen-Macaulay.

Theorem 1.1. *Let Z be a germ of an analytic space of pure dimension. (i) There is a natural number μ, only depending on Z, such that for any ideal $\mathfrak{a} = (a_1, \dots, a_m)$ in $\mathcal{O}_Z$ and $\phi \in \mathcal{O}_Z$,*

$$|\phi| \leq C|\mathfrak{a}|^{\mu+\ell-1} \tag{1.6}$$

implies that $\phi \in \mathfrak{a}^{\ell}$.

(ii) If for a given ideal $\mathfrak{a} = (a_1, \dots, a_m)$

$$\text{codim}\,(Z^r \cap Z^{\mathfrak{a}}) \geq m + 1 + r, \quad r \geq 0, \tag{1.7}$$

then for any $\phi \in \mathcal{O}_Z$,

$$|\phi| \leq C|\mathfrak{a}|^{m+\ell-1} \tag{1.8}$$

implies that $\phi \in \mathfrak{a}^\ell$.

Huneke's theorem (1.3) follows immediately from part (i) of Theorem 1.1, since even in the non-regular case $\phi \in \overline{(b)}$ immediately implies that $|\phi| \leq C|b|$. The less obvious implication $|\phi| \leq C|b| \Rightarrow \phi \in \overline{(b)}$ also holds, see, e.g., [20], and so Theorem 1.1 (i) is in fact equivalent to Huneke's theorem.

Example 1. If Z is the zero set of $z^p - w^2$ in $\mathbb{C}^2$, where $p > 2$ is a prime, then $|w| \leq |z|^{[p/2]}$ on Z, but w is not in (z). However, if $|\phi| \leq C|z|^{(p+1)/2}$, then $\phi \in (z)$, i.e., ϕ/z is strongly holomorphic on Z. □

Remark 1. The important point in Huneke's theorem is the uniformity in $\mathfrak{a}$ and ℓ. Notice that (1.3) implies the slightly weaker statement

$$\overline{\mathfrak{a}}^{\mu+\ell-1} \subset \mathfrak{a}^\ell. \tag{1.9}$$

It is quite easy to prove such an inclusion for fixed $\mathfrak{a}$ and ℓ. In fact, assume that Z is a germ of a subvariety in $\mathbb{C}^n$ and choose a tuple f such that $Z = \{f = 0\}$. Let $A = (A_1, \dots, A_m)$ and Φ denote fixed representatives in $\mathcal{O}_{\mathbb{C}^n}$ of $\mathfrak{a} = (a_1, \dots, a_m)$ and $\phi \in \overline{\mathfrak{a}}$. Then

$$|\Phi(z)| \leq Cd(z, Z^{\mathfrak{a}} \cap Z) \leq C'(|A| + |f|)^{1/M}$$

for some M by Lojasiewicz' inequality, and hence Φ^{Mn} is in the ideal $(A) + (f)$ by the usual Briançon-Skoda theorem in the ambient space. Thus $\phi^{Mn} \in \mathfrak{a}$ and therefore $\phi^{Mn\ell} \in \mathfrak{a}^\ell$. Thus $\overline{\mathfrak{a}}^{Mn\ell} \subset \mathfrak{a}^\ell$. □

From Theorem 1.1 (ii) we get:

Corollary 1.2. *If*

$$\text{codim}\, Z^r \geq m + 1 + r, \quad r \geq 0, \tag{1.10}$$

then (1.8) *implies that* $\phi \in \mathfrak{a}^\ell$ *for any* $\mathfrak{a}$ *with* m *generators.*

Assume that (1.10) holds for $m = 1$. The conclusion for $\ell = 1$ then is that each weakly holomorphic function is indeed holomorphic, i.e., Z (or equivalently $\mathcal{O}_Z$) is normal. In fact, if ϕ is weakly holomorphic, i.e., holomorphic on Z_{reg} and locally bounded, then it is meromorphic, so $\phi = g/h$ for some $g, h \in \mathcal{O}_Z$. The boundedness means that $|g| \leq C|h|$ and by the corollary thus ϕ is in $\mathcal{O}_Z$. One can check that (1.10) with $m = 1$ is equivalent to Serre's condition for normality of the local ring $\mathcal{O}_Z$ and therefore both necessary and sufficient.

The basic tool in our proof is the residue calculus developed in [1], [7], and [8], and we recall the necessary material in Section 2. Given an ideal sheaf $\mathcal{J}$ one can associate a current R such that a holomorphic function ϕ is in $\mathcal{J}$ as soon as $\phi R = 0$. We use such a current $R^{\mathfrak{a},\ell}$ associated with the ideal $\mathfrak{a}^\ell$. For $\ell = 1$ it is the current of Bochner-Martinelli type from [1], whereas for $\ell > 1$ we use a variant from [4]. Since we are to prove the membership on Z rather than on some ambient space, thinking of Z as embedded in some $\mathbb{C}^n$, we will also use a current R^Z associated to the radical ideal I of the embedding. For the analysis of this current we rely on results from [6], described in Section 3. It turns out that one can form the "product" $R^{\mathfrak{a},\ell} \wedge R^Z$ such that $\phi R^{\mathfrak{a},\ell} \wedge R^Z$ only depends on the values of ϕ on Z; moreover, if the hypotheses in Theorem 1.1 are fulfilled then it vanishes (Proposition 4.1), which in turn implies that ϕ belongs to the ideal $\mathfrak{a}$ modulo I. In the last section we present an integral formula that provides an explicit representation of the membership.

2. Currents obtained from locally free complexes

Let

$$0 \to E_N \xrightarrow{f_N} E_{N-1} \xrightarrow{f_{N-1}} \cdots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0 \tag{2.1}$$

be a generically exact holomorphic complex of Hermitian vector bundles over a complex manifold X, say a neighborhood of the origin in $\mathbb{C}^n$. We assume that E_0 is a trivial line bundle so that $\mathcal{O}(E_0) = \mathcal{O}$. There is an associated complex, like (1.4), of (locally) free sheaves of $\mathcal{O}$-modules, and we let $\mathcal{J} = f_1\mathcal{O}(E_1) \subset \mathcal{O}$ be the ideal sheaf generated by (the entries in) f_1. Let Z be the analytic set where (2.1) is not pointwise exact. In $X \setminus Z$ we let σ_k be the section of $\mathrm{Hom}\,(E_{k-1}, E_k)$ that vanishes on the orthogonal complement of the pointwise image of f_k and is the minimal left inverse of f_k on the image of f_k. If $E = \oplus E_k$, $f = \oplus f_k$, and $\sigma = \oplus \sigma_k$, then $\sigma f + f\sigma = I$, where I is the identity on E. Since E_0 is trivial we identify $\mathrm{Hom}\,(E_0, E)$ with E. Following [7], in $X \setminus Z$ we define the form-valued sections

$$u = \sum_{k=1}^{N} u_k, \quad u_k = (\bar\partial \sigma_k) \cdots (\bar\partial \sigma_2)\sigma_1, \tag{2.2}$$

of E. If $\nabla_f = f - \bar\partial$ we have that $\nabla_f u = 1$. It turns out that u has a current extension U to X as a principal value current: If F is a tuple of holomorphic functions such that $F = 0$ on Z, then $|F|^{2\lambda} u$ has a current-valued analytic continuation to $\mathrm{Re}\,\lambda > -\epsilon$ and U is the value at $\lambda = 0$. Alternatively one can take a smooth approximand of the characteristic function χ for $[1, \infty)$, and let $\chi_\delta = \chi(|F|^2/\delta^2)$. Then U is the weak limit of $\chi_\delta u$ when $\delta \to 0$ (see, e.g., the proofs of Theorems 16 and 21 in [24]). In this paper the latter definition will be more convenient. Clearly ∇_f also applies to currents, and

$$\nabla_f U = 1 - R, \tag{2.3}$$

where R is a residue current with support on Z; more precisely $R = \lim_{\delta\to 0} R^\delta$, where

$$R^\delta = R_0^\delta + R_1^\delta + R_2^\delta + \cdots = (1-\chi_\delta) + \bar\partial\chi_\delta\wedge u_1 + \bar\partial\chi_\delta\wedge u_2 + \cdots;$$

notice that R_k^δ is an E_k-valued $(0,k)$-current.

A basic observation is that the annihilator sheaf, $\operatorname{ann} R$, of R is contained in the sheaf $\mathcal{J}$, i.e.,

$$\operatorname{ann} R \subset \mathcal{J}. \tag{2.4}$$

In fact, if $\phi \in \mathcal{O}$ and $\phi R = 0$, then by (2.3), $\nabla_f(\phi U) = \phi - \phi R = \phi$. By solving a sequence of $\bar\partial$-equations, which is always possible locally at least, we get a holomorphic solution $\psi \in \mathcal{O}(E_1)$ to $f_1\psi = \phi$, which means that ϕ is in the ideal $\mathcal{J}$. One can also prove (2.4) by an integral formula that gives an explicit realization of the membership of ϕ in $\mathcal{J}$, see Section 5.

In general the converse inclusion is not true. However, if the associated sheaf complex is exact, i.e., a resolution of $\mathcal{O}/\mathcal{J}$, then indeed $\operatorname{ann} R = \mathcal{J}$ (Theorem 1.1 in [7]).

Example 2. Let $a_1, \dots, a_m$ be holomorphic functions in X. Choose a nonsense basis $\{e_1, \dots, e_m\}$ and consider $E_1 = sp\{e_j\}$ as a trivial vector bundle of rank m, let e_j^* be the dual basis, and consider $a = a_1e_1^* + \cdots + a_me_m^*$ as a section of the dual bundle E_1^*. If $E_k = \Lambda^k E_1$ we then get a complex (2.1), the Koszul complex, with the mappings f_k as interior multiplication δ_a with a. Following the recipe above (with the trivial metric on the E_k) we get, cf., [7] Example 1, the corresponding form

$$u^a = \sum_{k=1}^m \frac{(\sum_{j=1}^m \bar a_j e_j)\wedge(\sum_{j=1}^m \bar\partial\bar a_j\wedge e_j)^{k-1}}{|a|^{2k}} \tag{2.5}$$

outside $\{a = 0\}$ and the associated residue current $R^a = \lim_{\delta\to 0} R^{a,\delta}$ where $R^{a,\delta} = (1-\chi_\delta) + \bar\partial\chi_\delta\wedge u^a$ and $\chi_\delta = \chi(|a|^2/\delta^2)$. This current of so-called Bochner-Martinelli type was introduced already in [23], and its relation to the Koszul complex and division problems was noticed in [1]. Now (2.4) means that

$$\operatorname{ann} R^a \subset (a). \tag{2.6}$$

Except for the case when a is a complete intersection, in which case the Koszul complex provides a resolution of $\mathcal{O}/(a)$, the inclusion (2.6) is strict, see [28] and [19]. Nevertheless, the singularities of R^a reflect the characteristic varieties associated to the ideal, see [19] and [10], which are closely related to the integral closure of powers of (a), and therefore R^a is well suited for the Briançon-Skoda theorem.

A slight modification of the Koszul complex, derived from the so-called Eagon-Northcott complex, with associated ideal sheaf $\mathcal{J} = (a)^\ell$, was introduced in [4]. The associated form $u^{a,\ell}$ is a sum of terms like

$$\frac{\bar{a}_{I_1} \cdots \bar{a}_{I_\ell} \bar\partial \bar{a}_{I_{\ell+1}} \wedge \ldots \wedge \bar\partial \bar{a}_{I_{k+\ell-1}}}{|a|^{2(k+\ell-1)}}, \quad k \le m,$$

see the proof of Theorem 1.1 in [4] for a precise description of $u^{a,\ell}$ and the corresponding residue current $R^{a,\ell}$. It turns out that ϕ annihilates $R^{a,\ell}$ if (1.1) holds, and thus $\phi \in (a)^\ell$, so the classical Briançon-Skoda theorem follows. The most expedient way to prove this annihilation is to use a resolution of singularities where a is principal. However, it is not really necessary to define the current $R^{a,\ell}$ in itself; it is actually enough to make sure that $\phi R^{a,\ell,\delta} \to 0$ when $\delta \to 0$, and this can be proved essentially by integration by part in an ingenious way, thus providing a proof of the Briançon-Skoda theorem by completely elementary means, see [27]. □

In [8] was introduced the sheaf of *pseudomeromorphic* currents $\mathcal{PM}$. For the definition, see [8]. It is closed under $\bar\partial$ and multiplication with smooth forms. In particular, the currents U and R are pseudomeromorphic. The following fact (Corollary 2.4 in [8]) will be used repeatedly.

Proposition 2.1. *If $T \in \mathcal{PM}$ has bidegree (r, k) and the support of T is contained in a variety of codimension strictly larger than k, then $T = 0$.*

In particular, this means that if Z (the variety where (2.1) is not pointwise exact) has codimension p then $R = R_p + R_{p+1} + \cdots$.

As mentioned in the introduction, we need to form products of currents associated to complexes. Assume therefore that $(\mathcal{O}(E^g_\bullet), g_\bullet)$ and $(\mathcal{O}(E^h_\bullet), h_\bullet)$ are two complexes as above and $\mathcal{I}$ and $\mathcal{J}$ are the corresponding ideal sheaves. We can define a complex (2.1) with

$$E_k = \bigoplus_{i+j=k} E^g_i \otimes E^h_j, \tag{2.7}$$

and $f = g + h$, or more formally, $f = g \otimes I_{E^h} + I_{E^g} \otimes h$, such that

$$f(\xi \otimes \eta) = g\xi \otimes \eta + (-1)^{\deg \xi} \xi \otimes h\eta. \tag{2.8}$$

Notice that $E_0 = E^g_0 \otimes E^h_0 = \mathbb{C}$ and that $f_1 \mathcal{O}(E_1) = \mathcal{I} + \mathcal{J}$. One can extend (2.8) to form-valued or current-valued sections ξ and η and $\deg \xi$ then means total degree. It is natural to write $\xi \wedge \eta$ rather than $\xi \otimes \eta$, and we define $\eta \wedge \xi$ as $(-1)^{\deg \xi \deg \eta} \xi \wedge \eta$. Notice that

$$\nabla_f(\xi \otimes \eta) = \nabla_g \xi \otimes \eta + (-1)^{\deg \xi} \xi \otimes \nabla_h \eta. \tag{2.9}$$

Let u^g and u^h be the corresponding E^g-valued and E^h-valued forms, cf. (2.2). Then $u = u^h \wedge u^g$ is an E-valued form outside $Z^g \cup Z^h$. Following the proof of Proposition 2.1 in [8] we can define E-valued pseudomeromorphic currents

$$R^h\wedge R^g = \lim_{\delta\to 0} R^{h,\delta}\wedge R^g, \quad U^h\wedge R^g = \lim_{\delta\to 0} U^{h,\delta}\wedge R^g,$$

where $U^{h,\delta} = \chi_\delta u^h$ and $R^{h,\delta} = 1-\chi_\delta + \bar\partial\chi_\delta\wedge u^h$, and $\chi_\delta = \chi(|H|^2/\delta^2)$ as before. The "product" $R^h\wedge R^g$ so defined is *not* equal to $R^g\wedge R^h$ in general. It is also understood here that H only vanishes where it has to, i.e., on the set where the complex $(E^h_\bullet, h_\bullet)$ is not pointwise exact. If we use an H that vanishes on a larger set, the result will be affected. It is worth to point out that a certain component $R^h_k\wedge R^g$ may be nonzero even if R^h_k itself vanishes.

Proposition 2.2. *With the notation above we have that*

$$\nabla_f(U^g + U^h\wedge R^g) = 1 - R^h\wedge R^g. \tag{2.10}$$

Moreover, $\phi R^h\wedge R^g = 0$ implies that $\phi\in\mathcal{I}+\mathcal{J}$.

Proof. Recall that $\nabla_h U^{h,\delta} = 1-R^{h,\delta}$, $\nabla_g U^g = 1-R^g$ and $\nabla_g R^g = 0$. Therefore,

$$\nabla_f(U^g + U^{h,\delta}\wedge R^g) = 1 - R^g + (1-R^{h,\delta})\wedge R^g = 1 - R^{h,\delta}\wedge R^g.$$

Taking limits, we get (2.10). The second statement now follows in the same way as (2.4) above. □

3. The residue current associated to the variety Z

Consider a subvariety Z of a neighborhood of the origin in $\mathbb{C}^n$ with radical ideal sheaf $\mathcal{I}$ and let (1.4) be a resolution of $\mathcal{O}/\mathcal{I}$. Let R^Z be the associated residue current obtained as in the previous section. We then know that R^Z has support on Z and that $\operatorname{ann} R^Z = \mathcal{I}$. Outside the set Z_k, cf., Section 1, the mapping f_k has constant rank, and hence σ_k is smooth there. Outside Z_k we therefore have that

$$R^Z_{k+1} = \alpha_{k+1} R^Z_k \tag{3.1}$$

where $\alpha_{k+1} = \bar\partial\sigma_{k+1}$ is a smooth $\mathrm{Hom}(E_k, E_{k+1})$-valued $(0,1)$-form, cf., (2.2).

Locally on Z_{reg}, the current R^Z is essentially the integration current $[Z]$. We have the following more precise statement that gives a Dolbeault-Lelong-type representation, in the sense of [12], of the current R^Z. Let χ be a smooth regularization of the characteristic function of $[1,\infty)$ and $p = \operatorname{codim} Z$ as before.

Proposition 3.1. *For each given $x\in Z_{reg}$, there is a hypersurface $\{h=0\}$ in Z, avoiding x but containing Z_{sing} and intersecting Z properly, and E_k-valued $(n-p, k-p)$-forms β_k, smooth outside $\{h=0\}$, such that*

$$R^Z_k.(dz\wedge\xi) = \lim_{\epsilon\to 0}\int_Z \chi(|h|/\epsilon)\beta_k\wedge\xi, \quad \xi\in\mathcal{D}_{0,n-k}(X),$$

for $p \le k \le n$. Moreover, in a suitable resolution $\pi\colon \tilde{Z} \to Z$ the forms β_k locally have the form α_k/m_k, where α_k are smooth and m_k are monomials.

Here, $dz = dz_1 \wedge \cdots \wedge dz_n$.

Proof. Following Section 5 in [6] (the proof of Proposition 2.2) one can find, for each given $x \in Z_{reg}$, a holomorphic function h such that $h(x) \neq 0$ and h does not vanish identically on any component of Z_{reg}. Moreover, for $k \ge p$,

$$R_k^Z = \gamma_k \lrcorner [Z],$$

where γ_k is an E_k-valued and $(0, k-p)$-form-valued $(p,0)$-vector field that is smooth outside $\{h = 0\}$. Let ξ be a test form of bidegree $(0, n-k)$. The current R^Z has the so-called standard extension property, SEP, see [8] Section 5, which means that

$$R_k^Z.(\xi \wedge dz) = \lim_{\epsilon \to 0} \int \chi(|h|/\epsilon)\gamma_k \lrcorner [Z] \wedge \xi \wedge dz = \pm \lim_{\epsilon \to 0} \int_Z \chi(|h|/\epsilon)\xi \wedge \gamma_k \lrcorner dz.$$

Thus we can take $\beta_k = \pm \gamma_k \lrcorner dz$.

More precisely, according to the last paragraph of Section 5 in [6], γ_p is a meromorphic $(p,0)$-field (with poles where $h = 0$) composed by the orthogonal projection of E_p onto the orthogonal complement in E_p of the pointwise image of f_{p+1}. This projection is given by

$$I_{E_p} - f_{p+1}\sigma_{p+1}.$$

Furthermore, cf., (3.1),

$$\gamma_k = (\bar\partial \sigma_k) \cdots (\bar\partial \sigma_{p+1})\gamma_p$$

for $k > p$. Now choose a resolution of singularities $\tilde{Z} \to Z$ such that for each k the the determinant ideal of f_k is principal. On $\tilde{Z}$, then each σ_k (locally) is a smooth form over a monimial, see Section 2 in [7], and thus $\beta_k = \gamma_k \lrcorner dz$ has this form as well. $\square$

We can choose the resolution of singularities $\tilde{Z} \to Z$ so that also $\tilde{h} = \pi^* h$ is a monomial. By a partition of unity it follows that $R_k^Z.(dz \wedge \xi)$ is a finite sum of terms like

$$\lim_{\epsilon \to 0} \int_s \chi(|\tilde{h}|/\epsilon) \frac{ds_1 \wedge \ldots \wedge ds_\nu}{s_1^{\alpha_1+1} \cdots s_\nu^{\alpha_\nu+1}} \wedge \tilde{\xi} \wedge \psi, \tag{3.2}$$

where $s_1, \ldots, s_{n-p}$ are local holomorphic coordinates and $\nu \le n-p$, $\tilde{\xi} = \pi^*\xi$, and ψ is a smooth form with compact support. It is easily checked that this limit is the tensor product of the one-variable principal value currents $ds_i/s_i^{\alpha_i+1}$, $1 \le j \le \nu$, acting on $\tilde{\xi} \wedge \psi$. Therefore (3.2) is equal to (a constant times)

$$\int \frac{ds_1 \wedge \ldots \wedge ds_\nu}{s_1 \cdots s_\nu} \wedge \partial_s^\alpha (\tilde{\xi} \wedge \psi), \tag{3.3}$$

if $\partial_s^\alpha = \partial_{s_1}^{\alpha_1} \cdots \partial_{s_\nu}^{\alpha_\nu}$.

4. Proof of Theorem 1.1

To prove Theorem 1.1 we are going to apply the idea in Example 2 but performed on Z. To this end we assume that Z is embedded in $\mathbb{C}^n$ and we let R^Z be the current introduced in the previous section. Let $\mathfrak{a} = (a)$ be the ideal in $\mathcal{O}_Z$ and suppose for the moment that a also denotes representatives in $\mathcal{O}$ of the generators. If $R^{a,\ell} = \lim_{\delta\to 0} R^{a,\ell,\delta}$ denotes the current from Example 2 we can form, cf., the end of Section 2, the product

$$R^{a,\ell}\wedge R^Z = \lim_{\delta\to 0} R^{a,\ell,\delta}\wedge R^Z.$$

Since R^Z annihilates $\mathcal{I}$ it follows that $R^{a,\ell}\wedge R^Z$ only depends on $\mathfrak{a} \subset \mathcal{O}_Z$. For the same reason, $\phi R^{a,\ell}\wedge R^Z$ is well-defined for $\phi \in \mathcal{O}_Z$. We know from Proposition 2.2 that ϕ belongs to $\mathfrak{a}$ if it annihilates this current, and thus Theorem 1.1 follows from the following proposition.

Proposition 4.1. *If the hypotheses of Theorem 1.1 are fulfilled i.e., either* (1.6), *or* (1.8) *together with the geometric conditions* (1.7), *then* $\phi R^{a,\ell}\wedge R^Z = 0$.

Remark 2. It is natural to try to use the Lelong current $[Z]$ rather than R^Z. There is, see [5] Example 1, a holomorphic E_p-valued form ξ such that $[Z] = \xi \cdot R^Z_p$. Thus the hypotheses in Theorem 1.1 imply that $\phi R^a\wedge[Z] = 0$. However, this in turn does not imply that ϕ is in (a). In fact, if $m = 1$ so that a is just one function, then

$$0 = \phi R^a\wedge[Z] = \phi\bar\partial\frac{1}{a}\wedge[Z],$$

and this means that ϕ/a is in ω^0_Z introduced by Barlet, see, e.g., [16], and this class is wider than $\mathcal{O}_Z$ in general. □

Proof of Proposition 4.1. We first assume that (1.7) and (1.8) hold. Considering $\phi R^{a,\ell}$ as an intrinsic current on the submanifold Z_{reg} (cf. the beginning of this section) it follows from the residue proof of the Briançon-Skoda theorem in the regular case that $\phi R^{a,\ell}$ must vanish on Z_{reg} since (1.8) holds. Thus, $\phi R^{a,\ell}\wedge[Z]$ vanishes on Z_{reg} and so, in view of Proposition 3.1, it follows that the support of $\phi R^{a,\ell}\wedge R^Z$ is contained in Z_{sing}. On the other hand it is readily verified that $R^{a,\ell}\wedge R^Z$ must vanish if a is nonvanishing. Thus the support of $\phi R^{a,\ell}\wedge R^Z$ is contained in $Z_{sing} \cap Z^a$.

The current $R^{a,\ell}$ has (maximal) bidegree $(0, m)$ and hence $R^{a,\ell}\wedge R^Z_p$ has (maximal) bidegree $(0, m + p)$. Since it has support on $Z_{sing} \cap Z^a$ that has codimension $\geq p+m+1$ by (1.7), it follows that $\phi R^{a,\ell}\wedge R^Z_p = 0$. Outside Z_{p+1} we have that $R^Z_{p+1} = \alpha_{p+1}R^Z_p$ for a smooth form α_{p+1}, and hence

$$\phi R^{a,\ell}\wedge R^Z_{p+1} = \phi R^{a,\ell}\wedge\alpha_{p+1}R^Z_p = \alpha_{p+1}\phi R^{a,\ell}\wedge R^Z_p = 0$$

there. Thus $\phi R^{a,\ell}\wedge R^Z_{p+1}$ has support on $Z_{p+1}\cap Z^a$, and again for degree reasons we find that $\phi R^{a,\ell}\wedge R^Z_{p+1}=0$. Continuing in this way we can conclude that $\phi R^{a,\ell}\wedge R^Z=0$.

We now assume that (1.6) holds. We have to prove that $R^Z.(dz\wedge\xi)\to 0$ when $\delta\to 0$, for

$$\xi=\phi R^{a,\ell,\delta}\wedge\eta, \tag{4.1}$$

with test forms η of bidegree $(0,*)$. In view of the comments after the proof of Proposition 3.1 it is enough to prove that each term (3.3) tends to zero if (1.6) holds and μ is large enough (independently of (a) and ℓ). For this particular term we will see that we need $\mu\ge\mu_0$, where

$$\mu_0=|\alpha|+2\min(m,n-p). \tag{4.2}$$

For simplicity we omit all snakes from now on and write ϕ rather than $\tilde\phi$ etc. Moreover, we assume that $\ell=1$, the general case follows completely analogously. Since $\tilde Z$ is smooth, by the usual Briançon-Skoda theorem we have that

$$\phi\in(a)^{|\alpha|+\min(m,n-p)+1}. \tag{4.3}$$

Notice that

$$R_k^{a,\delta}=\chi'(|a|^2/\delta^2)\wedge\frac{\bar\partial|a|^2}{\delta^2}\wedge u_k^a,\quad k>0,$$

and thus $R_k^{a,\delta}$ is a sum of terms like

$$\chi'\frac{\bar\partial\bar a_{I_1}\wedge\ldots\wedge\bar\partial\bar a_{I_k}}{\delta^2|a|^{2k}}\bar a a\wedge\omega$$

for $|I|=k$, where in what follows a^r denotes a product of r factors a_i, and similarly with $\bar a^r$, and ω denotes a smooth form. For degree reasons $k\le\nu=\min(m,n-p)$. In view of (4.3) therefore $\phi R_k^{a,\delta}$ is a sum of terms like

$$\chi'\frac{\bar\partial\bar a_{I_1}\wedge\ldots\wedge\bar\partial\bar a_{I_\nu}}{\delta^2|a|^{2\nu}}\bar a a^{2+\nu+|\alpha|}\wedge\omega$$

plus lower order terms. A straight forward computation yields that $\partial_s^\alpha(\phi R_k^{a,\delta})$ is a finite sum of terms like

$$\chi^{(r+1)}\frac{\bar\partial\bar a_{I_1}\wedge\ldots\wedge\bar\partial\bar a_{I_\nu}}{\delta^{2(r+1)}|a|^{2(\nu+|\gamma|-r)}}\bar a^{1+|\gamma|}a^{2+\nu+|\gamma|}\wedge\omega,$$

where $\gamma\le\alpha$ and $r\le|\gamma|$, plus lower order terms.

We thus have to see that each

$$\int_s\frac{ds_1\wedge\ldots\wedge ds_\nu}{s_1\cdots s_\nu}\chi^{(r+1)}\frac{\bar\partial\bar a_{I_1}\wedge\ldots\wedge\bar\partial\bar a_{I_\nu}}{\delta^{2(r+1)}|a|^{2(\nu+|\gamma|-r)}}\bar a^{1+|\gamma|}a^{2+\nu+|\gamma|}\wedge\omega \tag{4.4}$$

tends to 0 when $\delta \to 0$. After a suitable further resolution we may assume that locally $a = a_0 a'$ where a_0 is holomorphic and a' is a non-vanishing tuple. Then

$$\bar\partial \bar a_{I_1}\wedge\ldots\wedge\bar\partial \bar a_{I_\nu} = \bar a_0^{\nu-1}\wedge\omega.$$

Also notice that the expression

$$\frac{ds_1\wedge\ldots\wedge ds_\nu}{s_1\cdots s_\nu} \tag{4.5}$$

becomes a sum of similar expressions in this new resolution. Altogether we end up with a finite sum of terms like

$$\int_s \frac{ds_1\wedge\ldots\wedge ds_\nu}{s_1\cdots s_\nu}\chi^{(r+1)}(|a|^2/\delta^2)\wedge\mathcal{O}(1),$$

and each such integral tends to zero by dominated convergence.

The term corresponding to $R_0^{a,\delta} = 1 - \chi(|a|^2/\delta^2)$ is handled in a similar but easier way. □

5. Integral representation of the membership

Finally we describe how one can obtain an explicit integral representation of the membership provided that the residue is annihilated. The starting point is the formalism in [2] to generate integral representations for holomorphic functions. Let δ_η denote interior multiplication with the vector field

$$2\pi i\sum_1^n(\zeta_j - z_j)\frac{\partial}{\partial\zeta_j}$$

and let $\nabla_\eta = \delta_\eta - \bar\partial$. A smooth form $g = g_0 + g_1 + \cdots + g_n$, where g_k has bidegree (k,k), is called a *weight* (with respect to z) if $\nabla_\eta g = 0$ and $g_0(z,z) = 1$. Notice that the product of two weights is again a weight.

Example 3. Let χ be a cutoff function that is identically 1 in a neighborhood of the closed unit ball, and let

$$s = \frac{1}{2\pi i}\frac{\partial|\zeta|^2}{|\zeta|^2 - \bar\zeta\cdot z}.$$

Then $\nabla_\eta s = 1 - \bar\partial s$ and therefore

$$g = \chi - \bar\partial\chi\wedge[s + s\wedge\bar\partial s + \cdots + s\wedge(\bar\partial s)^{n-1}]$$

is a weight with respect to z for each z in the ball, with compact support, and it depends holomorphically on z. □

If g is a weight with compact support and z is holomorphic on the support, then

$$\phi(z) = \int g\phi = \int g_n\phi.$$

Now consider a complex like (2.1) in Section 2, defined in a neighborhood of the closed ball, and let U^δ and R^δ be the associated E-valued forms. One can find, see [3] Proposition 5.3, holomorphic E_k^*-valued $(k,0)$-forms H_k^0 and $\mathrm{Hom}(E_k,E_1)$-valued $(k-1,0)$-forms H_k^1 such that $\delta_\eta H_k^0 = H_{k-1}^0 f_k(\zeta) - f_1(z)H_k^1$ and $H_j^j = \mathrm{Id}_{E_j}$. Using that $\nabla_f U^\delta = 1 - R^\delta$ one verifies that

$$f_1(z)HU^\delta + HR^\delta = 1 - \nabla_\eta\big(\sum H_k^0 U_k^\delta\big),$$

where

$$HU^\delta = \sum H_k^1 U_k^\delta, \quad HR^\delta = \sum H_k^0 R_k^\delta.$$

It follows that $g^\delta := f_1(z)HU^\delta + HR^\delta$ is a weight with respect to z. If g is, e.g., the weight from Example 3 we thus get the representation

$$\phi(z) = \int g^\delta \wedge g\phi = f_1(z)\int HU^\delta \wedge g\phi + \int HR^\delta \wedge g\phi.$$

Taking limits we obtain the interpolation-division formula

$$\phi(z) = f_1(z)\int HU\wedge g\phi + \int HR\wedge g\phi. \tag{5.1}$$

To be precise, the integrals here are the action of currents on smooth forms. In particular, (5.1) implies that ϕ belongs to the ideal generated by f_1 if $\phi R = 0$.

If we now choose as our complex the resolution of the sheaf $I = I_Z$, we get the formula

$$\phi(z) = \int g\wedge H^Z R^Z \phi, \quad z \in Z,$$

for $\phi \in \mathcal{O}_Z$. We then replace g by the weight $g^{a,\ell,\delta}\wedge g$, where

$$g^{a,\ell,\delta} = a(z)^\ell \cdot H^{a,\ell}U^{a,\ell,\delta} + H^{a,\ell}R^{a,\ell,\delta};$$

here $a(z)^\ell$ denotes the first mapping in the complex associated with $(a)^\ell$, cf., Example 2, so that its entries are elements in the ideal $(a)^\ell$. We get

$$\begin{aligned}\phi(z) \;&=\; a(z)^\ell \cdot \int_\zeta H^a U^{a,\ell,\delta}\wedge H^Z R^Z \phi \wedge g \\ &\quad + \int_\zeta H^a R^{a,\ell,\delta}\wedge H^Z R^Z \wedge g\phi.\end{aligned}$$

If the hypotheses in Theorem 1.1 are fulfilled, since H^Z, H^a and g are smooth, the second integral tends to zero when $\delta \to 0$, and the first integral on the right hand side converges to an $E_1^{a,\ell}$-valued holomorphic function. Thus we get the explicit representation

$$\phi(z) = a(z)^\ell \cdot \int_\zeta H^a U^{a,\ell}\wedge H^Z R^Z \phi \wedge g$$

of the membership.

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Department of Mathematics, Chalmers University of Technology and University of Gothenburg, S-412 96 GÖTEBORG, SWEDEN

E-mail address: matsa@chalmers.se, hasam@chalmers.se, sznajdma@chalmers.se

A BRIANÇON-SKODA TYPE RESULT FOR A NON-REDUCED ANALYTIC SPACE

JACOB SZNAJDMAN

Abstract. We present here an analogue of the Briançon-Skoda theorem for a germ of an analytic space Z at x, such that $\mathcal{O}_{Z,x}$ is Cohen-Macaulay, but not necessarily reduced. More precisely, we find a sufficient condition for membership of a function in a power of an arbitrary ideal $\mathfrak{a}^l \subset \mathcal{O}_{Z,x}$ in terms of size conditions of Noetherian differential operators applied to that function. This result generalizes a theorem by Huneke in the reduced case.

1. Introduction

The Briançon-Skoda theorem, [12], states that for any ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^n,0}$ generated by m germs, we have the inclusion $\overline{\mathfrak{a}^{\min(m,n)+l-1}} \subset \mathfrak{a}^l$, where $\overline{I}$ denotes the integral closure of I. The generalization to an arbitrary regular Noetherian ring was proven algebraically in [17], and for rational singularities in [1].

Huneke, [14], showed that for a quite general Noetherian reduced local ring S, there is an integer N such that $\overline{\mathfrak{a}^{N+l-1}} \subset \mathfrak{a}^l$, for all ideals $\mathfrak{a}$ and $l \geq 1$. In particular this applies when $S = \mathcal{O}_{V,x}$, the local ring of holomorphic functions of a germ of a reduced analytic space V. This case of Huneke's theorem was recently reproven analytically, [7]. Let $|\mathfrak{a}|^2 = \sum_1^m |a_i|^2$, which up to constants does not depend on the choice of the generators a_i. Since a function ϕ in $\overline{\mathfrak{a}^M}$ is characterized by the property that $|\phi| \leq C|\mathfrak{a}|^M$, [16], an equivalent formulation of the theorem is that ϕ belongs to $\mathfrak{a}^l$ whenever $|\phi| \leq C|\mathfrak{a}|^{N+l-1}$ (on V).

We will consider a germ of an analytic space, i.e. a pair $(Z, \mathcal{O}_{Z,x})$, or just Z for brevity, of a germ of an analytic variety Z at a point x and its local ring $\mathcal{O}_{Z,x} = \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ for some $\mathcal{J} \subset \mathcal{O}_{\mathbb{C}^n,0}$ such that $Z(\mathcal{J}) = Z$. The reduced space Z_{red} has the same underlying variety Z, but the structure ring is $\mathcal{O}_{Z,x}/\sqrt{0} = \mathcal{O}_{\mathbb{C}^n,0}/\sqrt{\mathcal{J}}$. We assume throughout this paper that $\mathcal{O}_{Z,x}$ is Cohen-Macaulay, which in particular gives that $\mathcal{J}$ has pure dimension.

The aim of this paper is to find an appropriate generalization of the Briançon-Skoda theorem to this setting – when $S = \mathcal{O}_{Z,x}$. Now that we have dropped the assumption that Z is reduced, the situation becomes different; the integral closure of any ideal contains the nilradical $\sqrt{0}$ by

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definition, so $\overline{\mathfrak{a}^N} \subset \mathfrak{a}$ can only hold if $\sqrt{0} \subset \mathfrak{a}$, or equivalently, if each element of $\mathcal{O}_{Z,x}$ that vanishes on Z belongs to $\mathfrak{a}$. Clearly this does not hold for any $\mathfrak{a}$. In the following example we consider the most simple non-reduced space. It will nevertheless help illustrate some general notions, and also our main result, Theorem 1.2.

Example 1.1. Consider the analytic space Z whose underlying space Z_{red} is $\{w = 0\} = \mathbb{C}^{n-1} \subset \mathbb{C}^n$, such that $\mathcal{O}_Z = \mathbb{C}[[z_1, \dots z_{n-1}, w]]/w^k$, $k > 2$. The nilradical is (w), and is not contained in $\mathfrak{a} = (w^2)$. It may be helpful to think of the space Z as $\mathbb{C}^{n-1}$ with an extra infinitesimal direction transversal to Z, and its structure sheaf being the k:th order Taylor expansions in that direction. For each $f \in \mathcal{O}_{Z,x}$ we have

$$f(z,w) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial w^i}(z,0) w^i, \quad \text{and}$$

$$\mathcal{O}_{Z,x} \simeq \bigoplus_0^{k-1} \mathcal{O}_{Z_{red},x}.$$

Although the function w is identically zero on Z (and therefore it most definitely satisfies $|w| \leq C|\mathfrak{a}|^M$ for any M), but the element w does not belong to $\mathfrak{a} = (w^2)$. Since Z is non-reduced, evaluating w, or any other element, as a function on Z does not give enough information to determine ideal memebership. We also have to take into account the transversal derivatives.

A germ of a holomorphic differential operator L is called *Noetherian* with respect to an ideal $\mathcal{J} \subset \mathcal{O}_{\mathbb{C}^n,0}$ if $L\phi \in \sqrt{\mathcal{J}}$ for all $\phi \in \mathcal{J}$. We say that $L_1, \dots, L_M$ is a *defining* set of Noetherian operators for $\mathcal{J}$, if $\phi \in \mathcal{J}$ if and only if $L_1\phi, \dots, L_M\phi \in \sqrt{\mathcal{J}}$. The existence of a defining set for any ideal $\mathcal{J}$ is due to Ehrenpreis [13] and Palamodov [19], see also [10], [15] and [18]. In the example above, a defining set is $1, \partial/\partial w, \dots, \partial^{k-1}/\partial w^{k-1}$.

If L is Noetherian with respect to $\mathcal{J}$, then $L\psi$ is a well-defined function on Z_{red} for any $\psi \in \mathcal{O}_{Z,x}$, and L induces an intrinsic mapping $L : \mathcal{O}_{Z,x} \to \mathcal{O}_{Z_{red},x}$. Let $\mathcal{N}(Z)$ be the set of all such mappings; this set does not depend on the choice of local embedding of Z. If (L_i) is a defining set for $\mathcal{J}$, then by definition any element $\phi \in \mathcal{O}_{Z,x}$ is determined uniquely by the tuple of functions $(L_i\phi)$ on Z_{red}, cf. Example 1.1. This fact indicates that it is natural to impose size conditions on the whole set $(L_i\phi)$ to generalize the Briançon-Skoda theorem:

Theorem 1.2. *Let Z be a germ of an analytic space such that $\mathcal{O}_{Z,x}$ is Cohen-Macaulay. Then there exists an integer N and operators $L_1, \dots, L_M \in \mathcal{N}(Z)$ such that for all ideals $\mathfrak{a} \subset \mathcal{O}_{Z,x}$ and all $l \geq 1$,*

$$|L_j\phi| \leq C|\mathfrak{a}|^{N+l-1}, \quad 1 \leq j \leq M, \tag{1}$$

implies that $\phi \in \mathfrak{a}^l$.

Remark 1.3. Let μ_0 be the maximal order of L_j for $1 \leq j \leq M$. If $\phi \in \mathfrak{a}^{l+\mu_0}$, then $L_j\phi \in \mathfrak{a}^l$, so for $N = -\mu_0 + 1$ and $l >> 0$, the inequalities (1) are necessary conditions for ϕ belonging to $\mathfrak{a}^l$. Note also that the special case $\mathfrak{a} = (0)$ of the theorem means that $\mathcal{L}_1, \ldots, \mathcal{L}_M$ is a defining set of Noetherian operators for $\mathcal{J}$, if $\mathcal{L}_j$ is any representative for L_j in the ambient space.

We will now prove Theorem 1.2 by elementary means in the case of Example 1.1:

Example 1.4. As we saw in the previous example, a set of defining differential operators for the ideal (w^k) consists of $L_j = \partial^j/\partial w^j$, $0 \leq j \leq k-1$. We assume that

$$|L_j\phi| \leq |\mathfrak{a}|^{r+(k-1)-j}, \tag{2}$$

for all $z \in \mathbb{C}^{n-1}, w = 0$, where $r = \min(n-1, m)$. We will allow ourselves to abuse notation; for example, we will write simply $\mathfrak{a}$ when we actually are referring to some element that belongs to $\mathfrak{a}$. Using Briançon-Skoda for $\mathbb{C}^{n-1}$ we get

$$\frac{\partial^j \phi}{\partial w^j} = \mathfrak{a}^{k-j} + k_j w, \quad k_j \in \mathcal{O}_{\mathbb{C}^n,0}. \tag{3}$$

We will show that

$$\phi = \sum_{i=0}^{p} w^i \mathfrak{a}^{k-i} + g_p w^{p+1}, \quad g_p \in \mathcal{O}_{\mathbb{C}^n,0} \tag{4}$$

holds for $p \leq k-1$. For $p = k-1$ it implies that $\phi \in \mathfrak{a}$ and for $p = 0$ it reduces to (3) with $j = 0$. Assume that (4) holds for some $p < k-1$. Let us differentiate (4) $p+1$ times with respect to w, and compare the result with (3) for $j = p+1$. This gives

$$g_p \in \sum_{i=0}^{p} w^i \mathfrak{a}^{k-i-p-1} + (w).$$

Now we substitute this back into (4), and get

$$\phi \in \sum_{i=0}^{p+1} w^i \mathfrak{a}^{k-i} + (w^{p+2}).$$

By induction (4) holds also for $p = k-1$. This proves the theorem for $l = 1$, and the same argument works for all l. Moreover we get that $N = r + k - 1$ works in Theorem 1.2. This is optimal as the following example shows, i.e. $N = r + k - 1$ is the Briançon-Skoda number for this particular analytic space. We get of course back the usual Briançon-Skoda number $N = r$ in the reduced case $k = 1$.

Example 1.5. To show that the example above is optimal, we need to find ϕ_p, for each $0 \leq p \leq k-1$, such that $\phi_p \notin \mathfrak{a}$ and $|\partial_w^j \phi_p| \leq |\mathfrak{a}|^{r+(k-1)-j}$ holds for $j \neq p$ and $|\partial_w^p \phi_p| \leq |\mathfrak{a}|^{r+(k-1)-p-1}$. Take $n = 2$

and $\mathfrak{a} = (z+w)$. Then $r = 1$, since $n-1 = m = 1$. A suitable choice is now $\phi_p = w^p z^{k-1-p}$. This does not belong to $\mathfrak{a}$, because if it did we would have $w^p z^{k-1-p} = (z+w)(a_0(z) + \cdots + a_{k-1}(z)w^{k-1})$, which would give $a_0 = \cdots = a_{p-1} = 0$ and $za_p = z^{k-1-p}$, $za_{p+1} = -a_p$, $za_{p+2} = -a_{p+1}$, etc, so $a_{k-1} = \pm 1/z$. This is a contradiction since a_{k-1} is holomorphic at $0 \in \mathbb{C}^n$.

In Section 5 we will apply the proof of the main theorem, with a minor modification, to the case where Z_{red} is smooth. We then recover the optimal result as in Examples 1.4 and 1.5 when $\mathcal{J} = (w^k)$.

Although the formulation of Theorem 1.2 is intrinsic, we will choose an embedding and work in the ambient space exclusively. If $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ annihilates a certain vector-valued residue current, $R^{a^l} \wedge R^Z$, then it turns out that it belongs to $\mathfrak{a}^l + \mathcal{J}$ (i.e. the image of ϕ in $\mathcal{O}_{Z,x}$ belongs to $\mathfrak{a}^l$); this follows by solving a certain sequence of $\overline{\partial}$-equations. Alternatively, one can also use a division formula to obtain an explicit integral representation of the membership. This way of proving ideal membership is described in [7] and goes back to [2] and [4]. We conclude that the proof is reduced to showing that $\phi R^{a^l} \wedge R^Z = 0$ whenever ϕ satisfies (1). The vanishing of this current is proved in Section 4. As a preparation, we begin to discuss Coleff-Herrera currents and the properties of R^{a^l}, R^Z and their product $R^{a^l} \wedge R^Z$.

2. Coleff-Herrera currents

Assume that Z is a germ of an analytic variety of pure codimension p in $\mathbb{C}^n$.

Definition 2.1. A Coleff-Herrera current on Z is a $\overline{\partial}$-closed current μ of bidegree $(0,p)$ with support on Z that is annihilated by $\overline{\phi}$ if $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on Z. One also requires that μ satisfies the standard extension property (SEP), that is, $\mu = \lim_{\varepsilon\to 0} \chi(|h|^2/\varepsilon^2)\mu$, if $h \in \mathcal{O}_{\mathbb{C}^n,0}$ does not vanish identically on any component of Z, and χ is a smooth cut-off function approximating $1_{[1,\infty)}$.

The set of all Coleff-Herrera currents on Z is an $\mathcal{O}$-module which we denote by $\mathcal{CH}_Z$. It is easy to see for any $\mu \in \mathcal{CH}_{Z,x}$, that $\operatorname{ann}\mu$ is a pure-dimensional ideal whose associated primes correspond to components of Z. There is a direct way, due to J-E. Björk [10], to obtain a defining set $L_1 \dots L_\nu$ for the ideal $\mathcal{J} = \operatorname{ann}\mu$. Furthermore, the operators L_j are related to μ by the formula (9). To obtain the formula (9), we need to consider

Björk's proof. Let ζ denote the first $n-p$ coordinates in $\mathbb{C}^n$ and η the p last. It follows from the local parametrization theorem one can find holomorphic functions $f_1 \dots f_p$ forming a complete intersection, such that Z is a union of a number of irreducible components of $V_f = \{f_1 =$

$\cdots = f_p = 0\}$. Moreover,

$$z = \zeta \qquad (5) \qquad w = f(\zeta, \eta)$$

are local holomorphic coordinates outside the hypersurface W defined by

$$h := \det \frac{\partial f}{\partial \eta}. \qquad (6)$$

By possibly rotating the coordinates (ζ, η), we can make sure h will not vanish identically on any component of Z. Since μ is a Coleff-Herrera current on the complete intersection V_f, we get by Theorem 4.2 in [5],

$$\mu = A \left[\overline{\partial} \frac{1}{f_1^{1+M_1}} \wedge \cdots \wedge \overline{\partial} \frac{1}{f_p^{1+M_p}} \right] \qquad (7)$$

for some integers M_i and strongly holomorphic function A. A basic fact is that for a $(n-p, n-p)$ test form ξ,

$$d\eta \wedge \left[\overline{\partial} \frac{1}{\eta_1^{1+M_1}} \wedge \cdots \wedge \overline{\partial} \frac{1}{\eta_p^{1+M_p}} \right] .\xi = \int_{\eta=0} \partial_{\eta_1}^{M_1} \ldots \partial_{\eta_p}^{M_p} \xi,$$

where the derivative symbols refer to Lie derivatives. Now let $d\eta \wedge \xi$ be an arbitrary $(n, n-p)$ test form supported outside W so that (z, w) are coordinates on its support. Then, since $d\eta \wedge \xi = \frac{1}{h} dw \wedge \xi$, the Leibniz rule gives that

$$\mu.d\eta \wedge \xi = \int_{w=0} \sum_{\alpha_j \leq M_j} \frac{c_\alpha}{h} \partial_w^{M-\alpha}(A) \partial_w^\alpha \xi, \qquad (8)$$

where $M = (M_1, \ldots, M_p)$. We now want to express $\phi\mu.d\eta \wedge \xi$ in terms of derivatives with respect to the variables η_i instead of w_i. By the chain rule,

$$\frac{\partial}{\partial w_j} = \frac{1}{h} \sum_k \gamma_{jk} \frac{\partial}{\partial \eta_k}.$$

Combining this with (8), we thus get operators Q_α so that

$$\phi\mu.d\eta \wedge \xi = \int_{w=0} \frac{1}{h} \sum_{\alpha_j \leq M_j} Q_\alpha(\phi) \partial_w^\alpha(\xi) =$$

$$= \int_Z \sum_{\alpha_j \leq M} \frac{1}{h^{N_0}} L_\alpha(\phi) K_\alpha(\xi), \qquad (9)$$

where N_0 is some (large) integer and $L_\alpha = h^{N_0} Q_\alpha$ and $K_\alpha = h^{N_0} \partial_w^\alpha$ are differential operators with respect to the variables η that are holomorphic across W.

Clearly, the values of $\partial_w^\alpha \xi$ can be prescribed on $\{w = 0\}$. Therefore $\phi\mu = 0$ on $Z \setminus W$ if and only if $L_\alpha(\phi) = 0$ on $Z \setminus W$ for all $\alpha \leq M$, but by continuity and SEP, these relations hold if and only if they hold across W. $\square$

Let $\mathcal{J} \subset \mathcal{O}_{\mathbb{C}^n,0}$, assume that $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ is Cohen-Macaulay, and let $Z = Z(\mathcal{J})$. It is well-known, that there is a finite set $\mu_1, \dots \mu_{\nu_0} \in \mathcal{CH}_Z$ such that $\mathcal{J} = \bigcap \operatorname{ann} \mu_j$. Hence we get a defining set for $\mathcal{J}$ by taking the union of the defining sets for $\operatorname{ann} \mu_j$. In the next section, we will use the construction in [9] of a set $\{\mu_j\}$ associated to $\mathcal{J}$ as above. It is proved in [6] that this set actually generates the $\mathcal{O}$-module of all $\mu \in \mathcal{CH}_Z$ such that $\mathcal{J}\mu = 0$.

Finally, we extend, for later convenience, the formula (9) across W, i.e. so that the support of ξ may intersect W. To simplify the argument we assume, without loss of generality, that $\phi = 1$. Then (9) agrees with

$$\mu.d\eta \wedge \xi = \lim_{\varepsilon \to 0} \int_Z \chi(|h|^2/\varepsilon^2) \frac{\mathcal{L}(\xi)}{h^{N_0}}, \tag{10}$$

for some holomorphic differential operator $\mathcal{L}$, since the cut-off function makes no difference when ξ has support in $Z \setminus W$ and ε is sufficiently small. We proceed to show that (10) continues to hold for general ξ. The right hand side of (10) defines a current, say τ, which must equal μ if it too has SEP with respect to h, so what we need to show is

$$\lim_{\delta \to 0} \chi(|h|^2/\delta^2)\tau - \tau = 0.$$

By expanding

$$\chi(|h|^2/\delta^2)\tau.\xi = \int_Z \frac{\mathcal{L}(\chi(|h|^2/\delta^2)\xi)}{h^{N_0}} \tag{11}$$

we get one term when all derivatives of $\mathcal{L}$ hit ξ, and clearly this term is precisely $\tau.\xi$ in the limit. We will now explain why all other contributions vanish; all these terms contain derivatives of $\chi(|h|^2/\delta^2)$ as a factor, and such a factor can be written as a sum of terms

$$\chi^{(k)}\left(\frac{|h|^2}{\delta^2}\right) \cdot \left(\frac{|h|^2}{\delta^2}\right)^k \frac{\sigma}{h^\kappa},$$

for some integers k and κ and a smooth function σ. Let us define $\chi_0(x) := x^k \chi^{(k)}(x)$. This function is identically zero in some neighbourhoods of 0 and infinity. After applying the following lemma to χ_0, we are done, since $\chi_0(\infty) = 0$.

Lemma 2.2. *If $\tilde{\chi}$ is any bounded function on $[0, \infty)$ that vanishes identically near 0 and has a limit at ∞, then*

$$\lim_{\delta \to 0} \tilde{\chi}(|h|^2/\delta^2)[\frac{1}{f}] \wedge [Z] = \tilde{\chi}(\infty)[\frac{1}{f}] \wedge [Z] \tag{12}$$

for any $h, f \in \mathcal{O}_{\mathbb{C}^n,0}$ that are generically non-vanishing on Z.

For a proof see [11], Lemma 2.

3. Residue currents associated to ideals

A summary of the machinery of residue currents needed to prove the Briançon-Skoda theorem on a singular variety, or in the present setting, appears in [7], Section 2. For the convenience of the reader, we give here an even shorter summary.

In [9] a method was presented for constructing currents R and U associated to any generically exact complex of hermitian vector bundles (E_k) with maps (f_k). These currents take values in the vector bundle $\mathrm{End}_E := \mathrm{End}(\bigoplus E_k)$. We define $f = \oplus f_j$ and $\nabla_f = f - \bar\partial$. The latter is an operator acting on End_E-valued forms and currents, and U is related to R by $\nabla_f U = 1 - R$. Since $\nabla_f^2 = 0$, it therefore follows that that $\nabla_f R = 0$. We restrict our attention to the case $\operatorname{rank} E_0 = 1$ and define the ideal $\mathcal{J} = Im(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$. Then $\operatorname{ann} R \subset \mathcal{J}$, but in general the inclusion is proper. However, a main result of [9] is that if $\mathcal{O}(E_k)$ with maps (f_k) is a resolution of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$, that is, an exact complex of sheaves, then $\operatorname{ann} R = \mathcal{J}$.

Now suppose that we start with any ideal $\mathcal{J} \subset \mathcal{O}_{\mathbb{C}^n,0}$ with the assumptions mentioned before, that is, $\mathcal{J}$ has pure codimension p and $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ is Cohen-Macaulay. We denote by R^Z the current obtained from a resolution so that $\operatorname{ann} R^Z = \mathcal{J}$. We have a decomposition $R^Z = \sum_{k=0}^N R_k^Z$ so that R_k^Z has bidegree $(0,k)$. The class of so called *pseudomeromorphic* currents $\mathcal{PM}$ was introduced in [8]. Since $R_k^Z \in \mathcal{PM}$ has support on $Z = Z(\mathcal{J})$, which has codimension p, R_k^Z must vanish for $k < p$ by Corollary 2.4 in [8], so $R^Z = R_p^Z + R_{p+1}^Z + \dots$. The same corollary implies that R^Z has SEP. The Cohen-Macaulay condition means that there is a resolution of length p of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$, so R_k^Z vanishes also for $k > p$, and thus $R^Z = R_p^Z$. Now $\nabla_f R^Z = 0$ implies that $\bar\partial R^Z = 0$ by a consideration of bidegrees. By Proposition 2.3 in [8], $\overline{I(Z)} = \sqrt{\mathcal{J}}$ annihilates R^Z. Collecting all information so far, we get that R^Z is a tuple of Coleff-Herrera currents, since its components satisfy the three required conditions. Thus $\mathcal{J} = \operatorname{ann} R^Z = \bigcap \operatorname{ann} \mu_j$, where μ_j are the components of R^Z, so by the proof of Björk's theorem in Section 2, we have a defining set of operators for $\mathcal{J}$, such that the current R^Z can be represented as a prinicipal value integral on Z in terms of these defining operators, (9).

Next we construct currents R^{a^l} and U^{a^l} associated to a complex that is the Koszul complex for $l = 1$ and a slight modification otherwise. The current R^{a^l} is of Bochner-Martinelli type and $\operatorname{ann} R^{a^l} \subset \mathfrak{a}^l$, but in general we do not have equality. The restriction u^{a^l} of U^{a^l} to the complement of $Z(\mathfrak{a})$ is displayed in (16), and R^{a^l} is obtained as the limit of $R_\delta^{a^l} = (1 - \chi_\delta^a) + \bar\partial \chi_\delta^a \wedge u^{a^l}$, where $\chi_\delta^a = \chi(|\mathfrak{a}|^2/\delta)$. We want to form the product of the currents R and R^{a^l}, which corresponds to restricting R^{a^l} to Z in a certain way. We thus define

$$R^{a^l} \wedge R^Z = \lim_{\delta \to 0} R^{a^l}_\delta \wedge R^Z.$$

This product takes values in the product of the two complexes of the two factors, see [7].

It follows by Proposition 2.2 in [7] that $\phi R^{a^l} \wedge R^Z = 0$ for $\phi \in \mathcal{O}_{\mathbb{C}^n,0}$ implies that $\phi \in \mathcal{J} + \mathfrak{a}^l$, that is, the image of ϕ in $\mathcal{O}_{Z,x}$ belongs to $\mathfrak{a}^l$. Although $R^{a^l} \wedge R^Z$ is a current in $\mathbb{C}^n$, $\phi R^{a^l} \wedge R^Z$ depends only on the image of ϕ. The proof of Theorem 1.2 is reduced to showing that indeed $\phi R^{a^l} \wedge R^Z = 0$ if ϕ satisfies (1), which we now begin to prove.

4. Proof of Theorem 1.2

Since R^Z is, under our assumptions, a tuple of Coleff-Herrera currents, we can use (9) with $\xi = \omega \wedge R^{a^l}_\delta$ to calculate $\phi R^{a^l}_\delta \wedge \mu.\omega$, where μ is a component of R^Z and ω is a test form. We now choose a resolution $X \xrightarrow{\pi} Z$ such that X is smooth and $\pi^* h$ is locally a monomial. Thus to show that $\phi R^{a^l} \wedge R^Z = 0$, it suffices by linearity to show that

$$I_\delta := \int_X \frac{\sigma}{s_1^{1+n_1} \cdot \dots \cdot s_{n-p}^{1+n_{n-p}}} \wedge \pi^*(L_i \phi) \pi^* \left(\partial^\alpha_\eta R^{a^l}_\delta \right) \to 0, \tag{13}$$

where σ is a smooth form with compact support. This integral is really a principal value integral with a smooth cut-off function as in (10). Recall that $|\alpha| \le |M| = \sum M_i$, see (7). We want to integrate (13) by parts. The reciprocal of the monomial occuring in this integral is just a tensor product of one variable distributions and

$$\frac{\partial}{\partial z} \left[\frac{1}{z^m} \right] = -m \left[\frac{1}{z^{m+1}} \right], \quad m \ge 1.$$

This yields indeed that

$$I_\delta = \int_X \frac{ds_1 \wedge \dots \wedge ds_{n-p}}{s_1 \cdot \dots \cdot s_{n-p}} \wedge \partial_s^{(n_1, \dots, n_{n-p})} \left(\sigma \wedge \pi^*(L_i \phi) \pi^* \left(\partial^\alpha_\eta R^{a^l}_\delta \right) \right). \tag{14}$$

We now extend π to a resolution $X' \xrightarrow{\gamma} X \xrightarrow{\pi} Z$ that principalizes $\mathfrak{a}$, and we call the generator in $\mathcal{O}_{X'}$ for a_0, i.e. $\gamma^* \pi^* a_j = a_0 a'_j$ and $|\gamma^* \pi^* \mathfrak{a}| \sim |a_0|$. Note that the form $\frac{ds_1 \wedge \dots \wedge ds_{n-p}}{s_1 \cdot \dots \cdot s_{n-p}}$ becomes a sum of similar forms when we pull it back to X'. If we show, for all integers $k_j \le n_j$, $1 \le j \le n-p$, that the form

$$\gamma^* \left[\partial_s^{(k_1, \dots, k_{n-p})} \left(\pi^*(L_i \phi) \pi^* \left(\partial^\alpha_\eta R^{a^l}_\delta \right) \right) \right] \tag{15}$$

is bounded on X' by a constant independent of δ, then an application of dominated convergence gives that $I_\delta \to 0$, and thereby concludes the proof. We assume that $k_j = n_j$ for $1 \le j \le n-p$. It will become apparent that this is the worst case.

The philosophy will be to write the expression in (15) as a product of factors of three types; factors whose modulii are equivalent to $|a_0|$, or to $|a_0|^{-1}$, and remaining factors, which we require to be bounded. We count the number of factors of the first type minus the number of factors of the second type, that is, the number of zeroes minus the number of singularities. Let us call this difference *homogeneity.* Since the factors of the third type may also be of the first type, the homogeneity depends on our factorization. However, we only want to find one factorization such that the homogeneity is non-negative, because then (15) is indeed bounded. Nothing will be lost if we instead get a finite number of terms which can be factorized in this way.

We examine first the factor $L_i\phi$. Then we consider the second factor, $\partial_\eta^\alpha R_\delta^{a^l}$, and add the two results.

Since X is a manifold, the usual Briançon-Skoda theorem and (1) gives that $\pi^*(L_i\phi) \in (\pi^*\mathfrak{a})^{N-\varrho+l}$, where $\varrho = \min(m, \dim Z)$. The conclusion is that the term in (15) for which no derivatives with respect to s hit $\pi^*(L_i\phi)$ gives a contribution of $N - \varrho + l$ to the homogeneity. A term for which $\pi^*(L_i\phi)$ is hit by k derivatives gives a contribution reduced by k, since $\partial\mathfrak{a}^{M+1} \subset \mathfrak{a}^M$ for any M.

We proceed now with the second factor. For simplicity we begin with the case $(n_1, \dots, n_{n-p}) = 0$, so that no derivatives appear with respect to the coordinates s. We will then consider the general case and see that the derivatives ∂_s have the same effect on homogeniety as ∂_η^α – namely each derivative decreases the homogeneity by one.

From [3] we know that

(16)
$$u^{a^l} = \sum_{\substack{1\le k\le\varrho \\ \beta_1+\cdots+\beta_p=k-1}} \left(\frac{\bigwedge_{i=1}^{l}(\sum_{j=1}^{m}\bar{a}_j e_j^i) \wedge \bigwedge_{i=1}^{l}(\sum_{j=1}^{m}\bar{\partial}\bar{a}_j \wedge e_j^i)^{\beta_i}}{|\mathfrak{a}|^{2(k+l-1)}} \right),$$

where e_*^i are frames of trivial bundles, and a_j, $1 \le j \le m$, are generators for $\mathfrak{a}$. For simplicity we consider only the most singular term of $R_\delta^{a^l} = (1 - \chi_\delta^a) - \bar{\partial}\chi_\delta^a \wedge u^{a^l}$, that is, the term of $\bar{\partial}\chi_\delta^a \wedge u^{a^l}$ for which $k = \varrho$ above. All other terms can be treated similarily, but easier. We denote the most singular term by ν_δ. Using the notation from Example 1.4, we get

$$\nu_\delta = \chi'\left(\frac{|\mathfrak{a}|^2}{\delta^2}\right)\left(\frac{\mathfrak{a}\bar{\partial}\bar{\mathfrak{a}}}{\delta^2}\right) \wedge \sum_{|\beta|=\varrho-1} \frac{\theta_\beta(a)}{|\mathfrak{a}|^{2(\varrho+l-1)}}, \tag{17}$$

where

$$\theta_\beta(a) = \bigwedge_{i=1}^{l}(\sum_{j=1}^{m}\bar{a}_j e_j^i) \wedge \bigwedge_{i=1}^{l}(\sum_{j=1}^{m}\bar{\partial}\bar{a}_j \wedge e_j^i)^{\beta_i}.$$

Lemma 4.1. *The homogeneity of $\gamma^*\pi^*\theta_\beta(a)$ is $\varrho + l - 1$.*

Proof. Recall that $\gamma^*\pi^*a_j = a_0 a'_j$. All terms for which $\overline{\partial}$ hits $\overline{a_0}$ vanish since they contain the wedge product of $(\sum_{j=1}^m \overline{a'_j} e_j^i)$ with itself, that is, the square of a 1-form. Thus all remaining terms are divisible by $a_0^{\rho+l-1}$. □

By differentiating (17), one sees that $\partial_\eta^\alpha \nu_\delta$ is a sum of terms like

$$\sigma\chi^{(1+|\alpha_1|)}\left(\frac{|\mathfrak{a}|^2}{\delta^2}\right)\left(\frac{\mathfrak{a}^{1-|\epsilon|+|\alpha_1|}}{\delta^{2|\alpha_1|+2}}\right)\wedge\sum_{|\beta|=\varrho-1}\frac{\overline{\mathfrak{a}}^{|\alpha_2|}\theta_\beta(a)}{|\mathfrak{a}|^{2(\varrho+l-1)+2|\alpha_2|}}, \tag{18}$$

where σ is a smooth function, and $\epsilon+\alpha_1+\alpha_2$ are multi-indices such that $\epsilon+\alpha_1+\alpha_2 \le \alpha$. It is readily seen that the homogeneity of (18) only depends on $|\epsilon|+|\alpha_1|+|\alpha_2|$ (for l and ρ fixed), except if $1-|\epsilon|+|\alpha_1|$ is negative, but this only improves the estimate. For the worst term it is therefore $-l-\varrho-|\alpha|$, due to Lemma 4.1. In other words, no matter which terms the derivatives hit, each one of them reduce the homogeneity by one (or zero). It remains to show that the homogeneity of $\gamma^*\partial_s^{\alpha_0}\pi^*\nu_\delta$ drops by at most $|\alpha_0|$, for any α_0. This is however quite immediate; we apply π^* to (18) and then differentiate with respect to s. The ideal $\mathfrak{a}$ is then replaced by $\pi^*\mathfrak{a}$ and since $\pi^*\theta_\beta(a)$ is anti-holomorphic, the differentiation follows exactly the same pattern as when we differentiated with respect to η – the derivatives can hit $\chi^{(k)}$, or a power of $\pi^*\mathfrak{a}$ or $|\pi^*\mathfrak{a}|^2$, or a smooth function, so we already know that each derivative decreases the homogeneity by no more than one.

The total homogeneity of (15) is the sum of $N-\varrho+l$ and $-l-\varrho-|\alpha|$ less the number of derivatives with respect both to s and to η, that is $N-2\varrho-|\alpha|-\sum_1^{n-p} n_j$. This is non-negative if $N \ge 2\min(m, \dim Z)+|\alpha|+\sum_1^{n-p} n_j$, so any such integer N has the desired property of Theorem 1.2, where $|\alpha|$ is maximized over the components of R^Z and $\sum_1^{n-p} n_j$ is maximized over an open covering of $\pi^{-1}(0)$ (recall that these numbers are constructed locally in the resolution).

5. An improvement for smooth analytic spaces

We return now to the case where Z_{red} is smooth. We then get a bound for N and the proof also simplifies significantly. Recall from Section 3 that $R^Z = (\mu_1, \ldots, \mu_\nu)$ is tuple of Coleff-Herrera currents, and each of its components therefore gives rise to holomorphic differential operators L_α, and all of these together form a defining set for $\mathcal{J}$ and satisfy (9). Note however that, since Z_{red} is smooth, we can take $w=\eta$ in (5) so that $h=1$ in (6). For each differential operator L we denote by $\operatorname{ord}(L)$ its order as a differential operator in the ambient space. This gives also a well-defined notion of order for the induced intrinsic mapping in $\mathcal{N}(Z)$.

Theorem 5.1. *Let Z be a germ of an analytic space such that Z_{red} is pure-dimensional and smooth (which implies that Z is Cohen-Macaulay).*

Let $L_1 \dots L_M$ be the Noetherian operators obtained from R as before and let d be the maximal order of these operators. Then if $l \geq 1$, and $\mathfrak{a} \subset \mathcal{O}_{Z,x}$ can be generated by m elements,

$$|L_j\phi| \leq C|\mathfrak{a}|^{\min(m,\dim Z)+d-\mathrm{ord}(L_j)+l-1}, \quad 1 \leq j \leq M \tag{19}$$

implies that $\phi \in \mathfrak{a}^l$.

Note that we have already seen (19) in a slightly different appearance in (2).

Proof. We can assume that $Z_{red} = \mathbb{C}^{n-p} \subset \mathbb{C}^n$, and we call the last p coordinates $w_1, \dots, w_p$. Clearly these functions form a complete intersection, so if μ is a component of R^Z, we have

$$\mu = A\left[\bar\partial\frac{1}{w_1^{1+M_1}} \wedge \cdots \wedge \bar\partial\frac{1}{w_p^{1+M_p}}\right],$$

for some holomorphic function A, cf. (7). We choose A so that the numbers M_i are minimal. Then the distribution order of μ is precisely $\sum M_i$, which, as we soon shall see, is also the highest order of the operators obtained from μ. By Example 1 in [6], the components of R^Z generate the $\mathcal{O}$-module of Coleff-Herrera currents on Z which are annihilated by $\mathcal{J}$. Hence, d is an invariant of Z. We proceed as in the general case to construct a set of defining Noetherian operators for the ideal $\mathcal{J}$ corresponding to Z. Since

$$\mu.dw \wedge \xi = \int_{w=0} \sum_{\alpha \leq M} \partial_w^{M-\alpha}(A)\partial_w^\alpha \xi,$$

where $M = (M_1, \dots, M_p)$, we get that

$$\phi\mu.dw \wedge \xi = \int_{w=0} \sum_{\alpha \leq M} L_\alpha(\phi)\partial_w^\alpha \xi, \tag{20}$$

and

$$L_\alpha = \sum_{\alpha \leq \gamma \leq M} C_{(\alpha,\gamma)}\partial_w^{M-\gamma}(A)\partial_w^{\gamma-\alpha},$$

for some combinatorial constants $C_{(\alpha,\gamma)}$. Clearly $\mathrm{ord}(L_\alpha) = |M| - |\alpha|$ and L_0 has the maximal order $\mathrm{ord}(L_0) = |M|$. As before, we need to substitue $\xi = \omega \wedge R_\delta^{a^l}$ into (20). By principalizing $\mathfrak{a}$ as before, it follows that the limit of (20) is zero when δ goes to zero, if the integrand can be factored in to two parts, one that is integrable, and one that is bounded. To see that the second factor is bounded we count its homogeneity as before. In the proof of the general case we used that

$$\bar\partial\chi\left(\frac{|\mathfrak{a}|^2}{\delta^2}\right) = \chi'\left(\frac{|\mathfrak{a}|^2}{\delta^2}\right)\left(\frac{\mathfrak{a}\bar\partial\bar{\mathfrak{a}}}{\delta^2}\right), \tag{21}$$

which was counted as a term of homogeneity -1. It is however well-known that $\partial a_0/|a_0|$ is integrable, so taking it as the first factor, it no longer counts into the homogeneity. Thus, the homogeneity contribution of (21) is now improved to 0. The most singular term of $\partial_w^\alpha(R_\delta^{a^l})$ has thus homogeneity $1-\varrho-l-|\alpha|$ (one unit more than in the non-smooth case). Since $1-\varrho-l-|\alpha| \geq 1-\varrho-l-d+\text{ord}(L_\alpha)$, the homogeneity is non-negative if $|L_\alpha(\phi)| \leq C|\mathfrak{a}|^{\varrho+d-\text{ord}(L_\alpha)+l-1}$, so we are done. □

Remark 5.2. In the non-smooth case the form in (21) has to be included into the bounded factor, since in that case we already had the form ds/s occupying the integrable factor. Note also that we avoided using Hironaka desingularization and applying the classical Briançon-Skoda theorem in the resolution, which previously cost us $\varrho-1$ units of homogeneity.

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APPENDIX: THE BRIANÇON-SKODA NUMBER OF A CUSP

We will now consider the cusp given by $z^p = w^q$ in $\mathbb{C}^2$ in order to better understand how to determine the Briançon-Skoda number for a singular variety. We compute this number for the cusp first by an elementary method and then by going through the proof of the main result of the second paper.

Lemma A.1. *Let p, q be relatively prime positive integers. The equation $ap + bq = n$ has a positive solution $a, b \geq 0$ for $n \geq (p-1)(q-1)$, but not for $n = (p-1)(q-1) - 1$.*

Theorem A.2. *Let $Z = Z(z^p - w^q) \subset \mathbb{C}^2$ be a cusp, where $p > q$ are relatively prime. Then the Briançon-Skoda number* $\mathrm{bs}(\mathcal{O}_{Z,x})$ *is* $\lceil \frac{(p-1)(q-1)}{q} \rceil + 1$.

Proof. Let $\phi : \mathbb{C} \to Z$ be the parametrization $z = t^q$, $w = t^p$ of the curve. We get an induced mapping (by pull-back) $\mathcal{O}_Z \to \mathbb{C}\{t\}$, whose image can be identified with the strongly holomorphic functions on Z. These functions are precisely those whose power series only contain exponents that belong to the semigroup $M \subset \mathbb{Z}$ generated by 1, p and q. The lemma above gives that M contains all numbers $\lambda \geq (p-1)(q-1)$. Assume that $\mathfrak{a} = (a_1, \dots, a_m)$. The condition $|\phi| \leq C|\mathfrak{a}|^{k+l-1}$ can be checked after pull-back. Since $|\pi^* a_i| \sim |t|_i^k$, we have $|\mathfrak{a}| \sim |a_i|$ for some $1 \leq i \leq m$. We can therefore assume that $\mathfrak{a} = (a)$ is principal. If t^n is the highest power of t that divides a and $|\phi| \leq C|a|^{k+l-1}$, then $t^{n(k+l-1)}$ divides ϕ. The weakly holomorphic function $u = \phi/a^l$ is divisible by $t^{n(k-1)}$. To see that u is in fact strongly holmorphic, we only need that $n(k-1) \geq (p-1)(q-1)$. This yields that $k \geq \frac{(p-1)(q-1)}{n} + 1$ is an upper bound for $\mathrm{bs}(\mathcal{O}_{Z,x})$. The worst case scenario is $n = q$, because this is the smallest element of M. By Example 0.1 in the introduction, this is also a lower bound for $\mathrm{bs}(\mathcal{O}_{Z,x})$. $\square$

We will now come to the same conclusion using the technique of the second paper. Hopefully, this will also shed some light on the general proof.

Assume that Z is the (p, q)-cusp above and that $\phi \in \mathcal{O}_{Z,x}$ satisfies $|\phi| \leq C|\mathfrak{a}|^{N+l-1}$ on Z, where N is the number in Theorem A.2, and $\mathfrak{a}$ is any ideal in $\mathcal{O}_{Z,x}$. Then we want to show that $\phi \in \mathfrak{a}$. As above we can assume that $\mathfrak{a} = (a)$ is principal. By the second paper it is sufficient that ϕ annihilates a certain product of residue currents, viz. $\phi R^{a^l} \wedge R^Z = 0$, or equivalently $\phi R^{\mathfrak{a}^l} \wedge R^Z \wedge dz \wedge dw = 0$. In the case we are considering, we have $R^{a^l} = \bar{\partial}\frac{1}{a^l}$ and $R^Z = \bar{\partial}\frac{1}{f}$, where $f = z^p - w^q$.

Thus our goal is to see that

$$\phi\bar\partial\frac{1}{a^l}\wedge\bar\partial\frac{1}{f}\wedge dz\wedge dw=0. \tag{1}$$

In view of Proposition 3.1 of the second paper, we need to find a generically smooth vector field γ such that $R^Z=\gamma\neg[Z]$. As $\frac{1}{2\pi i}\bar\partial\frac{1}{f}\wedge df=[Z]$, any γ, for which $\gamma\neg df=1$, will do. We can take for example

$$\gamma=\frac{K}{z^{p-1}}\frac{\partial}{\partial z},$$

where $K=2\pi i/p$. This works since $df=pz^{p-1}dz+qw^{q-1}dw$. Let now

$$\omega=\gamma\neg(dz\wedge dw)=K\frac{dw}{z^{p-1}}.$$

Then the proposition says that

$$\begin{aligned}R^Z.(dz\wedge dw\wedge\xi)&=\int\gamma\neg[Z]\wedge dz\wedge dw\wedge\xi\\&=\int[Z]\wedge\gamma\neg(dz\wedge dw)\wedge\xi=\int_Z\omega\wedge\xi.\end{aligned}$$

The pull-back of ω with the mapping π of Theorem A.2 is $\pi^*\omega=Kt^{-(p-1)(q-1)}dt$. Hence we get

$$R^Z.(dz\wedge dw\wedge\xi)=K\int_{\mathbb{C}}\frac{dt\wedge\pi^*\xi}{t^{(p-1)(q-1)}}. \tag{2}$$

We let

$$\eta_\delta=\phi\bar\partial\chi\left(|a|^2/\delta^2\right)\frac{1}{a^l}\wedge\xi,$$

and denote pull-backs in the sequel by ˜. Replacing ξ with η_δ in (2), we get

$$R^Z.(dz\wedge dw\wedge\eta_\delta)=K\int_{\mathbb{C}}\frac{dt\wedge\tilde\phi\bar\partial\chi\left(|\tilde a|^2/\delta^2\right)\wedge\tilde\xi}{\tilde a^l t^{(p-1)(q-1)}}.$$

In view of (1), we should now let δ tend to zero. We know that in the worst case we have $|\tilde a|=C|t^q|$, so $|\tilde\phi/\tilde a^l|\le C|t|^{q(N-1)}\le C|t|^{(p-1)(q-1)}$. This gives immediately that $\tilde\phi/\tilde a^l$ is divisible by $t^{(p-1)(q-1)}$, which cancels out the denominator above. We have then reduced (1) to

$$\int_{\mathbb{C}}\bar\partial\chi\left(|\tilde a|^2/\delta^2\right)\alpha=\int_{\mathbb{C}}\chi\left(|\tilde a|^2/\delta^2\right)\bar\partial\alpha\to\int_{\mathbb{C}}\bar\partial\alpha=0,$$

where α is a test form. The same limit follows also from dominated convergence.