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Modules and Hopf structures for (twisted) generalized Weyl algebras

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MODULES AND HOPF STRUCTURES FOR (TWISTED) GENERALIZED WEYL ALGEBRAS

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Abstract

This work consists of two papers on (twisted) generalized Weyl algebras.

In the first paper we present methods and explicit formulas for describing simple weight modules over twisted generalized Weyl algebras. When a certain commutative subalgebra is finitely generated over an algebraically closed field we obtain a classification of a class of locally finite simple weight modules as those induced from simple modules over a subalgebra isomorphic to a tensor product of noncommutative tori. As an application we describe simple weight modules over the quantized Weyl algebra.

In the second paper we derive necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain type. This construction generalizes many known Hopf algebras, for example $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$ and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. In a torsion-free case we describe the finite-dimensional simple modules, in particular their dimensions and prove a Clebsch-Gordan decomposition theorem for the tensor product of two simple modules. We construct a Casimir type operator and prove that any finite-dimensional weight module is semisimple.

Keywords: Generalized Weyl algebra, weight module, quantum Weyl algebra, ambiskew polynomial ring, Hopf algebra, quantum group

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Papers included in this thesis

Paper I. *Locally finite simple weight modules over twisted generalized Weyl algebras*, (submitted)

Paper II. *Hopf structures on ambiskew polynomial rings*, (submitted)

Papers not included in this thesis

Paper III. *Deformations of Lie algebras using σ-derivations*, (with D. Larsson and S.D. Silvestrov, to appear in Journal of Algebra)

1. Introduction

This thesis is about certain algebras and certain modules over them. Let A be a $\mathbb{C}$-algebra. A left A-module M is called *simple* if it has only itself and $\{0\}$ as submodules. The reason for calling them simple is that they are the building blocks of more complicated modules. However even the simple modules can be very complicated themselves.

Recall that a $\mathbb{C}$-algebra A is commutative if the multiplication satisfies $ab = ba$ for any $a, b \in A$. If A is a commutative $\mathbb{C}$-algebra, then one can show that any simple module over A which is finite-dimensional as a vector space must in fact be one-dimensional. If A is not commutative then, in general, the complexity of simple modules has no limits in some sense.

Therefore, to study non-commutative algebras and their modules, it is natural to restrict attention to some class of algebras which on the one hand is sufficiently big to include both important and already understood examples as well as new ones, while on the other hand is small enough to allow a detailed description or at least provides a setting for an interesting theory to be developed.

To motivate the class of algebras treated in this thesis we first consider a simpler structure. Assume that R is a commutative $\mathbb{C}$-algebra. To add some non-commutativity, we introduce a new abstract symbol X, and consider all expressions obtained from X and the elements of R using addition and multiplication. To get something more interesting we require in addition that for each $r \in R$ there is a unique $r' \in R$ such that

$$Xr = r'X.$$

It is an exercise to verify that the map $\sigma : R \to R$, $r \mapsto r'$ is a ring homomorphism. The resulting $\mathbb{C}$-algebra is called a skew polynomial ring and is denoted by $R[X;\sigma]$. The name comes from the fact that $R[X;\sigma]$ can be viewed as the set of all polynomials

$$\sum_{n=0}^{N} r_n X^n \tag{1.1}$$

with ordinary addition but multiplication is skew in the sense that

$$Xr = \sigma(r)X. \tag{1.2}$$

This means that when two sums of the form (1.1) are to be multiplied we must use (1.2) several times to get a sum of the form (1.1) again. If $\sigma(r) = r$ for all $r \in R$ then $R[X;\sigma]$ is the ordinary commutative ring of polynomials in one indeterminate X with coefficients in R.

2. Generalized Weyl algebras

We now define the main object of the thesis – generalized Weyl algebras.

Definition 2.1. Let R be a commutative $\mathbb{C}$-algebra, let σ be a automorphism of R and let $t \in R$. The *generalized Weyl algebra (GWA)* $R(\sigma, t)$ is defined as the ring extension of R by two generators X and Y modulo the following relations

$$YX = t, \quad XY = \sigma(t), \quad Xr = \sigma(r)X, \quad rY = Y\sigma(r) \tag{2.1}$$

for $r \in R$.

Example. Let

$$A = \mathbb{C}\langle x, \partial : \partial x - x\partial = 1\rangle.$$

Then A is isomorphic to the GWA $\mathbb{C}[t](\sigma, t)$ where $\sigma(t) = t - 1$ via the map $x \mapsto X$, $\partial \mapsto Y$.

The class of GWAs also includes the enveloping and quantum enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. The definition of GWA first appeared in Bavula [1]. Besides structural properties of GWAs, the author studied simple modules over a special type of GWA. More precisely, he gave a classification of all finite-dimensional simple modules up to indecomposable elements of a Euclidean ring. He also described necessary and sufficient conditions for the category of its finite-dimensional modules to be semisimple, i.e. for any finite-dimensional module to be a direct sum of simple ones.

In general however, the classification of finite-dimensional simple modules over a GWA is complicated. Moreover, most of the simple modules over a GWA are infinite-dimensional and there exist simple modules which depend on arbitrary many number of parameters. To obtain results, one needs to restrict the class of modules. One such restriction is to study so called weight modules.

Let V be a module over a GWA. A vector $v \in V$ is called a weight vector if it is annihilated by some maximal ideal of R. As an example, suppose $R = \mathbb{C}[x]$. Then any maximal ideal is of the form $(x - \alpha)$ where $\alpha \in \mathbb{C}$. Thus $v \in V$ is a weight vector iff $xv = \alpha v$ for some α, i.e. iff v is an eigenvector of (the linear operator represented by) x. A module V over a GWA is called a *weight module* if

$$V = \oplus_{\mathfrak{m} \in \mathrm{Max}(R)} V_{\mathfrak{m}}, \quad \text{where } V_{\mathfrak{m}} = \{v \in V : \mathfrak{m}v = 0\}.$$

Here $\mathrm{Max}(R)$ denote the set of all maximal ideals of R. Intuitively this means that V has a basis in which each $r \in R$ is represented by a diagonal matrix. Weight modules over GWAs have nice properties. For example

$$XV_{\mathfrak{m}} \subseteq V_{\sigma(\mathfrak{m})}, \quad \text{and} \quad YV_{\mathfrak{m}} \subseteq V_{\sigma^{-1}(\mathfrak{m})}.$$

Thus one can partly visualize such a module as a lattice of weights $\mathfrak{m}$ together with the action of X and Y on the weight components $V_{\mathfrak{m}}$. Also note that, if R is finitely generated, then any finite-dimensional simple module over a GWA is a weight module. In this sense the restriction to weight modules is not too restrictive.

In [5], Drozd, Guzner and Ovsienko described all simple and even indecomposable (i.e. those which are not direct sums of proper submodules) weight modules over a general GWA.

Higher rank analogues of GWAs were introduced in [3]. They are defined using $2n$ symbols X_i, Y_i, $i = 1, \ldots, n$ where each pair X_i, Y_i satisfies relations of the type (2.1) and in addition

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Y_j] = 0 \quad \text{for } i \neq j.$$

Indecomposable modules over such algebras were studied in [3]. They showed that, contrary to the results of [5], in most cases the description of all indecomposable weight modules over a higher rank GWA is very complicated (a so called wild problem).

3. Twisted generalized Weyl algebras

There are still some examples, like higher rank quantum Weyl algebras, which behave similarly as the higher rank GWAs but can not be realized as such in a natural way. To be able to include these examples, Mazorchuk and Turowska [10] introduced a twisted higher rank analogue of GWAs. Twisted means that more non-commutativity is allowed. More precisely the generators X_i, Y_j $(i \neq j)$ are allowed to not commute. The exact definition of twisted GWA is given in Paper I. The introduced class contains besides higher rank GWA also examples which are closely related to finite-dimensional simple Lie algebras (see [9]). In [10], [9] the authors began to study simple weight modules over twisted GWAs.

In Paper I we continue this study, describing a more general class of simple modules. The results can be applied to many examples and to illustrate how we give a classification of simple weight modules over the quantum Weyl algebra.

4. Hopf algebras and generalized Weyl algebras

Let A be a $\mathbb{C}$-algebra. The multiplication in A is a bilinear map from $A \times A$ to A. Hence it can be viewed as a linear map from $A \otimes A$ (by $\otimes$ we will mean $\otimes_{\mathbb{C}}$) to A. Dually, one defines a *coproduct* Δ on a linear space V to be a linear map $V \to V \otimes V$. This coproduct is called *coassociative* if

$$(\Delta \otimes \mathrm{Id})(\Delta(v)) = (\mathrm{Id} \otimes \Delta)(\Delta(v)), \quad \text{for any } v \in V.$$

A *Hopf algebra* is an algebra together with an associative coproduct Δ. The product and coproduct should be compatible in the sense that $\Delta : A \to A \otimes A$ is a homomorphism when $A \otimes A$ is given the algebra structure satisfying $(a \otimes b)(c \otimes d) = ac \otimes bd$ for $a, b, c, d \in A$. A Hopf algebra should also have a counit and an antipode (see Paper II for definition). The following example shows one way that Hopf algebras appear "in nature".

Example. Let G be a group and let $H = \mathcal{F}(G, \mathbb{C})$ be the set of all complex-valued functions on G. It is a commutative algebra under pointwise operations. We define

$$\Delta' : H \to \mathcal{F}(G \times G, \mathbb{C})$$

by

$$\Delta'(\varphi)(g, h) = \varphi(gh) \quad \text{for } \varphi \in H, g, h \in G.$$

Composing Δ' with the natural isomorphism $\mathcal{F}(G \times G, \mathbb{C}) \simeq H \otimes H$ defines a coproduct Δ on H. Using that G is associative one can check that Δ is coassociative. Similarly one can define a counit and antipode and verify that H is a Hopf algebra.

In the second paper we consider Hopf algebra structures on a subclass of GWAs, the so-called ambiskew polynomial rings. This subclass was studied by Jordan in [6], [7] and includes the down-up algebras defined by Benkart and Roby in [4]. Necessary conditions for a down-up algebra to be a Hopf algebra were given in [8].

One important property of Hopf algebras is that the coproduct allows one to define a module structure on the tensor product space $V \otimes W$ of two modules V and W. We give a formula for decomposing the tensor product of two simple modules into a direct sum of simple modules, generalizing the classical and quantum Clebsch-Gordan formula. We also prove that any weight module is semisimple.

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Paper I

LOCALLY FINITE SIMPLE WEIGHT MODULES OVER TWISTED GENERALIZED WEYL ALGEBRAS

JONAS T. HARTWIG

Abstract. We present methods and explicit formulas for describing simple weight modules over twisted generalized Weyl algebras. When a certain commutative subalgebra is finitely generated over an algebraically closed field we obtain a classification of a class of locally finite simple weight modules as those induced from simple modules over a subalgebra isomorphic to a tensor product of noncommutative tori. As an application we describe simple weight modules over the quantized Weyl algebra.

1. Introduction

Bavula defined in [1], [2] the notion of a generalized Weyl algebra (GWA) which is a class of algebras which include $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$, down-up algebras, and the Weyl algebra, as examples. In addition to various ring theoretic properties, the simple modules were also described for some GWAs in [2]. In [6] all simple and indecomposable weight modules of GWAs of rank (or degree) one were classified.

So called higher rank GWAs were defined in [2] and in [3] the authors studied indecomposable weight modules over certain higher rank GWAs.

In [8], with the goal to enrich the representation theory in the higher rank case, the authors defined the twisted generalized Weyl algebras (TGWA). This is a class of algebras which include all higher rank GWAs (if a certain subring R has no zero divisors) and also many algebras which can be viewed as twisted tensor products of rank one GWAs, for example certain Mickelsson step algebras and extended Orthogonal Gelfand-Zetlin algebras [7]. Under a technical assumption on the algebra formulated using a biserial graph, some torsion-free simple weight modules were described in [8]. Simple graded weight modules were studied in [7] using an analogue of the Shapovalov form.

In this paper we describe a more general class of locally finite simple weight modules over TGWAs using the well-known technique of considering the maximal graded subalgebra which preserves the weight spaces. It is known that under quite general assumptions (see Theorem 18 in [5]) any simple weight module over a TGWA is a unique quotient of a module which is induced from a simple module over this subalgebra. Our main results are the description of this subalgebra under various assumptions (Theorem 4.5 and Theorem 4.8) and the explicit formulas (Theorem 5.4) of the associated module of the TGWA. In contrast to [8], we do not assume that the orbits are torsion-free and we allow the modules to have some inner breaks, as long as they do not have any so called *proper* inner breaks (see Definition 3.7). The weight spaces will not in general be one-dimensional in our case, which was the case in [8], [7].

Moreover, as an application we classify the simple weight modules without proper inner breaks over a quantized Weyl algebra of rank two (Theorem 6.14).

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The paper is organized as follows. In Section 2 the definitions of twisted generalized Weyl constructions and algebras are given together with some examples. Weight modules and the subalgebra $B(\omega)$ are defined.

In Section 3 we first prove some simple facts and then define the class of simple weight modules with no proper inner breaks. We also show that this class properly contains all the modules studied in [8].

Section 4 is devoted to the description of the subalgebra $B(\omega)$. When the ground field is algebraically closed and a certain subalgebra R is finitely generated, we show that it is isomorphic to a tensor product of noncommutative tori for which the finite-dimensional irreducible representations are easy to describe.

In Section 5 we specify a basis and give explicit formulas for the irreducible quotient of the induced module.

Finally, in Section 6 we consider as an example the quantized Weyl algebra and determine certain important subsets of $\mathbb{Z}^n$ related to $B(\omega)$ and the support of modules as solutions to some systems of equations. In the rank two case we describe all simple weight modules with finite-dimensional weight spaces and no proper inner breaks.

2. Definitions

2.1. The TGWC and TGWA. Fix a positive integer n and set $\underline{n} = \{1, 2, \ldots, n\}$. Let K be a field, and let R be a commutative unital K-algebra, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be an n-tuple of pairwise commuting K-automorphisms of R, $\boldsymbol{\mu} = (\mu_{ij})_{i,j\in\underline{n}}$ be a matrix with entries from $K^* := K\backslash\{0\}$ and $\boldsymbol{t} = (t_1, \ldots, t_n)$ be an n-tuple of nonzero elements from R. The *twisted generalized Weyl construction* (TGWC) A' obtained from the data $(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ is the unital K-algebra generated over R by X_i, Y_i, $(i \in \underline{n})$ with the relations

$$X_i r = \sigma_i(r) X_i, \qquad Y_i r = \sigma_i^{-1}(r) Y_i, \qquad \text{for } r \in R, i \in \underline{n}, \tag{2.1}$$

$$Y_i X_i = t_i, \qquad X_i Y_i = \sigma_i(t_i), \qquad \text{for } i \in \underline{n}, \tag{2.2}$$

$$X_i Y_j = \mu_{ij} Y_j X_i, \qquad \text{for } i, j \in \underline{n}, i \neq j. \tag{2.3}$$

From the relations (2.1)–(2.3) follows that A' carries a $\mathbb{Z}^n$-gradation $\{A'_g\}_{g\in\mathbb{Z}^n}$ which is uniquely defined by requiring

$$\deg X_i = e_i, \quad \deg Y_i = -e_i, \quad \deg r = 0, \quad \text{for } i \in \underline{n}, r \in R,$$

where $e_i = (0, \ldots, \overset{i}{1}, \ldots, 0)$. The *twisted generalized Weyl algebra* (TGWA) $A = A(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ of *rank* n is defined to be A'/I, where I is the sum of all graded two-sided ideals of A' intersecting R trivially. Since I is graded, A inherits a $\mathbb{Z}^n$-gradation $\{A_g\}_{g\in\mathbb{Z}^n}$ from A'.

Note that from relations (2.1)–(2.3) follows the identity

$$X_i X_j t_i = X_j X_i \mu_{ji} \sigma_j^{-1}(t_i) \tag{2.4}$$

which holds for $i, j \in \underline{n}, i \neq j$. Multiplying (2.4) from the left by $\mu_{ij} Y_j$ we obtain

$$X_i\big(t_i t_j - \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)\big) = 0 \tag{2.5}$$

for $i, j \in \underline{n}, i \neq j$. One can show that the algebra A', hence A, is nontrivial if one assumes that $t_i t_j = \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)$ for $i, j \in \underline{n}, i \neq j$. Analogous identities exist for Y_i.

2.2. Examples. Some of the first motivating examples of a *generalized Weyl algebra* (GWA), i.e. a TGWC of rank 1, are $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$ and of course the Weyl algebra A_1. We refer to [2] for details.

We give some examples of TGWAs of higher rank.

2.2.1. *Quantized Weyl algebras.* Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ matrix with nonzero complex entries such that $\lambda_{ij} = \lambda_{ji}^{-1}$. Let $\bar{q} = (q_1, \dots, q_n)$ be an n-tuple of elements of $\mathbb{C}\backslash\{0,1\}$. The n:th quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ is the $\mathbb{C}$-algebra with generators x_i, y_i, $1 \le i \le n$, and relations

$$x_i x_j = q_i \lambda_{ij} x_j x_i, \qquad y_i y_j = \lambda_{ij} y_j y_i, \tag{2.6}$$

$$x_i y_j = \lambda_{ji} y_j x_i, \qquad x_j y_i = q_i \lambda_{ij} y_i x_j, \tag{2.7}$$

$$x_i y_i - q_i y_i x_i = 1 + \sum_{k=1}^{i-1} (q_k - 1) y_i x_i, \tag{2.8}$$

for $1 \le i < j \le n$. Let $R = \mathbb{C}[t_1, \dots, t_n]$ be the polynomial algebra in n variables and σ_i the $\mathbb{C}$-algebra automorphisms defined by

$$\sigma_i(t_j) = \begin{cases} t_j, & j < i, \\ 1 + q_i t_i + \sum_{k=1}^{i-1}(q_k - 1)t_k, & j = i, \\ q_i t_j, & j > i. \end{cases} \tag{2.9}$$

One can check that the σ_i commute. Let $\boldsymbol{\mu} = (\mu_{ij})_{i,j \in \underline{n}}$ be defined by $\mu_{ij} = \lambda_{ji}$ and $\mu_{ji} = q_i \lambda_{ij}$ for $i < j$. Let also $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\boldsymbol{t} = (t_1, \dots, t_n)$. One can show that the maximal graded ideal of the TGWC $A'(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ is generated by the elements

$$X_i X_j - q_i \lambda_{ij} X_j X_i, \; Y_i Y_j - \lambda_{ij} Y_j Y_i, \quad 1 \le i < j \le n.$$

Thus $A_n^{\bar{q},\Lambda}$ is isomorphic to the TGWA $A(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ via $x_i \mapsto X_i$, $y_i \mapsto Y_i$.

2.2.2. *Q_{ij}-CCR.* Let $(Q_{ij})_{i,j=1}^d$ be an $d \times d$ matrix with complex entries such that $Q_{ij} = Q_{ji}^{-1}$ if $i \ne j$ and A_d be the algebra generated by elements a_i, a_i^*, $1 \le i \le d$ and relations

$$a_i^* a_i - Q_{ii} a_i a_i^* = 1, \qquad a_i^* a_j = Q_{ij} a_j a_i^*,$$
$$a_i a_j = Q_{ji} a_j a_i, \qquad a_i^* a_j^* = Q_{ij} a_j^* a_i^*,$$

where $1 \le i, j \le d$ and $i \ne j$. Let $R = \mathbb{C}[t_1, \dots, t_d]$ and define the automorphisms σ_i of R by $\sigma_i(t_j) = t_j$ if $i \ne j$ and $\sigma_i(t_i) = 1 + Q_{ii} t_i$. Let $\mu_{ij} = Q_{ji}$ for all i, j. Then A_d is isomorphic to the TGWA $A(R, (\sigma_1, \dots, \sigma_n), (t_1, \dots, t_n), \boldsymbol{\mu})$.

2.2.3. *Mickelsson and OGZ algebras.* In both the above examples the generators X_i and X_j commute up to a multiple of the ground field. This need not be the case as shown in [7], where it was shown that Mickelsson step algebras and extended orthogonal Gelfand-Zetlin algebras are TGWAs.

2.3. **Weight modules.** Let A be a TGWC or a TGWA. Let $\mathrm{Max}(R)$ denote the set of all maximal ideals in R. A module M over A is called a *weight module* if

$$M = \oplus_{\mathfrak{m} \in \mathrm{Max}(R)} M_{\mathfrak{m}},$$

where

$$M_{\mathfrak{m}} = \{v \in M \mid \mathfrak{m} v = 0\}.$$

The *support*, $\mathrm{supp}(M)$, of M is the set of all $\mathfrak{m} \in \mathrm{Max}(R)$ such that $M_{\mathfrak{m}} \ne 0$. A weight module is *locally finite* if all the weight spaces $M_{\mathfrak{m}}$, $\mathfrak{m} \in \mathrm{supp}(M)$, are finite-dimensional over the ground field K.

Since the σ_i are pairwise commuting, the free abelian group $\mathbb{Z}^n$ acts on R as a group of K-algebra automorphisms by

$$g(r) = \sigma_1^{g_1} \sigma_2^{g_2} \dots \sigma_n^{g_n}(r) \tag{2.10}$$

for $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$ and $r \in R$. Then $\mathbb{Z}^n$ also acts naturally on $\mathrm{Max}(R)$ by $g(\mathfrak{m}) = \{g(r) \mid r \in \mathfrak{m}\}$. Note that

$$X_i M_{\mathfrak{m}} \subseteq M_{\sigma_i(\mathfrak{m})} \quad \text{and} \quad Y_i M_{\mathfrak{m}} \subseteq M_{\sigma_i^{-1}(\mathfrak{m})} \tag{2.11}$$

for any $\mathfrak{m} \in \mathrm{Max}(R)$. If $a \in A$ is homogeneous of degree $g \in \mathbb{Z}^n$, then by using (2.1) and (2.11) repeatedly one obtains the very useful identities

$$a \cdot r = g(r) \cdot a, \quad r \cdot a = a \cdot (-g)(r), \tag{2.12}$$

for $r \in R$ and

$$aM_{\mathfrak{m}} \subseteq M_{g(\mathfrak{m})} \tag{2.13}$$

for $\mathfrak{m} \in \mathrm{Max}(R)$.

2.4. **Subalgebras leaving the weight spaces invariant.** Let $\omega \subseteq \mathrm{Max}(R)$ be an orbit under the action of $\mathbb{Z}^n$ on $\mathrm{Max}(R)$ defined in (2.10). Let

$$\mathbb{Z}^n_\omega = \mathbb{Z}^n_{\mathfrak{m}} = \{g \in \mathbb{Z}^n \mid g(\mathfrak{m}) = \mathfrak{m}\} \tag{2.14}$$

where $\mathfrak{m}$ is some point in ω. Since $\mathbb{Z}^n$ is abelian, $\mathbb{Z}^n_\omega$ does not depend on the choice of $\mathfrak{m}$ from ω. Define

$$B(\omega) = \oplus_{g \in \mathbb{Z}^n_\omega} A_g. \tag{2.15}$$

Since A is $\mathbb{Z}^n$-graded and since $\mathbb{Z}^n_\omega$ is a subgroup of $\mathbb{Z}^n$, $B(\omega)$ is a subalgebra of A and $R = A_0 \subseteq B(\omega)$. Let $\mathfrak{m} \in \omega$ and suppose that M is a simple weight A-module with $\mathfrak{m} \in \mathrm{supp}(M)$. Since M is simple we have $\mathrm{supp}(M) \subseteq \omega$. Using (2.13) it follows that $B(\omega)M_{\mathfrak{m}} \subseteq M_{\mathfrak{m}}$ and by definition $M_{\mathfrak{m}}$ is annihilated by $\mathfrak{m}$ hence also by the two-sided ideal $(\mathfrak{m})$ in $B(\omega)$ generated by $\mathfrak{m}$. Thus $M_{\mathfrak{m}}$ is naturally a module over the algebra

$$B_{\mathfrak{m}} := B(\omega)/(\mathfrak{m}). \tag{2.16}$$

By Proposition 7.2 in [7] (see also Theorem 18 in [5] for a general result), $M_{\mathfrak{m}}$ is a simple $B_{\mathfrak{m}}$-module, and any simple $B_{\mathfrak{m}}$-module occurs as a weight space in a simple weight A-module. Moreover, two simple weight A-modules M, N are isomorphic if and only if $M_{\mathfrak{m}}$ and $N_{\mathfrak{m}}$ are isomorphic as $B_{\mathfrak{m}}$-modules. Therefore we are led to study the algebra $B_{\mathfrak{m}}$ and simple modules over it.

3. Preliminaries

3.1. **Reduced words.** Let $L = \{X_i\}_{i \in \underline{n}} \cup \{Y_i\}_{i \in \underline{n}}$. By a *word* $(a; Z_1, \dots, Z_k)$ in A we will mean an element a in A which is a product of elements from the set L, together with a fixed tuple $(Z_1, \dots, Z_k)$ of elements from L such that $a = Z_1 \cdot \ldots \cdot Z_k$. When referring to a word we will often write $a = Z_1 \dots Z_k \in A$ to denote the word $(a; Z_1, \dots, Z_k)$ or just write $a \in A$, suppressing the fixed representation of a as a product of elements from L.

Set $X_i^* = Y_i$ and $Y_i^* = X_i$. For a word $a = Z_1 \dots Z_k \in A$ we define

$$a^* := Z_k^* \cdot Z_{k-1}^* \cdot \ldots \cdot Z_1^*.$$

In the special case when $\mu_{ij} = \mu_{ji}$ for all i, j then by (2.1)-(2.3) there is an anti-involution $*$ on A' defined by $X_i^* = Y_i$, and $r^* = r$ for $r \in R$. Since $I^* = I$ this anti-involution carries over to A.

Definition 3.1. A word $Z_1 \dots Z_k$ will be called *reduced* if

$$Z_i \neq Z_j^* \text{ for } i, j \in \underline{k}$$

and

$$Z_i \in \{X_r\}_{r \in \underline{n}} \Longrightarrow Z_j \in \{X_r\}_{r \in \underline{n}} \; \forall j \geq i.$$

For example $Y_1Y_2Y_1X_3$ is reduced whereas $Y_1Y_2X_1$ and $Y_1X_2Y_3$ are not. The following lemma and corollary explains the importance of the reduced words.

Lemma 3.2. *Any word b in A can be written $b = a \cdot r = r' \cdot a$, where a is a reduced word, and $r, r' \in R$.*

Proof. If a and r has been found we can take $r' = (\deg a)(r)$, according to (2.12). Thus we concentrate on finding a and r. Let $b = Z_1 \dots Z_k$ be an arbitrary word in A. We prove the statement by induction on k. If $k = 1$, then b is necessarily reduced so take $a = b, r = 1$. When $k > 1$, use the induction hypothesis to write

$$Z_1 \dots Z_{k-1} = Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} \cdot r',$$

where $1 \le i_u, j_v \le n$ and $i_u \ne j_v$ for any u, v. Consider first the case when $Z_k = Y_j$ for some $j \in \underline{n}$. Then

$$Z_1 \dots Z_k = Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} Y_j \cdot \sigma_j(r').$$

If $j_v \ne j$ for $v = 1, \dots, m$ we are done because using relation (2.3) repeatedly we obtain,

$$Z_1 \dots Z_k = Y_{i_1} \dots Y_{i_l} Y_j X_{j_1} \dots X_{j_m} \cdot \mu\sigma_j(r')$$

for some $\mu \in K^*$. Otherwise, let $v \in \{1, \dots, m\}$ be maximal such that $j_v = j$. Then

$$\begin{aligned} Z_1 \dots Z_k &= Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_v} Y_j X_{j_{v+1}} \dots X_{j_m} \mu\sigma_j(r') = \\ &= Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_{v-1}} X_{j_{v+1}} \dots X_{j_m} w(t_j)\mu\sigma_j(r') \end{aligned}$$

for some $\mu \in K^*$ and some $w \in W$. It remains to consider the case $Z_k = X_j$ for some $j \in \underline{n}$. But using that

$$Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} = X_{j_1} \dots X_{j_m} Y_{i_1} \dots Y_{i_l} \mu$$

for some $\mu \in K^*$, it is clear that this case is analogous. □

Corollary 3.3. *Each A_g, $g \in W$, is generated as a right (and also as a left) R-module by the reduced words of degree g.*

Lemma 3.4. *Suppose $*$ defines an anti-involution on A. Let $\mathfrak{p}$ be a prime ideal of R. Let $g \in \mathbb{Z}^n$ and let $a \in A_g$. If $ba \notin \mathfrak{p}$ for some $b \in A_{-g}$ then $a^*a \notin \mathfrak{p}$.*

Proof. Since $\mathfrak{p}$ is prime, and $ba \in R$ we have

$$\mathfrak{p} \not\ni (ba)^2 = (ba)^*ba = a^*b^*ba = a^*a \cdot (-\deg a)(b^*b)$$

so in particular $a^*a \notin \mathfrak{p}$. □

Remark 3.5. If we assume a and b to be words in the formulation of Lemma 3.4, one can easily show that the statement remains true without the restriction on $*$ to be an anti-involution.

3.2. **Inner breaks and canonical modules.** Let A be a TGWC or a TGWA and let M be a simple weight module over A. In [8] Remark 1 it was noted that the problem of describing simple weight modules over a TGWC is wild in general. This is a motivation for restricting attention to some subclass which has nice properties. In [8] the following definition was made.

Definition 3.6. The support of M has *no inner breaks* if for all $\mathfrak{m} \in \operatorname{supp}(M)$,

$$t_i \in \mathfrak{m} \Longrightarrow \sigma_i(\mathfrak{m}) \notin \operatorname{supp}(M), \text{ and}$$
$$\sigma_i(t_i) \in \mathfrak{m} \Longrightarrow \sigma_i^{-1}(\mathfrak{m}) \notin \operatorname{supp}(M).$$

We introduce the following property.

Definition 3.7. We say that M has *no proper inner breaks* if for any $\mathfrak{m} \in \operatorname{supp}(M)$ and any word a with $aM_{\mathfrak{m}} \neq 0$ we have $a^*a \notin \mathfrak{m}$.

Observe that whether or not $a^*a \in \mathfrak{m}$ for a word a does not depend on the particular representation of a as a product of generators. Note also that to prove that a simple weight module M has no proper inner breaks, it is sufficient to find for any $\mathfrak{m} \in \operatorname{supp}(M)$ and any word a with $aM_{\mathfrak{m}} \neq 0$ a word $b \in A$ of degree $-\deg a$ such that $ba \notin \mathfrak{m}$ because then $a^*a \notin \mathfrak{m}$ automatically by Remark 3.5. In fact one can show that a simple weight module M has no proper inner breaks if (and only if) there exists an $\mathfrak{m} \in \operatorname{supp}(M)$ such that for any reduced word $a \in A$ with $aM_{\mathfrak{m}} \neq 0$ and $aM_{\mathfrak{m}} \subseteq M_{\mathfrak{m}}$ there is a word b of degree $-\deg a$ such that $ba \notin \mathfrak{m}$. However we will not use this result.

The choice of terminology in Definition 3.7 is motivated by the following proposition.

Proposition 3.8. *If M has no inner breaks, then M has no proper inner breaks either.*

Proof. Let $\mathfrak{m} \in \operatorname{supp}(M)$ and $a = Z_1 \dots Z_k \in A$ be a word such that $aM_{\mathfrak{m}} \neq 0$. Thus $Z_i \dots Z_k M_{\mathfrak{m}} \neq 0$ for $i = 1, \dots, k+1$ so (2.13) implies that

$$(\deg Z_i \dots Z_k)(\mathfrak{m}) \in \operatorname{supp}(M).$$

If M has no inner breaks, it follows that $Z_i^* Z_i \notin (\deg Z_{i+1} \dots Z_k)(\mathfrak{m})$ for $i = 1, \dots, k$. Now using (2.12),

$$a^*a = Z_k^* \dots Z_1^* Z_1 \dots Z_k = Z_k^* \dots Z_2^* Z_2 \dots Z_k (-\deg Z_2 \dots Z_k)(Z_1^* Z_1) =$$

$$= \dots = \prod_{i=1}^{k} (-\deg Z_{i+1} \dots Z_k)(Z_i^* Z_i) \notin \mathfrak{m}. \tag{3.1}$$

Thus M has no proper inner breaks. □

In [8], a simple weight module M was defined to be *canonical* if for any $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ there is an automorphism σ of R of the form

$$\sigma = \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k}, \quad \varepsilon_j = \pm 1 \text{ and } 1 \leq i_j \leq n, \text{ for } j = 1, \dots, k,$$

such that $\sigma(\mathfrak{m}) = \mathfrak{n}$ and such that for each $j = 1, \dots, k$,

$$t_{i_j} \notin \sigma_{i_{j+1}}^{\varepsilon_{j+1}} \dots \sigma_{i_k}^{\varepsilon_k}(\mathfrak{m}) \quad \text{if } \varepsilon_j = 1, \text{ and} \tag{3.2}$$

$$\sigma_{i_j}(t_{i_j}) \notin \sigma_{i_{j+1}}^{\varepsilon_{j+1}} \dots \sigma_{i_k}^{\varepsilon_k}(\mathfrak{m}) \quad \text{if } \varepsilon_j = -1. \tag{3.3}$$

This definition can be reformulated as follows.

Proposition 3.9. *M is canonical iff for any $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ there is a word $a \in A$ such that $aM_{\mathfrak{m}} \subseteq M_{\mathfrak{n}}$ and $a^*a \notin \mathfrak{m}$.*

Proof. Suppose M is canonical, and let $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$. Let σ be as in the definition of canonical module. Define $a = Z_1 \dots Z_k$ where $Z_j = X_{i_j}$ if $\varepsilon_j = 1$ and $Z_j = Y_{i_j}$ otherwise. Using (2.13) we see that $aM_{\mathfrak{m}} \subseteq M_{\mathfrak{n}}$. Also, (3.2) and (3.3) translates into

$$Z_j^* Z_j \notin (\deg Z_{j+1} \dots Z_k)(\mathfrak{m})$$

for $j = 1, \dots, k$. Using the calculation (3.1) and that $\mathfrak{m}$ is prime we deduce that $a^*a \notin \mathfrak{m}$.

Conversely, given a word $a = Z_1 \dots Z_k \in A$ with $aM_{\mathfrak{m}} \subseteq M_{\mathfrak{n}}$ and $a^*a \notin \mathfrak{m}$, we define $\varepsilon_i = 1$ if $Z_i = X_i$ and $\varepsilon_i = -1$ otherwise. Then from $a^*a \notin \mathfrak{m}$ follows that $\sigma := \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k}$ satisfies (3.2) and (3.3) by the same reasoning as above. □

Corollary 3.10. *If M has no proper inner breaks, then M is canonical.*

Proof. We only need to note that since M is a simple weight module there is for each $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ a word a such that $0 \neq aM_{\mathfrak{m}} \subseteq M_{\mathfrak{n}}$. □

Under the assumptions in [8] any canonical module has no inner breaks (see [8], Proposition 1). However we have the following example of a TGWA A and a simple weight module M over A which has no proper inner breaks, and thus is canonical by Corollary 3.10, but nonetheless has an inner break.

Example 3.11. Let $R = \mathbb{C}[t_1, t_2]$ and define the $\mathbb{C}$-algebra automorphisms σ_1 and σ_2 of R by $\sigma_i(t_j) = -t_j$ for $i, j = 1, 2$. Let $\boldsymbol{\mu} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Let $A' = A'(R, \boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{\mu})$ be the associated TGWC, where $\boldsymbol{t} = (t_1, t_2), \boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. Then one can check that $I = \langle X_1X_2 + X_2X_1, Y_1Y_2 + Y_2Y_1 \rangle$. Let M be a vector space over $\mathbb{C}$ with basis $\{v, w\}$ and define an A'-module structure on M by letting $X_1M = Y_1M = 0$ and

$$\begin{aligned} X_2v &= w, & X_2w &= v, \\ Y_2v &= w, & Y_2w &= -v. \end{aligned}$$

It is easy to check that the required relations are satisfied and that $IM = 0$, hence M becomes an A-module. Let $\mathfrak{m} = (t_1, t_2 + 1)$ and $\mathfrak{n} = (t_1, t_2 - 1)$. Then

$$M = M_{\mathfrak{m}} \oplus M_{\mathfrak{n}}, \quad \text{where } M_{\mathfrak{m}} = \mathbb{C}v, M_{\mathfrak{n}} = \mathbb{C}w$$

so M is a weight module. Any proper nonzero submodule of M would also be a weight module by standard results. That no such submodule can exist is easy to check, so M is simple. One checks that M has no proper inner breaks. But $t_1 \in \mathfrak{m}$ and $\sigma_1(\mathfrak{m}) = \mathfrak{n} \in \operatorname{supp}(M)$ so $\mathfrak{m}$ is an inner break.

4. The weight space preserving subalgebra and its irreducible representations

In this section, let A be a TGWC, $\mathfrak{m} \in \operatorname{Max}(R)$ and let ω be the $\mathbb{Z}^n$-orbit of $\mathfrak{m}$. Recall the set $\mathbb{Z}^n_\omega$ defined in (2.14). Define the following subsets of $\mathbb{Z}^n$:

$$\tilde{G}_{\mathfrak{m}} = \{g \in \mathbb{Z}^n \mid a^*a \notin \mathfrak{m} \text{ for some word } a \in A_g\} \quad \text{and} \quad G_{\mathfrak{m}} = \tilde{G}_{\mathfrak{m}} \cap \mathbb{Z}^n_\omega. \tag{4.1}$$

Let also $\varphi_{\mathfrak{m}} : A \to A/(\mathfrak{m})$ denote the canonical projection, where $(\mathfrak{m})$ is the two-sided ideal in A generated by $\mathfrak{m}$, and let $R_{\mathfrak{m}} = R/\mathfrak{m}$ be the residue field of R at $\mathfrak{m}$.

Lemma 4.1. *Let $g \in \tilde{G}_{\mathfrak{m}}$. Then*

$$\varphi_{\mathfrak{m}}(A_g) = R_{\mathfrak{m}} \cdot \varphi_{\mathfrak{m}}(a) = \varphi_{\mathfrak{m}}(a) \cdot R_{\mathfrak{m}} \tag{4.2}$$

for any word $a \in A_g$ with $a^*a \notin \mathfrak{m}$.

Proof. Let $b \in A_g$ be any element and $a \in A_g$ a word such that $a^*a \notin \mathfrak{m}$, We must show that there is an $r \in R$ such that $\varphi_{\mathfrak{m}}(b) = \varphi_{\mathfrak{m}}(r)\varphi_{\mathfrak{m}}(a)$. Since $a^*a \notin \mathfrak{m}$ and $\mathfrak{m}$ is maximal, $1 - r_1a^*a \in \mathfrak{m}$ for some $r_1 \in R$. Set $r = br_1a^*$. Then $r \in R$ and

$$b - ra = b(1 - r_1a^*a) \in (\mathfrak{m}).$$

The last equality in (4.2) is immediate using (2.12). □

The following result was proved in [8] Lemma 8 for simple weight modules with so called regular support which in particular means that they have no inner breaks. It is still true in the more general situation when M has no proper inner breaks. Recall the ideal I from the definition of a $TGWA$.

Proposition 4.2. *Suppose A is a TGWC. If M is a simple weight A-module with no proper inner breaks, then $IM = 0$. Hence M is naturally a module over the associated TGWA A/I.*

Proof. Since I is graded and M is a weight modules, it is enough to show that $(I\cap A_g)M_{\mathfrak{m}}=0$ for any $g\in\mathbb{Z}^n$ and any $\mathfrak{m}\in\operatorname{supp}(M)$. Assume that $a\in I\cap A_g$ and $av\neq 0$ for some $v\in M_{\mathfrak{m}}$. Then $a_1v\neq 0$ for some word a_1 in a. Since M has no proper inner breaks, $a_1^*a_1\notin\mathfrak{m}$ so by Lemma 4.1 there is an $r\in R$ such that $av=a_1rv$. Thus $0\neq a_1^*a_1rv=a_1^*av$ which implies that $a_1^*a\in R\backslash\{0\}$. This contradicts that $a\in I$. □

We fix now for each $g\in\tilde{G}_{\mathfrak{m}}$ a word $a_g\in A_g$ such that $a_g^*a_g\notin\mathfrak{m}$. For $g=0$ we choose $a_g=1$.

Lemma 4.3. *For any $g\in\tilde{G}_{\mathfrak{m}}, h\in G_{\mathfrak{m}}$ we have*
a) $(a_ga_h^*)^*a_ga_h^*\notin\mathfrak{m}$ so in particular $g-h\in\tilde{G}_{\mathfrak{m}}$ and $G_{\mathfrak{m}}$ is a subgroup of $\mathbb{Z}^n_\omega$,
b) $\varphi_{\mathfrak{m}}(A_g)\varphi_{\mathfrak{m}}(A_h)=\varphi_{\mathfrak{m}}(A_gA_h)=\varphi_{\mathfrak{m}}(A_{g+h})$,
c) $A_{g+h}M_{\mathfrak{m}}=A_gM_{\mathfrak{m}}$.

Proof. a) We have

$$(a_ga_h^*)^*a_ga_h^*=a_ha_g^*a_ga_h^*=a_ha_h^*h(a_g^*a_g). \tag{4.3}$$

Now $a_g^*a_g\notin\mathfrak{m}$ so $h(a_g^*a_g)\notin h(\mathfrak{m})=\mathfrak{m}$. And

$$\mathfrak{m}\not\ni(a_h^*a_h)^2=a_h^*(a_ha_h^*)a_h=a_h^*a_h\cdot(-h)(a_ha_h^*)$$

so $a_ha_h^*\notin h(\mathfrak{m})=\mathfrak{m}$. Since $\mathfrak{m}$ is maximal the right hand side of (4.3) does not belong to $\mathfrak{m}$. Since $\deg(a_ga_h^*)=g-h$ we obtain $g-h\in\tilde{G}_{\mathfrak{m}}$. If in addition $g\in G_{\mathfrak{m}}$ then $g-h\in\mathbb{Z}^n_\omega$ also since $\mathbb{Z}^n_\omega$ is a group. Thus $g-h\in G_{\mathfrak{m}}$ so $G_{\mathfrak{m}}$ is a subgroup of $\mathbb{Z}^n_\omega$.

b) Since $\varphi_{\mathfrak{m}}$ is a homomorphism, the first equality holds. By part a), $-h\in G_{\mathfrak{m}}$ so by part a) again, $(a_ga_{-h}^*)^*a_ga_{-h}^*\notin\mathfrak{m}$. Hence by Lemma 4.1, we have

$$\varphi_{\mathfrak{m}}(A_{g+h})=R_{\mathfrak{m}}\cdot\varphi_{\mathfrak{m}}(a_ga_{-h}^*)\subseteq\varphi_{\mathfrak{m}}(A_gA_h).$$

The reverse inclusion holds since $\{A_g\}_{g\in\mathbb{Z}^n}$ is a gradation of A.

c) By part a), $g+h=g-(-h)\in\tilde{G}_{\mathfrak{m}}$. Thus by part b),

$$A_{g+h}M_{\mathfrak{m}}=\varphi_{\mathfrak{m}}(A_{g+h})M_{\mathfrak{m}}=\varphi_{\mathfrak{m}}(A_gA_h)M_{\mathfrak{m}}=A_gA_hM_{\mathfrak{m}}\subseteq A_gM_{h(\mathfrak{m})}=A_gM_{\mathfrak{m}}.$$

By part a), the same calculation holds if we replace g by $g+h$ and and h by $-h$, which gives the opposite inclusion. □

Lemma 4.4. *Let $g\in\mathbb{Z}^n\backslash\tilde{G}_{\mathfrak{m}}$. Then $A_gM_{\mathfrak{m}}=0$ for any simple weight module M over A with no proper inner breaks.*

Proof. Let $a\in A_g$ be any word. Then $a^*a\in\mathfrak{m}$ and hence if M is a simple weight module over A with no proper inner breaks, $aM_{\mathfrak{m}}=0$. Since the words generate A_g as a left R-module, it follows that $A_gM_{\mathfrak{m}}=0$. □

4.1. **General case.** Recall that $(\mathfrak{m})$ denotes the two-sided ideal in A generated by $\mathfrak{m}$. Since $(\mathfrak{m})$ is a graded ideal in A, there is an induced $\mathbb{Z}^n$-gradation of the quotient $A/(\mathfrak{m})$ and $\varphi_{\mathfrak{m}}(A_g)=(A/(\mathfrak{m}))_g$. Corresponding to the decomposition $\mathbb{Z}^n_\omega$ into the subset $G_{\mathfrak{m}}$ and its complement are two K-subspaces of the algebra $B_{\mathfrak{m}}=B(\omega)/\big(B(\omega)\cap(\mathfrak{m})\big)$ which will be denoted by $B_{\mathfrak{m}}^{(1)}$ and $B_{\mathfrak{m}}^{(0)}$ respectively. In other words, $B_{\mathfrak{m}}=B_{\mathfrak{m}}^{(1)}\oplus B_{\mathfrak{m}}^{(0)}$, where

$$B_{\mathfrak{m}}^{(1)}=\bigoplus_{g\in G_{\mathfrak{m}}}(A/(\mathfrak{m}))_g\quad\text{and}\quad B_{\mathfrak{m}}^{(0)}=\bigoplus_{g\in\mathbb{Z}^n_\omega\backslash G_{\mathfrak{m}}}(A/(\mathfrak{m}))_g.$$

By Lemma 4.3a), $G_{\mathfrak{m}}$ is a subgroup of the free abelian group $\mathbb{Z}^n$, hence is free abelian itself of rank $k\leq n$. Let $s_1,\dots,s_k$ denote a basis for $G_{\mathfrak{m}}$ over $\mathbb{Z}$ and let $b_i=\varphi_{\mathfrak{m}}(a_{s_i})$ for $i=1,\dots,k$. Note also that $R_{\mathfrak{m}}$ is an extension field of K and that

$\mathbb{Z}_\omega^n$ acts naturally on $R_\mathfrak{m}$ as a group of K-automorphisms. Let $\{\rho_j\}_{j\in J}$ be a basis for $R_\mathfrak{m}$ over K.

Theorem 4.5. *a)* $B_\mathfrak{m}^{(0)} M_\mathfrak{m} = 0$ *for any simple weight module M over A with no proper inner breaks, and*
b) the b_i are invertible and as a K-linear space, $B_\mathfrak{m}^{(1)}$ has a basis

$$\{\rho_j b_1^{l_1} \dots b_k^{l_k} \mid j \in J \text{ and } l_i \in \mathbb{Z} \text{ for } 1 \le i \le k\} \tag{4.4}$$

and the following commutation relations hold

$$b_i \lambda = s_i(\lambda) b_i, \quad i = 1, \dots, k, \lambda \in R_\mathfrak{m}, \tag{4.5}$$

$$b_i b_j = \lambda_{ij} b_j b_i, \quad i, j = 1, \dots, k \tag{4.6}$$

for some nonzero $\lambda_{ij} \in R_\mathfrak{m}$.

Proof. a) Let $g \in \mathbb{Z}_\omega^n \backslash G_\mathfrak{m}$. By Lemma 4.4, $A_g M_\mathfrak{m} = 0$ and thus $\varphi_\mathfrak{m}(A_g) M_\mathfrak{m} = 0$.

b) Since $s_i \in G_\mathfrak{m}$, $\varphi_\mathfrak{m}(a_{s_i}^*) b_i \in R_\mathfrak{m} \backslash \{0\}$ and by Lemma 4.3a) with $g = 0$ and $h = s_i$ we have $b_i \varphi_\mathfrak{m}(a_{s_i}^*) \in R_\mathfrak{m} \backslash \{0\}$. So the b_i are invertible. The relation (4.5) follows from (2.12). Next we prove (4.6). From Lemma 4.3a) and Lemma 4.1 we have $\varphi(A_{s_i+s_j}) = R_\mathfrak{m} b_i b_j$. Switching i and j it follows that (4.6) must hold for some nonzero $\lambda_{ij} \in R_\mathfrak{m}$.

Finally we prove that (4.4) is a basis for $B_\mathfrak{m}^{(1)}$ over K. Linear independence is clear. Let $g \in G_\mathfrak{m}$ and write $g = \sum_i l_i s_i$. By repeated use of Lemma 4.3b) we obtain that

$$\varphi_\mathfrak{m}(A_g) = \varphi_\mathfrak{m}(A_{\operatorname{sgn}(l_1)s_1})^{|l_1|} \dots \varphi_\mathfrak{m}(A_{\operatorname{sgn}(l_k)s_k})^{|l_k|}.$$

For $l_i = 0$ the factor should be interpreted as $R_\mathfrak{m}$. By Lemma 4.1,

$$\varphi_\mathfrak{m}(A_{\pm s_i})^l = R_\mathfrak{m} b_i^{\pm l}$$

for $l > 0$ so using (4.5) we get

$$\varphi_\mathfrak{m}(A_g) = R_\mathfrak{m} b_1^{l_1} \dots b_k^{l_k}.$$

The proof is finished. □

4.2. **Restricted case.** In this subsection we will assume that K is algebraically closed. Moreover we will assume that the K-algebra inclusion $K \hookrightarrow R_\mathfrak{m}$ is onto which is the case when R is finitely generated as a K-algebra by the (weak) Nullstellensatz. Then $\mathbb{Z}_\omega^n$ acts trivially on $R_\mathfrak{m}$. The structure of $B_\mathfrak{m}^{(1)}$ given in Theorem 4.5 is then simplified in the following way.

Corollary 4.6. *Let $k = \operatorname{rank} G_\mathfrak{m}$ and let $b_i = \varphi_\mathfrak{m}(a_{s_i})$ for $i = 1, \dots, k$ where $\{s_1, \dots, s_k\}$ is a $\mathbb{Z}$-basis for $G_\mathfrak{m}$. Then $B_\mathfrak{m}^{(1)}$ is the K-algebra with invertible generators $b_1, \dots, b_k$ and the relation*

$$b_i b_j = \lambda_{ij} b_j b_i, \quad 1 \le i, j \le k.$$

Using the normal form of a skew-symmetric integral matrix we will now show that $B_\mathfrak{m}^{(1)}$ can be expressed as a tensor product of noncommutative tori. Consider the matrix $(\lambda_{ij})_{1 \le i,j \le k}$ from (4.6).

Claim 4.7. *If $B_\mathfrak{m}^{(1)}$ has a nontrivial irreducible finite-dimensional representation, then all the λ_{ij} are roots of unity.*

Proof. Indeed, let N be a finite-dimensional simple module over $B_\mathfrak{m}^{(1)}$ and let $i \in \{1, \dots, k\}$. Since K is algebraically closed, b_i has an eigenvector $0 \ne v \in N$ with eigenvalue μ, say. Since b_i is invertible, $\mu \ne 0$. Let $j \ne i$ and consider the vector

b_jv. It is also nonzero, since b_j is invertible, and it is an eigenvector of b_i with eigenvalue $\lambda_{ij}\mu$. Repeating the process, we obtain a sequence

$$\mu,\ \lambda_{ij}\mu,\ \lambda_{ij}^2\mu,\ \ldots$$

of eigenvalues of b_i. Since N is finite-dimensional, they cannot all be pairwise distinct, and thus λ_{ij} is a root of unity. □

For $\lambda \in K$, let T_λ denote the K-algebra with two invertible generators a and b satisfying $ab = \lambda ba$. T_λ (or its C^*-analogue) is sometimes referred to as a noncommutative torus.

Theorem 4.8. *Let $k = \operatorname{rank} G_{\mathfrak{m}}$. If all the λ_{ij} in (4.6) are roots of unity, then there is a root of unity λ, an integer r with $0 \le r \le \lfloor k/2 \rfloor$ and positive integers $p_i, i = 1, \ldots, r$ with $1 = p_1|p_2|\ldots|p_r$ such that*

$$B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda^{p_1}} \otimes T_{\lambda^{p_2}} \otimes \cdots \otimes T_{\lambda^{p_r}} \otimes L$$

where L is a Laurent polynomial algebra over K in $k - 2r$ variables.

Proof. If $k = 1$, then $B_{\mathfrak{m}}^{(1)} \simeq K[b_1, b_1^{-1}]$ and $r = 0$. If $k > 1$, let p be the smallest positive integer such that $\lambda_{ij}^p = 1$ for all i, j. Using that K is algebraically closed, we fix a primitive p:th root of unity $\varepsilon \in K$. Then there are integers θ_{ij} such that

$$\lambda_{ij} = \varepsilon^{\theta_{ij}}$$

and

$$\theta_{ji} = -\theta_{ij}. \tag{4.7}$$

Equation (4.7) means that $\Theta = (\theta_{ij})$ is a $k \times k$ skew-symmetric integer matrix. Next, consider a change of generators of the algebra $B_{\mathfrak{m}}^{(1)}$:

$$b_i \mapsto b_i' = b_1^{u_{i1}} \cdots b_k^{u_{ik}} \tag{4.8}$$

Such a change of generators can be done if we are given an invertible $k \times k$ integer matrix $U = (u_{ij})$. The new commutation relations are

$$\begin{aligned} b_i'b_j' &= b_1^{u_{i1}} \cdots b_k^{u_{ik}} b_1^{u_{j1}} \cdots b_k^{u_{jk}} = \\ &= \lambda_{11}^{u_{1i}u_{1j}} \ldots \lambda_{k1}^{u_{ki}u_{1j}} \cdot \ldots \\ &\quad \cdot \lambda_{1k}^{u_{1i}u_{kj}} \ldots \lambda_{kk}^{u_{ki}u_{kj}} \cdot b_j'b_i' = \\ &= \varepsilon^{\sum_{p,q}\theta_{pq}u_{pi}u_{qj}} b_j'b_i' \end{aligned}$$

Hence $\Theta' = U^T\Theta U$. By Theorem IV.1 in [9] there is a U such that Θ' has the skew normal form

$$\begin{pmatrix} 0 & \theta_1 & & & & & & \\ -\theta_1 & 0 & & & & & & \\ & & 0 & \theta_2 & & & & \\ & & -\theta_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & \theta_r & \\ & & & & & -\theta_r & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix}$$

where $r \le \lfloor k/2 \rfloor$ is the rank of Θ, the θ_i are nonzero integers, $\theta_i|\theta_{i+1}$ and $\mathbf{0}$ is a $k - 2r$ by $k - 2r$ zero matrix. Set $\lambda = \varepsilon^{\theta_1}$ and $p_i = \theta_i/\theta_1$ for $i = 1, \ldots, r$. The claim follows. □

The following result, describing simple modules over the tensor product of noncommutative tori, is more or less well-known, but we provide a proof for convenience.

Proposition 4.9. *Let M be a finite dimensional simple module over*

$$T := T_{\lambda_1} \otimes \cdots \otimes T_{\lambda_r},$$

where the λ_i are roots of unity in K. Then there are simple modules M_i over T_{λ_i} such that, as T-modules,

$$M \simeq M_1 \otimes \cdots \otimes M_r.$$

Proof. Denote the generators of T_{λ_i} by a_i and b_i. We will view T_{λ_i} as subalgebras of T. Since the elements $a_i, i = 1, \ldots, r$ commute and M is finite dimensional and K is algebraically closed, there is a nonzero common eigenvector $w \in M$ of the a_i:

$$a_i w = \mu_i w, \quad i = 1, \ldots, r, \tag{4.9}$$

where $\mu_i \in K^*$ because a_i is invertible. Let n_i be the order of λ_i. Then $b_i^{n_i}$ acts as a scalar by Schur's Lemma. By simplicity of M, any element of M has the form (using the commutation relations and (4.9))

$$\sum_{j \in \mathbb{Z}^r,\ 0 \le j_i < n_i} \rho_j b_1^{j_1} \ldots b_r^{j_r} \cdot w, \tag{4.10}$$

where $\rho_j \in K$. This shows that

$$\dim_K M \le n_1 \cdot \ldots \cdot n_r.$$

But the terms in (4.10) all belong to different weight spaces with respect to the commutative subalgebra generated by $a_1, \ldots, a_r$:

$$a_i \cdot b_1^{j_1} \ldots b_r^{j_r} w = \lambda_i^{j_i} \mu_i \cdot b_1^{j_1} \ldots b_r^{j_r} w, \quad i = 1, \ldots, r,$$

and

$$(\lambda_1^{j_1} \mu_1, \ldots, \lambda_r^{j_r} \mu_r) \neq (\lambda_1^{l_1} \mu_1, \ldots, \lambda_r^{l_r} \mu_r)$$

if $j, l \in \mathbb{Z}^r, 0 \le j_i, l_i < n_i$ and $j \neq l$. Hence by standard results they must be linearly independent. Thus

$$\dim_K M = n_1 \cdot \ldots \cdot n_r. \tag{4.11}$$

Next, set $M_i = T_{\lambda_i} \cdot w$. Then $M_i = \oplus_{j=0}^{n_i - 1} K b_i^j \cdot w$ and

$$\dim_K M_i = n_i. \tag{4.12}$$

Finally, define

$$\psi : M_1 \otimes \ldots \otimes M_r \to M$$

by

$$\psi(w \otimes \ldots \otimes w) = w$$

and by requiring that ψ is a T-module homomorphism. This is possible since $M_1 \otimes \ldots \otimes M_r$ is generated by $w \otimes \ldots \otimes w$ as a T-module. Then ψ is surjective, since M is simple. Also the dimensions on both sides agree, so ψ is an isomorphism of T-modules. □

5. Explicit formulas for the induced modules

In this section we show explicitly how one can obtain simple weight modules with no proper inner breaks over a TGWA (equivalently over a TGWC by Proposition 4.2) from the structure of its weight spaces as $B(\omega)$-modules.

Since the $B(\omega)$-modules were described in the restricted case in Subsection 4.2, we obtain in particular a description of all simple weight modules over A with no proper inner breaks and finite-dimensional weight spaces if R is finitely generated over an algebraically closed field K.

5.1. **A basis for M.** Let $\{v_i\}_{i\in I}$ be a basis for $M_\mathfrak{m}$ over K. By Lemma 4.3a), $\tilde{G}_\mathfrak{m}$ is the union of some cosets in $\mathbb{Z}^n/G_\mathfrak{m}$. Let $S\subseteq\mathbb{Z}^n$ be a set of representatives of these cosets. For $g\in\tilde{G}_\mathfrak{m}$, choose $r_g\in R$ such that $a_g':=r_g a_g^*$ satisfies $\varphi_\mathfrak{m}(a_g')\varphi_\mathfrak{m}(a_g)=1$.

Theorem 5.1. *The set $C=\{a_g v_i\mid g\in S, i\in I\}$ is a basis for M over K.*

Proof. First we show that C is linearly independent over K. Assume that

$$\sum_{g,i}\lambda_{gi}a_g v_i=0.$$

Then $\sum_i\lambda_{gi}a_g v_i=0$ for each g since the elements belong to different weight spaces. Hence $0=a_g'\sum_i\lambda_{gi}a_g v_i=\sum_i\lambda_{gi}v_i$ for each g. Since v_i is a basis over K, all the λ_{gi} must be zero.

Next we prove that C spans M over K. Since M is simple and $M_\mathfrak{m}\neq 0$,

$$M=AM_\mathfrak{m}=\sum_{g\in\mathbb{Z}^n}A_gM_\mathfrak{m}=\sum_{g\in\tilde{G}_\mathfrak{m}}A_gM_\mathfrak{m}=\sum_{h\in S}\sum_{g\in h+G_\mathfrak{m}}A_gM_\mathfrak{m}=\sum_{h\in S}A_hM_\mathfrak{m}$$

by Lemma 4.4 and Lemma 4.3c). □

Corollary 5.2. $\operatorname{supp}(M)=\{g(\mathfrak{m})\mid g\in S\}$ *and* $g(\mathfrak{m})\neq h(\mathfrak{m})$ *if* $g,h\in S$, $g\neq h$.

Corollary 5.3. $\dim M=|S|\cdot\dim M_\mathfrak{m}$ *with natural interpretation of* ∞.

5.2. **The action of A.** Our next step is to describe the action of the X_i,Y_i on the basis C for M. Let $\zeta:\tilde{G}_\mathfrak{m}\to S$ be the function defined by requiring $g-\zeta(g)\in G_\mathfrak{m}$.

Theorem 5.4. *Let $g\in S$ and let $v\in M_\mathfrak{m}$. Then*

$$X_ia_gv=\begin{cases}a_h\cdot b_{g,i}v & \text{if } g+e_i\in\tilde{G}_\mathfrak{m},\\ 0 & \text{otherwise,}\end{cases}$$

where $h=\zeta(g+e_i)$ and

$$b_{g,i}=(-h)(X_ia_ga'_{g+e_i-h}a'_h)\cdot a_{g+e_i-h}$$

and

$$Y_ia_gv=\begin{cases}a_k\cdot c_{g,i}v & \text{if } g-e_i\in\tilde{G}_\mathfrak{m},\\ 0 & \text{otherwise,}\end{cases}$$

where $k=\zeta(g-e_i)$ and

$$c_{g,i}=(-k)(Y_ia_ga'_{g-e_i-k}a'_k)\cdot a_{g-e_i-k}.$$

Remark 5.5. Note that

$$\deg X_ia_ga'_{g+e_i-h}a'_h=\deg Y_ia_ga'_{g-e_i-k}a'_k=0$$

so the action of $\mathbb{Z}^n$ on these elements is well defined. Thus we see that $\deg b_{g,i}\in G_\mathfrak{m}$ and $\deg c_{g,i}\in G_\mathfrak{m}$, i.e. that $b_{g,i}$ and $c_{g,i}$ belong to $B(\omega)$. Therefore the action of these elements on a basis element v_i of $M_\mathfrak{m}$ can be determined if we know the structure of $M_\mathfrak{m}$ as an $B(\omega)$-module. In the restricted case this was described in Subsection 4.2. Expanding the result in the basis $\{v_i\}$ again and acting by a_h or a_k we obtain a linear combination of basis elements from the set C.

Proof. Assume $g+e_i\in\tilde{G}_\mathfrak{m}$. Let $h=\zeta(g+e_i)$. Then

$$\begin{aligned}X_ia_gv&=X_ia_ga'_{g+e_i-h}a_{g+e_i-h}v=\\&=(X_ia_ga'_{g+e_i-h}a'_h)a_ha_{g+e_i-h}v=\\&=a_h\cdot(-h)(X_ia_ga'_{g+e_i-h}a'_h)\cdot a_{g+e_i-h}v.\end{aligned}$$

If $g+e_i\notin\tilde{G}_\mathfrak{m}$, then $X_ia_gv=0$ by Lemma 4.4.

Assume $g - e_i \in \tilde{G}_{\mathfrak{m}}$. Let $k = \zeta(g - e_i)$. Then

$$\begin{aligned} Y_i a_g v &= Y_i a_g a'_{g-e_i-k} a_{g-e_i-k} v = \\ &= (Y_i a_g a'_{g-e_i-k} a'_k) a_k a_{g-e_i-k} v = \\ &= a_k \cdot (-k)(Y_i a_g a'_{g-e_i-k} a'_k) \cdot a_{g-e_i-k} v. \end{aligned}$$

If $g - e_i \notin \tilde{G}_{\mathfrak{m}}$, then $Y_i a_g v = 0$ by Lemma 4.4. □

Note that we do not need the technical assumptions in the proof of Theorem 1 in [8] under which the exact formulas for simple weight modules were obtained.

6. Application to quantized Weyl algebras

In this final part we will apply the methods developed in the previous sections to the problem of describing representations of the quantized Weyl algebra, defined in Section 2.2. As mentioned there, it is naturally a TGWA.

First we find the isotropy group and the set $\tilde{G}_{\mathfrak{m}}$ expressed as solution of systems of linear equations (see Proposition 6.3 and Proposition 6.4). These sets are directly related to the structure of the subalgebra $B(\omega)$ (Theorem 4.5) and the support of a module (Corollary 5.2).

Then in Section 6.2 we give a complete classification of all locally finite simple weight modules with no proper inner breaks over a quantized Weyl algebra of rank two. The parameters q_1 and q_2 are allowed to be any numbers from $\mathbb{C}\backslash\{0,1\}$. Example 6.7 shows that the assumption that the modules have no proper inner breaks is not superfluous.

6.1. **The isotropy group and $\tilde{G}_{\mathfrak{m}}$.** Let $R = \mathbb{C}[t_1, \dots, t_n]$ and fix $\mathfrak{m} = (t_1 - \alpha_1, \dots, t_n - \alpha_n) \in \mathrm{Max}(R)$. Let ω be the orbit of $\mathfrak{m}$ under the action (2.10) of $\mathbb{Z}^n$. Set $[k]_q = \sum_{j=0}^{k-1} q^i$ for $k \in \mathbb{Z}$ and $q \in \mathbb{C}$. Recall the definition (2.9) of the automorphisms σ_i of R.

Proposition 6.1. *Let $(g_1, \dots, g_n) \in \mathbb{Z}^n$. Then*

$$\begin{aligned} &\sigma_1^{g_1} \dots \sigma_n^{g_n}(\mathfrak{m}) = \\ &\qquad \Big([g_1]_{q_1} + q_1^{g_1} t_1 - \alpha_1, \quad [g_2]_{q_2}\big(1 + (q_1 - 1)\alpha_1\big) + q_1^{g_1} q_2^{g_2} t_2 - \alpha_2, \dots \\ &\qquad \dots, [g_j]_{q_j}\big(1 + \sum_{r=1}^{j-1}(q_r - 1)\alpha_r\big) + q_1^{g_1} \dots q_j^{g_j} t_j - \alpha_j, \dots \\ &\qquad \dots, [g_n]_{q_n}\big(1 + \sum_{r=1}^{n-1}(q_r - 1)\alpha_r\big) + q_1^{g_1} \dots q_n^{g_n} t_n - \alpha_n \Big). \end{aligned}$$

Proof. Induction. □

For notational brevity we set $\beta_i = (q_i - 1)\alpha_i$ and $\gamma_i = 1 + \beta_1 + \beta_2 + \ldots + \beta_i$. We also set $\gamma_0 = 1$. The numbers γ_i will play an important role in the next statements. By a *j-break* we mean an ideal $\mathfrak{n} \in \mathrm{Max}(R)$ such that $t_j \in \mathfrak{n}$.

Corollary 6.2. *For $j = 1, \dots, n$ we have*

$$t_j \in \sigma_1^{g_1} \dots \sigma_n^{g_n}(\mathfrak{m}) \Longleftrightarrow \gamma_j = q_j^{g_j} \gamma_{j-1}.$$

Thus ω contains a j-break iff $\gamma_j = q_j^k \gamma_{j-1}$ for some integer k.

Proof. By Proposition 6.1,

$$t_j \in \sigma_1^{g_1} \dots \sigma_n^{g_n}(\mathfrak{m})$$

iff

$$[g_j]_{q_j}\big(1+\sum_{r=1}^{j-1}(q_r-1)\alpha_r\big)=\alpha_j.$$

Multiply both sides with q_j-1 to get

$$(q_j^{g_j}-1)(1+\beta_1+\ldots+\beta_{j-1})=\beta_j.$$

□

The next Proposition describes the isotropy subgroup $\mathbb{Z}^n_\omega$ defined in (2.14).

Proposition 6.3. *We have*

$$\mathbb{Z}^n_\omega=\{g\in\mathbb{Z}^n\mid (q_1^{g_1}\ldots q_j^{g_j}-1)\gamma_j=0\ \forall j=1,\ldots,n\}. \tag{6.1}$$

Proof. From Proposition 6.1, $\sigma_1^{g_1}\ldots\sigma_n^{g_n}(\mathfrak{m})=\mathfrak{m}$ iff

$$\begin{aligned}
\alpha_1&=[g_1]_{q_1}+q_1^{g_1}\alpha_1\\
\alpha_2&=[g_2]_{q_2}\big(1+(q_1-1)\alpha_1\big)+q_1^{g_1}q_2^{g_2}\alpha_2\\
&\ \vdots\\
\alpha_n&=[g_n]_{q_n}\big(1+(q_1-1)\alpha_1+\ldots+(q_{n-1}-1)\alpha_{n-1}\big)+q_1^{g_1}\ldots q_n^{g_n}\alpha_n
\end{aligned}$$

Multiply the i:th equation by q_i-1. Then the system can be written

$$\begin{aligned}
\beta_1&=q_1^{g_1}-1+q_1^{g_1}\beta_1\\
\beta_2&=(q_2^{g_2}-1)(1+\beta_1)+q_1^{g_1}q_2^{g_2}\beta_2\\
&\ \vdots\\
\beta_n&=(q_n^{g_n}-1)(1+\beta_1+\ldots+\beta_{n-1})+q_1^{g_1}\ldots q_n^{g_n}\beta_n
\end{aligned}$$

or equivalently

$$\begin{aligned}
1+\beta_1&=q_1^{g_1}(1+\beta_1)\\
1+\beta_1+\beta_2&=q_2^{g_2}(1+\beta_1)+q_1^{g_1}q_2^{g_2}\beta_2\\
&\ \vdots\\
1+\beta_1+\ldots+\beta_n&=q_n^{g_n}(1+\beta_1+\ldots+\beta_{n-1})+q_1^{g_1}\ldots q_n^{g_n}\beta_n
\end{aligned}$$

Now for i from 1 to $n-1$, replace the expression $1+\beta_1+\ldots+\beta_i$ in the right hand side of the $i+1$:th equation by the right hand side of the i:th equation. After simplification, the claim follows. □

Note that it follows from (6.1) that the subgroup

$$Q=\{g\in\mathbb{Z}^n\mid q_j^{g_j}=1 \text{ for } j=1,\ldots,n\} \tag{6.2}$$

of $\mathbb{Z}^n$ is always contained in $\mathbb{Z}^n_\omega$ for any orbit ω. Moreover $\mathbb{Z}^n_\omega=Q$ if ω (viewed as a subset of $\mathbb{C}^n$) does not intersect the union of the hyperplanes in $\mathbb{C}^n$ defined by the equations $1+(q_1-1)x_1+\ldots+(q_j-1)x_j=0$ $(1\le j\le n)$. Of course the group Q can be trivial. This is the case for example when all the q_j are positive real numbers.

Another case of interest is when for any j, $q_1^{g_1}\ldots q_j^{g_j}=1$ implies $g_1=\ldots=g_j=0$. If for instance the q_j are pairwise distinct prime numbers this hold. Then $\mathbb{Z}^n_\omega=\{0\}$ unless $1+\beta_1+\ldots+\beta_j=0$ for all j, i.e. unless ω contains the point

$$\mathfrak{n}_0=(t_1-(1-q_1)^{-1},t_2,\ldots,t_n).$$

So in this very special case we have $\omega=\{\mathfrak{n}_0\}$ and $\mathbb{Z}^n_\omega=\mathbb{Z}^n$.

We now turn to the set $\tilde{G}_{\mathfrak{m}}$ defined in (4.1) which can here be described explicitly in terms of $\mathfrak{m}$ in the following way.

Proposition 6.4.

$$\tilde{G}_{\mathfrak{m}} = \tilde{G}_{\mathfrak{m}}^{(1)} \times \ldots \times \tilde{G}_{\mathfrak{m}}^{(n)},$$

where

$$\tilde{G}_{\mathfrak{m}}^{(j)} = \{k \geq 0 \mid \gamma_j \neq q_j^i \gamma_{j-1} \ \forall i = 0, 1, \ldots, k-1\} \cup \\ \cup \{k < 0 \mid \gamma_j \neq q_j^i \gamma_{j-1} \ \forall i = -1, -2, \ldots, k\}.$$

Proof. From the relations of the algebra follows that the subspace spanned by the words in A_g is one-dimensional. Thus $g \in \tilde{G}_{\mathfrak{m}}$ iff

$$Z_n^{-g_n} \ldots Z_1^{-g_1} Z_1^{g_1} \ldots Z_n^{g_n} \notin \mathfrak{m} \tag{6.3}$$

where $Z_i^k = X_i^k$ if $k \geq 0$ and $Z_i^k = Y_i^{-k}$ if $k < 0$. Since $\sigma_i(t_j) = t_j$ for $j < i$, (6.3) is equivalent to

$$Z_n^{-g_n} Z_n^{g_n} \ldots Z_1^{-g_1} Z_1^{g_1} \notin \mathfrak{m}.$$

Since $\mathfrak{m}$ is prime, this holds iff $Z_j^{-g_j} Z_j^{g_j} \notin \mathfrak{m}$ for each j. If $g_j = 0$ this is true. If $g_j > 0$ we have

$$Z_j^{-g_j} Z_j^{g_j} = Y_j^{g_j} X_j^{g_j} = Y_j^{g_j-1} X_j^{g_j-1} \sigma_j^{-g_j+1}(t_j) = \ldots = t_j \sigma_j^{-1}(t_j) \ldots \sigma_j^{-g_j+1}(t_j),$$

while if $g_j < 0$

$$Z_j^{-g_j} Z_j^{g_j} = X_j^{-g_j} Y_j^{-g_j} = X_j^{-g_j-1} Y_j^{-g_j-1} \sigma_j^{-g_j}(t_j) = \ldots = \sigma_j(t_j) \ldots \sigma_j^{-g_j}(t_j).$$

Since $\mathfrak{m}$ is prime, $g \in \tilde{G}_{\mathfrak{m}}$ iff for all $j = 1, \ldots, n$

$$t_j \notin \sigma_j^i(\mathfrak{m}), \quad i = 0, \ldots, g_j - 1 \text{ if } g_j \geq 0,$$

and

$$t_j \notin \sigma_j^i(\mathfrak{m}), \quad i = -1, -2 \ldots, g_j \text{ if } g_j < 0.$$

The claim now follows from Corollary 6.2. □

Corollary 6.5. *If $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{Q}(q_1, \ldots, q_n)$, then $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^n$.*

6.2. **Description of simple weight modules over rank two algebras.** Assume from now on that A is a quantized Weyl algebra of rank two. In this section we will obtain a list of all locally finite simple weight A-modules with no proper inner breaks.

We consider first some families of ideals in $\mathrm{Max}(R)$. Define for $\lambda \in \mathbb{C}$,

$$\mathfrak{n}_\lambda^{(1)} = \big(t_1 - (1-\lambda)(1-q_1)^{-1}, t_2 - \lambda(1-q_2)^{-1}\big),$$
$$\mathfrak{n}_\lambda^{(2)} = \big(t_1 - (1-q_1)^{-1}, t_2 - \lambda\big),$$

and set $\mathfrak{n}_0 = \mathfrak{n}_0^{(1)} = \mathfrak{n}_0^{(2)}$. The following lemma will be useful.

Lemma 6.6. *For $\lambda \in \mathbb{C}$ and integers k, l we have*

$$\sigma_1^k \sigma_2^l (\mathfrak{n}_\lambda^{(1)}) = \mathfrak{n}_{\lambda q_1^{-k}}^{(1)}, \tag{6.4}$$

$$\sigma_1^k \sigma_2^l (\mathfrak{n}_\lambda^{(2)}) = \mathfrak{n}_{\lambda q_1^{-k} q_2^{-l}}^{(2)}. \tag{6.5}$$

Proof. Follows from Proposition 6.1 or by direct calculation using the definition (2.9) of the σ_i. □

The following example shows the existence of locally finite simple weight modules M over A which have some proper inner breaks.

Example 6.7. Assume that $q_1\lambda_{12}$ is a root of unity of order r. Let M be a vector space of dimension r and let $\{v_0, v_1, \dots, v_{r-1}\}$ be a basis for M. Define an action of A on M as follows.

$$X_1 v_k = \begin{cases} v_{k+1}, & k < r-1 \\ v_0, & k = r-1 \end{cases} \qquad X_2 v_k = (q_1\lambda_{12})^{-k} v_k$$

$$Y_1 v_k = \begin{cases} (1-q_1)^{-1} v_{k-1}, & k > 0 \\ (1-q_1)^{-1} v_{r-1}, & k = 0 \end{cases} \qquad Y_2 v_k = 0$$

It is easy to check that (2.6)–(2.8) hold so this defines a module over A. It is immediate that $M = M_{\mathfrak{m}}$ where $\mathfrak{m} = \mathfrak{n}_0 = (t_1 - (1-q_1)^{-1}, t_2)$ so M is a weight module and M is simple by standard arguments. However, recalling Definition 3.7, M has some proper inner breaks in the sense that $\mathfrak{m} \in \operatorname{supp}(M)$, $X_2 M_{\mathfrak{m}} \neq 0$ but $Y_2 X_2 M_{\mathfrak{m}} = 0$.

We will describe the isotropy groups of the different ideals in $\operatorname{Max}(R)$. Let K_1 and K_2 denote the kernels of the group homomorphisms from $\mathbb{Z}\times\mathbb{Z}$ to the multiplicative group $\mathbb{C}\backslash\{0\}$ which map (k,l) to q_1^k and $q_1^k q_2^l$ respectively. Then $Q = K_1 \cap K_2$ where Q was defined in (6.2). For $\mathfrak{m} \in \operatorname{Max}(R)$, recall that $\mathbb{Z}^2_{\mathfrak{m}} = \{g \in \mathbb{Z}^2 \mid g(\mathfrak{m}) = \mathfrak{m}\}$. The following corollary describes the isotropy group $\mathbb{Z}^2_{\mathfrak{m}}$ of any $\mathfrak{m} \in \operatorname{Max}(R)$.

Corollary 6.8. *Let $\lambda \in \mathbb{C}\backslash\{0\}$ and $\mathfrak{n} \in \operatorname{Max}(R)\backslash\{\mathfrak{n}^{(i)}_\mu \mid \mu \in \mathbb{C}, i = 1,2\}$. Then we have the following equalities in the lattice of subgroups of $\mathbb{Z}^2$.*

$$\begin{tikzcd}
 & \mathbb{Z}^2_{\mathfrak{n}_0} = \mathbb{Z}^2 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Z}^2_{\mathfrak{n}^{(1)}_\lambda} = K_1 \arrow[dr, dash] & & \mathbb{Z}^2_{\mathfrak{n}^{(2)}_\lambda} = K_2 \arrow[dl, dash] \\
 & \mathbb{Z}^2_{\mathfrak{n}} = Q &
\end{tikzcd}$$

Proof. The family of ideals $\{\mathfrak{n}^{(1)}_\lambda \mid \lambda \in \mathbb{C}\}$ are precisely those for which $\gamma_2 = 0$. And $\{\mathfrak{n}^{(2)}_\lambda \mid \lambda \in \mathbb{C}\}$ are exactly those such that $\gamma_1 = 0$. Thus the claim follows from Proposition 6.3. $\square$

Let M be a simple weight A-module with no proper inner breaks and finite dimensional weight spaces, $\mathfrak{m} = (t_1 - \alpha_1, t_2 - \alpha_2) \in \operatorname{supp} M$ and let ω be the orbit of $\mathfrak{m}$. We consider four main cases separately: $\mathfrak{m} = \mathfrak{n}_0$, $\mathfrak{m} = \mathfrak{n}^{(1)}_\lambda$ for some $\lambda \neq 0$, $\mathfrak{m} = \mathfrak{n}^{(2)}_\lambda$ for some $\lambda \neq 0$ and $\mathfrak{m} \notin \{\mathfrak{n}^{(i)}_\mu \mid \mu \in \mathbb{C}, i = 1,2\}$. Some of these cases will contain subcases. In each case we will proceed along the following steps, which also illustrate the procedure for a general TGWA.

1. Find the sets $\mathbb{Z}^n_{\mathfrak{m}}$ and $\tilde{G}_{\mathfrak{m}}$ using Corollary 6.8 and Proposition 6.4. Write down $G_{\mathfrak{m}} = \mathbb{Z}^n_{\mathfrak{m}} \cap \tilde{G}_{\mathfrak{m}}$ and choose a basis $\{s_1, \dots, s_k\}$ for $G_{\mathfrak{m}}$ over $\mathbb{Z}$.
2. For each $g \in \tilde{G}_{\mathfrak{m}}$, choose a word a_g of degree g such that $a_g^* a_g \notin \mathfrak{m}$.
3. Using Corollary 4.6, describe $B^{(1)}_{\mathfrak{m}}$ and the finite-dimensional simple $B^{(1)}_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$.
4. Choose a set of representatives S for $\tilde{G}_{\mathfrak{m}}/G_{\mathfrak{m}}$. By Theorem 5.1 we know then a basis C for M.
5. Calculate the action of X_i, Y_i on the basis using either relations (2.6)–(2.8) or Theorem 5.4.

We will use the following notation: $Z_j^k = X_j^k$ if $k \geq 0$ and $Z_j^k = Y_j^{-k}$ if $k < 0$. Note that the k in Z_j^k should only be regarded as an upper index, not as a power. The choice of a_g in step two above is more or less irrelevant for a quantized Weyl algebra because each A_g is one-dimensional. Therefore we will always choose $a_g = Z_1^{g_1} Z_2^{g_2}$ where $g = (g_1, g_2)$.

6.3. **The case $\mathfrak{m} = \mathfrak{n}_0$.** Here $\alpha_1 = (1-q_1)^{-1}$, $\alpha_2 = 0$ so that $\gamma_1 = \gamma_2 = 0$. By Corollary 6.8 we have $\mathbb{Z}^2_\mathfrak{m} = \mathbb{Z}^2$ and from Proposition 6.4 one obtains that $\tilde{G}_\mathfrak{m} = \mathbb{Z} \times \{0\}$. Thus $G_\mathfrak{m} = \mathbb{Z} \times \{0\} = \mathbb{Z} \cdot s_1$ with $s_1 = (1, 0)$. Since $G_\mathfrak{m}$ has rank one, Corollary 4.6 implies that $B^{(1)}_\mathfrak{m}$ is isomorphic to the Laurent polynomial algebra $\mathbb{C}[T, T^{-1}]$ in one variable. Therefore $M_\mathfrak{m}$ is one-dimensional, say $M_\mathfrak{m} = \mathbb{C}v_0$ and $b_1 = \varphi_\mathfrak{m}(Z_1^1) = \varphi_\mathfrak{m}(X_1)$, hence X_1, acts in $M_\mathfrak{m}$ as some nonzero scalar ρ. And

$$Y_1 v_0 = \rho^{-1} Y_1 X_1 v_0 = \rho^{-1}(1-q_1)^{-1} v_0.$$

Here $S = \{(0,0)\}$ and $C = \{v_0\}$ is a basis for M with the following action:

$$\begin{aligned} X_1 v_0 &= \rho v_0, & X_2 v_0 &= 0, \\ Y_1 v_0 &= \rho^{-1}(1-q_1)^{-1} v_0, & Y_2 v_0 &= 0. \end{aligned} \tag{6.6}$$

That $Z_2^{\pm 1} v_0 = 0$ follows from Theorem 5.4 since $(0, \pm 1) \notin \tilde{G}_\mathfrak{m}$.

6.4. **The case $\mathfrak{m} = \mathfrak{n}^{(1)}_\lambda$, $\lambda \neq 0$.** Here $\alpha_1 = (1-\lambda)(1-q_1)^{-1}$ and $\alpha_2 = \lambda(1-q_1)^{-1}$ so $\gamma_1 = \lambda$ and $\gamma_2 = 0$. By Proposition 6.4, $\tilde{G}^{(2)}_\mathfrak{m} = \mathbb{Z}$ and

$$\tilde{G}^{(1)}_\mathfrak{m} = \{k \geq 0 \mid \lambda \neq q_1^i \ \forall i = 0, 1, \ldots, k-1\} \cup \{k < 0 \mid \lambda \neq q_1^i \ \forall i = -1, -2, \ldots, k\}.$$

We consider four subcases according to whether ω contains a 1-break or not and whether q_1 is a root of unity or not.

6.4.1. *The case $\mathfrak{m} = \mathfrak{n}^{(1)}_\lambda$, $\lambda \neq 0$, ω contains a 1-break and q_1 is a root of unity.* By Corollary 6.2 $\lambda = q_1^k$ for some $k \in \mathbb{Z}$. Let o_1 be the order of q_1. Then $\mathbb{Z}^2_\mathfrak{m} = K_1 = (o_1\mathbb{Z}) \times \mathbb{Z}$. We can further assume that $k \in \{0, 1, \ldots, o_1 - 1\}$.

Note that $X_1^k M_\mathfrak{m} \neq 0$ because $\deg X_1^k = (k, 0) \in \tilde{G}_\mathfrak{m}$ so $Y_1^k X_1^k \notin \mathfrak{m}$. Hence $\sigma_1^k(\mathfrak{m}) \in \mathrm{supp}(M)$. By Lemma 6.6, $\sigma_1^k(\mathfrak{m}) = \mathfrak{n}^{(1)}_{q_1^k q_1^{-k}} = \mathfrak{n}^{(1)}_1$. We can thus change notation and let $\mathfrak{m} = \mathfrak{n}^{(1)}_1$. Then by Proposition 6.4 we have

$$\tilde{G}_\mathfrak{m} = \{0, -1, -2, \ldots, -o_1 + 1\} \times \mathbb{Z}.$$

And $G_\mathfrak{m} = \tilde{G}_\mathfrak{m} \cap \mathbb{Z}^2_\mathfrak{m} = \{0\} \times \mathbb{Z}$. By Corollary 4.6, $B^{(1)}_\mathfrak{m}$ is a Laurent polynomial algebra in one variable. Thus $M_\mathfrak{m}$ is one dimensional with a basis vector, say v_0. X_2 acts by some nonzero scalar ρ on v_0 and $Y_2 X_2 v_0 = (1-q_2)^{-1} v_0$. X_1 and $Y_1^{o_1}$ act as zero on $M_\mathfrak{m}$ by Lemma 4.4 because their degrees $(1, 0)$ and $(-o_1, 0)$ does not belong to $\tilde{G}_\mathfrak{m}$.

As a set of representatives for $\tilde{G}_\mathfrak{m}/G_\mathfrak{m}$ we choose

$$S = \{(0,0), (-1,0), (-2,0), \ldots, (-o_1+1, 0)\}.$$

By Corollary 5.2 we obtain that

$$\mathrm{supp}(M) = \{\mathfrak{n}^{(1)}_1, \mathfrak{n}^{(1)}_{q_1^{-1}}, \ldots, \mathfrak{n}^{(1)}_{q_1^{-o_1+1}}\}.$$

By 5.1, the set

$$C = \{v_j := Y_1^j v_0 \mid j = 0, 1, \ldots, o_1 - 1\}$$

is a basis for M. The following picture shows the support of the module and how the X_i act on it. Since the Y_i just act in the opposite direction of the X_i we do not draw their arrows.

$$\begin{tikzcd}
\bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"'] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"'] & \bullet \arrow[loop above, "X_2"] & \cdots\cdots & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"'] & \bullet \arrow[loop above, "X_2"]
\end{tikzcd}$$

Using Lemma 6.6,

$$X_1 v_j = X_1 Y_1^j v_0 = Y_1^{j-1} \sigma_1^j(t_1) v_0 = [j]_{q_1} v_{j-1}$$

and from relations (2.6)–(2.8) follow that

$$X_2 v_j = q_1^j \lambda_{12}^j Y_1^j X_2 v_0 = \rho \lambda_{12}^j q_1^j v_j,$$

$$Y_2 v_j = \lambda_{21}^j Y^j Y_2 v_0 = (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j.$$

Thus the action on the basis $\{v_0, \dots, v_{o_1-1}\}$ is

$$\begin{aligned}
X_1 v_j &= \begin{cases} 0, & j = 0, \\ [j]_{q_1} v_{j-1}, & 0 < j \le o_1 - 1, \end{cases} \\
Y_1 v_j &= \begin{cases} v_{j+1}, & 0 \le j < o_1 - 1, \\ 0, & j = o_1 - 1, \end{cases} \\
X_2 v_j &= \rho \lambda_{12}^j q_1^j v_j, \\
Y_2 v_j &= (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j.
\end{aligned} \tag{6.7}$$

6.4.2. *The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω contains a 1-break and q_1 is not a root of unity.* Now there is a unique integer $k \in \mathbb{Z}$ such that $\lambda = q_1^k$. If $k \ge 0$, then $\tilde{G}_\mathfrak{m}^{(1)}$ is the set of all integers $\le k$ while if $k < 0$, then $\tilde{G}_\mathfrak{m}^{(1)}$ is all integers $\ge k+1$.

If $k \ge 0$, $X_1^k M_\mathfrak{m} \neq 0$ because $(k, 0) \in \tilde{G}_\mathfrak{m}$ so $Y_1^k X_1^k \notin \mathfrak{m}$. Therefore $\sigma_1^k(\mathfrak{m}) = \mathfrak{n}_1^{(1)} \in \mathrm{supp}(M)$. We change notation and let $\mathfrak{m} = \mathfrak{n}_1^{(1)}$. Then $\tilde{G}_\mathfrak{m}^{(1)} = \{\dots, -2, -1, 0\}$ and $G_\mathfrak{m} = \{0\} \times \mathbb{Z}$. We choose $S = \{(i, 0) \mid i \le 0\}$. $Y_2 X_2 = (1-q_2)^{-1}$ on $M_\mathfrak{m}$ so $M_\mathfrak{m} = \mathbb{C} v_0$, for a basis vector v_0, and $X_2 v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$. The set $C = \{v_j := Y_1^j v_0 \mid j \le 0\}$ is a basis for M and we have the following picture of $\mathrm{supp}(M)$.

$$\begin{tikzcd}
\cdots\cdots & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"]
\end{tikzcd}$$

One easily obtains the following action on the basis $\{v_j \mid j \le 0\}$:

$$\begin{aligned}
X_1 v_j &= \begin{cases} 0, & j = 0, \\ [j]_{q_1} v_{j-1}, & j \ge 1, \end{cases} \\
Y_1 v_j &= v_{j+1}, \\
X_2 v_j &= \rho \lambda_{12}^j q_1^j v_j, \\
Y_2 v_j &= (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j.
\end{aligned} \tag{6.8}$$

The case $k < 0$ is analogous and yields a lowest weight representation with $\mathfrak{m} = \mathfrak{n}_{q_1^{-1}}^{(1)}$ as its lowest weight. A basis for M is then

$$C = \{v_j := X_1^j v_0 \mid j \ge 0\},$$

where $M_{\mathfrak{m}} = \mathbb{C}v_0$ and the action is given by

$$
\begin{aligned}
X_1 v_j &= v_{j+1},\\
Y_1 v_j &= \begin{cases} 0, & j = 0,\\ [-j]_{q_1} v_{j-1}, & j > 0,\end{cases}\\
X_2 v_j &= (q_1\lambda_{12})^{-j}\rho v_j,\\
Y_2 v_j &= \lambda_{12}^{j}(1-q_2)^{-1}\rho^{-1} v_j.
\end{aligned}
\tag{6.9}
$$

6.4.3. *The case* $\mathfrak{m} = \mathfrak{n}_{\lambda}^{(1)}$, $\lambda \neq 0$, ω *contains no* 1*-break and* q_1 *is a root of unity.* By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. So by Proposition 6.4, $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^2$. $G_{\mathfrak{m}} = (o_1\mathbb{Z})\times\mathbb{Z}$ and we can choose $S = \{0, 1, \dots, o_1 - 1\} \times \{0\}$. From

$$X_1^{o_1} X_2 = (q_1\lambda_{12})^{o_1} X_2 X_1^{o_1} = \lambda_{12}^{o_1} X_2 X_1^{o_1}$$

and Corollary 4.6 follows that $B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda_{12}^{o_1}}$. It can only have finite-dimensional irreducible representations if $\lambda_{12}^{o_1}$ is a root of unity. Assuming this, any such representation is r-dimensional, where r is the order of $\lambda_{12}^{o_1}$, and is parametrized by $\mathbb{C}^* \times \mathbb{C}^* \ni (\rho, \mu)$ with basis

$$M_{\mathfrak{m}} = \operatorname{Span}\{v_j := X_2^j v_0 \mid j = 0, 1, \dots, r-1\},$$

where $X_1^{o_1} v_0 = \rho v_0$ and relations

$$
\begin{aligned}
X_1^{o_1} v_j &= \lambda_{12}^{o_1 j}\rho v_j,\\
X_2 v_j &= \begin{cases} v_{j+1}, & 0 \le j < r-1,\\ \mu v_0, & j = r-1.\end{cases}
\end{aligned}
$$

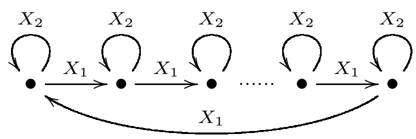
Therefore by Theorem 5.1,

$$M = \operatorname{Span}\{w_{ij} = X_1^i v_j \mid 0 \le i < o_1, 0 \le j < r\}.$$

Using the commutation relations and the formulas in Lemma 6.6 we can write down the action as follows.

$$
\begin{aligned}
X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & 0 \le i < o_1 - 1,\\ \lambda_{12}^{o_1 j}\rho w_{0,j}, & i = o_1 - 1,\end{cases}\\
Y_1 w_{ij} &= \begin{cases} (1-\lambda)(1-q_1)^{-1}\lambda_{12}^{-o_1 j}\rho^{-1} w_{o_1-1,j}, & i = 0,\\ (1-\lambda q_1^{-i})(1-q_1)^{-1} w_{i-1,j}, & 0 < i \le o_1 - 1,\end{cases}\\
X_2 w_{ij} &= \begin{cases} q_1^{-i}\lambda_{21}^{i} w_{i,j+1}, & 0 \le j < r-1,\\ q_1^{-i}\lambda_{21}^{i}\mu w_{i,0}, & j = r-1,\end{cases}\\
Y_2 w_{ij} &= \begin{cases} \lambda_{12}^{i}\mu^{-1}\lambda(1-q_2)^{-1} w_{i,r-1}, & j = 0,\\ \lambda_{12}^{i}\lambda(1-q_2)^{-1} w_{i,j-1}, & 0 < j \le r-1.\end{cases}
\end{aligned}
\tag{6.10}
$$

The action can be illustrated in the following way.



6.4.4. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω *contains no* 1*-break and* q_1 *is not a root of unity.* By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. Now $\mathbb{Z}_\mathfrak{m}^2 = \{0\} \times \mathbb{Z}$ so $G_\mathfrak{m} = \{0\} \times \mathbb{Z}$. $M_\mathfrak{m}$ is one-dimensional with basis v_0, say, and $X_2 = \rho$ on $M_\mathfrak{m}$ while $Y_2 X_2 = \lambda(1-q_2)^{-1} \neq 0$ on $M_\mathfrak{m}$. We choose $S = \mathbb{Z} \times \{0\}$. Then a basis for M is

$$C = \{v_j := X_1^j v_0 \mid j \geq 0\} \cup \{v_j := \zeta_j Y_1^{-j} v_0 \mid j < 0\},$$

where we determine ζ_j by requiring that $X_1 v_j = v_{j+1}$ for all j. Explicitly we have for $j < 0$,

$$\zeta_j = \frac{(1-q_1)^{-j}}{(1-\lambda q_1^{-j})(1-\lambda q_1^{-j-1})\dots(1-\lambda q_1)}.$$

Using the commutation relations and the formulas in Lemma 6.6 we get the action on $M = \mathrm{Span}\{v_j \mid j \in \mathbb{Z}\}$.

$$\begin{aligned} X_1 v_j &= v_{j+1}, & X_2 v_j &= q_1^{-j} \lambda_{12}^{-j} \rho v_j, \\ Y_1 v_j &= \frac{1-\lambda q_1^{-j+1}}{1-q_1} v_{j-1}, & Y_2 v_j &= \lambda_{12}^j \lambda (1-q_2)^{-1} \rho^{-1} v_j, \end{aligned} \tag{6.11}$$

and a corresponding diagram

$$\begin{tikzcd} \cdots\cdots & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] & \cdots\cdots \end{tikzcd} .$$

6.5. **The case** $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$**,** $\lambda \neq 0$**.** Here $\gamma_1 = 0$ while $\gamma_2 = \lambda(q_2 - 1)$. By Corollary 6.2, ω does not contain any breaks. We have $\tilde{G}_\mathfrak{m} = \mathbb{Z}^2$ and $G_\mathfrak{m} = \mathbb{Z}_\mathfrak{m}^2 = K_2$.

We will need some lemmas in order to proceed.

Lemma 6.9. *For* $k, l \in \mathbb{Z}$ *we have*

$$Z_1^k Z_2^l = q_1^{k\bar{l}} \lambda_{12}^{kl} Z_2^l Z_1^k, \tag{6.12}$$

where $\bar{l} = \max\{0, l\}$.

Proof. Relations (2.6)–(2.8) can be rewritten in the more compact form

$$Z_1^k Z_2^l = q_1^{k\delta_{l,1}} \lambda_{12}^{kl} Z_2^l Z_1^k, \quad k, l = \pm 1,$$

where $\delta_{l,1}$ is the Kronecker symbol. After repeated application of this, (6.12) follows. $\square$

By Lemma 6.6 we have for $k, l \in \mathbb{Z}$,

$$\sigma_1^k \sigma_2^l(t_1) = (1-q_1)^{-1} \mod \mathfrak{m}, \tag{6.13}$$

$$\sigma_1^k \sigma_2^l(t_2) = \lambda q_1^k q_2^l \mod \mathfrak{m}. \tag{6.14}$$

Lemma 6.10. *Let* $k, l \in \mathbb{Z}$ *and let* $m = \min\{|k|, |l|\}$. *Then, as operators on* $M_\mathfrak{m}$*, we have*

$$Z_1^k Z_1^l = \begin{cases} Z_1^{k+l}, & kl \geq 0, \\ (1-q_1)^{-m} Z_1^{k+l}, & kl < 0, \end{cases} \tag{6.15}$$

$$Z_2^k Z_2^l = \begin{cases} Z_2^{k+l}, & kl \geq 0, \\ \lambda^m q_2^{(1-2l+(\operatorname{sgn} l)m)m/2} Z_2^{k+l}, & kl < 0. \end{cases} \tag{6.16}$$

Proof. Direct calculation using (6.13) and (6.14). For example if $k > 0$ and $l < 0$ we have

$$\begin{aligned} Z_2^k Z_2^l &= X_2^k Y_2^{-l} = X_2^{k-1}\sigma_2(t_2)Y_2^{-l-1} = \\ &= X_2^{k-1}Y_2^{-l-1}\sigma_2^{-l}(t_2) = X_2^{k-1}Y_2^{-l-1}\lambda q_2^{-l} = \ldots = \\ &= \lambda q_2^{-l}\lambda q_2^{-l-1}\ldots\lambda q_2^{-l-(m-1)}Z_2^{k+l} = \\ &= \lambda^m q_2^{-lm-m(m-1)/2}Z_2^{k+l}. \end{aligned}$$

□

Lemma 6.11. *Let $k,l \in \mathbb{Z}$ and let $m = \min\{|k|,|l|\}$. Then, as operators on $M_\mathfrak{m}$,*

$$Z_1^k Z_1^l = Z_1^l Z_1^k, \tag{6.17}$$

and

$$Z_2^k Z_2^l = c(k,l) Z_2^l Z_2^k, \tag{6.18}$$

where

$$c(k,l) = \begin{cases} 1, & kl \geq 0, \\ q_2^{(k-l)m-(\operatorname{sgn} k - \operatorname{sgn} l)m^2/2}, & kl < 0. \end{cases} \tag{6.19}$$

Proof. Follows directly from Lemma 6.10. □

Lemma 6.12. *Let $g = (g_1,g_2) \in \mathbb{Z}^2 = \tilde{G}_\mathfrak{m}$ and set $r_g = \varphi_\mathfrak{m}(a_g^* a_g)^{-1}$ where $\varphi_\mathfrak{m}$ is the projection $R \to R/\mathfrak{m} = K$. Then*

$$r_g = (1-q_1)^{|g_1|}(\lambda^{-1}q_2^{(g_2-1)/2})^{|g_2|} \tag{6.20}$$

and $(a_g)^{-1} = r_g a_g^* = r_g Z_2^{-g_2} Z_1^{-g_1}$ as operators on $M_\mathfrak{m}$.

Proof. We have

$$a_g^* a_g = (Z_1^{g_1}Z_2^{g_2})^* Z_1^{g_1}Z_2^{g_2} = Z_2^{-g_2}Z_1^{-g_1}Z_1^{g_1}Z_2^{g_2} = Z_1^{-g_1}Z_1^{g_1}Z_2^{-g_2}Z_2^{g_2},$$

by Lemma 6.9. Thus by Lemma 6.10,

$$\varphi_\mathfrak{m}(a_g^* a_g) = (1-q_1)^{-|g_1|}\lambda^{|g_2|}q_2^{(1-2g_2+g_2)|g_2|/2}$$

which proves the formula. The last statement is immediate. □

We consider the three subcases corresponding to the rank of the free abelian group K_2.

6.5.1. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 0$. $G_\mathfrak{m} = K_2 = \{0\}$ so $B_\mathfrak{m}^{(1)} = R$ which is commutative, hence $M_\mathfrak{m} = \mathbb{C}v_0$ for some v_0, and $S = \mathbb{Z}^2$. Thus $C = \{a_g v_0 \mid g \in \mathbb{Z}^2\}$ is a basis for M and using Lemma 6.10 and Lemma 6.9 we obtain that the action of X_i is given by

$$\begin{aligned} X_1 a_g v_0 &= \begin{cases} a_{g+e_1}v_0, & g_1 \geq 0, \\ (1-q_1)^{-1}a_{g+e_1}v_0, & g_1 < 0, \end{cases} \\ X_2 a_g v_0 &= \begin{cases} (q_1\lambda_{12})^{-g_1}a_{g+e_2}v_0, & g_2 \geq 0, \\ (q_1\lambda_{12})^{-g_1}\lambda q_2^{-g_2}a_{g+e_2}v_0, & g_2 < 0. \end{cases} \end{aligned} \tag{6.21}$$

The action of Y_i on the basis is deduced uniquely from

$$\begin{aligned} Y_1 X_1 a_g v_0 &= (1-q_1)^{-1}a_g v_0, \\ Y_2 X_2 a_g v_0 &= \lambda q_1^{-g_1} q_2^{-g_2} a_g v_0, \end{aligned} \tag{6.22}$$

which hold by (6.13) and (6.14).

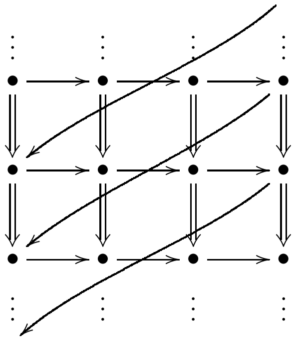


FIGURE 1. Example of a weight diagram for M when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$ and $\operatorname{rank} K_2 = 1$. Here $a = 4$, $b = -2$. The action of X_1 is indicated by $\rightarrow$ arrows, while $\Rightarrow$ arrows are used for X_2.

6.5.2. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 1$. Let (a, b) be a basis element. Since $G_\mathfrak{m} = K_2$ which is of rank one, $B_\mathfrak{m}^{(1)} \simeq \mathbb{C}[T, T^{-1}]$ by Corollary 4.6 so $M_\mathfrak{m}$ is one-dimensional. As before we let $M_\mathfrak{m} = \mathbb{C}v_0$. Then $Z_1^a Z_2^b v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$.

We assume $a \neq 0$. The case $b \neq 0$ can be treated similarly. By changing basis, we can assume that $a > 0$. Choose $S = \{0, 1, \ldots, a-1\} \times \mathbb{Z}$. The corresponding basis for M is

$$C = \{w_{ij} := X_1^i Z_2^j v_0 \mid 0 \leq i \leq a-1, j \in \mathbb{Z}\}.$$

We now aim to apply Theorem 5.4. If $0 \leq i < a-1$ then clearly $X_1 w_{ij} = w_{i+1,j}$. And

$$X_1 w_{a-1,j} = X_1^a Z_2^j v_0 \in \mathbb{C} Z_2^{j-b} v_0 = \mathbb{C} w_{0,j-b}.$$

We want to compute the coefficient of $w_{0,j-b}$. Similarly to the proof of Theorem 5.4 we have, using Lemma 6.12, Lemma 6.9 and (6.16),

$$\begin{aligned} X_1 w_{a-1,j} &= Z_1^a Z_2^j v_0 = (Z_1^a Z_2^j r_{(a,b)} Z_2^{-b} Z_1^{-a}) Z_1^a Z_2^b v_0 = \\ &= r_{(a,b)} (q_1 \lambda_{12})^{ja} q_1^{a \cdot \overline{-b}} \lambda_{12}^{-ab} Z_2^j Z_2^{-b} Z_1^a Z_1^{-a} \rho v_0 = \\ &= (\lambda^{-1} q_2^{(b-1)/2})^{|b|} q_1^{a(j + \overline{-b})} \lambda_{12}^{a(j-b)} \rho C_0 w_{0,j-b}, \end{aligned}$$

where

$$C_0 = \begin{cases} 1, & b \leq 0, \\ \lambda^{\min\{j,b\}} q_2^{(1+2b-\min\{j,b\})\min\{j,b\}/2}, & b > 0. \end{cases}$$

Using Lemma 6.9 one easily get the action of X_2 on the basis. We conclude that

$$\begin{aligned} X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & 0 \leq i < a-1, \\ (\lambda^{-1} q_2^{(b-1)/2})^{|b|} q_1^{a(j+\overline{-b})} \lambda_{12}^{a(j-b)} \rho C_0 w_{0,j-b}, & i = a-1, \end{cases} \\ X_2 w_{ij} &= \begin{cases} q_1^{-i} \lambda_{21}^i w_{i,j+1}, & j \geq 0, \\ q_1^{-i} \lambda_{21}^i \lambda q_2^j w_{i,j+1}, & j < 0. \end{cases} \end{aligned} \tag{6.23}$$

The action of the Y_i is uniquely determined by

$$\begin{aligned} Y_1 X_1 v_{ij} &= (1 - q_1)^{-1} v_{ij}, \\ Y_2 X_2 v_{ij} &= \lambda q_1^{-i} q_2^{-j} v_{ij}, \end{aligned} \tag{6.24}$$

which hold by (6.13)–(6.14). See Figure 1 for a visual representation.

6.5.3. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 2$. Let $s_1 = \mathbf{a} = (a_1, a_2)$, $s_2 = \mathbf{b} = (b_1, b_2)$ be a basis for $G_\mathfrak{m} = K_2$ over $\mathbb{Z}$. We can assume that $a_1, b_1 \geq 0$ and that $d := \left|\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right| > 0$.

By Corollary 4.6, $B_\mathfrak{m}^{(1)} \simeq T_\nu$ for some ν which we will now determine. Using Lemma 6.9 and Lemma 6.11 we have, as operators on $M_\mathfrak{m}$,

$$Z_1^{a_1} Z_2^{a_2} Z_1^{b_1} Z_2^{b_2} = q_1^{-b_1 \overline{a_1}} \lambda_{12}^{-b_1 a_2} c(a_2, b_2) Z_1^{b_1} Z_1^{a_1} Z_2^{b_2} Z_2^{a_2} =$$
$$= q_1^{a_1 \overline{b_2} - b_1 \overline{a_2}} \lambda_{12}^{a_1 b_2 - b_1 a_2} c(a_2, b_2) Z_1^{b_1} Z_2^{b_2} Z_1^{a_1} Z_2^{a_2}.$$

We conclude that $B_\mathfrak{m}^{(1)} \simeq T_\nu$ where

$$\nu = \lambda_{12}^d q_1^{a_1 \overline{b_2} - b_1 \overline{a_2}} c(a_2, b_2). \tag{6.25}$$

The function c was defined in (6.19), $d = a_1 b_2 - b_1 a_2$ and $\overline{k} := \max\{0, k\}$ for $k \in \mathbb{Z}$. For $M_\mathfrak{m}$ to be finite-dimensional it is thus necessary that this ν is a root of unity. Assume this and let r denote its order. Then $\dim M_\mathfrak{m} = r$. Let

$$\{v_0, v_1, \dots, v_{r-1}\} \tag{6.26}$$

be a basis such that

$$Z_1^{a_1} Z_2^{a_2} v_j = \nu^j \rho v_j, \tag{6.27}$$

$$Z_1^{b_1} Z_2^{b_2} v_j = \begin{cases} v_{j+1}, & 0 \leq j < r-1, \\ \mu v_0, & j = r-1, \end{cases} \tag{6.28}$$

where $\rho, \mu \in \mathbb{C}^*$.

The next step is to determine a set $S \subseteq \tilde{G}_\mathfrak{m} = \mathbb{Z}^2$ of representatives for the set of cosets $\tilde{G}_\mathfrak{m}/G_\mathfrak{m} = \mathbb{Z}^2/K_2$ which makes it possible to write down the action of the algebra later. We proceed as follows.

Recall that $K_2 = \mathbb{Z} \cdot (a_1, a_2) \oplus \mathbb{Z} \cdot (b_1, b_2)$. Let d_1 be the smallest positive integer such that $(d_1, 0) \in K_2$. We claim that $d_1 = d / \operatorname{GCD}(a_2, b_2)$. Indeed d_1 must be of the form $ka_1 + lb_1$ where $k, l \in \mathbb{Z}$ and $ka_2 + lb_2 = 0$ with $\operatorname{GCD}(k, l) = 1$. For such k, l, $k | b_2$, $l | a_2$ and $b_2/k = -a_2/l =: p > 0$. Then $\operatorname{GCD}(a_2/p, b_2/p) = 1$ which implies that $\operatorname{GCD}(a_2, b_2) = p$. Thus $d_1 = ka_1 + lb_1 = (b_2 a_1 - a_2 b_1)/p = d / \operatorname{GCD}(a_2, b_2)$ as claimed.

Next, let d_2 denote the smallest positive integer such that some K_2-translation of $(0, d_2)$ lies on the x-axis, i.e. such that

$$\big((0, d_2) + K_2\big) \cap \mathbb{Z} \times \{0\} \neq \emptyset.$$

Such an integer exists because if we write $\operatorname{GCD}(a_2, b_2) = ka_2 + lb_2$, then

$$(0, ka_2 + lb_2) - k(a_1, a_2) - l(b_1, b_2) = (-ka_1 - lb_1, 0).$$

On the other hand if $(0, d_2) + k\mathbf{a} + l\mathbf{b} \in \mathbb{Z} \times \{0\}$, i.e. if $d_2 = ka_2 + lb_2$, then $\operatorname{GCD}(a_2, b_2) | d_2$. Therefore $d_2 = \operatorname{GCD}(a_2, b_2)$.

We also see that for any point in $\mathbb{Z}^2$ of the form (x, d_2) there is a $g \in K_2$ such that $(x, d_2) + g \in \mathbb{Z} \times \{0\}$. Also, $(d_1, 0) \in K_2$ so for any point of the form (d_1, y) there is a $g \in K_2$ (namely $(-d_1, 0)$) such that $(d_1, y) + g \in \{0\} \times \mathbb{Z}$.

Suppose now that for some $k, l \in \mathbb{Z}$,

$$k(a_1, a_2) + l(b_1, b_2) \in K_2 \cap \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\}.$$

Then we would have $(0, ka_2 + lb_2) - (k\mathbf{a} + l\mathbf{b}) \in \mathbb{Z} \times \{0\}$ and $ka_2 + lb_2 \in \{0, 1, \dots, d_2 - 1\}$ which contradicts the minimality of d_2 unless $ka_2 + lb_2 = 0$. But in this case $(ka_1 + lb_1, 0) \in K_2$ which contradicts the minimality of d_1 unless $ka_1 + lb_1 = 0$. Hence $K_2 \cap \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\} = \{(0, 0)\}$. We have shown that

$$S := \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\}$$

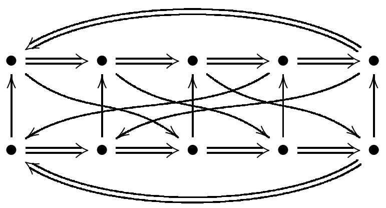


FIGURE 2. An example of the action on $\mathrm{supp}(M)$ when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$ and $\mathrm{rank}\, K_2 = 2$. Here $\mathbf{a} = (2, -2)$, $\mathbf{b} = (3, 2)$, $d_1 = 5, d_2 = 2$ and $s = 2$. The $\Rightarrow$ arrows indicate the action of X_1 and the $\rightarrow$ arrows show the action of X_2.

is a set of representatives for $\mathbb{Z}^2/K_2$. In particular we get from Corollary 5.3 that $\dim M$ is finite and

$$\dim M / \dim M_{\mathfrak{m}} = |S| = d_1 d_2 = a_1 b_2 - b_1 a_2.$$

We fix now integers a_2', b_2' such that

$$d_2 = \mathrm{GCD}(a_2, b_2) = a_2' a_2 + b_2' b_2 \tag{6.29}$$

and such that $-a_2' a_1 - b_2' b_1 \in \{0, 1, \ldots, d_1 - 1\}$. This can be done because for any $p \in \mathbb{Z}$, $(a_2'', b_2'') := (a_2' + p b_2/d_2, b_2' - p a_2/d_2)$ also satisfies $a_2'' a_2 + b_2'' b_2 = d_2$ but now

$$-a_2'' a_1 - b_2'' b_1 = -(a_2' + p b_2/d_2) a_1 - (b_2' - p a_2/d_2) b_1 = -a_2' a_1 - b_2' b_1 - p d_1.$$

We set

$$s = -a_2' a_1 - b_2' b_1. \tag{6.30}$$

Let $(i, j) \in S$. We have the following reductions in $\mathbb{Z}^2$ modulo K_2.

$$(1,0) + (i,j) = \begin{cases} (i+1, j), & 0 \leq i < d_1 - 1, \\ (0, j), & i = d_1 - 1, \end{cases}$$

$$(0,1) + (i,j) = \begin{cases} (i, j+1), & 0 \leq j < d_2 - 1, \\ (i+s, 0), & j = d_2 - 1, i + s \leq d_1 - 1, \\ (i+s-d_1, 0), & j = d_2 - 1, j + s > d_1 - 1. \end{cases}$$

From this we can understand how the X_i act on the support of M, see Figure 2 for an example. By Theorem 5.1 the set

$$C = \{w_{ijk} := X_1^i X_2^j v_k \mid 0 \leq i < d_1, 0 \leq j < d_2, 0 \leq k < r\}$$

is a basis for M where v_k is the basis (6.26) for $M_{\mathfrak{m}}$.

If $0 \leq i < d_1 - 1$ we clearly have $X_1 w_{ijk} = w_{i+1,j,k}$. Suppose $i = d_1 - 1$. Then by Lemma 6.9,

$$X_1 w_{ijk} = X_1^{d_1} X_2^j v_k = q_1^{d_1 j} \lambda_{12}^{d_1 j} X_2^j X_1^{d_1} v_k.$$

Thus we must express $X_1^{d_1}$ in terms of $Z_1^{a_1} Z_2^{a_2}$ and $Z_1^{b_1} Z_2^{b_2}$. Since $(d_1, 0) = b_2/d_2 \mathbf{a} - a_2/d_2 \mathbf{b}$ we have

$$(Z_1^{a_1} Z_2^{a_2})^{b_2/d_2} (Z_1^{b_1} Z_2^{b_2})^{-a_2/d_2} = C_1^{-1} X_1^{d_1} \tag{6.31}$$

as operators on $M_{\mathfrak{m}}$ for some constant C_1^{-1} which we must calculate.

Lemma 6.13. *The constant C_1 defined in (6.31) is given by*

$$C_1^{-1} = r_{\mathbf{a}}^{\overline{-b_2/d_2}} (q_1^{-a_1 \overline{a_2}} \lambda_{12}^{-a_1 a_2})^{\frac{b_2}{d_2}(\frac{b_2}{d_2} - 1)/2} \cdot r_{\mathbf{b}}^{\overline{a_2/d_2}} (q_1^{-b_1 \overline{b_2}} \lambda_{12}^{-b_1 b_2})^{\frac{a_2}{d_2}(\frac{a_2}{d_2} + 1)/2} \cdot \\ \cdot q_1^{b_1 a_2 \overline{a_2 b_2}/d_2^2} \lambda_{12}^{b_1 a_2^2 b_2/d_2^2} r_{(0, -b_2 a_2/d_2)}^{-1} C_1', \tag{6.32}$$

where the r_g, $g \in \mathbb{Z}^2$ *are given by (6.20),*

$$C_1' = \begin{cases} (1-q_1)^{-\min\{|a_1 b_2/d_2|,|b_1 a_2/d_2|\}}, & a_2 b_2 > 0,\\ 1, & a_2 b_2 \le 0, \end{cases}$$

$\overline{k} = \max\{0,k\}$ *for* $k \in \mathbb{Z}$ *and* $d_2 = \mathrm{GCD}(a_2, b_2)$.

Proof. If $b_2 \ge 0$ for example, we have by Lemma 6.9

$$\begin{aligned}(Z_1^{a_1} Z_2^{a_2})^{b_2/d_2} &= q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1 a_2} \cdot (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1 a_2})^2 \cdot \ldots \\ &\ldots \cdot (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1 a_2})^{b_2/d_2 - 1} Z_1^{a_1 b_2/d_2} Z_2^{a_2 b_2/d_2} = \\ &= (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1 a_2})^{\frac{b_2}{d_2}(\frac{b_2}{d_2}-1)/2} Z_1^{a_1 b_2/d_2} Z_2^{a_2 b_2/d_2}\end{aligned}$$

When $b_2 < 0$ we get a similar calculation where $r_{\mathbf{a}}^{-b_2/d_2}$ appears by Lemma 6.12. $(Z_1^{b_1} Z_2^{b_2})^{-a_2/d_2}$ can analogously be expressed as a multiple of $Z_1^{-b_1 a_2/d_2} Z_2^{-b_2 a_2/d_2}$. We then commute $Z_2^{a_2 b_2/d_2}$ and $Z_1^{-b_1 a_2/d_2}$ using Lemma 6.9. As a last step we use Lemma 6.10 and obtain two more factors. □

We conclude that

$$X_1 w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1,\\ q_1^{j d_1} \lambda_{12}^{j d_2} C_1 \nu^{b_2/d_2 k_1''} \rho^{b_2/d_2} \mu^{k_1'} w_{0,j,k_1''}, & i = d_1 - 1. \end{cases}$$

Here

$$k - a_2/d_2 = r k_1' + k_1'' \quad \text{with } 0 \le k_1'' < r. \tag{6.33}$$

Next we turn to the description of how X_2 acts on the basis C. If $0 \le j < d_2 - 1$ we have $X_2 w_{ijk} = q_1^{-i} \lambda_{12}^{-i} w_{i,j+1,k}$ by Lemma 6.9. Suppose $j = d_2 - 1$. Then, as in the first step of the proof of Theorem 5.4,

$$X_2 w_{ijk} = q_1^{-i} \lambda_{12}^{-i} X_1^i X_2^{d_2} v_k = q_1^{-i} \lambda_{12}^{-i} X_1^i (X_2^{d_2} r_{(-s,d_2)} Z_2^{-d_2} Z_1^s)(Z_1^{-s} Z_2^{d_2}) v_k. \tag{6.34}$$

By (6.16) and (6.20),

$$\begin{aligned} X_2^{d_2} r_{(-s,d_2)} Z_2^{-d_2} Z_1^s &= r_{(-s,d_2)} r_{(0,-d_2)}^{-1} Z_1^s = \\ &= (1-q_1)^s (\lambda^{-1} q_2^{(d_2-1)/2})^{d_2} (\lambda^{-1} q_2^{(-d_2-1)/2})^{d_2} Z_1^s = \\ &= (1-q_1)^s (\lambda^2 q_2)^{-d_2} Z_1^s. \end{aligned} \tag{6.35}$$

We must express $Z_1^{-s} Z_2^{d_2}$ in the generators of the algebra $B_{\mathfrak{m}}^{(1)}$ in order to calculate its action on v_k.

$$(Z_1^{a_1} Z_2^{a_2})^{a_2'} (Z_1^{b_1} Z_2^{b_2})^{b_2'} = C_2^{-1} Z_1^{-s} Z_2^{d_2}, \tag{6.36}$$

for some $C_2 \in \mathbb{C}^*$ since the degree on both sides are equal by (6.29) and (6.30). Similarly to the proof of Lemma 6.13,

$$\begin{aligned} C_2^{-1} = r_{\mathbf{a}}^{\overline{-a_2'}} (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1 a_2})^{a_2'(a_2'-1)/2} \cdot r_{\mathbf{b}}^{\overline{-b_2'}} (q_1^{-b_1\overline{b_2}} \lambda_{12}^{-b_1 b_2})^{b_2'(b_2'-1)/2} \cdot \\ \cdot q_1^{-b_1 b_2' \overline{a_2 a_2'}} \lambda^{-b_1 b_2' a_2 a_2'} C_2' C_2'', \end{aligned} \tag{6.37}$$

and

$$C_2' = \begin{cases} 1, & a_2' b_2' \ge 0,\\ (1-q_1)^{-\min\{|a_1 a_2'|,|b_1 b_2'|\}}, & a_2' b_2' < 0, \end{cases}$$

$$C_2'' = \begin{cases} 1, & a_2 a_2' b_2 b_2' \ge 0,\\ \lambda^{m'} q_2^{(1-2b_2 b_2' + (\operatorname{sgn} b_2 b_2') m') m'/2}, & a_2 a_2' b_2 b_2' < 0, \end{cases}$$

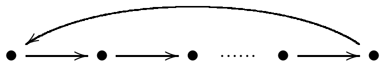


FIGURE 3. Weight diagram when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$, $\operatorname{rank} K_2 = 2$ and $q_1 = q_2$.

where $m' = \min\{|a_2 a_2'|, |b_2 b_2'|\}$. Furthermore, letting

$$b_2' + k = r k_2' + k_2'', \quad \text{where } 0 \le k_2'' < r \tag{6.38}$$

we have by (6.27)–(6.28),

$$(Z_1^{a_1} Z_2^{a_2})^{a_2'} (Z_1^{b_1} Z_2^{b_2})^{b_2'} v_k = \nu^{a_2' k_2''} \rho^{a_2'} \mu^{k_2'} v_{k_2''}. \tag{6.39}$$

If $i + s \le d_1 - 1$ we can now write down the action of X_2 on w_{ijk} by combining (6.34)–(6.37), (6.39) to get a multiple of $w_{i+s,0,k_2''}$. However if $i + s > d_1 - 1$, we must reduce further because then $(i+s, 0) \notin S$. Let

$$k_2'' - a_2/d_2 = r k_3' + k_3'', \quad \text{where } 0 \le k_3'' < r. \tag{6.40}$$

Then by the calculations for the action of $X_1^{d_1}$ on $M_\mathfrak{m}$,

$$X_1^{d_1} v_{k_2''} = X_1^{i+s-d_1} X_1^{d_1} v_{k_2''} = C_1 \mu^{k_3'} \nu^{k_3'' b_2/d_2} \rho^{b_2/d_2} w_{i+s-d_1,0,k_3''}.$$

Summing up, M has a basis

$$\{w_{ijk} \mid 0 \le i < d_1, 0 \le j < d_2, 0 \le k < r\}$$

and X_1, X_2 act on this basis as follows.

(6.41)

$$X_1 w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1, \\ q_1^{jd_1} \lambda_{12}^{jd_2} C_1 \nu^{b_2/d_2 k_1''} \rho^{b_2/d_2} \mu^{k_1'} w_{0,j,k_1''}, & i = d_1 - 1. \end{cases}$$

$$X_2 w_{ijk} = (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1,k}, \\ \qquad \text{if } 0 \le j < d_2 - 1, \\ (1-q_1)^s (\lambda^2 q_2)^{-d_2} C_2 \nu^{a_2' k_2''} \rho^{a_2'} \mu^{k_2'} w_{i+s,0,k_2''}, \\ \qquad \text{if } j = d_2 - 1 \text{ and } i + s \le d_1 - 1, \\ (1-q_1)^s (\lambda^2 q_2)^{-d_2} C_2 \nu^{a_2' k_2'' + k_3'' b_2/d_2} \rho^{a_2' + b_2/d_2} \mu^{k_2' + k_3'} C_1 w_{i+s-d_1,0,k_3''}, \\ \qquad \text{if } j = d_2 - 1 \text{ and } i + s > d_1 - 1, \end{cases}$$

where C_1 is given by (6.32), C_2 by (6.37) and ν by (6.25). The parameters ρ and μ comes from the action (6.27), (6.28) of $B_\mathfrak{m}^{(1)}$ on $M_\mathfrak{m}$ and k_i', k_i'' are defined in (6.33), (6.38) and (6.40).

The action of the Y_i is uniquely determined by

$$\begin{aligned} Y_1 X_1 w_{ijk} &= (1-q_1)^{-1} w_{ijk}, \\ Y_2 X_2 w_{ijk} &= \lambda q_1^{-i} q_2^{-j} w_{ijk}. \end{aligned} \tag{6.42}$$

We remark that the case $q_1 = q_2$ corresponds to $\mathbf{a} = (a_1, a_2) = (1, -1)$. Then $d_2 = 1$, $d_1 = d = |b_1 + b_2|$ and $s = 1$. X_1 and X_2 will act on the support in the same direction, cyclically as in Figure 3. The explicit action can be deduced from the above more general case noting that here $k_2'' = k$, $k_2' = 0$ and

$$k_1' = k_3' = \begin{cases} 0, & k < r-1, \\ 1, & k = r-1, \end{cases} \qquad k_1'' = k_3'' = \begin{cases} k, & k < r-1, \\ 0, & k = r-1. \end{cases}$$

6.6. **The case $\mathfrak{m} \notin \{\mathfrak{n}_\mu^{(i)} \mid \mu \in \mathbb{C}, i = 1,2\}$.** This is the generic case. We have $\mathbb{Z}^2_{\mathfrak{m}} = Q$ by Corollary 6.8. Our statements here generalize without any problem to the case of arbitrary rank.

Assume first that the q_i are roots of unity of orders o_i $(i = 1,2)$ and that ω does not contain any 1-breaks or 2-breaks. Then by Corollary 6.2 and Proposition 6.4 we have $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^2$. Thus $G_{\mathfrak{m}} = (o_1\mathbb{Z}) \times (o_2\mathbb{Z})$. Moreover,

$$X_1^{o_1} X_2^{o_2} = \lambda_{12}^{o_1 o_2} X_2^{o_2} X_1^{o_1}$$

so $B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda_{12}^{o_1 o_2}}$ by Corollary 4.6. This algebra has only finite dimensional representations if $\lambda_{12}^{o_1 o_2}$ is a root of unity. Assuming this, let r be the order of $\lambda_{12}^{o_1 o_2}$. Then there are $\rho, \mu \in \mathbb{C}^*$ and $M_{\mathfrak{m}}$ has a basis $v_0, v_1, \ldots, v_{r-1}$ such that

$$\begin{aligned} X_1^{o_1} v_i &= \lambda_{12}^{i o_1 o_2} \rho v_i \\ X_2^{o_1} v_i &= \begin{cases} v_{i+1} & 0 \le i < p-1 \\ \mu v_0 & i = p-1 \end{cases} \end{aligned}$$

Choose $S = \{0, 1, \ldots, o_1 - 1\} \times \{0, 1, \ldots, o_2 - 1\}$. The corresponding basis for M is $C = \{w_{ijk} := X_1^i X_2^j v_k \mid 0 \le i < o_1, 0 \le j < o_2, 0 \le k < r\}$. The following formulas are easily deduced using (2.6)–(2.8).

$$\begin{aligned} X_1 w_{ijk} &= \begin{cases} w_{i+1,j,k}, & k < o_1 - 1, \\ \lambda_{12}^{o_1(o_2 k + j)} \rho w_{0jk}, & k = o_1 - 1, \end{cases} \\ X_2 w_{ijk} &= (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1,l}, & l < o_2 - 1, \\ w_{i,0,l+1}, & l = o_2 - 1, i < r-1, \\ \mu w_{i00}, & l = o_2 - 1, i = r - 1. \end{cases} \end{aligned} \tag{6.43}$$

The action of Y_1, Y_2 is determined by

$$\begin{aligned} Y_1 X_1 w_{ijk} &= q_1^{-i}(\alpha_1 - [i]_{q_1}) w_{ijk}, \\ Y_2 X_2 w_{ijk} &= q_1^{-i} q_2^{-j} (\alpha_2 - [j]_{q_2}(1 + (q_1 - 1)\alpha_1)) w_{ijk}. \end{aligned} \tag{6.44}$$

In all other cases one can show using the same argument that $\dim M_{\mathfrak{n}} = 1$ for all $\mathfrak{n} \in \operatorname{supp}(M)$ and that M can be realized in a vector space with basis $\{w_{ij}\}_{(i,j)\in I}$, where $I = I_1 \times I_2$ is one of the following sets

$$\begin{gathered} \mathbb{N}_{d_1} \times \mathbb{N}_{d_2}, \quad \mathbb{N}_{d_1} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{N}_{d_2}, \quad \mathbb{Z} \times \mathbb{Z}, \\ \mathbb{Z}^{\pm} \times \mathbb{Z}, \quad \mathbb{Z} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{Z}^{\mp}, \end{gathered}$$

where $\mathbb{N}_d = \{0, 1, \ldots, d-1\}$, $\mathbb{Z}^{\pm} = \{k \in \mathbb{Z} \mid \pm k \ge 0\}$ and d_i is the order of q_i if finite. The action of the generators is given by the following formulas.

$$\begin{aligned} X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & (i+1, j) \in I, \\ \rho \lambda_{12}^{d_1 j} w_{0,j}, & (i+1, j) \notin I, I_1 = \mathbb{N}_{d_1} \text{ and } \alpha_1 \neq [i]_{q_1}, \\ 0, & \text{otherwise}, \end{cases} \\ X_2 w_{ij} &= (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1}, & (i, j+1) \in I, \\ \mu w_{i,0}, & (i, j+1) \notin I, I_2 = \mathbb{N}_{d_2} \\ & \qquad \text{and } \alpha_2 \neq [j]_{q_2}(1 + (q_1 - 1)\alpha_1), \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{6.45}$$

$$Y_1 w_{ij} = q_1^{-i+1}(\alpha_1 - [i-1]_{q_1})\cdot$$
$$\cdot\begin{cases} w_{i-1,j}, & (i-1,j)\in I,\\ (\rho\lambda_{12}^{d_1 j})^{-1} w_{d_1-1,j}, & (i-1,j)\notin I, I_1=\mathbb{N}_{d_1} \text{ and } \alpha_1\neq [i-1]_{q_1},\\ 0, & \text{otherwise,}\end{cases}$$

$$Y_2 w_{ij} = \lambda_{12}^{-i} q_2^{-j+1}(\alpha_2 - [j-1]_{q_2}(1+(q_1-1)\alpha_1))\cdot \tag{6.46}$$
$$\cdot\begin{cases} w_{i,j+1}, & (i,j+1)\in I,\\ \mu^{-1} w_{i,d_2-1}, & (i,j+1)\notin I, I_1=\mathbb{N}_{d_2}\\ & \quad\text{ and } \alpha_2\neq [j-1]_{q_2}(1+(q_1-1)\alpha_1),\\ 0, & \text{otherwise.}\end{cases}$$

Thus we have proved the following result.

Theorem 6.14. *Let A be a quantized Weyl algebra of rank two with arbitrary parameters $q_1, q_2 \in \mathbb{C}\backslash\{0,1\}$. Then any simple weight A-module with no proper inner breaks is isomorphic to one of the modules defined by formulas (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.21-6.22), (6.23-6.24), (6.41-6.42), (6.43-6.44) or (6.45-6.46).*

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Paper II

HOPF STRUCTURES ON AMBISKEW POLYNOMIAL RINGS

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ABSTRACT. We derive necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain type. This construction generalizes many known Hopf algebras, for example $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$ and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. In a torsion-free case we describe the finite-dimensional simple modules, in particular their dimensions and prove a Clebsch-Gordan decomposition theorem for the tensor product of two simple modules. We construct a Casimir type operator and prove that any finite-dimensional weight module is semisimple.

1. Introduction

In [4], the authors define a four parameter deformation of the Heisenberg (oscillator) Lie algebra $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$ and study its representations. Moreover by requiring this algebra to be invariant under $q \to q^{-1}$, they define a Hopf algebra structure on $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$ generalizing several previous results.

The quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ has by definition the structure of a Hopf algebra. In [6], an extension of this quantum group to an associative algebra denoted by $U_q(f(H,K))$ (where f is a Laurent polynomial in two variables) is defined and finite-dimensional representations are studied. The authors show that under certain conditions on f, a Hopf algebra structure can be introduced. Among these Hopf algebras is for example the Drinfeld double $\mathcal{D}(\mathfrak{sl}_2)$.

All of the mentioned algebras fall (after suitable mathematical formalization in the case of $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$) into the class of so called ambiskew polynomial rings (see Section 2 for the definition). Motivated by these examples of similar classes of algebras, all of which can be equipped with Hopf algebra structures, we consider a certain type of Hopf structures on a class of ambiskew polynomial rings.

In Section 2, we recall some definitions and fix notation. We present the conditions for a certain Hopf structure on an ambiskew polynomial ring in Section 3, while Section 4 is devoted to examples. In Section 5 we introduce some convenient notation and state some useful formulas for viewing R as an algebra of functions on its set of maximal ideals. Finite-dimensional simple modules are studied in Section 6. Those have already been classified in [7], but we focus on describing the dimensions in terms of the highest weights. The main result is stated in Theorem 6.17. The classical Clebsch-Gordan theorem for $U(\mathfrak{sl}_2)$ is generalized in Section 7 to the present more general setting, using the results of the previous section. Finally, in Section 8 we first construct a kind of Casimir operator and prove that it can be used to distinguish non-isomorphic simple modules. This is then used to prove that any weight module is semisimple.

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2. Preliminaries

Throughout, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All algebras are associative and unital $\mathbb{K}$-algebras.

By a *Hopf structure* on an algebra A we mean a triple (Δ, ε, S) where the *coproduct* $\Delta : A \to A \otimes A$ is a homomorphism, ($A \otimes A$ is given the tensor product algebra structure) the *counit* $\varepsilon : A \to \mathbb{K}$ is a homomorphism, and the *antipode* $S : A \to A$ is an anti-homomorphism such that

$$(\mathrm{Id} \otimes \Delta)(\Delta(x)) = (\Delta \otimes \mathrm{Id})(\Delta(x)), \qquad \text{(Coassociativity)} \tag{2.1}$$

$$(\mathrm{Id} \otimes \varepsilon)(\Delta(x)) = x = (\varepsilon \otimes \mathrm{Id})(\Delta(x)), \qquad \text{(Counit axiom)} \tag{2.2}$$

$$m\Big((S \otimes \mathrm{Id})(\Delta(x))\Big) = \varepsilon(x) = m\Big((\mathrm{Id} \otimes S)(\Delta(x))\Big), \qquad \text{(Antipode axiom)} \tag{2.3}$$

for all $x \in A$. Here $m : A \otimes A \to A$ denotes the multiplication map of A. A *Hopf algebra* is an algebra equipped with a Hopf structure. An element $x \in A$ of a Hopf algebra A is called *grouplike* if $\Delta(x) = x \otimes x$ and *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. In the former case it follows from the axioms that $\varepsilon(x) = 1$, x is invertible and $S(x) = x^{-1}$ while in the latter $\varepsilon(x) = 0$ and $S(x) = -x$.

If V_i $(i = 1, 2)$ are two modules over a Hopf algebra H, then $V_1 \otimes V_2$ becomes an H-module in the following way

$$a(v_1 \otimes v_2) = \sum_i (a_i' v_1) \otimes (a_i'' v_2) \tag{2.4}$$

for $v_i \in V_i$ $(i = 1, 2)$ if $a \in H$ with $\Delta(a) = \sum_i a_i' \otimes a_i''$. From (2.1) it follows that if V_i $(i = 1, 2, 3)$ are modules over H then the natural vector space isomorphism $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$ is an isomorphism of H-modules. From (2.2) follows that the one-dimensional module $\mathbb{K}_\varepsilon$ associated to the representation ε of H is a tensor unit, i.e. $\mathbb{K}_\varepsilon \otimes V \simeq V \simeq V \otimes \mathbb{K}_\varepsilon$ as H-modules for any H-module V.

Let R be a finitely generated commutative algebra over $\mathbb{K}$. Let σ be an automorphism of R, $\mathsf{h} \in R$ and $\xi \in \mathbb{K}\backslash\{0\}$. Then we define the algebra $A = A(R, \sigma, \mathsf{h}, \xi)$ as the associative $\mathbb{K}$-algebra formed by adjoining to R two symbols X_+, X_- subject to the relations

$$X_\pm a = \sigma^{\pm 1}(a) X_\pm \text{ for } a \in R, \tag{2.5}$$

$$X_+ X_- = \mathsf{h} + \xi X_- X_+. \tag{2.6}$$

This algebra is called an *ambiskew polynomial ring*. Its structure and representations were studied by Jordan [8] (see also references therein).

We recall the definition of a *generalized Weyl algebra (GWA)* (see [1] and references therein). If B is a ring, σ an automorphism of B, and $t \in B$ a central element, then the generalized Weyl algebra $B(\sigma, t)$ is the ring extension of B generated by two elements x_+, x_- subject to the relations

$$\begin{gathered} x_\pm a = \sigma^{\pm 1}(a) x_\pm, \quad \text{for } a \in B, \\ x_- x_+ = t, \quad \text{and} \quad x_+ x_- = \sigma(t). \end{gathered} \tag{2.7}$$

The relation between these two constructions is the following. Let $A = A(R, \sigma, \mathsf{h}, \xi)$ be an ambiskew polynomial ring. Denote by $R[t]$ be the polynomial ring in one variable t with coefficients in R and let us extend the automorphism σ of R to a $\mathbb{K}$-algebra automorphism of $R[t]$ satisfying

$$\sigma(t) = \mathsf{h} + \xi t. \tag{2.8}$$

Then A is isomorphic to the GWA $R[t](\sigma, t)$.

3. The Hopf structure

Let $A = A(R, \sigma, \mathsf{h}, \xi)$ be a skew polynomial ring and assume that R has been equipped with a Hopf structure. In this section we will extend the Hopf structure on R to A. We make the following anzats, guided by [4] and [6]:

$$\Delta(X_\pm) = X_\pm \otimes r_\pm + l_\pm \otimes X_\pm, \tag{3.1}$$

$$\varepsilon(X_\pm) = 0, \tag{3.2}$$

$$S(X_\pm) = s_\pm X_\pm. \tag{3.3}$$

The elements $r_\pm, l_\pm$ and $s_\pm$ will be assumed to belong to R.

Theorem 3.1. *Formulas (3.1)-(3.3) define a Hopf algebra structure on A which extends that of R iff*

$$(\sigma \otimes \mathrm{Id}) \circ \Delta|_R = \Delta \circ \sigma|_R = (\mathrm{Id} \otimes \sigma) \circ \Delta|_R, \tag{3.4a}$$

$$S \circ \sigma|_R = \sigma^{-1} \circ S|_R, \tag{3.4b}$$

$$\Delta(\mathsf{h}) = \mathsf{h} \otimes r_+ r_- + l_+ l_- \otimes \mathsf{h}, \tag{3.5a}$$

$$\varepsilon(\mathsf{h}) = 0, \tag{3.5b}$$

$$S(\mathsf{h}) = -(l_+ l_- r_+ r_-)^{-1} \mathsf{h}, \tag{3.5c}$$

$$r_\pm \textit{ and } l_\pm \textit{ are grouplike, i.e. } \Delta(x) = x \otimes x \textit{ for } x \in \{r_\pm, l_\pm\}, \tag{3.6a}$$

$$\sigma(l_\pm) \otimes \sigma(r_\mp) = \xi l_\pm \otimes r_\mp, \tag{3.6b}$$

$$(s_\pm)^{-1} = -l_\pm \sigma^{\pm 1}(r_\pm). \tag{3.7}$$

Proof. From (2.5)-(2.6) we see that ε extends to a homomorphism $A \to \mathbb{K}$ satisfying (3.2) if and only if (3.5b) holds. Assume for a moment that Δ extends to a homomorphism $A \to A \otimes A$. From (3.1)-(3.2) it follows that ε is a counit iff

$$\varepsilon(r_+) = \varepsilon(r_-) = \varepsilon(l_+) = \varepsilon(l_-) = 1. \tag{3.8}$$

Δ is coassociative iff (dropping the $\pm$)

$$(\mathrm{Id} \otimes \Delta)(\Delta(X)) = (\Delta \otimes \mathrm{Id})(\Delta(X))$$

which is equivalent to

$$X \otimes \Delta(r) + l \otimes X \otimes r + l \otimes l \otimes X = X \otimes r \otimes r + l \otimes X \otimes r + \Delta(l) \otimes X,$$

or

$$X \otimes (\Delta(r) - r \otimes r) = (\Delta(l) - l \otimes l) \otimes X. \tag{3.9}$$

From (2.5)-(2.6) follows that A has a $\mathbb{Z}$-gradation defined by requiring that $\deg r = 0$ for $r \in R$, $\deg X_\pm = \pm 1$. This also induces a $\mathbb{Z}^2$-gradation on $A \otimes A$ in a natural way. The left and right hand sides of equation (3.9) are homogenous of different $\mathbb{Z}^2$-degrees, namely $(\pm 1, 0)$ and $(0, \pm 1)$ respectively. Hence, since homogenous elements of different degrees must be linearly independent, (3.9) is equivalent to both sides being zero which holds iff $r_\pm$ and $l_\pm$ are grouplike.

Δ respects (2.5) iff (again dropping $\pm$)

$$\begin{gathered}\Delta(X)\Delta(a)=\Delta(\sigma(a))\Delta(X),\\ (X\otimes r+l\otimes X)\Delta(a)=\Delta(\sigma(a))(X\otimes r+l\otimes X),\\ (\sigma\otimes 1)(\Delta(a))\cdot(X\otimes r)+(1\otimes\sigma)(\Delta(a))\cdot(l\otimes X)=\Delta(\sigma(a))(X\otimes r+l\otimes X),\\ \big((\sigma\otimes 1)(\Delta(a))-\Delta(\sigma(a))\big)\cdot(X\otimes r)+\big((1\otimes\sigma)(\Delta(a))-\Delta(\sigma(a))\big)\cdot(l\otimes X)=0.\end{gathered}$$

As before the two terms in the last equation have different $\mathbb{Z}^2$-degrees and therefore must be zero. So Δ respects (2.5) iff (3.4a) holds.

It is straightforward to check that Δ respects (2.6) iff

$$\begin{aligned}\mathsf{h}\otimes r_+r_-+l_+l_-\otimes\mathsf{h}-\Delta(\mathsf{h})+&\\ +\Big(l_+\otimes\sigma(r_-)-\xi\sigma^{-1}(l_+)\otimes r_-\Big)X_-\otimes X_++&\\ +\Big(\sigma(l_-)\otimes-\xi l_-\otimes\sigma^{-1}(r_+)\Big)X_+\otimes X_-=0.&\end{aligned}\tag{3.10}$$

Again these three terms have different degrees so each of them must be zero. Hence (3.5a) holds. Multiply the second term by $X_+\otimes X_-$ from the right:

$$\big(l_+\otimes\sigma(r_-)-\xi\sigma^{-1}(l_+)\otimes r_-\big)t\otimes\sigma(t)=0.$$

Here we use the extension (2.8) of σ to $R[t]$ where $t=X_-X_+$. If we apply $e_1\otimes e_1'$ to this equation, where e_r (e_r') for $r\in R$ is the evaluation homomorphism $R[t]\to R$ which maps t ($\sigma(t)$) to r, we get

$$l_+\otimes\sigma(r_-)=\xi\sigma^{-1}(l_+)\otimes r_-.$$

Applying $\sigma\otimes 1$ to this we obtain one of the relations in (3.6b). Similarly the vanishing of the third term in (3.10) implies the other.

Assuming that S is an anti-homomorphism $A\to A$ satisfying (3.3), we obtain that S is a an antipode on A iff

$$S(X_\pm)r_\pm+S(l_\pm)X_\pm=0=X_\pm S(r_\pm)+l_\pm S(X_\pm),$$

which is equivalent to (3.7), using that $r_\pm$ and $l_\pm$ are grouplike. And S extends to a well-defined anti-homomorphism $A\to A$ iff

$$S(a)S(X_\pm)=S(X_\pm)S(\sigma^{\pm1}(a)),\quad\text{for } a\in R,\tag{3.11}$$

$$S(X_-)S(X_+)=S(\mathsf{h})+\xi S(X_+)S(X_-).\tag{3.12}$$

Using (3.7) and that $r_\pm,l_\pm$ are invertible, (3.11) holds iff (3.4b) holds. And (3.12) holds iff

$$\begin{aligned}0&=s_-X_-s_+X_+-S(\mathsf{h})-\xi s_+X_+s_-X_-=\\ &=s_-\sigma^{-1}(s_+)X_-X_+-S(\mathsf{h})-s_+\xi\sigma(s_-)X_+X_-=\\ &=-S(\mathsf{h})-s_+\sigma(s_-)\xi\mathsf{h}+\\ &\quad+\big(s_-\sigma^{-1}(s_+)-s_+\sigma(s_-)\xi^2\big)t.\end{aligned}$$

Applying e_0 and e_1 we obtain

$$\begin{gathered}S(\mathsf{h})=-\xi s_+\sigma(s_-)\mathsf{h},\\ s_-\sigma^{-1}(s_+)=\xi^2 s_+\sigma(s_-).\end{gathered}$$

Substituting (3.7) in these equations and using (3.6b), the first is equivalent to (3.5c), while the other already holds. □

4. Examples

Many Hopf algebras known in the literature can be viewed as one defined in the previous section.

4.1. **Heisenberg algebra.** Let $R = \mathbb{C}[c]$ with c primitive, and $\sigma(c) = c$. Choose $\mathsf{h} = c$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then A is the universal enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra.

4.2. $U(\mathfrak{sl}_2)$ **and its quantizations.**

4.2.1. $U(\mathfrak{sl}_2)$. Let $R = \mathbb{C}[H]$ with Hopf algebra structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$, $S(H) = -H$. Define $\sigma(H) = H - 1$. Choose $\mathsf{h} = H$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then $A \simeq U(\mathfrak{sl}_2)$ as Hopf algebras.

4.2.2. $U_q(\mathfrak{sl}_2)$. Let $R = \mathbb{C}[K, K^{-1}]$ with Hopf structure defined by requiring that K is grouplike. Define $\sigma(K) = q^{-2}K$, where $q \in \mathbb{C}, q^2 \neq 1$, and choose $\mathsf{h} = \frac{K-K^{-1}}{q-q^{-1}}$, $\xi = r_- = l_+ = 1$ and $r_+ = K$, $l_- = K^{-1}$. Then the equations in Theorem 3.1 are satisfied giving a Hopf algebra A which is isomorphic to $U_q(\mathfrak{sl}_2)$.

4.2.3. $\breve{U}_q(\mathfrak{sl}_2)$. For the definition of this algebra, see for example [10]. Let $q \in \mathbb{C}, q^4 \neq 1$. Let $R = \mathbb{C}[K, K^{-1}]$ with K grouplike. Define $\sigma(K) = q^{-1}K$, $\mathsf{h} = \frac{K^2-K^{-2}}{q-q^{-1}}$, $\xi = 1$, $r_+ = r_- = K$, $l_+ = l_- = K^{-1}$. Then $A = A(R, \sigma, \mathsf{h}, \xi)$ is a Hopf algebra isomorphic to $\breve{U}_q(\mathfrak{sl}_2)$.

4.3. $U_q(f(H,K))$. Let $R = \mathbb{C}[H, H^{-1}, K, K^{-1}]$, $\sigma(H) = q^2 H$, $\sigma(K) = q^{-2}K$. Let $\alpha \in \mathbb{K}$ and $M, p, r, s, t, p', r', s', t' \in \mathbb{Z}$ such that $M = m - n = m' - n' = p + t - r - s$, $s - t = s' - t'$ and $p - r = p' - r'$. Set $\mathsf{h} = \alpha(K^m H^n - K^{-m'} H^{-n'})$, $\xi = 1$, $r_+ = K^p H^r$, $l_+ = K^s H^t$, $r_- = K^{-s'} H^{-t'}$, $l_- = K^{-p'} H^{-r'}$. Then A is the Hopf algebra described in [6], Theorem 3.3.

4.4. **Down-up algebras.** The down-up algebra $A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{C}$, was defined in [2] and studied by many authors, see for example [3], [5], [8], [9], and references therein. It is the algebra generated by u, d and relations

$$\begin{aligned} ddu &= \alpha dud + \beta udd + \gamma d, \\ duu &= \alpha udu + \beta uud + \gamma u. \end{aligned}$$

In [8] it is proved that if σ is allowed to be any endomorphism, not necessarily invertible, then any down-up algebra is an ambiskew polynomial ring. Here we consider the down-up algebra $B = A(0, 1, 1)$. Thus B is the $\mathbb{C}$-algebra with generators u, d and relations

$$d^2 u = ud^2 + d, \quad du^2 = u^2 d + u. \tag{4.1}$$

Let $R = \mathbb{C}[\mathsf{h}]$, $\sigma(\mathsf{h}) = \mathsf{h} + 1$ and $\xi = -1$. Then B is isomorphic to the ambiskew polynomial ring $A(R, \sigma, \mathsf{h}, \xi)$ via $d \mapsto X_+$ and $u \mapsto X_-$.

One can show that B is isomorphic to the enveloping algebra of the Lie super algebra $\mathfrak{osp}(1, 2)$ and hence has a *graded* Hopf structure. A question was raised in [9] whether there exists a Hopf structure on B. We do not answer this question here but we show the existence of a Hopf structure on a larger algebra B_q giving us a formula for the tensor product of weight (in particular finite-dimensional) modules over B.

Let $q \in \mathbb{C}^*$ and fix a value of $\log q$. By q^a we always mean $e^{a \log q}$. Let B_q be the ambiskew polynomial ring $B_q = A(R, \sigma, \mathsf{h}, \xi)$ where $R = \mathbb{C}[\mathsf{h}, w, w^{-1}]$, $\sigma(\mathsf{h}) = \mathsf{h} + 1$, $\sigma(w) = qw$, and $\xi = -1$.

Theorem 4.1. *For any $\rho, \lambda \in \mathbb{Z}$ such that $q^{\rho-\lambda} = -1$ and $q^{2\rho} = 1$, the algebra B_q has a Hopf algebra structure given by*

$$\Delta(X_\pm) = X_\pm \otimes w^{\pm\rho} + w^{\pm\lambda} \otimes X_\pm, \quad \varepsilon(X_\pm) = 0,$$
$$S(X_\pm) = -w^{\mp\lambda} X_\pm w^{\mp\rho} = -q^\rho w^{\mp(\rho+\lambda)} X_\pm,$$

and

$$\Delta(w) = w \otimes w, \quad \varepsilon(w) = 1, \quad S(w) = w^{-1},$$
$$\Delta(\mathsf{h}) = \mathsf{h} \otimes 1 + 1 \otimes \mathsf{h}, \quad \varepsilon(\mathsf{h}) = 0, \quad S(\mathsf{h}) = -\mathsf{h}.$$

Proof. The subalgebra $\mathbb{C}[\mathsf{h}, w, w^{-1}]$ of B_q has a unique Hopf structure given by the maps above. We must verify (3.4)-(3.7) with $\mathsf{h} = v$, $\xi = -1$, $r_\pm = w^{\pm\rho}$, $l_\pm = w^{\pm\lambda}$, and $s_\pm = -q^\rho w^{\mp(\rho+\lambda)}$. This is straightforward. □

This gives us a tensor structure on the category of modules over B_q. Next aim is to show how using the Hopf structure on B_q one can define a tensor structure on the category of weight modules over B.

In general, if C is a commutative subalgebra of an algebra A, we say that an A-module V is a *weight module* with respect to C if

$$V = \oplus_{\mathfrak{m} \in \mathrm{Max}(C)} V_\mathfrak{m}, \qquad V_\mathfrak{m} = \{v \in V | \mathfrak{m} v = 0\},$$

where $\mathrm{Max}(C)$ denotes the set of all maximal ideals of C. When C is finitely generated this is equivalent to V having a basis in which each $c \in C$ acts diagonally.

By weight modules over B (B_q) we mean weight modules with respect to the subalgebra $\mathbb{C}[\mathsf{h}]$ ($\mathbb{C}[\mathsf{h}, w, w^{-1}]$). We need a simple lemma.

Lemma 4.2. *Any finite-dimensional module V over B is a weight module.*

Proof. By Proposition 5.3 in [8], any finite-dimensional B-module is semisimple. Since direct sums of weight modules are weight modules we can assume that V is simple. Since V is finite-dimensional, the commutative subalgebra $\mathbb{C}[\mathsf{h}]$ has a common eigenvector $v \neq 0$, i.e. $\mathfrak{m} v = 0$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{C}[\mathsf{h}]$. Acting on this weight vector by $X_\pm$ produces another weight vector: $\sigma^{\pm 1}(\mathfrak{m}) X_\pm v = X_\pm \mathfrak{m} v = 0$. Since B is generated by $\mathbb{C}[\mathsf{h}]$ and $X_\pm$, any vector in the B-submodule of V generated by v is a sum of weight vectors. But V was simple so $V = \oplus_\mathfrak{m} V_\mathfrak{m}$. □

Let $\mathcal{W}(B)$ denote the category of weight B-modules and similarly for B_q.

Theorem 4.3. *The category of weight modules over B can be embedded into the category of weight modules over B_q, i.e. there exist functors*

$$\mathcal{W}(B) \xrightarrow{\mathcal{E}} \mathcal{W}(B_q) \xrightarrow{\mathcal{R}} \mathcal{W}(B)$$

whose composition is the identity functor. In particular, the category of finite-dimensional B-modules can be embedded in $\mathcal{W}(B_q)$.

Proof. $\mathcal{R}$ is given by restriction. It takes weight modules to weight modules. Next we define $\mathcal{E}$. Let V be a weight module over B and define

$$wv = q^\alpha v \qquad \text{for } v \in V_{(\mathsf{h}-\alpha)} \text{ and } \alpha \in \mathbb{C}. \tag{4.2}$$

It is immediate that w commutes with h. Let $v \in V_{(\mathsf{h}-\alpha)}$ be arbitrary. Then

$$X_+wv = X_+q^\alpha v = q^\alpha X_+v.$$

On the other hand, since $\mathsf{h}X_+v = X_+(\mathsf{h}-1)v = (\alpha-1)X_+v$ which shows that $X_+v \in V_{(\mathsf{h}-(\alpha-1))}$, we have

$$qwX_+v = qq^{\alpha-1}X_+v = q^\alpha X_+v.$$

Thus $X_+w = qwX_+$. Similarly $X_-w = q^{-1}wX_-$ on V. Thus V becomes a module over B_q. That V is a weight module with respect to $\mathbb{C}[\mathsf{h}, w, w^{-1}]$ is clear. We define $\mathcal{E}(V)$ to be the same space V with additional action (4.2). If $\varphi : V \to W$ is a morphism of weight B-modules then $\varphi(wv) = w\varphi(v)$ for weight vectors v, since $\varphi(V_\mathfrak{m}) \subseteq W_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathbb{C}[\mathsf{h}]$. But then $\varphi(wv) = w\varphi(v)$ for all $v \in V$ since V is a weight module. Thus φ is automatically a morphism of B_q-modules and we set $\mathcal{E}(\varphi) = \varphi$. It is clear that the composition of the functors is the identity on objects and morphisms. □

Note that

$$\mathcal{E}\big(\mathcal{W}(B)\big) = \big\{V \in \mathcal{W}(B_q) \mid \operatorname{Supp}(V) \subseteq \{\mathfrak{m} = (\mathsf{h}-\alpha, w-q^\alpha) \mid \alpha \in \mathbb{C}\}\big\}. \tag{4.3}$$

It is not difficult to see that

$$\mathcal{E}(V_1) \otimes \mathcal{E}(V_2) \in \mathcal{E}\big(\mathcal{W}(B)\big)$$

and hence there is a unique $V_3 \in \mathcal{W}(B)$ such that

$$\mathcal{E}(V_1) \otimes \mathcal{E}(V_2) = \mathcal{E}(V_3).$$

Thus we can define

$$V_1 \otimes V_2 := V_3$$

and this will make $\mathcal{W}(B)$ into a tensor category.

4.5. **Non Hopf ambiskew polynomial rings.** There are many examples of ambiskew polynomial rings which do not have any Hopf structure. One example is the Weyl algebra $W = \langle a, b | ab - ba = 1\rangle$ which can have no counit ε. Indeed, a counit is in particular a homomorphism $\varepsilon : W \to \mathbb{C}$ so we would have $1 = \varepsilon(1) = \varepsilon(a)\varepsilon(b) - \varepsilon(b)\varepsilon(a) = 0$. Moreover all down-up algebras are ambiskew polynomial rings (see [8]) and [9] contains necessary conditions for the existence of a Hopf structure on a down-up algebra in terms of the parameters α, β, γ. More precisely, they show that if $A = A(\alpha, \beta, \gamma)$ is a Noetherian down-up algebra that is a Hopf algebra, then $\alpha + \beta = 1$. Moreover if $\gamma = 0$, then $(\alpha, \beta) = (2, -1)$ and as algebras, A is isomorphic to the universal enveloping algebra of the three-dimensional Heisenberg Lie algebra, while if $\gamma \neq 0$, then $-\beta$ is not an nth root of unity for $n \geq 3$. It would be of interest to generalize such a result to a more general class of ambiskew polynomial rings and also to other GWAs.

5. R as functions on a group

From now on we assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an algebra of the form defined in Section 3 and that conditions (3.4)-(3.7) hold so that A becomes a Hopf algebra with R as a Hopf subalgebra. Let G denote the set of all maximal ideals in R. Since $\mathbb{K}$ is algebraically closed and R is finitely generated, the inclusion map $i_\mathfrak{m} : \mathbb{K} \to R/\mathfrak{m}$ is onto for any $\mathfrak{m} \in G$ and we let $\varphi_\mathfrak{m} : R \to \mathbb{K}$ denote the composition of the projection $R \to R/\mathfrak{m}$ and $i_{\mathfrak{m}}^{-1}$. Thus $\varphi_{\mathfrak{m}}(a)$ is the unique element of $\mathbb{K}$ such that $a - \varphi_{\mathfrak{m}}(a) \in \mathfrak{m}$. We define the *weight sum* of $\mathfrak{m}, \mathfrak{n} \in G$ to be

$$\mathfrak{m} + \mathfrak{n} := \ker(m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta|_R).$$

This is the kernel of a $\mathbb{K}$-algebra homomorphisms $R \to \mathbb{K}$, hence $\mathfrak{m} + \mathfrak{n} \in G$. We will never use the usual addition of ideals so $+$ should not cause any confusion. Using that Δ is coassociative, ε is a counit and S is an antipode, one easily deduces that $+$ is associative, that $\underline{0} := \ker \varepsilon$ is a unit element and $S(\mathfrak{m})$ is the inverse of $\mathfrak{m}$. Thus G is a group under $+$. If R is cocommutative, G is abelian.

Example 5.1. Let $R = \mathbb{C}[H]$. Then $G = \{(H - \alpha) \,|\, \alpha \in \mathbb{C}\}$. Give R the Hopf structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$ and $S(H) = -H$. Then the operation $+$ will be

$$(H - \alpha) + (H - \beta) = (H - (\alpha + \beta)),$$

i.e. the correspondence $\mathbb{C} \ni \alpha \mapsto (H - \alpha) \in G$ is an additive group isomorphism.

If $R = \mathbb{C}[K, K^{-1}]$ then $G = \{(K - \alpha) \,|\, \alpha \in \mathbb{C}^*\}$. With the Hopf structure $\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$ and $S(K) = K^{-1}$, the operation $+$ will be

$$(K - \alpha) + (K - \beta) = (K - \alpha\beta)$$

for $\alpha, \beta \neq 0$. Thus $G \simeq \langle \mathbb{C}^*, \cdot \rangle$.

We will often think of elements from R as $\mathbb{K}$-valued functions on G and for $x \in R$ and $\mathfrak{m} \in G$ we will use the notation $x(\mathfrak{m})$ for $\varphi_{\mathfrak{m}}(x)$. Note however that different elements $x, y \in R$ can represent the same function. In fact one can check that the map from R to functions on G is a homomorphism of $\mathbb{K}$-algebras with kernel equal to the radical $\mathrm{Rad}(R) := \cap_{\mathfrak{m} \in G} \mathfrak{m}$.

Define a map

$$\zeta : \mathbb{Z} \to G, \; n \mapsto \underline{n} := \sigma^n(\underline{0}). \tag{5.1}$$

Lemma 5.2. *Let $\mathfrak{m}, \mathfrak{n} \in G$. Then for any $a \in R$,*

$$\sigma(a)(\mathfrak{m}) = a\big(\sigma^{-1}(\mathfrak{m})\big), \tag{5.2}$$

$$a(\mathfrak{m} + \mathfrak{n}) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta(a) = \sum_{(a)} a'(\mathfrak{m}) a''(\mathfrak{n}), \tag{5.3}$$

$$\mathfrak{m} + \underline{1} = \sigma(\mathfrak{m}) = \underline{1} + \mathfrak{m}. \tag{5.4}$$

Thus ζ is a group homomorphism and its image is contained in the center of G.

Proof. Since for any $a \in R$ we have

$$\sigma(a)(\mathfrak{m}) - a = \sigma^{-1}\big(\sigma(a)(\mathfrak{m}) - \sigma(a)\big) \in \sigma^{-1}(\mathfrak{m}),$$

(5.2) holds. Similarly,

$$a(\mathfrak{m} + \mathfrak{n}) - a \in \mathfrak{m} + \mathfrak{n}$$

so applying the map $m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta$ to $a(\mathfrak{m} + \mathfrak{n}) - a$ yields zero. This gives (5.3). Finally we have for any $a \in \mathfrak{m}$,

$$\begin{aligned}\sigma(a)(\mathfrak{m} + \underline{1}) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{1}}) \circ \Delta(\sigma(a)) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{1}}) \circ (1 \otimes \sigma)\Delta(a) = \\ = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{0}}) \circ \Delta(a) = a(\mathfrak{m} + \underline{0}) = a(\mathfrak{m}) = 0.\end{aligned}$$

Here we used (5.3) in the first and the fourth equality, (3.4a) in the second and (5.2) in the third. Thus $\sigma(\mathfrak{m}) \subseteq \mathfrak{m} + \underline{1}$ and then equality holds since both sides are maximal ideals. The proof of the other equality in (5.4) is symmetric. □

Example 5.3. If $R = \mathbb{C}[K, K^{-1}]$ with $\Delta(K) = K \otimes K, \varepsilon(K) = 1, S(K) = K^{-1}$ and $\sigma(K) = q^{-2}K$, then $\ker \varepsilon = (K - 1)$ so

$$\underline{n} = \sigma^n(\underline{0}) = \sigma^n((K-1)) = (q^{-2n}K - 1) = (K - q^{2n}).$$

From (5.3) follows that if $x \in R$ is grouplike, then viewed as a function $G \to \mathbb{K}$ it is a multiplicative homomorphism. Using (5.3) and (3.5a)-(3.5c), the following formulas are satisfied by h as a function on G.

$$\begin{aligned} \mathsf{h}(\mathfrak{m}+\mathfrak{n}) &= \mathsf{h}(\mathfrak{m})r(\mathfrak{n}) + l(\mathfrak{m})\mathsf{h}(\mathfrak{n}), \\ \mathsf{h}(\underline{0}) &= 0, \\ \mathsf{h}(-\mathfrak{m}) &= -r^{-1}l^{-1}\mathsf{h}(\mathfrak{m}), \end{aligned} \tag{5.5}$$

where $r = r_+r_-$ and $l = l_+l_-$.

6. Finite-dimensional simple modules

In this section we consider finite-dimensional simple modules over the algebra A. The main theorem is Theorem 6.17 where we, under the torsion-free assumption (6.1), characterize the finite-dimensional simple modules of a given dimension in terms of their highest weights. This result will be used in Section 7 to prove a Clebsch-Gordan decomposition theorem.

Throughout the rest of the paper we will assume that

$$\sigma^n(\mathfrak{m}) \neq \mathfrak{m} \text{ for any } n \in \mathbb{Z}\backslash\{0\} \text{ and any } \mathfrak{m} \in G. \tag{6.1}$$

By (5.4), this condition holds iff $\underline{1}$ has infinite order in G.

6.1. **Weight modules, Verma modules and their finite-dimensional simple quotients.** In this section we define weight modules, Verma modules and derive an equation for the dimension of its finite-dimensional simple quotients.

Let V be an A-module. We call $\mathfrak{m} \in G$ a *weight* of V if $\mathfrak{m}v = 0$ for some nonzero $v \in V$. The *support* of V, denoted $\mathrm{Supp}(V)$, is the set of weights of V. To a weight $\mathfrak{m}$ we associate its *weight space*

$$V_\mathfrak{m} = \{v \in V \mid \mathfrak{m}v = 0\}.$$

Elements of $V_\mathfrak{m}$ are called *weight vectors of weight* $\mathfrak{m}$. A module V is a *weight module* if $V = \oplus_\mathfrak{m} V_\mathfrak{m}$. A *highest weight vector* $v \in V$ of weight $\mathfrak{m}$ is a weight vector of weight $\mathfrak{m}$ such that $X_+v = 0$. A module V is called a *highest weight module* if it is generated by a highest weight vector. From the defining relations of A it follows that

$$X_\pm V_\mathfrak{m} \subseteq V_{\sigma^{\pm 1}(\mathfrak{m})}. \tag{6.2}$$

Equation (6.2) implies that a highest weight module is a weight module.

Let $\mathfrak{m} \in G$. The *Verma module* $M(\mathfrak{m})$ is defined as the left A-module $A/I(\mathfrak{m})$ where $I(\mathfrak{m})$ is the left ideal $AX_+ + A\mathfrak{m} \subseteq A$. From relations (2.5),(2.6) follows that

$$\{v_n := X_-^n + I(\mathfrak{m}) \mid n \geq 0\}$$

is a basis for $M(\mathfrak{m})$. It is clear that $M(\mathfrak{m})$ is a highest weight module generated by v_0. We also see that the vectors v_n ($n \geq 0$) are weight vectors of weights $\sigma^n(\mathfrak{m})$ respectively. By (6.1) we conclude $\dim M(\mathfrak{m})_\mathfrak{m} = 1$. Therefore the sum of all its proper submodules is proper and equals the unique maximal submodule $N(\mathfrak{m})$ of $M(\mathfrak{m})$. Thus $M(\mathfrak{m})$ has a unique simple quotient $L(\mathfrak{m})$. Since it is easy to see that any highest weight module over A of highest weight $\mathfrak{m}$ is a quotient of $M(\mathfrak{m})$ we deduce that $L(\mathfrak{m})$ is the unique irreducible highest weight module over A with given highest weight $\mathfrak{m} \in G$. We set

$$G_f := \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) < \infty\}.$$

Proposition 6.1. *Any finite-dimensional simple module over A is isomorphic to $L(\mathfrak{m})$ for some $\mathfrak{m} \in G_f$.*

Proof. Let V be a finite-dimensional simple A-module. Since $\mathbb{K}$ is algebraically closed, R has a common eigenvector $v \neq 0$, i.e. there exists $\mathfrak{n} \in G$ such that $\mathfrak{n}v = 0$. From (2.5) it follows that $\sigma^n(\mathfrak{n})(X_+)^n v = 0$ for any $n \geq 0$. By (6.1), the set $\{X_+^n v \mid n \geq 0\}$ is a set of weight vectors of different weights. Since V is finite-dimensional it follows that $(X_+)^n v = 0$ for some $n > 0$. This proves the existence of a highest weight vector of weight $\mathfrak{m}$ in V for some weight $\mathfrak{m}$. Thus $V = L(\mathfrak{m})$. □

Corollary 6.2. *Let V be a finite-dimensional weight module over A. Then*

$$\mathrm{Supp}(V) \subseteq G_f + \underline{\mathbb{Z}} = \{\mathfrak{m} + \underline{n} \mid \mathfrak{m} \in G_f, n \in \mathbb{Z}\}.$$

Proof. Let $\mathfrak{m} \in \mathrm{Supp}(V)$ and let $0 \neq v \in V_{\mathfrak{m}}$. Then $(X_+)^n v = 0$ for some smallest $n > 0$. But then $(X_+)^{n-1} v$ is a highest weight vector so its weight $\sigma^{n-1}(\mathfrak{m}) = \mathfrak{m} + \underline{n-1}$ must belong to G_f. Thus $\mathfrak{m} = \mathfrak{m} + \underline{n-1} - \underline{n-1} \in G_f + \underline{\mathbb{Z}}$. □

The following lemma was essentially proved in [7], Proposition 2.3, and the general result was mentioned in [8]. We give a proof for completeness.

Proposition 6.3. *The dimension of $L(\mathfrak{m})$ is the smallest positive integer n such that*

$$\sum_{k=0}^{n-1} \xi^{n-1-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = 0.$$

Proof. Let $e^{\mathfrak{m}}$ be a highest weight vector in $L(\mathfrak{m})$. Let $n > 0$ be the smallest positive integer such that $X_-^n e^{\mathfrak{m}} = 0$. Then the set spanned by the vectors $X_-^j e^{\mathfrak{m}}$, $0 \leq j < n$, is invariant under X_-, under R using (2.5), and under X_+, using (2.6). Hence it is a nonzero submodule and so coincides with $L(\mathfrak{m})$ since the latter is simple. Therefore $n = \dim L(\mathfrak{m})$. Let $k > 0$. Then $X_-^k e^{\mathfrak{m}} = 0$ implies that $X_+^k X_-^k e^{\mathfrak{m}} = 0$. Conversely, suppose $X_+^k X_-^k e^{\mathfrak{m}} = 0$. Then $X_+^{k-1} X_-^k e^{\mathfrak{m}}$ generates a proper submodule and thus is zero. Repeating this argument we obtain $X_-^k e^{\mathfrak{m}} = 0$. Hence $\dim L(\mathfrak{m})$ is the smallest positive integer n such that $X_+^n X_-^n e^{\mathfrak{m}} = 0$. Using induction it is easy to deduce the formulas

$$X_+ X_-^n = X_-^{n-1}\Big(\xi^n X_- X_+ + \sum_{k=0}^{n-1} \xi^{n-1-k} \sigma^k(\mathsf{h})\Big),$$

$$X_+^n X_-^n = \prod_{m=1}^{n} \Big(\xi^m X_- X_+ + \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(\mathsf{h})\Big). \tag{6.3}$$

Applying both sides of this equality to the vector $e^{\mathfrak{m}}$ gives

$$X_+^n X_-^n e^{\mathfrak{m}} = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(h) e^{\mathfrak{m}}. \tag{6.4}$$

Using that $e^{\mathfrak{m}}$ is a weight vector of weight $\mathfrak{m}$ and formula (5.2) we have

$$\sigma^k(\mathsf{h})e^{\mathfrak{m}} = \sigma^k(\mathsf{h})(\mathfrak{m})e^{\mathfrak{m}} = \mathsf{h}(\mathfrak{m} - \underline{k})e^{\mathfrak{m}}.$$

Substituting this into (6.4) we obtain

$$X_+^n X_-^n e^{\mathfrak{m}} = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k}\mathsf{h}(\mathfrak{m} - \underline{k})e^{\mathfrak{m}}.$$

The smallest positive n such that this is zero must be the one such that the last factor is zero. The claim is proved. $\square$

Corollary 6.4. *If* $\mathfrak{m}, \mathfrak{m}_0 \in G$ *where* $\mathsf{h}(\mathfrak{m}_0) = 0$, *then*

$$\dim L(\mathfrak{m}_0 + \mathfrak{m}) = \dim L(\mathfrak{m}) = \dim L(\mathfrak{m} + \mathfrak{m}_0).$$

Proof. Note that (5.5) implies that $\mathsf{h}(\mathfrak{n}+\mathfrak{m}_0) = \mathsf{h}(\mathfrak{n})r(\mathfrak{m}_0)$ and $\mathsf{h}(\mathfrak{m}_0+\mathfrak{n}) = l(\mathfrak{m}_0)\mathsf{h}(\mathfrak{n})$ for any $\mathfrak{n} \in G$, recall that r and l are invertible and use Proposition 6.3. $\square$

6.2. **Dimension and highest weights.** The goal in this subsection is to prove Theorem 6.17 which describes in detail the relationship between the dimension of a finite-dimensional simple module and its highest weight.

We begin with a few useful lemmas. Recall that $r = r_+r_-$ and $l = l_+l_-$. For brevity we set $r_1 = r(\underline{1})$ and $l_1 = l(\underline{1})$. Since $r_\pm, l_\pm$ are grouplike so are r and l and thus r_1, l_1 are nonzero scalars.

Lemma 6.5. *We have*
a) $\xi^2 r_1 l_1 = 1$,
b) $\mathsf{h}(-\underline{k}) = -r_1^{-k}l_1^{-k}\mathsf{h}(\underline{k})$ *for any* $k \in \mathbb{Z}$,
c) for any $k \in \mathbb{Z}$ *and* $\mathfrak{m} \in G$ *we have*

$$\xi^k\mathsf{h}(\mathfrak{m} + \underline{k}) + \xi^{-k}\mathsf{h}(\mathfrak{m} - \underline{k}) = \big((\xi r_1)^k + (\xi r_1)^{-k}\big)\mathsf{h}(\mathfrak{m}). \tag{6.5}$$

Proof. For a), multiply the two equations in (3.6b) and apply the multiplication map to both sides to obtain

$$\sigma(l_+l_-r_+r_-) = \xi^2 l_+l_-r_+r_-.$$

Evaluate both sides at $\underline{1}$ to get

$$1 = lr(\underline{0}) = lr(\sigma^{-1}(\underline{1})) = \sigma(lr)(\underline{1}) = \xi^2 lr(\underline{1}) = \xi^2 l_1 r_1.$$

Next (5.5) gives for any $k \in \mathbb{Z}$,

$$0 = \mathsf{h}(\underline{k} - \underline{k}) = \mathsf{h}(\underline{k})r_1^{-k} + l_1^k\mathsf{h}(-\underline{k}),$$

hence b) follows. Finally, using (5.5) again, we have

$$\begin{aligned}\xi^k\mathsf{h}(\mathfrak{m} + \underline{k}) + \xi^{-k}\mathsf{h}(\mathfrak{m} - \underline{k}) &= \xi^k\mathsf{h}(\mathfrak{m})r_1^k + \xi^k l(\mathfrak{m})\mathsf{h}(\underline{k}) + \\ &\quad + \xi^{-k}\mathsf{h}(\mathfrak{m})r_1^{-k} + \xi^{-k}l(\mathfrak{m})\mathsf{h}(-\underline{k}) = \\ &= \mathsf{h}(\mathfrak{m})\big((\xi r_1)^k + (\xi r_1)^{-k}\big) + \\ &\quad + l(\mathfrak{m})\mathsf{h}(\underline{k})(\xi^k - \xi^{-k}r_1^{-k}l_1^{-k}).\end{aligned}$$

In the last equality we used part b). Now the second term in the last expression vanishes due to part a). Thus c) follows. $\square$

In what follows, we will treat the two cases when $\mathsf{h}(\underline{1}) = 0$ and $\mathsf{h}(\underline{1}) \neq 0$ separately. The algebras satisfying the former condition have a representation theory which reminds of that of the enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra, while the latter case includes $U(\mathfrak{sl}_2)$ and other algebras with similar structure of representations.

6.2.1. *The case* $\mathsf{h}(\underline{1}) = 0$.

Proposition 6.6. *If* $\mathsf{h}(\underline{1}) = 0$*, then*

$$\xi^2 = r_1^2 = 1,\ \sigma(\mathsf{h}) = r_1\mathsf{h},\ \sigma(r) = r_1 r,\ \text{and } \sigma(l) = r_1 l. \tag{6.6}$$

In particular, $\langle X_+, X_-, \mathsf{h}\rangle$ *is a subalgebra of* A *with relations*

$$\begin{aligned}
&[X_+, X_-] = h, && [h, X_\pm] = 0, && \text{if } \xi = 1,\ r_1 = 1,\\
&[X_+, X_-] = h, && \{h, X_\pm\} = 0, && \text{if } \xi = 1,\ r_1 = -1,\\
&\{X_+, X_-\} = h, && [h, X_\pm] = 0, && \text{if } \xi = -1,\ r_1 = 1,\\
&\{X_+, X_-\} = h, && \{h, X_\pm\} = 0, && \text{if } \xi = -1,\ r_1 = -1,
\end{aligned}$$

respectively, where $\{\cdot,\cdot\}$ *denotes anti-commutator.*

Proof. Suppose $\mathsf{h}(\underline{1}) = 0$. Then, by Lemma 6.5b), $\mathsf{h}(-\underline{1}) = 0$. This means that $\mathsf{h} \in \underline{-1} = \sigma^{-1}(\underline{0}) = \sigma^{-1}(\ker\varepsilon)$. Thus $\varepsilon(\sigma(\mathsf{h})) = 0$. Using (2.2), (3.4a) and (3.5a) we deduce

$$\begin{aligned}
\sigma(\mathsf{h}) &= (\varepsilon\otimes 1)(\Delta(\sigma(\mathsf{h}))) = (\varepsilon\otimes 1)(\sigma\otimes 1)(\Delta(\mathsf{h})) =\\
&= \varepsilon(\sigma(\mathsf{h}))\otimes r + \varepsilon(\sigma(l))\otimes \mathsf{h} = \varepsilon(\sigma(l))\mathsf{h}.
\end{aligned}$$

Analogously one proves $\sigma(\mathsf{h}) = \varepsilon(\sigma(r))\mathsf{h}$. Hence $\varepsilon(\sigma(r)) = \varepsilon(\sigma(l))$. But

$$\varepsilon(\sigma(r)) = \sigma(r)(\ker\varepsilon) = \sigma(r)(\underline{0}) = r(-\underline{1}) = r_1^{-1}$$

and similarly for l. So $r_1 = l_1$. From Lemma 6.5a) we obtain $(\xi r_1)^2 = 1$. Now

$$S(\sigma(\mathsf{h})) = S(r_1^{-1}\mathsf{h}) = -r_1^{-1}\mathsf{h}, \quad \text{and} \quad \sigma^{-1}(S(\mathsf{h})) = \sigma^{-1}(-h) = -r_1\mathsf{h},$$

so (3.4b) implies that $r_1^2 = 1$. A similar calculation as above shows that $\sigma(r) = r_1^{-1}r = r_1 r$ and $\sigma(l) = l_1^{-1}l = r_1 l$. □

We leave it to the reader to prove the following statement.

Proposition 6.7. *All finite-dimensional simple modules over an algebra* $A(R, \sigma, \mathsf{h}, \xi)$ *satisfying* (6.6) *and one of the commutation relations above are either one- or two-dimensional.*

Remark 6.8. The algebra $U(\mathfrak{h}_3)$ is an ambiskew polynomial ring, as shown in Section 4.1. For this algebra we have $\mathsf{h}(\underline{1}) = 0$ and $\xi = r_1 = 1$.

6.2.2. *The case* $\mathsf{h}(\underline{1}) \neq 0$. In this section, we consider the more complicated case when $\mathsf{h}(\underline{1}) \neq 0$. We prove Theorem 6.17 which describes the dimensions of $L(\mathfrak{m})$ in terms of $\mathfrak{m}$. The following two subsets of G will play a vital role:

$$G_0 = \{\mathfrak{m} \in G \mid \mathsf{h}(\mathfrak{m}) = 0\}, \tag{6.7}$$

$$G_{1/2} = \{\mathfrak{m} \in G \mid \mathsf{h}(\mathfrak{m} - \underline{1}) + \xi\mathsf{h}(\mathfrak{m}) = 0\}. \tag{6.8}$$

The reason for this notation is that when $A = U(\mathfrak{sl}_2)$ as in Section 4.2.1 then we have $G_0 = \{(H-0)\}$ and $G_{1/2} = \{(H - \frac{1}{2})\}$. From (5.5) it is immediate that G_0 is a subgroup of G. By Proposition 6.3 we have

$$G_0 = \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) = 1\}. \tag{6.9}$$

The following analogous result holds for $G_{1/2}$.

Proposition 6.9.

$$G_{1/2} = \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) = 2\}. \tag{6.10}$$

Proof. If $\mathfrak{m} \in G_{1/2}$, then by Proposition 6.3, $\dim L(\mathfrak{m}) \leq 2$. But if $\dim L(\mathfrak{m}) = 1$, then $\mathsf{h}(\mathfrak{m}) = 0$ so using $\mathfrak{m} \in G_{1/2}$ we get $\mathsf{h}(\mathfrak{m} - \underline{1}) = 0$ also. Since G_0 is a group we deduce that $\underline{1} \in G_0$, i.e. $\mathsf{h}(\underline{1}) = 0$ which is a contradiction. So $\dim L(\mathfrak{m}) = 2$. The converse inclusion is immediate from Proposition 6.3. ☐

Set

$$N = \begin{cases} \text{order of } \xi r_1 & \text{if } (\xi r_1)^2 \neq 1 \text{ and } \xi r_1 \text{ is a root of unity,} \\ \infty & \text{otherwise.} \end{cases} \tag{6.11}$$

We also set

$$N' = \begin{cases} N, & \text{if } N \text{ is odd,} \\ N/2, & \text{if } N \text{ is even,} \\ \infty, & \text{if } N = \infty. \end{cases}$$

The next statement describes the intersection of G_0 and $G_{1/2}$ with $\underline{\mathbb{Z}}$.

Proposition 6.10. *We have*

$$G_0 \cap \underline{\mathbb{Z}} = \begin{cases} \{\underline{0}\}, & \text{if } N = \infty, \\ \underline{N'\mathbb{Z}}, & \text{otherwise,} \end{cases} \tag{6.12}$$

and

$$G_{1/2} \cap \underline{\mathbb{Z}_{>0}} = \begin{cases} \emptyset, & \text{if } N = \infty, \\ \{\underline{n} \in \underline{\mathbb{Z}_{>0}} : N \big| 2n-1\}, & \text{otherwise.} \end{cases} \tag{6.13}$$

Remark 6.11. The set $G_{1/2} \cap \underline{\mathbb{Z}_{\leq 0}}$ can be understood using (6.13) and Lemma 6.14a).

Proof. We first prove (6.12). Let $n \in \mathbb{Z}$. The right hand side of (6.12) is invariant under $n \mapsto -n$. By Lemma 6.5b) so is the left hand side. Moreover since $\mathsf{h}(\underline{0}) = 0$, the ideal $\underline{0}$ belongs to both sides of the equality. Thus we can assume $n > 0$.

Using (5.5) and that r and l, viewed as functions $G \to \mathbb{K}$, are multiplicative homomorphisms it follows by induction that

$$\mathsf{h}(\underline{n}) = \mathsf{h}(\underline{1}) \sum_{i=0}^{n-1} r_1^i l_1^{n-1-i}.$$

By Lemma 6.5a), $r_1/l_1 = (\xi r_1)^2/(\xi^2 r_1 l_1) = (\xi r_1)^2$, so we can rewrite this as

$$\mathsf{h}(\underline{n}) = \mathsf{h}(\underline{1}) l_1^{n-1} \sum_{i=0}^{n-1} (\xi r_1)^{2i}. \tag{6.14}$$

If $N=\infty$ and $(\xi r_1)^2\neq 1$ then by (6.14) we have $\underline{n}\in G_0\cap\underline{\mathbb{Z}}$ iff $(\xi r_1)^{2n}=1$, which is false. If $(\xi r_1)^2=1$, then (6.14) implies that $\underline{n}\notin G_0\cap\underline{\mathbb{Z}}$. If $N<\infty$, then $(\xi r_1)^2\neq 1$ so by (6.14), $\mathsf{h}(\underline{n})=0$ iff $(\xi r_1)^{2n}=1$ i.e. iff $N\big|2n$. This is equivalent to $N'\big|n$.

Next we prove (6.13). Suppose $n\in\mathbb{Z}_{>0}$. By definition, $n\in G_{1/2}$ iff

$$\mathsf{h}(\underline{n}-\underline{1})+\xi\mathsf{h}(\underline{n})=0.$$

Using (6.14) on both terms and dividing by $\mathsf{h}(\underline{1})\xi l_1^{n-1}$, this is equivalent to

$$\xi^{-1}l_1^{-1}\sum_{k=0}^{n-2}(\xi r_1)^{2k}+\sum_{k=0}^{n-1}(\xi r_1)^{2k}=0.$$

But $\xi^{-1}l_1^{-1}=\xi r_1$ by Lemma 6.5a) so this can be rewritten as

$$\sum_{k=0}^{2n-2}(\xi r_1)^k=0. \tag{6.15}$$

Thus $(\xi r_1)^2\neq 1$ and multiplying by ξr_1-1 we get $(\xi r_1)^{2n-1}=1$. Therefore $N<\infty$ and $N\big|2n-1$. Conversely, if $N<\infty$ and $N\big|2n-1$ then $(\xi r_1)^2\neq 1$ and $(\xi r_1)^{2n-1}=1$ which implies (6.15). This proves (6.13). □

Proposition 6.12. *Suppose* $\mathsf{h}(\underline{1})\neq 0$ *and* $G_{1/2}\neq\emptyset$. *Then*
a) $\xi r_1\neq -1$, *and*
b) $G_{1/2}$ *is a left and right coset of* G_0 *in* G.

Proof. Let $\mathfrak{m}_{1/2}\in G_{1/2}$. To prove a), suppose that $\xi r_1=-1$. Then

$$\begin{aligned}0&=\mathsf{h}(\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})=\\&=\mathsf{h}(\mathfrak{m}_{1/2})r(-\underline{1})+l(\mathfrak{m}_{1/2})\mathsf{h}(-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})=\\&=\mathsf{h}(\mathfrak{m}_{1/2})(r_1^{-1}+\xi)+l(\mathfrak{m}_{1/2})\mathsf{h}(-\underline{1})=\\&=-l(\mathfrak{m}_{1/2})r_1^{-1}l_1^{-1}\mathsf{h}(\underline{1}),\end{aligned}$$

where we used Lemma 6.5b) in the last equality. Since l is invertible we deduce that $\mathsf{h}(\underline{1})=0$ which is a contradiction.

To prove part b), we will show that

$$G_{1/2}=G_0+\mathfrak{m}_{1/2}.$$

One proves $G_{1/2}=\mathfrak{m}_{1/2}+G_0$ in an analogous way. Let $\mathfrak{m}\in G_0$ be arbitrary. Then using (5.5) twice,

$$\mathsf{h}(\mathfrak{m}+\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}+\mathfrak{m}_{1/2})=l(\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})\big)=0.$$

Since l is invertible we get $\mathfrak{m}+\mathfrak{m}_{1/2}\in G_{1/2}$.

Conversely, suppose $\mathfrak{m}\in G_{1/2}$. Then

$$\begin{aligned}\mathsf{h}(\mathfrak{m}-\underline{1})+\xi\mathsf{h}(\mathfrak{m})=0,\\ \mathsf{h}(\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})=0.\end{aligned}$$

Multiply the first equation by $r(-\mathfrak{m}_{1/2})$ and the second by $-r(-\mathfrak{m}_{1/2})l(-\mathfrak{m}_{1/2})l(\mathfrak{m})$ and add them together. Then we get

$$\begin{aligned}&\big((\mathsf{h}(\mathfrak{m})r_1^{-1}+l(\mathfrak{m})\mathsf{h}(-\underline{1})\big)r(-\mathfrak{m}_{1/2})-\\&\qquad r(-\mathfrak{m}_{1/2})l(-\mathfrak{m}_{1/2})l(\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}_{1/2}r_1^{-1}+l(\mathfrak{m}_{1/2})\mathsf{h}(-\underline{1})\big)+\xi\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2})=0,\end{aligned}$$

or equivalently,

$$\mathsf{h}(\mathfrak{m})r_1^{-1}r(-\mathfrak{m}_{1/2}) - r(-\mathfrak{m}_{1/2})l(-\mathfrak{m}_{1/2})l(\mathfrak{m})\mathsf{h}(\mathfrak{m}_{1/2})r_1^{-1} + \xi\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2}) = 0.$$

Using (5.5) this can be written

$$r_1^{-1}(1+\xi r_1)\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2}) = 0.$$

Since $\xi r_1 \neq -1$ by part a), we conclude that $\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2}) = 0$. This shows that $\mathfrak{m} \in G_0 + \mathfrak{m}_{1/2}$. □

The following lemma will be useful.

Lemma 6.13. *Let $j \in \mathbb{Z}$. If $\mathfrak{m}_0 \in G_0$, then*

$$\mathfrak{m}_0 + \underline{j} \in G_0 \Longleftrightarrow \underline{j} \in G_0, \tag{6.16}$$

and if $\mathsf{h}(\underline{1}) \neq 0$ and $\mathfrak{m}_{1/2} \in G_{1/2}$, then

$$\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2} \Longleftrightarrow \underline{j} \in G_0. \tag{6.17}$$

Proof. (6.16) is immediate since G_0 is a subgroup of G. If $\underline{j} \in G_0$, then $\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2}$ by Proposition 6.12. Conversely, if $\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2}$ then by Proposition 6.12, $G_0 \ni \mathfrak{m}_{1/2} + \underline{j} - \mathfrak{m}_{1/2} = \underline{j}$. □

The next statements will be needed in Section 8.

Lemma 6.14. *Suppose $\mathsf{h}(\underline{1}) \neq 0$ and let $\mathfrak{m}, \mathfrak{n} \in G_{1/2}$. Then*

a) *$\underline{1} - \mathfrak{m} \in G_{1/2}$, and*
b) *$\mathfrak{m} + \mathfrak{n} - \underline{1} \in G_0$.*

Proof. Part a) follows from the calculation

$$\begin{aligned}\mathsf{h}(\underline{1}-\mathfrak{m}-\underline{1}) + \xi\mathsf{h}(\underline{1}-\mathfrak{m}) &= -l(-\mathfrak{m})r(-\mathfrak{m})\mathsf{h}(\mathfrak{m}) - \xi l(\underline{1}-\mathfrak{m})(r(\underline{1}-\mathfrak{m})\mathsf{h}(\mathfrak{m}-\underline{1}) = \\ &= -l(-\mathfrak{m})r(-\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}) + \xi r_1 l_1 \mathsf{h}(\mathfrak{m}-\underline{1})\big) = \\ &= -l(-\mathfrak{m})r(-\mathfrak{m})\xi^{-1}\big(\xi\mathsf{h}(\mathfrak{m}) + \mathsf{h}(\mathfrak{m}-\underline{1})\big) = 0.\end{aligned}$$

For part b), use that $\dim L(\underline{1}-\mathfrak{n}) = 2$ by part a), and thus $\mathfrak{m}+\mathfrak{n}-\underline{1} = \mathfrak{m}-(\underline{1}-\mathfrak{n}) \in G_0$ by Proposition 6.12b). □

The formulas provided by the following technical lemma are the key to proving our main theorem.

Lemma 6.15. *Let $\mathfrak{m} \in G$ and $j \in \mathbb{Z}_{\geq 0}$. If $n = 2j+1$ then*

$$\sum_{k=0}^{n-1} \xi^{n-1-k}\mathsf{h}(\mathfrak{m}-\underline{k}) = r_1^{-j}\mathsf{h}(\mathfrak{m}-\underline{j})\sum_{k=0}^{n-1}(\xi r_1)^k \tag{6.18}$$

and if $n = 2j+2$ then

$$\sum_{k=0}^{n-1} \xi^{n-1-k}\mathsf{h}(\mathfrak{m}-\underline{k}) = r_1^{-j}\big(\mathsf{h}(\mathfrak{m}-\underline{j}-\underline{1}) + \xi\mathsf{h}(\mathfrak{m}-\underline{j})\big)\sum_{k=0}^{n/2-1}(\xi r_1)^{2k}. \tag{6.19}$$

Proof. If $n = 2j+1$, we make the change of index $k \mapsto j-k$, then factor out ξ^j and apply formula (6.5):

$$\sum_{k=0}^{2j} \xi^{2j-k}\mathsf{h}(\mathfrak{m}-\underline{k}) = \sum_{k=-j}^{j} \xi^{j+k}\mathsf{h}(\mathfrak{m}-\underline{j}+\underline{k}) = \xi^j\mathsf{h}(\mathfrak{m}-\underline{j})\sum_{k=-j}^{j}(\xi r_1)^k.$$

Factoring out $(\xi r_1)^{-j}$ and changing index from k to $k-j$ yields (6.18).

For the $n=2j+2$ case we first split the sum in the left hand side of (6.19) into two sums corresponding to odd and even k:

$$\sum_{k=0}^{j}\xi^{2j-2k}\mathsf{h}(\mathfrak{m}-\underline{2k}-\underline{1})+\sum_{k=0}^{j}\xi^{2j+1-2k}\mathsf{h}(\mathfrak{m}-\underline{2k})$$

Then we make the change of summation index $k\mapsto -k+j/2$ in both sums

$$\xi^j\sum_{k=-j/2}^{j/2}\xi^{2k}\mathsf{h}(\mathfrak{m}-\underline{j}-\underline{1}+\underline{2k})+\xi^{j+1}\sum_{k=-j/2}^{j/2}\xi^{2k}\mathsf{h}(\mathfrak{m}-\underline{j}+\underline{2k})$$

and use (6.5) on each of them to get

$$\big(\mathsf{h}(\mathfrak{m}-\underline{j}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}-\underline{j})\big)\xi^j\sum_{k=-j/2}^{j/2}(\xi r_1)^{2k}.$$

If we factor out $(\xi r_1)^{-j}$ and change summation index from k to $k-j/2$ we obtain (6.19). □

We now come to the main results in this section.

Main Lemma 6.16. *Assume that* $\mathsf{h}(\underline{1})\neq 0$ *and let* $\mathfrak{m}\in G$. *Then*

a) $\dim L(\mathfrak{m})\leq N$,
b) *if* $\dim L(\mathfrak{m})=n<N$ *then* $\mathfrak{m}\in G_{\frac{i-1}{2}}+\underline{j}$ *where* $n=2j+i$, $i\in\{1,2\}$, $j\in\mathbb{Z}_{\geq 0}$, *and*
c) *if* $i\in\{1,2\}$, $j\in\mathbb{Z}_{\geq 0}$, $2j+i\leq N$ *and* $\mathfrak{m}\in G_{\frac{i-1}{2}}$ *then*

$$\dim L(\mathfrak{m}+\underline{j})=2j+i. \tag{6.20}$$

d) *If* $N'<\infty$ *then* $\dim L(\mathfrak{m}+\underline{N'j})=\dim L(\mathfrak{m})$ *for any* $j\in\mathbb{Z}$.

Proof. Part a) is trivial when $N=\infty$. If N is finite and odd, Proposition 6.3 and (6.18) imply that $\dim L(\mathfrak{m})\leq N$. If N is finite and even, then $(\xi r_1)^N=1$ and $(\xi r_1)^2\neq 1$ so $\sum_{k=0}^{N/2-1}(\xi r_1)^{2k}=0$. Hence Proposition 6.3 and (6.19) implies $\dim L(\mathfrak{m})\leq N$ in this case as well.

Next we turn to part b). Suppose first that $\dim L(\mathfrak{m})=n=2j+1<N$. Then by Proposition 6.3 and (6.18) the right hand side of (6.18) is zero. The definition of N implies that $\mathsf{h}(\mathfrak{m}-\underline{j})=0$, i.e. $\mathfrak{m}\in G_0+\underline{j}$. If instead $\dim L(\mathfrak{m})=2j+2<N$, Proposition 6.3 and (6.19) similarly implies that $\mathfrak{m}\in G_{1/2}+\underline{j}$.

To prove (6.20), we proceed by induction on j. For $j=0$ it follows from (6.9) and (6.10). Suppose it holds for $j=0,1,\dots,k-1$, where $k>0$ and $2k+i\leq N$. We first show that $\dim L(\mathfrak{m}+\underline{k})\leq 2k+i$. If $i=1$ then by (6.18),

$$\sum_{l=0}^{2k}\xi^{2k-l}\mathsf{h}(\mathfrak{m}+\underline{k}-\underline{l})=r_1^{-k}\mathsf{h}(\mathfrak{m})\sum_{l=0}^{2k}(\xi r_1)^l=0$$

since $\mathfrak{m}\in G_0$. Similarly, if $i=2$, then (6.19) gives

$$\sum_{l=0}^{2k+1}\xi^{2k+1-l}\mathsf{h}(\mathfrak{m}+\underline{k}-\underline{l})=r_1^{-k}\big(\mathsf{h}(\mathfrak{m}-\underline{1})+\xi\mathsf{h}(\mathfrak{m})\big)\sum_{l=0}^{k}(\xi r_1)^{2l}=0$$

since $\mathfrak{m} \in G_{1/2}$ in this case. Thus $\dim L(\mathfrak{m}+\underline{j}) \leq 2j+i$ by Proposition 6.3. Write $\dim L(\mathfrak{m}+\underline{k}) = 2k'+i'$ where $k' \geq 0$, $i' \in \{1,2\}$ and assume that $2k'+i' < 2k+i$. By part b) we have $\mathfrak{m}+\underline{k} \in G_{\frac{i'-1}{2}} + \underline{k'}$ which implies that $\dim L(\mathfrak{m}+\underline{k}-\underline{k'}) = i'$ by (6.9) and (6.10). This contradicts the induction hypothesis unless $k'=0$. Assuming $k'=0$ we get $\mathfrak{m}+\underline{k} \in G_{\frac{i'-1}{2}}$. If $i=i'$ then from Lemma 6.13 follows that $\underline{k} \in G_0$. Since $0 < k < \frac{2k+i}{2} \leq N/2 \leq N'$ this contradicts 6.12. We now show that $i \neq i'$ is also impossible. If $i=1$ and $i'=2$, then $\mathfrak{m} \in G_0$ and $\mathfrak{m}+\underline{k} \in G_{1/2}$ so by Proposition 6.12b), $\underline{k} \in G_{1/2} \cap \underline{\mathbb{Z}_{>0}}$. By (6.13) we get $N \big| 2k-1$ which is absurd because $0 < 2k-1 < 2k+1 \leq N$. If $i=2$ and $i'=1$ then $\mathfrak{m} \in G_{1/2}$ and $\mathfrak{m}+\underline{k} \in G_0$. By Proposition 6.12b) we have $\underline{-k} = \mathfrak{m}-(\mathfrak{m}+\underline{k}) \in G_{1/2}$. By Lemma 6.14a), $\underline{1+k} \in G_{1/2}$ so (6.13) implies that $N \big| 2(1+k)-1 = 2k+1$. This is impossible since $0 < 2k+1 < 2k+2 \leq N$. We have proved that the assumption $2k'+i' < 2k+i$ is false and hence that $\dim L(\mathfrak{m}+\underline{k}) = 2k+i$, which proves the induction step.

Finally, part d) follows from Corollary 6.4 and Proposition 6.10. □

Theorem 6.17. *Let $\mathfrak{m} \in G$.*

- *If $N = \infty$, then*

$$\dim L(\mathfrak{m}) < \infty \iff \mathfrak{m} \in (G_0 + \underline{\mathbb{Z}_{\geq 0}}) \cup (G_{1/2} + \underline{\mathbb{Z}_{\geq 0}}) \tag{6.21}$$

and

$$\dim L(\mathfrak{m}_0 + \underline{j}) = 2j+1, \quad \textit{for } \mathfrak{m}_0 \in G_0 \textit{ and } j \in \mathbb{Z}_{\geq 0}, \tag{6.22}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = 2j+2, \quad \textit{for } \mathfrak{m}_{1/2} \in G_{1/2} \textit{ and } j \in \mathbb{Z}_{\geq 0}. \tag{6.23}$$

- *If $N < \infty$ and N is even, then*

$$\dim L(\mathfrak{m}) < \infty \iff \mathfrak{m} \in (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}}) \tag{6.24}$$

and

$$\dim L(\mathfrak{m} + \underline{(N/2)j}) = \dim L(\mathfrak{m}), \quad \textit{for any } \mathfrak{m} \in G \textit{ and } j \in \mathbb{Z}, \tag{6.25}$$

and for $\mathfrak{m}_0 \in G_0$ and $\mathfrak{m}_{1/2} \in G_{1/2}$ we have

$$\dim L(\mathfrak{m}_0 + \underline{j}) = 2j+1, \quad \textit{if } 0 \leq j < N/2, \tag{6.26}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = 2j+2, \quad \textit{if } 0 \leq j < N/2. \tag{6.27}$$

- *If $N < \infty$ and N is odd, then*

$$\dim L(\mathfrak{m}) < \infty \iff \mathfrak{m} \in G_0 + \underline{\mathbb{Z}} = G_{1/2} + \underline{\mathbb{Z}} \tag{6.28}$$

and

$$\dim L(\mathfrak{m} + \underline{Nj}) = \dim L(\mathfrak{m}), \quad \textit{for any } \mathfrak{m} \in G \textit{ and } j \in \mathbb{Z}, \tag{6.29}$$

and for $\mathfrak{m}_0 \in G_0$ and $\mathfrak{m}_{1/2} \in G_{1/2}$ we have

$$\dim L(\mathfrak{m}_0 + \underline{j}) = \begin{cases} 2j+1, & \textit{if } 0 \leq j < \frac{N+1}{2}, \\ 2j+1-N, & \textit{if } \frac{N+1}{2} \leq j < N, \end{cases} \tag{6.30}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \begin{cases} 2j+2, & \textit{if } 0 \leq j < \frac{N-1}{2}, \\ 2j+2-N, & \textit{if } \frac{N-1}{2} \leq j < N. \end{cases} \tag{6.31}$$

Proof. When $N = \infty$, relations (6.21)-(6.23) are immediate from Lemma 6.16b) and c).

Suppose N is finite and even. The $\Rightarrow$ implication in (6.24) holds by Lemma 6.16b). And (6.25) follows from (6.12) and Corollary 6.4. Assume that $\mathfrak{m} \in (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}})$. Using (6.25) we can assume that $\mathfrak{m} = \mathfrak{m}' + \underline{j}$ where $\mathfrak{m}' \in G_0 \cup G_{1/2}$ and $0 \leq j < N/2$. Then, if $i \in \{1, 2\}$ we have $2j + i \leq N$ and Lemma 6.16c) implies (6.26)-(6.27) and therefore $\dim L(\mathfrak{m}) < \infty$ so (6.24) is also proved.

Assume that N is finite and odd. By (6.13) we have $\underline{(N+1)/2} \in G_{1/2}$. Therefore $G_0 + \underline{\mathbb{Z}} = G_0 + \underline{(N+1)/2} + \underline{\mathbb{Z}} = G_{1/2} + \underline{\mathbb{Z}}$ since $G_{1/2}$ is a right coset of G_0 in G by Proposition 6.12. As before, Lemma 6.16b) implies the $\Rightarrow$ case in (6.28) and (6.29) holds by virtue of (6.12) and Corollary 6.4. If $\mathfrak{m} \in G_0 + \underline{\mathbb{Z}}$ we can assume by (6.29) that $\mathfrak{m} \in G_0 + \underline{j}$ where $0 \leq j < N$. If $j < \frac{N+1}{2}$, then $2j + 1 < N + 2$ so since N is odd we have $2j + 1 \leq N$. By Lemma 6.16c) we deduce that $\dim L(\mathfrak{m}) = 2j + 1$. If instead $j \geq \frac{N+1}{2}$, then $\mathfrak{m} = \underline{(N+1)/2} + \mathfrak{m} - \underline{(N+1)/2} \in G_{1/2} + \underline{k}$ where $k = j - \frac{N+1}{2}$ so $0 \leq k < \frac{N-1}{2}$. Thus $2k + 2 \leq N$ so Lemma 6.16c) implies that $\dim L(\mathfrak{m}) = 2k + 2 = 2j + 1 - N$. This proves (6.30) and the $\Leftarrow$ implication in (6.28). Finally (6.31) is equivalent to (6.30) in the following sense. Let $0 \leq j < N$ and $\mathfrak{m}_{1/2} \in G_{1/2}$. Then

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j'}),$$

where $j' = j + (N+1)/2$ and $\mathfrak{m}_0 = \mathfrak{m}_{1/2} - \overline{(N+1)/2}$. Now $\mathfrak{m}_0 \in G_0$ since $G_{1/2}$ is a coset of G_0 in G. If $0 \leq j < \frac{N-1}{2}$, then $\frac{N+1}{2} \leq j' < N$ so by (6.30) we have

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j'}) = 2j' + 1 - N = 2j + 2.$$

And if $\frac{N-1}{2} \leq j < N$, then $0 \leq j' - N < \frac{N+1}{2}$ and hence

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j' - N}) = 2(j' - N) + 1 = 2j + 1 - N.$$

The proof is finished. $\square$

Corollary 6.18. *If $N = \infty$ and $\mathfrak{m} \in G_0 \cup G_{1/2}$, then $L(\mathfrak{m} + \underline{j})$ is infinite-dimensional for any $j \in \mathbb{Z}_{<0}$.*

Proof. If the dimension of $L(\mathfrak{m} + \underline{j})$ were finite and odd (even), then $\dim L(\mathfrak{m} + \underline{j - k}) = 1$ (2) for some $k \geq 0$ by Lemma 6.16b). By Lemma 6.16c), $L(\mathfrak{m})$ has then dimension $2(j - k) + 1$ $(2(j - k) + 2)$ and thus $j = k$ which is absurd. $\square$

Corollary 6.19. *Suppose $N = \infty$ and let $\mathfrak{m} \in G_f$. Then $L(\mathfrak{m})$ is the unique finite-dimensional quotient of $M(\mathfrak{m})$.*

Proof. It is enough to prove that the unique maximal proper submodule $N(\mathfrak{m})$ of $M(\mathfrak{m})$ is simple. By Theorem 6.17 we can write $\mathfrak{m} = \mathfrak{n} + \underline{j}$ where $\mathfrak{n} \in G_0 \cup G_{1/2}$ and $j \in \mathbb{Z}_{\geq 0}$. From the proof of Proposition 6.3 we have

$$\mathrm{Supp}(L(\mathfrak{m})) = \{\mathfrak{n} + \underline{j}, \mathfrak{n} + \underline{j} - \underline{1}, \ldots, \mathfrak{n} - \underline{j}\}.$$

Thus $N(\mathfrak{m})$ is a highest weight module of highest weight $\mathfrak{n} - \underline{j} - \underline{1}$. So $N(\mathfrak{m})$ is a quotient of $M(\mathfrak{n} - \underline{j} - \underline{1})$. But $M(\mathfrak{n} - \underline{j} - \underline{1})$ is simple, otherwise it would have a finite-dimensional simple quotient, i.e. $L(\mathfrak{n} - \underline{j} - \underline{1})$ would be finite-dimensional, contradicting Corollary 6.18. Thus $N(\mathfrak{m})$ is also simple. $\square$

Remark 6.20. We finish this section by remarking that there exist algebras in the class studied in this paper which do not have even-dimensional simple modules as for example the algebra B_q from Section 4.4. Indeed, in this case we have $\xi r_1 = -1$ and so $N = \infty$ by definition. By Proposition 6.12, $G_{1/2} = \emptyset$ so by Theorem 6.17, there can exist no even-dimensional simple modules.

7. Tensor products and a Clebsch-Gordan formula

As we have seen in Section 2 the existence of a Hopf structure on an algebra allows one to define tensor product of its representations by (2.4). The aim of this section is to prove a formula which decomposes the tensor product of two simple A-modules into a direct sum of simple modules. It generalizes the classical Clebsch-Gordan formula for modules over $U(\mathfrak{sl}_2)$. We will assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an ambiskew polynomial ring and that it carries a Hopf structure of the type considered in Section 3. We will also assume (6.1) and that $N = \infty$.

Lemma 7.1. *Let V and W be two A-modules. Then*

$$V_{\mathfrak{m}} \otimes W_{\mathfrak{n}} \subseteq (V \otimes W)_{\mathfrak{m}+\mathfrak{n}} \tag{7.1}$$

for any $\mathfrak{m}, \mathfrak{n} \in G$. Hence if V and W are weight modules, then so is $V \otimes W$ and

$$\mathrm{Supp}(V \otimes W) = \{\mathfrak{m} + \mathfrak{n} \mid \mathfrak{m} \in \mathrm{Supp}(V), \mathfrak{n} \in \mathrm{Supp}(W)\}.$$

Proof. Let $v \in V_{\mathfrak{m}}, w \in W_{\mathfrak{n}}$. Then for any $r \in R$,

$$r(v \otimes w) = \sum_{(r)} r'v \otimes r''w = \sum_{(r)} r'(\mathfrak{m})v \otimes r''(\mathfrak{n})w =$$

$$= \sum_{(r)} r'(\mathfrak{m})r''(\mathfrak{n})v \otimes w = r(\mathfrak{m}+\mathfrak{n})v \otimes w$$

by (5.3), proving (7.1). Thus if V, W are weight modules,

$$V \otimes W = (\oplus_{\mathfrak{m}} V_{\mathfrak{m}}) \otimes (\oplus_{\mathfrak{n}} W_{\mathfrak{n}}) = \oplus_{\mathfrak{m},\mathfrak{n}} V_{\mathfrak{m}} \otimes W_{\mathfrak{n}} = \oplus_{\mathfrak{m}} \big(\oplus_{\mathfrak{m}_1+\mathfrak{m}_2=\mathfrak{m}} V_{\mathfrak{m}_1} \otimes W_{\mathfrak{m}_2} \big).$$

□

Theorem 7.2. *Let $\mathfrak{m}, \mathfrak{n} \in G_f$. We have the following isomorphism*

$$L(\mathfrak{m}) \otimes L(\mathfrak{n}) \simeq L(\mathfrak{m}+\mathfrak{n}) \oplus L(\mathfrak{m}+\mathfrak{n}-\underline{1}) \oplus \ldots \oplus L(\mathfrak{m}+\mathfrak{n}-\underline{s}+\underline{1}) \tag{7.2}$$

where $s = \min\{\dim L(\mathfrak{m}), \dim L(\mathfrak{n})\}$.

Proof. Let $e^{\mathfrak{m}}, e^{\mathfrak{n}}$ denote highest weight vectors in $L(\mathfrak{m})$, $L(\mathfrak{n})$ respectively and set $e^{\mathfrak{m}}_j := (X_-)^j e^{\mathfrak{m}}$ for $j \in \mathbb{Z}_{\geq 0}$ and similarly for $\mathfrak{n}$. Set $V = L(\mathfrak{m}) \otimes L(\mathfrak{n})$. By Lemma 7.1 we have

$$V_{\mathfrak{m}+\mathfrak{n}-\underline{k}} = \oplus_{i+j=k} \mathbb{K} e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_j$$

for $k \in \mathbb{Z}_{\geq 0}$. Fix $0 \leq k \leq s-1$. We will prove that

$$\dim \ker X_+|_{V_{\mathfrak{m}+\mathfrak{n}-\underline{k}}} = 1. \tag{7.3}$$

From the calculations in the proof of Proposition 6.3 follows that when $j > 0$, $X_+ e^{\mathfrak{m}}_j$ is a nonzero multiple of $e^{\mathfrak{m}}_{j-1}$. Let $\nu^{\mathfrak{m}}_j$ denote this multiple. Let

$$u = \sum_{i=0}^{k} \lambda_i e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_{k-i}$$

be an arbitrary vector in $V_{\mathfrak{m}+\mathfrak{n}-\underline{k}}$. Then

$$\begin{aligned}X_+u &= \sum_{i=0}^{k}\lambda_i(X_+e_i^{\mathfrak{m}}\otimes r_+e_{k-i}^{\mathfrak{n}} + l_+e_i^{\mathfrak{m}}\otimes X_+e_{k-i}^{\mathfrak{n}}) =\\ &= \sum_{i=0}^{k-1}\big[\lambda_{i+1}\nu_{i+1}^{\mathfrak{m}}r_+(\mathfrak{n}-\underline{k}+\underline{i}+\underline{1}) + \lambda_i l_+(\mathfrak{m}-\underline{i})\nu_{k-i}^{\mathfrak{n}}\big]e_i^{\mathfrak{m}}\otimes e_{k-1-i}^{\mathfrak{n}}.\end{aligned}$$

Setting

$$\begin{aligned}c_i &= l_+(\mathfrak{m}-\underline{i})\nu_{k-i}^{\mathfrak{n}},\\ c_i' &= \nu_i^{\mathfrak{m}}r_+(\mathfrak{n}-\underline{k}+\underline{i}),\end{aligned}$$

the condition for u to be a highest weight vector can hence be written as

$$\begin{bmatrix}c_0 & c_1' & & & \\ & c_1 & c_2' & & \\ & & \ddots & \ddots & \\ & & & c_{k-1} & c_k'\end{bmatrix}\begin{bmatrix}\lambda_0\\ \lambda_1\\ \vdots\\ \lambda_k\end{bmatrix} = 0. \tag{7.4}$$

Since r_+ and l_+ are grouplike, they are invertible and hence $c_i \neq 0 \neq c_{i+1}'$ for any $i = 0,1,\ldots,k-1$. Therefore the space of solutions to (7.4) is one-dimensional. Thus (7.3) is proved.

From the definition of Verma modules, it follows that for $k = 0,1,\ldots,s-1$, there is a nonzero A-module morphism

$$M(\mathfrak{m}+\mathfrak{n}-\underline{k}) \to L(\mathfrak{m})\otimes L(\mathfrak{n})$$

which maps a highest weight vector in $M(\mathfrak{m}+\mathfrak{n}-\underline{k})$ to a highest weight vector in $L(\mathfrak{m})\otimes L(\mathfrak{n})$ of weight $\mathfrak{m}+\mathfrak{n}-\underline{k}$. But $L(\mathfrak{m})\otimes L(\mathfrak{n})$ is finite-dimensional so this morphism must factor through $L(\mathfrak{m}+\mathfrak{n}-\underline{k})$ by Corollary 6.19. Taking direct sums of these morphisms we obtain an A-module morphism

$$\varphi : L(\mathfrak{m}+\mathfrak{n})\oplus L(\mathfrak{m}+\mathfrak{n}-\underline{1})\oplus\ldots\oplus L(\mathfrak{m}+\mathfrak{n}-\underline{s}+\underline{1}) \to L(\mathfrak{m})\otimes L(\mathfrak{n}).$$

We claim it is injective. Indeed, the projection of the kernel of φ to any term $L(\mathfrak{m}+\mathfrak{n}-\underline{i})$ must be zero, because it is a proper submodule of the simple module $L(\mathfrak{m}+\mathfrak{n}-\underline{i})$.

To conclude we now calculate the dimensions of both sides. Write $\dim L(\mathfrak{m}) = 2j_1 + i_1$ and $\dim L(\mathfrak{n}) = 2j_2 + i_2$ where $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ and $i_1, i_2 \in \{1,2\}$. By Lemma 6.16b), $\dim L(\mathfrak{m}-\underline{j_1}) = i_1$ and $\dim L(\mathfrak{n}-\underline{j_2}) = i_2$. First note that

$$\dim L(\mathfrak{m}-\underline{j_1}+\mathfrak{n}-\underline{j_2}) = i_1 + i_2 - 1.$$

When $i_1 = i_2 = 1$, this is true because G_0 is a subgroup of G. When one of i_1, i_2 is 1 and the other 2, it follows from Proposition 6.12b). And if $i_1 = i_2 = 2$, it follows from Lemma 6.14b) and Theorem 6.17.

From Theorem 6.17 also follows that $\dim L(\mathfrak{m}+\underline{k}) = \dim L(\mathfrak{m})+2k$ if $\dim L(\mathfrak{m}) < \infty$ and $k \in \mathbb{Z}_{\geq 0}$. Hence, recalling that $s = \min\{\dim L(\mathfrak{m}), \dim L(\mathfrak{n})\}$, we have

$$\begin{aligned}\sum_{k=0}^{s-1} \dim L(\mathfrak{m}+\mathfrak{n}-\underline{k}) &= \sum_{k=0}^{s-1} \dim L(\mathfrak{m}-\underline{j_1}+\mathfrak{n}-\underline{j_2}+\underline{j_1}+\underline{j_2}-\underline{k}) = \\ &= \sum_{k=0}^{s-1} \big(i_1+i_2-1+2(j_1+j_2-k)\big) = \\ &= s(i_1+i_2-1+2j_1+2j_2) - s(s-1) = \\ &= s(\dim L(\mathfrak{m}) + \dim L(\mathfrak{n}) - s) = \\ &= \dim L(\mathfrak{m}) \dim L(\mathfrak{n}) = \dim \big(L(\mathfrak{m}) \otimes L(\mathfrak{n})\big).\end{aligned}$$

This completes the proof of the theorem. □

Under some conditions it is possible to introduce a $*$-structure on A. In this connection it would be interesting to study Clebsch-Gordan coefficients and the relation with special functions. This will be a subject for future investigation.

8. Casimir operators and semisimplicity

Arguing as in the proof of Lemma 4.2, it is easy to see that any finite-dimensional semisimple module over $A = A(R, \sigma, \mathsf{h}, \xi)$ is a weight module. In this section we will prove the converse, that any finite-dimensional weight module over A is semisimple. Note that in general not all finite-dimensional modules over our algebra A are semisimple. The corresponding example is constructed in [6] for the algebra from Section 4.3. A necessary and sufficient condition for all finite-dimensional modules over an ambiskew polynomial ring to be semisimple was given in [8], Theorem 5.1.

In this section we assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an ambiskew polynomial ring with a Hopf structure of the type introduced in Section 3 such that (6.1) holds. We also assume that $N = \infty$.

Let V be a finite-dimensional weight module over A. We will first treat the case when $\mathrm{Supp}(V) \subseteq \mathfrak{m} + \underline{\mathbb{Z}}$ where $\mathfrak{m} \in G_0$ is fixed. Define a linear map

$$C_V : V \to V$$

by requiring

$$C_V v = \sigma^j(t)v, \quad \text{for } v \in V_{\mathfrak{m}+\underline{j}} \text{ and } j \in \mathbb{Z}.$$

Here σ denotes the extended automorphism (2.8). More explicitly we have (if $j \geq 0$)

$$C_V v = \sigma^j(t)v = \Big(\xi^j t + \sum_{k=0}^{j-1} \xi^k \sigma^{j-1-k}(\mathsf{h})\Big)v = \xi^j t v + \sum_{k=0}^{j-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1})v$$

and similarly when $j < 0$. It is easy to check that C_V is a morphism of A-modules. Hence it is constant on each finite-dimensional simple module V by Schur's Lemma. Moreover if $\varphi : V \to W$ is a morphism of weight A-modules with support in $\mathfrak{m} + \underline{\mathbb{Z}}$, then $\varphi C_V = C_W \varphi$.

Proposition 8.1. *Let $j_1, j_2 \in \mathbb{Z}_{\geq 0}$. If $C_{L(\mathfrak{m}+\underline{j_1})} = C_{L(\mathfrak{m}+\underline{j_2})}$, then $j_1 = j_2$.*

Proof. By applying $C_{L(\mathfrak{m}+\underline{j})}$ to the highest weight vector of $L(\mathfrak{m}+\underline{j})$, ($j \in \mathbb{Z}_{\geq 0}$) we get

$$C_{L(\mathfrak{m}+\underline{j})} = \sum_{k=0}^{j-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}). \tag{8.1}$$

We can assume $j_1 < j_2$. By assumption we have

$$0 = \sum_{k=0}^{j_2-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) - \sum_{k=0}^{j_1-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) = \sum_{k=j_1}^{j_2-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) =$$
$$= \xi^{j_1} \sum_{k=0}^{j_2-j_1-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{j_2} - \underline{(j_2-j_1)} + \underline{k+1}).$$

By Proposition 6.3 this means that $\dim L(\mathfrak{m} + \underline{j_2}) \leq j_2 - j_1$. But this contradicts Theorem 6.17 which says that $\dim L(\mathfrak{m} + \underline{j_2}) = 2j_2 + 1$. □

Theorem 8.2. *Let V be a finite-dimensional weight module over A with support in $G_0 + \underline{\mathbb{Z}}$. Then V is semisimple.*

Proof. We follow the idea of the proof of Proposition 12 in [10], Chapter 3. Writing

$$V = \oplus_{\mathfrak{m}\in G_0} \big(\oplus_{j\in\mathbb{Z}} V_{\mathfrak{m}+\underline{j}} \big)$$

and noting that $\oplus_{j\in\mathbb{Z}} V_{\mathfrak{m}+\underline{j}}$ are submodules, we can reduce to the case when $\mathrm{Supp}(V)$ is contained in $\mathfrak{m} + \mathbb{Z}$ for a fixed $\mathfrak{m} \in G_0$.

Let $\lambda_1, \ldots, \lambda_k$ be the generalized eigenvalues of the Casimir operator C_V, i.e. the elements of the set

$$\{\lambda \in \mathbb{K} \mid \ker(C_V - \lambda \,\mathrm{Id})^p \neq 0 \text{ for some } p > 0\}.$$

Then each generalized eigenspace $\sum_p \ker(C_V - \lambda_i \,\mathrm{Id})^p$ is invariant under A, hence they are submodules. It suffices to prove that each such submodule is semisimple. Let V be one of them. Let $V_1 = \{v \in V \mid X_+ v = 0\}$. Then V_1 is invariant under R and since V is a weight module, $V_1 = \oplus_{\mathfrak{n}\in G}(V_1 \cap V_{\mathfrak{n}})$. Now if $0 \neq v \in V_1 \cap V_{\mathfrak{n}}$, then v is a highest weight vector of V and generates a submodule isomorphic to $L(\mathfrak{n})$. Hence if $V_1 \cap V_{\mathfrak{n}} \neq 0$ for more than one $\mathfrak{n} \in G$, C_V will have two different eigenvalues by Proposition 8.1 which is impossible. Here we used that the restriction of C_V to a submodule W coincides with C_W. Hence V_1 is contained in a single weight space, say $V_{\mathfrak{n}}$. Let $v_1, \ldots, v_k$ be a basis for V_1. Then each v_i generates a simple submodule isomorphic to $L(\mathfrak{n})$. We will show that the sum of these submodules is direct. Vectors of different weights are linearly independent so it suffices to show that if

$$\sum_{i=1}^{k} \lambda_i (X_-)^m v_k = 0$$

then all $\lambda_i = 0$. Assume the sum was nonzero and act by X_+ m times. In each step we get a nonzero result because we have not reached the highest weight $\mathfrak{n}$ yet. But then, using (6.3), we have a linear relation among the v_k – a contradiction. We have shown that V contains the direct sum V' of k copies of $L(\mathfrak{n})$. Now X_+ acts injectively on V/V'. This is only possible in a torsion-free finite-dimensional weight A-module if it is 0-dimensional. Thus V is semisimple. □

We now turn to the general case. Assume now that A has an even-dimensional irreducible representation. By Lemma 6.16b), $G_{1/2} \neq \emptyset$. We fix $\mathfrak{m}_{1/2} \in G$. Then $G_{1/2} = G_0 + \mathfrak{m}_{1/2}$ by Proposition 6.12.

Theorem 8.3. *Any finite-dimensional weight module V over A is semisimple.*

Proof. By Corollary 6.2 and Theorem 6.17,

$$\mathrm{Supp}(V) \subseteq (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}})$$

Thus we have a decomposition

$$V = \Big(\bigoplus_{\mathfrak{m} \in G_0} V_{\mathfrak{m}+\underline{\mathbb{Z}}} \Big) \oplus \Big(\bigoplus_{\mathfrak{m} \in G_0} V_{\mathfrak{m}+\mathfrak{m}_{1/2}+\underline{\mathbb{Z}}} \Big)$$

where $V_{\mathfrak{n}+\mathbb{Z}} := \oplus_{j\in\mathbb{Z}} V_{\mathfrak{n}+\underline{j}}$ for $\mathfrak{n} \in G$ are submodules. It remains to prove that a weight module V with support in $\mathfrak{m} + \mathfrak{m}_{1/2} + \mathbb{Z}$ is semisimple. By Lemma 7.1,

$$\mathrm{Supp}\,\big(V \otimes L(\mathfrak{m}_{1/2})\big) \subseteq \mathfrak{m} + \mathfrak{m}_{1/2} + \mathfrak{m}_{1/2} + \mathbb{Z} = \mathfrak{m}' + \mathbb{Z}$$

where $\mathfrak{m}' := \mathfrak{m} + \mathfrak{m}_{1/2} + \mathfrak{m}_{1/2} - 1 \in G_0$ by Lemma 6.14b). Hence $V \otimes L(\mathfrak{m}_{1/2})$ is semisimple by Theorem 8.2. By the Clebsch-Gordan formula (7.2), the tensor product of two semisimple modules is semisimple again. Therefore $V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2})$ is semisimple, where $\dim L(\underline{1} - \mathfrak{m}_{1/2}) = 2$ by Lemma 6.14a). On the other hand, by (7.2) again we have

$$V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2}) \simeq V \otimes \big(L(0) \oplus L(\mathfrak{m})\big) \simeq \big(V \otimes L(0)\big) \oplus \big(V \otimes L(\mathfrak{m})\big).$$

Finally, it is easy to verify the isomorphism $V \simeq V \otimes L(0)$, $v \mapsto v \otimes e$ where $0 \neq e \in L(0)$ is fixed. Thus V is isomorphic to a submodule of the semisimple module $V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2})$ and is therefore itself semisimple. $\square$

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