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Sequences and games generalizing the combinatorial game of Wythoff Nim

Urban Larsson

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Chalmers University of Technology and Göteborg University
Göteborg, Sweden

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URBAN LARSSON

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Department of Mathematical Sciences
Chalmers University of Technology and Göteborg University
412 96 Göteborg
Sweden
Telephone: +46 (0)31-772 1000

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SEQUENCES AND GAMES GENERALIZING THE COMBINATORIAL GAME OF WYTHOFF NIM

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Abstract. One single Queen is placed on an arbitrary starting position of a (large) Chess board. Two players alternate in moving the Queen as in a game of Chess but with the restriction that the L^1 distance to the lower left corner, position $(0, 0)$, must decrease. The player who moves there wins.

Let $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio. In 1907 W. A. Wythoff proved that the second player wins if and only if the coordinates of the starting position are of the form $\{a_n, b_n\}$, where $a_n = \lfloor n\phi \rfloor, b_n = a_n + n$ for some non-negative integer n.

Here, we introduce the game of *Imitation Nim*, a *move-size dynamic* restriction on the classical game of (2-pile) Nim. We prove that this game is a 'dual' of *Wythoff Nim* in the sense that the latter has the same solution/P-positions as the former.

On the one hand we define extensions and restrictions to Wythoff Nim—including the classical generalizations by I.G. Connell (1959) and A.S. Fraenkel (1982)—and Imitation Nim. All our games are purely combinatorial, so there are no 'hidden cards' and no 'chance device'. In fact we only study so-called *Impartial games* where the set of options does not depend on whose turn it is. In particular we introduce rook-type and bishop-type *blocking manoeuvres/Muller twists* to Wythoff Nim: For each move, the previous player may 'block off' a predetermined number of next player options. We study the solutions of the new games and for each blocking manoeuvre give non-blocking dual game rules.

On the other hand, observing that the pair of sequences (a_n) and (b_n)—viewed as a permutation of the natural numbers which takes a_n to b_n and b_n to a_n—may be generated by a 'greedy' algorithm, we study extensive generalizations to these. We also give interpretations of our sequences as so-called *Interspersion arrays* and/or *Beatty sequences*.

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Key words and phrases. Blocking manoeuvre, Beatty sequence, Combinatorial game, Complementary sequences, Impartial game, Interspersion array, Muller twist, Nim, Permutation of the natural numbers, Stolarsky array, Wythoff Nim.

PREFACE

We present three papers:

I: Permutations of the natural numbers with prescribed difference multisets (published in Integers, Volume 6 (2006), article A3),

II: 2-pile Nim with a restricted number of move-size dynamic imitations (accepted for publication in Integers, Volume 9 (2009), article G4),

III: Restrictions of m-Wythoff Nim and p-complementary Beatty sequences (accepted for publication in Games of no Chance 2008).

The inspiration for this thesis is to be found in the last section of my masters thesis [LaKn], written together with Jonas Knape.

The paper [BHKLS05] also originates from our masters thesis. But I have choosen not to include it here since the main interest of this work is on combinatorial games.

ACKNOWLEDGMENTS

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To Hanna

1. Introduction

Our games are purely combinatorial, that is there are no 'hidden cards' and no 'chance device'. Although there is an infinite number of possible starting positions, a game always terminates in a finite number of moves. The games belong to the family of 2-player Impartial games, that is, two players move alternately according to identical move rules (unlike, for example, Chess where the players usually have different options). We play so-called normal play, that is the player who moves to a terminal/final position wins.

We follow the convention of denoting our players as the next player (the player whose turn it is) and the previous player. Let G be (a position of) an impartial game. Then G is P (a P-position) if the previous player wins from G. It is N if the next player wins. Let $\mathcal{P}(G)$ denote the set of P-positions of G.

We will now give a brief summary of each paper. Although the first paper is mostly on combinatorial number theory, here we will put an emphasis on the game theory that will be revisited in paper II and III.

Denote by $\mathbb{N}$ the positive integers and by $\mathbb{N}_0$ the non-negative integers.

Paper I: Permutations of the natural numbers with prescribed difference multisets. Consider the permutation $\pi = \pi_g$ of the natural numbers defined inductively as follows :

(i) $\pi(1) = 1$,

(ii) for each $n > 1$, $\pi(n) := t$, where t is the least natural number not already appearing among $\pi(1), ..., \pi(n-1)$ and such that $t - n \neq \pi(i) - i$, for any $1 \leq i < n$.

Informally, we say that π chooses numbers greedily under the restriction that differences $\pi(n) - n$ may not be repeated. The permutation π exhibits a rich variety of beautiful properties. It is an involution of $\mathbb{N}$ and its asymptotics are given by

$$\lim_{n \in A} \frac{\pi(n)}{n} = \phi = \frac{1+\sqrt{5}}{2}, \tag{1}$$

the golden ratio, where $A = \{n : \pi(n) \geq n\}$.

1.1. *Beatty sequences and Wythoff's game.* Let r, s be any positive real numbers such that

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{2}$$

The famous Beatty's theorem [Ray94, Bea26, HyOs27] states that the sets $X = \{\lfloor nr \rfloor : n \in \mathbb{N}\}$, $Y = \{\lfloor ns \rfloor : n \in \mathbb{N}\}$ form a partition of $\mathbb{N}$ if and only if r and s are irrationals.

With notation as in (1) and (2), choose $r = \phi, s = \phi + 1 = \phi^2$. It is well-known that (2) is satisfied and that π is completely described by

$$\pi(1) = 1, \quad \pi = \pi^{-1}, \quad \pi(\lceil nr \rceil) = \lceil ns \rceil, \quad \forall\, n \geq 1. \tag{3}$$

Wythoff Nim is an impartial game (also known as Corner the Queen or Wythoff's game). One single Queen is placed on an arbitrary starting position of a (large) Chess board. Two players alternate in moving the Queen

$\pi(n)-n$	0	1	-1	2	3	-2	4	-3	5	6	-4
$\pi(n)$	1	3	2	6	8	4	11	5	14	16	7
n	1	2	3	4	5	6	7	8	9	10	11

TABLE 1. The first few entries of π_g together with $\pi(n)-n$. Let $A = \{n \mid \pi(n) - n \geq 0\}$ and $B = \mathbb{N} \setminus A$. We prove that π is the unique permutation of the natural numbers such that $\{\pi(n) - n \mid n \in \mathbb{N}\} = \mathbb{Z}$ and both sequences $(\pi(n))_{n\in A}$ and $(\pi(n))_{n\in B}$ are strictly increasing. The P-positions of the combinatorial game of Wythoff Nim are $(n-1, \pi(n)-1)$.

as in a game of Chess but with the restriction that the L^1 distance to the lower left corner, position $(0,0)$, must decrease. The player who moves there wins. In other words:

From any given position (k,l), the allowed moves are

TYPE I: $(k,l) \to (k',l)$ for any $0 \leq k' < k$.
TYPE II: $(k,l) \to (k,l')$ for any $0 \leq l' < l$.
TYPE III: $(k,l) \to (k-s, l-s)$ for any $0 \leq s \leq \min\{k,l\}$.

In [Wyt07] W. A. Wythoff proved that a position of this game is P if and only if it is of the form

$$(n-1, \pi(n)-1), \tag{4}$$

for some $n \in \mathbb{N}$.

In paper I we study permutations $\pi = \pi_g^M$ of $\mathbb{N}$, defined by a greedy choice procedure, under the restriction that the differences $\pi(n) - n$ belong to some assigned, but otherwise arbitrary, (multi)subset M of $\mathbb{Z}$. Hence the above discussion relates to the case $M = \mathbb{Z}$.

1.2. *A. S. Fraenkel's game.* In [Fra82] A. S. Fraenkel studied the following generalization of Wythoff Nim. Fix $m \in \mathbb{N}$. The game of m-Wythoff Nim, or W_m, is played according to the same rules as Wythoff Nim (for this game we simply write $W = W_1$) except that we expand the set of allowed moves of TYPE III as follows: from a position (k,l) one can move to any position $(k-s, l-t)$ such that $0 \leq s \leq k$, $0 \leq t \leq l$ and $|\, s-t \,| < m$.

Let $\pi_{W_m} = \pi_g^{m\mathbb{Z}}$ be the permutation of $\mathbb{N}$ constructed by the same greedy choice procedure as π, but with the restriction that $\pi_{W_m}(i) - i$ must be a multiple of m for all $i \in \mathbb{N}$. It is easy to see that the P-positions for m-Wythoff Nim are just the pairs

$$(n-1, \pi_{W_m}(n) - 1),$$

for all $n \in \mathbb{N}$. Fraenkel showed that these positions can be written in terms of Beatty sequences:

If we choose

$$r = r_m := \frac{2 - m + \sqrt{m^2+4}}{2}, \quad s = s_m := r_m + m, \tag{5}$$

then (3) holds, with π replaced by π_{W_m}.

$\pi_{W_2}(n) - n$	0	2	4	-2	6	8	-4	10	12	14	-6
$\pi_{W_2}(n) - 1$	0	3	6	1	10	13	2	17	20	23	4
$n - 1$	0	1	2	3	4	5	6	7	8	9	10

TABLE 2. The P-positions of 2-Wythoff Nim are $(n-1, \pi_{W_2}(n) - 1)$.

$\pi_{1,2}(n) - n$	0	0	1	-1	1	-1	2	2	-2	-2	3
$\pi_{1,2}(n) - 1$	0	1	3	2	5	4	8	9	6	7	13
$n - 1$	0	1	2	3	4	5	6	7	8	9	10

TABLE 3. The P-positions of 2-blocking 1-Wythoff Nim are the pairs $(n-1, \pi_{1,2}(n) - 1)$ for $n \geq 1$.

1.3. *p-Blocking m-Wythoff Nim.* Fix $m, p \in \mathbb{N}$. In section 5 of paper I we define the game of p-Blocking m-Wythoff Nim, or $W_{m,p}$. The P-positions are precisely the pairs $(n-1, \pi_{m,p}(n) - 1)$ for $n \geq 1$, where $\pi_{m,p} = \pi_{W_{m,p}}$ is the relaxation of π_m where, as differences $\pi_{m,p}(i) - i$, p repetitions of every multiple of m are permitted.

Hence for this game we consider a multisubset of $\mathbb{Z}$ which consists of precisely p copies of each multiple of m.

The rules of the game are just as in the m-Wythoff game, with one exception. Before each move, the previous player is allowed to 'block' some of the possible moves of TYPE III. When the move is carried out, any blocking manoeuvre is forgotten and has no further impact on the game. More precisely, if the current configuration is (k, l), then before the next move is made, the previous player is allowed to choose up to $p - 1$ distinct, positive integers $c_1, ..., c_{p-1} \leq \min\{k, l\}$ and declare that the next player may not move to any configuration $(k - c_i, l - c_i)$.

The interest of this game lies in it being a Muller twist, in the sense of [SmSt02], of m-Wythoff Nim. We also explore a 'move-size dynamic' 'dual' of this game, called (m, p)-Imitation Nim, in our second paper.

The asymptotic behaviour of $\pi_{m,p}$ is given by

$$\lim_{n \in A} \frac{\pi_{m,p}(n)}{n} = \frac{m + \sqrt{m^2 + 4p^2}}{2p},$$

which is the positive root of the equation

$$x^2 - \frac{m}{p}x - 1 = 0,$$

where $A = \{n : \pi_{m,p}(n) \geq n\}$.

We note that it is not in general possible to express the pairs $(n, \pi_{m,p}(n))$ as $(\lceil nr \rceil, \lceil ns \rceil)$ for any real r and s satisfying (2), and depending only on m and p.

1.4. *Some results for more general multisets M and 'greedy' permutations π_g^M.* For each $n \in \mathbb{Z}$, let $\zeta_n \in \mathbb{N}_0 \cup \{\infty\}$. The sequence $M := (\zeta_n)_{n=-\infty}^{\infty}$ is called a multisubset of $\mathbb{Z}$, or simply a multiset. We think of ζ_n as the number of occurrences of the integer n in M.

If $\zeta_n = \zeta_{-n}$ for all integers n, then M is said to be symmetric. The asymptotic density of a multiset M is defined by

$$d(M) := \lim_{n\to\infty} \frac{1}{2n+1}\left(\sum_{k=-n}^{n} \zeta_k\right),$$

whenever this limit exists. Observe that $d(M) = \infty$ whenever $\zeta_n = \infty$ for some n. Hence, this concept is only really interesting if $\zeta_n \in \mathbb{N}_0$ for all n. Such a multiset is called finitary. Further, M is said to be a greedy multiset if either

- M is finitary, or
- there is at most one non-negative n and at most one non-positive n for which $\zeta_n = \infty$ and
  - if $n \geq 0$ and $\zeta_n = \infty$ then $\zeta_{n'} = 0$ for all $n' > n$,
  - if $n \leq 0$ and $\zeta_n = \infty$, then $\zeta_{n'} = 0$ for all $n' < n$.

Given a greedy multiset M, an injective mapping $\pi_g = \pi_g^M : \mathbb{N} \to \mathbb{N}$ can be constructed by means of a 'greedy algorithm' : for each $n \in \mathbb{N}$, $\pi_g(n)$ is defined inductively to be the least positive integer t not equal to $\pi_g(k)$ for any $k < n$ and, satisfying the additional condition that

$$\#\{k < n : \pi_g(k) - k = t - n\} < \zeta_{t-n}.$$

Then π_g is also surjective, hence a permutation of $\mathbb{N}$.

We have an associated partition of the positive integers $\mathbb{N} = A \sqcup B$ where

$$A = A_M := \{n \in \mathbb{N} : \pi_g(n) - n \geq 0\}, \quad B = B_M := \{n \in \mathbb{N} : \pi_g(n) - n < 0\}.$$

Next we turn to asymptotics. Let

$$L := \limsup_{n\in\mathbb{N}} \frac{\pi(n)}{n} = \limsup_{n\in A} \frac{\pi(n)}{n},$$
$$l := \liminf_{n\in\mathbb{N}} \frac{\pi(n)}{n} = \liminf_{n\in B} \frac{\pi(n)}{n}.$$

We seek sufficient conditions for both L and l to be finite limits, over $n \in A$ and $n \in B$ respectively.

The main results in the first section of paper I include the following items:

- π_g is the only permutation of the natural numbers with certain properties, see also Table 1;
- if M is symmetric, then π_g is an involution of $\mathbb{N}$;
- the asymptotics of π_g can always be computed provided M is 'sufficiently nice': namely provided the positive and negative parts of M each have an asymptotic density.

1.5. *Stolarsky's Interspersion arrays.* An array $A = (a_{ij})_{i,j>0}$ of natural numbers is called an interspersion array (see [KimS95]) if the following properties are satisfied:

(i) each natural number appears exactly once in the array,
(ii) each row of the array is an infinite increasing sequence,
(iii) each column is an increasing sequence (the number of rows may or may not be finite),
(iv) for any $i, j, p, q > 0$ with $i \neq j$, if $a_{i,p} < a_{j,q} < a_{i,p+1}$, then $a_{i,p+1} < a_{j,q+1} < a_{i,p+2}$.

1	2	3	5	8	13	21	34	55	89	...
4	7	11	18	29	47	76	123	199	322	...
6	10	16	26	42	68	110	178	288	466	...
9	15	24	39	63	102	165	267	432	699	...
12	20	32	52	84	136	220	356	576	932	...

TABLE 4. The first five rows of the Wythoff Array. The next two entries to be filled in are: 14 in the position $(6, 1)$, because it is the least number not yet in the array, and $14 + 9 = 23$ in position $(6, 2)$, since 9 is the least difference not already occupying as $a_{i,2t} - a_{i,2t-1}$, $1 \le i \le 5$. Suppose the whole array has been filled in, in accordance with this rule and (6). Take any pair x and y of consecutive entries, from row i say. Suppose $j \ne i$. Then, provided $a_{j,1} > x$, there is precisely one t such that $x < a_{j,t} < y$.

1	2	3	5	8	13	21	34	55	89	...
4	6	10	16	26	42	68	110	178	288	...
7	11	18	29	47	76	123	199	322	521	...
9	14	23	37	60	97	157	254	311	565	...
12	19	31	50	81	131	212	343	555	878	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 5. The dual of the Wythoff Array. Here the Wythoff-pairs are $(a_{1,1} - 1, a_{1,1} - 1), (a_{1,2} - 1, a_{1,3} - 1), \ldots,$ $(a_{2,1} - 1, a_{2,2} - 1), (a_{2,3} - 1, a_{2,4} - 1), \ldots,$ $(a_{3,1} - 1, a_{3,2} - 1), (a_{3,3} - 1, a_{3,4} - 1), \ldots,$ and so on.

A Stolarsky interspersion ([KimS95, Sto77]) satisfies the following additional property :

(v) every row of the array is a Fibonacci sequence, i.e.: for any $i \ge 1$ and $j \ge 3$, we have that

$$a_{i,j} = a_{i,j-1} + a_{i,j-2}. \tag{6}$$

There are two Stolarsky arrays naturally associated with the permutation π_W. The first is called the Wythoff array ([Mor80]) or the Zeckendorff array ([KimZ95]). It is known that the pairs $\{a_{i,2t-1}, a_{i,2t}\}$, for all $i, t > 0$, in the Wythoff array, constitute the complete set of P-positions for Wythoff Nim. The second, the dual Wythoff array ([KimS95]), is constructed in a very similar manner to the first.

We shall generalize the constructions of these arrays to all greedy and symmetric multisets M and their associated permutations π_g^M. In the case when $M = \mathbb{Z}$ we show that our arrays coincide with the Wythoff array and its dual.

We only discuss interspersion arrays in the first paper. The reason for this is that the sequences we study in the second paper are a subset of those encountered in the first while those in the third paper cannot, in their general form, be interpreted as interspersion arrays.

In the final section of paper I we give a precise description of π_g in a somewhat more general context then we have presented here.

Paper II: 2-pile Nim with a restricted number of move-size dynamic imitations. The purpose of this paper is to explore a variation of 2-pile Nim with the same P-positions as p-blocking m-Wythoff Nim.

A possible extension of any impartial game is to adjoin the P-positions of the original game as moves in the new game. Clearly this will alter the P-positions of the original game. Indeed, if we adjoin the P-positions of 2-pile Nim as moves, we get the game of Wythoff Nim.

However there is also another way to alter the P-positions of a game, namely, using an instance of a technique that in general is known as move-size dynamics, see [BeCoGu82, Col05, HoReRu03, HoRe05]. Namely, from the original game, remove the next-player winning strategy. For 2-pile Nim this means that we remove the possibility to imitate the previous player's move— we call the new game Imitation Nim—where imitate has the following interpretation:

Definition 1. Given two piles, A and B, where $\#A \leq \#B$ and the number of tokens in the respective pile is counted before the previous player's removal of tokens, then, if the previous player removed tokens from pile A, the next player *imitates* the previous player's move if he removes the same number of tokens from pile B as the previous player removed from pile A.

Example 1. Suppose the game is Imitation Nim and the position is $(1, 3)$. If this is an a initial position, then there is no 'dynamic' restriction on the next move so that the set $\{(1, 2), (1, 1), (1, 0), (0, 3)\}$ of Nim options is identical to the set of Imitation Nim options. But this holds also, if the previous player's move was

$$(1, x) \to (1, 3),$$

or

$$(x, 3) \to (1, 3) \tag{7}$$

where $x \geq 4$. For these cases, the imitation rule does not apply since the previous player removed tokens from the larger of the two piles.

If, on the other hand, the previous move was as in (7) with $x \in \{2, 3\}$ then, by the imitation rule, the option $(1, 3) \to (1, 3 - x + 1)$ is prohibited.

Let $m \in \mathbb{N}$. We relax the notion of an imitation to an m-imitation (or just imitation) by saying: provided the previous player removed x tokens from pile A, with notation as in Definition 1, then the next player m-imitates the previous player's move if he removes $y \in [x, x + m - 1]$ tokens from pile B.

Definition 2. Let $m, p \in \mathbb{N}$. We denote by (m, p)-*Imitation Nim* the game where no p consecutive m-imitations are allowed by one and the same player.

Clearly, this new rule removes the winning strategy from 2-pile Nim if and only if the number of tokens in each pile is $\geq p$.

Suppose the parameters m and p are given (as in (m, p)-Wythoff Nim). For the statement of our main results we need two more definitions.

Definition 3. Let $a, b \in \mathbb{N}_0$. Then

$$\xi(a,b) = \xi_{m,p}\big((a,b)\big) := \#\big\{\, (i,j) \in \mathcal{P}(W_{m,p}) \mid\ j - i = b - a,\ i < a \,\big\}.$$

Definition 4. Let (a,b) be a position of a game of (m,p)-Imitation Nim. Put

$$L(a,b) = L_{m,p}((a,b)) := p - 1$$

if

(i) (a,b) is the starting position, or
(ii) $(c,d) \to (a,b)$ was the most recent move and (c,d) was the starting position, or
(iii) The previous move was $(e,f) \to (c,d)$ but the move (or option) $(c,d) \to (a,b)$ is not an m-imitation.

Otherwise, with notation as in (iii), put

$$L(a,b) = L(e,f) - 1.$$

We may state our main result.

Theorem 1. Let $0 \le a \le b$ be integers and suppose the game is (m,p)-Imitation Nim. Then (a,b) is P if and only if

(i) $(a,b) \in \mathcal{P}(W_{m,p})$ and $0 \le \xi(a,b) \le L(a,b)$, or
(ii) there is a $a \le c < b$ such that $(a,c) \in \mathcal{P}(W_{m,p})$ but $-1 \le L(a,c) < \xi(a,c) \le p - 1$.

Then, as an easy corollary we get that if (a,b) is a starting position of (m,p)-Imitation Nim it is P if and only if it is a P-position of (m,p)-Wythoff Nim.

Paper III: Restrictions of m-Wythoff Nim and p-complementary Beatty sequences. Fix $m, p \in \mathbb{N}$. In this paper we study three restrictions of m-Wythoff Nim. We generalize the solution of m-Wythoff Nim to a new pair of Beatty sequences, denoted by $a = (a_n)_{n\ge0}$ and $b = (b_n)_{n\ge0}$, where for all n,

$$a_n = a_n^{m,p} = \left\lfloor \frac{n\phi_{mp}}{p} \right\rfloor \tag{8}$$

and

$$b_n = b_n^{m,p} = \left\lfloor \frac{n(\phi_{mp} + mp)}{p} \right\rfloor . \tag{9}$$

Notice that, for fixed m and p and for all n, $b_n - a_n = mn$.

These sequences also generalize the solution of another classical variation of Wythoff Nim, studied by I.G. Connell in [Con59].

Our results depend on a generalization of Beatty's theorem (see also [Bry02]) to so-called p-complementary sequences (our terminology).

Definition 5. Let $p \in \mathbb{N}$. Two sequences (a_i) and (b_i) of non-negative integers are *p-complementary* if, for any $n \in \mathbb{N}_0$,

$$\#\{i \mid a_i = n\} + \#\{i \mid b_i = n\} = p.$$

A 1-complementary pair of sequences is simply denoted complementary, see for example [Fra69].

Theorem 2 (K. O'Bryant)**.** Let $0 < \alpha < \beta$ be irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

and let $p \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$,

$$p = \#\left\{i \in \mathbb{N} \mid n = \left\lfloor \frac{i\alpha}{p} \right\rfloor\right\} + \#\left\{i \in \mathbb{N} \mid n = \left\lfloor \frac{i\beta}{p} \right\rfloor\right\}.$$

In other words, the sequences $(\lfloor \frac{i\alpha}{p} \rfloor)$ and $(\lfloor \frac{i\beta}{p} \rfloor)$ are p-complementary.

By this result it is not hard to show that, for arbitrary m and p, a and b are p-complementary.

Fix $p \in \mathbb{N}$. We define three new 'p-generalizations' of m-Wythoff Nim—here we give a rough outline:

The first variation has a blocking manoeuvre on the rook-type (Type I, II) options, namely the previous player may block off up to $p-1$ of the next player's rook-type options as long as they are not (expanded) Type III options;

The second variation has a certain 'congruence' restriction on the rook-type options, namely a rook-type move is restricted to jumps of length $mpn+j$, for some $n \in \mathbb{N}$ and $j \in \{0, 1, \ldots, m-1\}$. We also consider a similar game where the rook-type move is simply a jump of pn and where $\gcd(m, p) = 1$.

Fix an $l \in \{0, 1, \ldots, m-1\}$. The third variation is as (mp)-Wythoff Nim, but before the game starts, the second player may determine a certain game constant $l \in \{0, 1, \ldots, k-1\}$. Then the rectangle with base ml and height $m(p-l)$ is removed from the lower left corner of the game-board. In this way, on the new board, there are two final positions, namely $(ml, 0)$ and $(0, m(p-l))$, except if the second player chooses $l = 0$. For this case the terminal position is simply $(0, 0)$.

We prove that, in terms of game complexity, our pair a and b resolve each game in polynomial time, namely a position is P if and only if it is of the form $\{a_n, b_n\}$.

For arbitrary m and $p > 1$, we also prove that our new pair of sequences is unique in the sense that it is the only pair of p-complementary Beatty sequences of which one of the sequences is strictly increasing.

It is well-known that the solution of Wythoff Nim satisfies the so-called complementary equation (see for example [Kim08])

$$x_{x_n} = y_n - 1.$$

For arbitrary m and p, we generalize this formula to a 'p-complementary equation',

$$x_{\varphi_n} = y_n - 1,$$

where $\varphi_n = \frac{x_n + (mp-1)y_n}{m}$ and show that a solution is given by $x = a$ and $y = b$. We also prove that one may also obtain a and b by three minimal exclusive algorithms, in various ways generalizing a construction given in [Fra82].

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E-mail address: *urban.larsson@chalmers.se*

Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden

PERMUTATIONS OF THE NATURAL NUMBERS WITH PRESCRIBED DIFFERENCE MULTISETS

Peter Hegarty
Chalmers University of Technology, 41296 Göteborg, Sweden
hegarty@math.chalmers.se

Urban Larsson
Chalmers University of Technology, 41296 Göteborg, Sweden
md0larur@student.chalmers.se

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Abstract

We study permutations π of the natural numbers for which the numbers $\pi(n)$ are chosen greedily under the restriction that the differences $\pi(n)-n$ belong to a given (multi)subset M of $\mathbf{Z}$ for all $n \in S$, a given subset of $\mathbf{N}$. Various combinatorial properties of such permutations (for quite general M and S) are exhibited and others conjectured. Our results generalise to a large extent known facts in the case $M = \mathbf{Z}, S = \mathbf{N}$, where the permutation π arises in the study of the game of Wythoff Nim.

1. Introduction

Consider the permutation $\pi = \pi_g$ of the natural numbers defined inductively as follows:

(i) $\pi(1) = 1$,

(ii) for each $n > 1$, $\pi(n) := t$, where t is the least natural number not already appearing among $\pi(1), ..., \pi(n-1)$ and such that $t - n \neq \pi(i) - i$, for any $1 \leq i < n$.

Informally, we say that π chooses numbers greedily under the restriction that differences $\pi(n) - n$ may not be repeated. The permutation π exhibits a rich variety of beautiful properties, which may be said to be well-known. It is an involution of $\mathbf{N}$ and its asymptotics are given by

$$\lim_{n \in A} \frac{\pi(n)}{n} = \phi = \frac{1+\sqrt{5}}{2},$$

the golden ratio, where $A = \{n : \pi(n) \geq n\}$. In the literature it is usually studied in one of several different contexts, for example in the game of Wythoff Nim, in connection with Beatty sequences and with so-called Stolarsky interspersion arrays. This material is reviewed below.

Our idea for this paper was to study permutations $\pi = \pi_g^{M,S}$ of $\mathbf{N}$, defined by a greedy choice procedure, under the restriction that the differences $\pi(n) - n$ belong to some assigned, but otherwise arbitrary, (multi)subset M of $\mathbf{Z}$, whenever $n \in S$, some assigned subset of $\mathbf{N}$. Hence the above discussion relates to the case $S = \mathbf{N}$ and $M = \mathbf{Z}$. We were motivated by the observation that some of the attractive properties of $\pi_g^{\mathbf{Z},\mathbf{N}}$ can be naturally generalised, and the purpose of the paper (and perhaps others to follow) is to carry out this generalisation as far as possible.

An outline of our results will be presented in the next section. First we wish to recall in some more detail, for the sake of the uninitiated reader, the properties of $\pi_g^{\mathbf{Z},\mathbf{N}}$ referred to above. An exposition of this material, including a detailed list of references, can be found in, for example, [9]. To ease notation, and to emphasise the connection with the game of Wythoff Nim, we henceforth denote our permutation as π_W.

Wythoff Nim (a.k.a. Corner the Queen) The positions of this 2-person impartial game, first studied by Wythoff [12], consist of pairs (k, l) of non-negative integers. From any given such position, the allowed moves are

Type I: $(k, l) \to (k', l)$ for any $0 \leq k' < k$.

Type II: $(k, l) \to (k, l')$ for any $0 \leq l' < l$.

Type III: $(k, l) \to (k - s, l - s)$ for any $0 \leq s \leq \min\{k, l\}$.

It is not dificult to see that the P-positions for this game, that is those starting positions from which the previous player has a winning strategy, are precisely the pairs $(n-1, \pi_W(n) - 1)$, for all $n \in \mathbf{N}$.

Beatty Sequences Let r, s be any positive irrational numbers such that

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{1}$$

Beatty discovered [1] that the sets $X = \{\lfloor nr \rfloor : n \in \mathbf{N}\}$, $Y = \{\lfloor ns \rfloor : n \in \mathbf{N}\}$ form a partition of $\mathbf{N}$. Now choose $r = \phi, s = \phi + 1 = \phi^2$. One readily checks that (1) is satisfied. It is well-known that π_W is completely described by

$$\pi_W(1) = 1, \quad \pi_W = \pi_W^{-1}, \quad \pi_W(\lceil nr \rceil) = \lceil ns \rceil, \quad \forall\, n \geq 1. \tag{2}$$

The point is that this gives a much more precise description of π_W than just knowing its asymptotic behaviour.

Stolarsky interspersion arrays An array $A = (a_{ij})_{i,j>0}$ of natural numbers is called an *interspersion array* if the following properties are satisfied:

(i) each natural number appears exactly once in the array,
(ii) each row of the array is an infinite increasing sequence,
(iii) each column is an increasing sequence (the number of rows may or may not be finite),
(iv) for any $i, j, p, q > 0$ with $i \neq j$, if $a_{i,p} < a_{j,q} < a_{i,p+1}$, then $a_{i,p+1} < a_{j,q+1} < a_{i,p+2}$.

A *Stolarsky interspersion* satisfies the following additional property:

(v) every row of the array is a Fibonacci sequence, i.e.: for any $i \geq 1$ and $j \geq 3$, we have that

$$a_{i,j} = a_{i,j-1} + a_{i,j-2}. \tag{3}$$

(Note that such an interspersion array must necessarily have infinitely many rows). There are two Stolarsky arrays naturally associated with the permutation π_W. The first is called the *Wythoff array* or the *Zeckendorff array*. Its first row is the 'usual' Fibonacci sequence determined by $a_{1,1} = 1$, $a_{1,2} = 2$. The remaining rows are determined inductively as follows:

(i) $a_{i,1}$ is the least natural number not already appearing in the preceeding rows
(ii) $a_{i,2}$ is the so-called *Zeckendorff right-shift* $\mathcal{Z}$ of $a_{i,1}$. That is, $a_{i,1}$ is written in terms of the base for $\mathbf{N}$ provided by the first row and then each basis element is replaced by its successor. So, for example, $a_{2,1} = 4 = 1 + 3$, in terms of the Fibonacci base, so that $a_{2,2} = \mathcal{Z}(1) + \mathcal{Z}(3) = 2 + 5 = 7$.
(iii) for every $j \geq 3$, the relation (3) is satisfied.

It was shown by Kimberling [7] that the pairs $(a_{i,2t-1}, a_{i,2t})$, for all $i, t > 0$, in the Wythoff array, constitute the complete set of P-positions for Wythoff Nim.

The second array, known in the literature as the *dual Wythoff array*, is constructed in a very similar manner to the first. The only difference is in the choice of $a_{i,2}$, for each $i > 1$. Since $a_{i,1}$ is the least positive integer not already appearing in the preceeding rows, there is a unique pair (k, j), with $k < i$, such that $a_{i,1} = a_{k,j} + 1$. In the dual array, we set $a_{i,2} := a_{k,j+1} + 1$. Here, of course, the fact that the dual array is an interspersion is already non-trivial, since one needs to prove that, for each $i > 1$, $a_{i,2}$ has not yet appeared in the preceeding rows. This fact is contained in the following well-known characterisation (see [6], section 5) of the permutation π_W:

$$\pi_W(1) = 1, \quad \pi_W = \pi_W^{-1}, \quad \pi_W(a_{1,2t}) = a_{1,2t+1} \;\; \forall\, t \geq 1,$$
$$\pi_W(a_{i,2t-1}) = a_{i,2t} \;\; \forall\, i \geq 2, t \geq 1.$$

We close this introduction by observing that in [2] Fraenkel studied the following nice generalisation of Wythoff Nim. Let m be a natural number. The game of *m-Wythoff Nim* (our terminology) is played according to the same rules as ordinary Wythoff Nim (the case $m = 1$) except that we expand the set of allowed moves of TYPE III as follows: from a position (k, l) one can move to any position $(k - s, l - t)$ such that $0 \leq s \leq k$, $0 \leq t \leq l$ and $|s - t| < m$.

Let $\pi_{W_m} = \pi_g^{m\mathbf{Z},\mathbf{N}}$ be the permutation of $\mathbf{N}$ constructed by the same greedy choice procedure as π_W, but with the restriction that $\pi_{W_m}(i) - i$ must be a multiple of m for all $i \in \mathbf{N}$. It is easy to see that the P-positions for m-Wythoff Nim are just the pairs $(n-1, \pi_{W_m}(n)-1)$, for all $n \in \mathbf{N}$. Fraenkel showed that these positions can be written in terms of Beatty sequences. If we choose

$$r = r_m := \frac{2 - m + \sqrt{m^2+4}}{2}, \qquad s = s_m := r_m + m, \tag{4}$$

then (2) holds, with π_W replaced by π_{W_m}. In particular, the asymptotic behaviour of π_{W_m} is given by

$$\lim_{n \in A} \frac{\pi_{W_m}(n)}{n} = \frac{s_m}{r_m} = \frac{m + \sqrt{m^2+4}}{2},$$

which is the positive root of the equation $x^2 - mx - 1 = 0$, where $A = \{n : \pi_{W_m}(n) \geq n\}$. The natural generalisations of the Wythoff and dual Wythoff interspersion arrays are also implicitly contained in Fraenkel's paper.

Finally, it is worth noting that a good deal of work has been done on various wide-ranging generalisations of Wythoff Nim: see, for example, [4] and [11] for some recent material. One application of our results, to be discussed in the next section, will involve an apparently novel generalisation of the game.

2. Notation, terminology and summary of results

The following standard notations will be adhered to throughout the paper:

Given two sequences $(f_n)_1^\infty$ and $(g_n)_1^\infty$ of positive real numbers, we write $f_n = \Theta(g_n)$ if there exist positive constants $c_1 < c_2$ such that $c_1 < f_n/g_n < c_2$ for all n. We write $f_n \sim g_n$ if $f_n/g_n \to 1$ as $n \to \infty$, $f_n \gtrsim g_n$ if $\liminf f_n/g_n \geq 1$ and $f_n = o(g_n)$ if $f_n/g_n \to 0$.

We now specify our principal notations and terminology.

For each $n \in \mathbf{Z}$, let $\zeta_n \in \mathbf{N}_0 \cup \{\infty\}$. The sequence $M := (\zeta_n)_{n=-\infty}^\infty$ is called a *multisubset* of $\mathbf{Z}$, or simply a *multiset*. We think of ζ_n as the number of occurrences of the integer n in M. If $\sum_{n \geq 0} \zeta_n = \infty$ we say that M is *injective*. If $\sum_{n \leq 0} \zeta_n = \infty$, we say that M is *surjective*. A multiset which is both injective and surjective will be called *bijective*. If $\zeta_n = \zeta_{-n}$ for all integers n, then M is said to be *symmetric*. The *asymptotic density* of a multiset M is defined by

$$d(M) := \lim_{n \to \infty} \frac{1}{2n+1} \left(\sum_{k=-n}^{n} \zeta_k \right),$$

whenever this limit exists. Observe that $d(M) = \infty$ whenever $\zeta_n = \infty$ for some n. Hence, this concept is only really interesting if $\zeta_n \in \mathbf{N}_0$ for all n. Such a multiset is called *finitary*.

M is said to be a *greedy multiset* if either M is finitary or the following holds: there is at most one non-negative n and at most one non-positive n for which $\zeta_n = \infty$. If $n \geq 0$ and $\zeta_n = \infty$ then $\zeta_{n'} = 0$ for all $n' > n$. If $n \leq 0$ and $\zeta_n = \infty$, then $\zeta_{n'} = 0$ for all $n' < n$.

The *positive* (resp. *negative*) *part* of a multiset M, denoted M_+ (resp. M_-), is the multiset (ζ'_n) such that $\zeta'_n = 0$ for all $n < 0$ (resp. $n \geq 0$) and $\zeta'_n = \zeta_n$ for all $n \geq 0$ (resp. $n < 0$). Finally, let $M_1 = (\zeta_{1,n})$ and $M_2 = (\zeta_{2,n})$ be any two multisets. We write $M_1 \leq M_2$ if $\zeta_{1,n} \leq \zeta_{2,n}$ for all $n \in \mathbf{Z}$.

Let S be a subset of $\mathbf{N}$ and $f : \mathbf{N} \to \mathbf{N}$ be any function. For $n \in \mathbf{N}$ we denote $d(n) = d_f(n) := f(n) - n$. The *difference multiset* of f *with respect to* S, denoted $D_{f,S}$, is defined by $D_{f,S} = (\zeta_n)_{-\infty}^{\infty}$ where

$$\zeta_n = \#\{k \in S : d(k) = n\}.$$

If $S = \mathbf{N}$ we drop the second subscript and write simply D_f.

Suppose $S = \mathbf{N}$. If f is an injective function, then D_f must be an injective multiset. For otherwise, $d_f(n) < 0$ for all but finitely many n. Thus there exists an integer $n_0 \geq 1$ such that $d_f(n) < 0$ for all $n \geq n_0$. Let $n_0 \leq T := \max\{f(n) : 1 \leq n < n_0\}$. Then $f(n) \leq T$ for all $1 \leq n \leq T + 1$, contradicting injectivity of f. By a similar argument, if f is surjective, then so is D_f. Hence if f is a permutation, then D_f is bijective. For the remainder of this paper, all multisets are assumed to be bijective.

Let $M = (\zeta_n)$ and S be given. An injective mapping $\pi_g = \pi_g^{M,S} : \mathbf{N} \to \mathbf{N}$ such that $D_{\pi_g,S} \leq M$ can be constructed by means of a 'greedy algorithm': for each $n \in \mathbf{N}$, $\pi_g(n)$ is defined inductively to be the least positive integer t not equal to $\pi_g(k)$ for any $k < n$ and, if $n \in S$, satisfying the additional condition that $\#\{k < n : k \in S \text{ and } d_{\pi_g}(k) = t - n\} < \zeta_{t-n}$.

It is easy to see that π_g is also surjective (since M is), hence a permutation of $\mathbf{N}$.

We have an associated partition of the natural numbers $\mathbf{N} = A \sqcup B \sqcup C$ where

$$A = A_{M,S} := \{n \in S : d_{\pi_g}(n) \geq 0\}, \quad B = B_{M,S} := \{n \in S : d_{\pi_g}(n) < 0\},$$
$$C = C_S := \mathbf{N} \backslash S.$$

We also fix the following notation: for each $k \in \mathbf{Z}$, $n \geq 1$, set

$$\Xi_{n,k} = \Xi_{n,k,M,S} := \#\{j : 1 \leq j \leq n, j \in S \text{ and } d_{\pi_g}(j) = k\}.$$

Note that the permutation π_{W_m} discussed in Section 1 corresponds to the pair $S = \mathbf{N}$ and $M = m\mathbf{Z}$, i.e.: $\zeta_n = 1$ if $m|n$ and $\zeta_n = 0$ otherwise.

The rest of the paper is organised as follows:

In Section 3 we begin by verifying some very general properties of these 'greedy difference' permutations (Proposition 3.1). Some are valid for any M and S, others only for certain S, including the most natural case $S = \mathbf{N}$. In particular, when $S = \mathbf{N}$ then π_g always satisfies a certain 'uniqueness property' not immediately obvious from its definition, and if furthermore M is symmetric, then π_g is an involution of $\mathbf{N}$. The main result of this section (Theorem 3.3) shows how the asymptotics of π_g can always be computed provided that M and S are 'sufficiently nice': more precisely, provided S and both the positive and negative parts of M have an asymptotic density. We also prove a converse result in the case of $S = \mathbf{N}$ and symmetric M (Proposition 3.4).

In the next two sections, it is assumed that $S = \mathbf{N}$. In Section 4 we illustrate that, for any symmetric M, there are two natural ways to arrange the pairs $\{n, \pi_g(n)\}$ in an interspersion array. These generalise the Wythoff array and its dual respectively.

The reader who seeks further motivation for our investigations, before ploughing into the rather technical material in Sections 3 and 4, might profitably read Section 5 first. In this section, we further study the multisets which we denote by $\mathcal{M}_{m,p}$, i.e.: $\zeta_n = p$ if $m|p$ and $\zeta_n = 0$ otherwise. This thus generalises the material in Fraenkel's paper [2] (the case $p = 1$). We describe a beautifully simple generalisation of the Wythoff Nim game for which the P-positions are just the pairs $(n-1, \pi_g^{\mathcal{M}_{m,p},\mathbf{N}}(n) - 1)$. The idea is to introduce a type of blocking manoeuvre, or so-called Muller twist, into the game. Our game does not seem to be studied in the existing literature either on combinatorial games with Muller twists (see [10], for example), or on Wythoff Nim (see [4], [11]).

This section is closed with a conjecture which suggests a close relationship between the values $\pi_g^{\mathcal{M}_{m,p},\mathbf{N}}(n)$ and certain Beatty sequences, which partly generalises the known results when $p = 1$. It is this aspect of the classical framework which seems to be the most difficult to generalise, which is not surprising since it concerns a very precise 'algebraic' description of the permutations $\pi_g^{M,\mathbf{N}}$, which is certainly not going to be possible for very general M. Neverthless, in some cases like $M = \mathcal{M}_{m,p}$, there is numerical evidence to suggest a very close relationship with Beatty sequences.

In Section 6, we return to the setting of more general S. We prove a quite technical theorem (Theorem 6.1) about the permutation $\pi_g^{\mathbf{Z},2\mathbf{N}}$, which establishes a very close relationship between it and a certain Beatty sequence. We close the paper with a wide-ranging conjecture which further generalises that in Section 5.

3. General properties and asymptotics

Proposition 3.1 Let M be a bijective multisubset of $\mathbf{Z}$, S a subset of $\mathbf{N}$, $\pi := \pi_g^{M,S}$, $D := D_{\pi,S}$, $A := A_{M,S}$, $B := B_{M,S}$, $C := C_S$.

(i) For any M and S, π satisfies the following properties:

$\mathcal{U}1$: The difference function d is non-decreasing on A and non-increasing on B,

$\mathcal{U}2$: π is strictly increasing on each of A and $B \cup C$.

(ii) D is a greedy multiset and, if S is infinite, then $D = M$ if and only if M is greedy.

Now suppose $S = \mathbf{N}$ (hence C is the empty set). Then

(iii) π is the unique permutation of $\mathbf{N}$ with difference multiset D which satisfies $\mathcal{U}1$ and $\mathcal{U}2$.

(iv) π is an involution, i.e.: $\pi = \pi^{-1}$, if and only if D is symmetric. If M is symmetric and greedy, then π is the unique involution on $\mathbf{N}$ with difference multiset M, which satisfies $\mathcal{U}1$ and $\mathcal{U}2$.

Proof: Fix M and S. We begin by establishing the following stronger form of property $\mathcal{U}1$:

$\mathcal{U}$: Let $n \geq 1$. Let $\Delta_n := \min\{k : k \geq 0 \text{ and } \Xi_{n-1,k} < \zeta_k\}$, $\delta_n := \max\{k : k \leq 0 \text{ and } \Xi_{n-1,k} < \zeta_k\}$. Then $d(n) \in \{\delta_n, \Delta_n\}$ if $n \in S$.

We can establish $\mathcal{U}$ by induction on n. It holds trivially for $n = 1$, so suppose it holds for $1 \leq n' < n$. If $n \in C$, there is nothing to prove. If $n \in A$, then $\mathcal{U}$ implies that no number $\geq n + \Delta_n$ has yet been chosen by π. But since π chooses greedily, it is thus clear that $\pi(n) = n + \Delta_n$, so that $\mathcal{U}$ continues to hold in this case.

Suppose $n \in B$. Now $\mathcal{U}$ guarantees that $\pi(n) \leq n + \delta_n$. It suffices to establish a contradiction to the assumption that $\pi(n) < n+\delta_n$. Let $k = k_1 < n$ be such that $\pi(n)-k_1 = \delta_n$. If $k_1 \in S$ then $\mathcal{U}$ implies that $\pi(k_1) > \pi(n)$, contradicting the definition of π. So $k_1 \in C$ and $\pi(k_1) < k_1 + \delta_n$. Let $k_2 < k_1$ be such that $\pi(k_2) - k_1 = \delta_n$. Run through the same argument again to obtain the desired contradiction unless $k_2 \in C$ and $\pi(k_2) < k_2 + \delta_n$. But now we may iterate the same argument indefinitely and thereby obtain an infinite decreasing sequence of elements of C, which is ridiculous.

Thus we have established $\mathcal{U}$, from which $\mathcal{U}1$ follows immediately, plus the fact that π is increasing on A. Suppose $m, n \in B \cup C$, with $m < n$ and $\pi(m) > \pi(n)$. Then $m \in B$. Let $z = z_1 := m - [\pi(m) - \pi(n)]$. If $z_1 \in B$ then, since $m \in B$, $\mathcal{U}$ implies that $\pi(z_1) > \pi(n)$, contradicting the definition of π. So $z_1 \in C$ and hence $\pi(z_1) < \pi(n)$. We set $z_2 := z_1 - [\pi(n) - \pi(z_1)]$ and run through the same argument to obtain a contradiction unless $z_2 \in C$ and $\pi(z_2) < \pi(z_1)$. Iterating indefinitely we obtain, as above, an infinite decreasing sequence of elements of C, which is absurd. Thus we've established $\mathcal{U}2$ and hence part (i) of the proposition. Part (ii) follows easily from $\mathcal{U}$ and previous arguments.

Turning to (iii), let τ be a permutation of $\mathbf{N}$ with $D_\tau = D_\pi$ which satisfies $\mathcal{U}1$ and $\mathcal{U}2$. Suppose $\pi \neq \tau$ and let n_0 be the smallest integer such that $\pi(n_0) \neq \tau(n_0)$. First suppose $\pi(n_0) < n_0$. Since τ is surjective, there exists $n_1 > n_0$ such that $\tau(n_1) = \pi(n_0)$. Thus $d_\tau(n_1) < d_\pi(n_0)$. But since $D_\tau = D_\pi$ and τ satisfies $\mathcal{U}1$, this implies the existence of some $n_2 \in (n_0, n_1)$ such that $d_\tau(n_2) = d_\pi(n_0)$. But then $\tau(n_2) > \tau(n_1)$, contradicting the assumption that τ satisfies $\mathcal{U}2$.

Finally, suppose $\pi(n_0) > n_0$. Then $\mathcal{U}1$ forces $\tau(n_0) < n_0$. But then, by $\mathcal{U}1$ again, we have a contradiction, since a greedy choice algorithm would rather have chosen $\tau(n_0)$ in position n_0.

Finally, it is trivial that if π is an involution, then D is symmetric. The rest of (iv) follows from (ii) and (iii) since, if π satisfies $\mathcal{U}1$ and $\mathcal{U}2$, then so does π^{-1}.

Remark 1 Suppose $S = \mathbf{N}$. For many multisets M, one can strengthen part (iii) of Proposition 3.1 to the following statement:

π is the unique permutation of $\mathbf{N}$ with difference multiset D which satisfies $\mathcal{U}1$; in particular, $\mathcal{U}1$ implies $\mathcal{U}2$.

Indeed, it is easily seen from the proof of (iii) that this is true for any multiset $M = (\zeta_n)$ satisfying: if $\zeta_n \neq 0$ and $n < m < 0$ then $\zeta_m \neq 0$. A full classification of those M for which this stronger statement of (iii) holds seems a rather messy exercise, however.

Remark 2 We now give an example to illustrate the more significant, if rather simple, fact that parts (iii) and (iv) of Proposition 3.1 do not hold for general S, that is, π_g will in general neither be the unique permutation satisfying properties $\mathcal{U}1$ and $\mathcal{U}2$, nor an involution when M is symmetric. We leave aside the issue of determining for which S such a generalisation does hold.

EXAMPLE: Let $M = \mathbf{Z}$, $S = 2\mathbf{N}$. The first few values of π_g are given by

n	1	2	3	4	5	6	7	8	9	10	11
$\pi_g(n)$	1	2	3	5	4	8	6	7	9	13	10
$d(n)$	0	0	0	1	-1	2	-1	-1	0	3	-1

from which we immediately see that π_g is not an involution. In addition, if σ is the permutation of $\mathbf{N}$ which begins

n	1	2	3	4	5	6	7	8	9	10	11
$\sigma(n)$	1	2	3	5	4	8	6	11	7	9	10
$d(n)$	0	0	0	1	-1	2	-1	3	-2	-1	-1

and then continues to choose greedily for all $n \geq 12$, then σ will also satisfy properties $\mathcal{U}1$ and $\mathcal{U}2$.

Next we turn to asymptotics. Let

$$L := \limsup_{n \in \mathbf{N}} \frac{\pi(n)}{n} = \limsup_{n \in A} \frac{\pi(n)}{n},$$
$$l := \liminf_{n \in \mathbf{N}} \frac{\pi(n)}{n} = \liminf_{n \in B \cup C} \frac{\pi(n)}{n}.$$

We seek sufficient conditions for both L and l to be finite limits, over $n \in A$ and $n \in B \cup C$ respectively. First we need a technical lemma. For $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{C})$ we denote by

$\mu_T : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ the Möbius transformation $\mu_T(z) := Tz := \frac{az+b}{cz+d}$. Recall that T is said to be *hyperbolic* if the fixed-point equation $Tz = z$ has two distinct real solutions.

Lemma 3.2 Let $r, s \in \mathbf{R}_{>0}$, $\delta \in (0,1]$ and set

$$a = a(\delta, r, s) := \left(1 + \frac{1}{rs} - \frac{1}{s}\right) + \left(1 - \frac{1}{r}\right)\left(\frac{1-\delta}{s}\right) = 1 - \frac{\delta}{s}\left(1 - \frac{1}{r}\right),$$

$$b = b(\delta, s) := \frac{1}{s} - \frac{1-\delta}{s} = \frac{\delta}{s},$$
$$c = c(r, s) := \frac{1}{r} + \frac{1}{rs} - \frac{1}{s},$$
$$d = d(s) := 1 + \frac{1}{s}.$$

Let $T = T_{\delta,r,s} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for any choice of r, s and δ, the following hold:

(i) $\det(T) = 1 - \frac{1-\delta}{s}$. T is hyperbolic with a unique fixed point $\alpha = \alpha_T$ in $[0, \delta]$ which is neither 0 nor δ.
(ii) Let $(x_n)_{n=1}^{\infty}$ be a sequence of positive real numbers such that for any $\epsilon > 0$, $x_n \in (0, \delta + \epsilon)$ for all sufficiently large n, and suppose the x_n satisfy a recurrence

$$x_{n+1} = T_n x_n, \qquad n \geq 1,$$

where $T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in GL_2(\mathbf{R})$ is such that $c_n = c$, $d_n = d$ for all n and $a_n \to a$, $b_n \to b$ as $n \to \infty$. Then $x_n \to \alpha$.

Proof: That $\det(T) = 1 - \frac{1-\delta}{s}$ is easily verified. Next,

$$(\mathrm{tr}\ T)^2 - 4(\det T) = \left(\frac{\delta - 1}{s}\right)^2 + \frac{2\delta(1-\delta)}{rs^2} + \left(2 + \frac{\delta}{rs}\right)^2 - 4 > 0,$$

which proves that T is hyperbolic. Finally, it is a tedious but straightforward exercise in high-school algebra to verify that exactly one fixed point lies in $[0, \delta]$ and is neither 0 nor δ.

(ii) This is probably a simple exercise for anyone familiar with the (elementary) theory of iteration of Möbius transformations, but we give a proof for the sake of completeness.

For convenience, we let all suffixes n range over $\mathbf{N} \cup \{\infty\}$, where $n = \infty$ refers to the matrix T, its entries, fixpoints etc. Denote the other fixpoint of T by β, so $\beta \in \mathbf{R} \backslash [0, \delta]$. Without loss of generality, each T_n is hyperbolic with fixpoints $\tau_{1,n}, \tau_{2,n} \in \mathbf{R} \cup \{\infty\}$ such that $\tau_{1,n} \in (0, \delta)$ and $\tau_{2,n} \notin [0, \delta]$ for all n and $\tau_{1,n} \to \alpha$, $\tau_{2,n} \to \beta$. Let $P_n := \begin{pmatrix} 1 & -\tau_{1,n} \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -\tau_{1,n} \\ 1 & -\tau_{2,n} \end{pmatrix}$ according as $\tau_{2,n} = \infty$ or otherwise. Note that, since the c-entry of T_n is fixed, then $\beta = \infty \Leftrightarrow c = 0 \Leftrightarrow \tau_{2,n} = \infty$ for all n.

For all $z \in \mathbf{C} \cup \{\infty\}$, $\mu_{P_n T_n P_n^{-1}}(z) = \kappa_n z$ for some $\kappa_n \in \mathbf{R}_{>0} \backslash \{1\}$ such that $\kappa_n \to \kappa_\infty$. There are two cases, namely $\kappa_\infty < 1$ and $\kappa_\infty > 1$. In either case, we may also assume without loss of generality that each κ_n satisfies the same inequality and that there exists $\epsilon > 0$ such that $|\kappa_n - 1| > \epsilon$ for all n.

Now $x_n \to \alpha$ if and only if $P_n x_n \to 0$. We have

$$\begin{aligned} P_{n+1}x_{n+1} &= P_n x_{n+1} + (P_{n+1}x_{n+1} - P_n x_{n+1}) \\ &= P_n T_n x_n + (P_{n+1}x_{n+1} - P_n x_{n+1}) \\ &= (P_n T_n P_n^{-1})(P_n x_n) + (P_{n+1}x_{n+1} - P_n x_{n+1}) \\ &= \kappa_n (P_n x_n) + (P_{n+1}x_{n+1} - P_n x_{n+1}) \,. \end{aligned} \tag{5}$$

Note that since $P_n \to P_\infty$ and the x_n are assumed to be bounded, it follows that $|P_{n+1}x_{n+1} - P_n x_{n+1}| \to 0$. First suppose $\kappa_\infty < 1$. Applying the triangle inequality to (5) gives

$$|P_{n+1}x_{n+1}| \le (1-\epsilon)|P_n x_n| + \delta_n,$$

where $\delta_n \to 0$, from which it is easily deduced that $P_n x_n \to 0$, as desired. Finally, suppose $\kappa_\infty > 1$. This time, the triangle inequality gives

$$|P_{n+1}x_{n+1}| \ge (1+\epsilon)|P_n x_n| - \delta_n,$$

which is easily seen to leave only two possibilities: either $P_n x_n \to 0$ or $P_n x_n \to \infty$. But the latter would imply that $x_n \to \beta$, which is impossible, since $\lim x_n$, if it exists, must by hypothesis lie in $[0, \delta]$. This completes the proof.

We now come to the main result of this section:

Theorem 3.3 Let M be a finitary multiset and suppose $d(M_+)$ and $d(M_-)$ both exist in $(0, \infty)$, say equal to $r/2$ and $s/2$ respectively. Let $S \subseteq \mathbf{N}$ be a set with asymptotic density $\delta/2 > 0$ (considered as a multisubset of $\mathbf{Z}$ also). Let $\alpha \in (0, \delta)$ be a fixpoint of $T_{\delta,r,s}$ as in Lemma 3.2. Then the following hold for $\pi := \pi_g^{M,S}$:

$$L = \lim_{n \in A} \frac{\pi(n)}{n}, \;\; \text{i.e., the limit exists,}$$
$$l = \lim_{n \in B \cup C} \frac{\pi(n)}{n}, \;\; \text{i.e., the limit exists,}$$

$$L = 1 + \frac{\alpha}{r}, \tag{6}$$
$$l = 1 - \frac{\delta - \alpha}{s}. \tag{7}$$

Proof: The main point is to prove that the limits exist - eqs. (6) and (7) will then follow easily.

We denote $M_+ = (\mu_n)_0^\infty$, $M_= (\nu_n)_{-1}^{-\infty}$ and, for each $n \geq 1$, $a_n := \max\{A \cap [1,n]\}$, $b_n := \max\{B \cap [1,n]\}, c_n := \max\{C \cap [1,n]\}$ and

$$\alpha_n := \frac{|A \cap [1,n]|}{n}.$$

The main task will be to show that $\alpha_n \to \alpha$ as $n \to \infty$. We begin by establishing a couple of claims.

Claim 1: $a_n \sim n$ and $b_n \sim n$.

First consider a_n. We have $(a_n, n] \subset B \cup C$. Property $\mathcal{U}2$ implies that d_π is constant on $(a_n, n]$, with value $d_0 < 0$, say. Then unless $a_n \sim n$ we'll get the contradiction that $\nu_{d_0} = \Theta(d_0)$, as the assumption that S has positive density guarantees that a positive proportion of the interval $(a_n, n]$ lies in B.

Next, consider b_n. Suppose, on the contrary, that we can find a sequence $n_l \to \infty$ such that $n_l - b_{n_l} = \Theta(n_l)$. Let us assume that $c_{n_l} \sim n_l$ as otherwise the argument becomes much simpler (note that such a situation can only arise a priori if $\delta = 1$). The aim now will be to produce a subsequence $(l') \subseteq (l)$ and intervals $I_{l'} \subseteq [1, n_{l'}]$ such that

(i) $|I_{l'}| = \Theta(n_{l'})$,

(ii) $|I_{l'} \cap \pi(A)| \gtrsim \delta |I_{l'}|$.

First suppose we have such a sequence of intervals - we can obtain a contradiction from this. Fix l'. Let

$$\pi(q) := \min\{I_{l'} \cap \pi(A)\},$$
$$\pi(Q) := \max\{I_{l'} \cap \pi(A)\}.$$

Let $K_{l'} := [q, Q] \subseteq [1, n_{l'}]$. Then $\mathcal{U}1$ implies that $|K_{l'} \cap A| = |I_{l'} \cap \pi(A)|$. In particular, $|K_{l'}| = \Theta(n_{l'})$. That $d(M_+) = \frac{r}{2} > 0$ implies that (as $l' \to \infty$)

$$[\pi(Q) - Q] - [\pi(q) - q] \sim \frac{|I_{l'} \cap \pi(A)|}{r} \gtrsim \frac{\delta}{r} |I_{l'}|,$$

hence

$$|K_{l'}| = 1 + (Q - q) \lesssim |I_{l'}| \left(1 - \frac{\delta}{r}\right).$$

It follows that

$$\frac{|K_{l'} \cap A|}{|K_{l'}|} \gtrsim \frac{\delta \cdot |I_{l'}|}{\left(1 - \frac{\delta}{r}\right) \cdot |I_{l'}|} = \delta + |\Theta(1)|.$$

But since $|K_{l'}| = \Theta(n_{l'})$, this contradicts the fact that S has density $\delta/2$.

So it remains to find the intervals $I_{l'}$. We divide the analysis into two cases:

Case I: $[\pi(c_{n_l}) - \pi(b_{n_l})] \sim (c_{n_l} - b_{n_l})$.

In this case, take $I_l := (\pi(b_{n_l}), \pi(c_{n_l})]$, so that (i) is satisfied. By $\mathcal{U}2$, $I_l \cap \pi(B) = \phi$. But

$$|I_l \cap \pi(C)| = |(b_{n_l}, c_{n_l}] \cap C| \sim (1-\delta)(c_{n_l} - b_{n_l}) \sim (1-\delta)|I_l|,$$

so (ii) is also satisfied.

Case II: We can find a sequence $(l') \subseteq (l)$ such that $[\pi(c_{n_{l'}}) - \pi(b_{n_{l'}})] \lesssim (1-\Theta(1))(c_{n_{l'}} - b_{n_{l'}})$.

Then

$$c_{n_{l'}} = d(b_{n_{l'}}) + \pi(c_{n_{l'}}) + \Theta(n_{l'}).$$

Let $\tau_{l'}$ be the smallest integer such that $\Xi_{b_{n_{l'}}, d(b_{n_{l'}}) - \tau_{l'}} < \nu_{d(b_{n_{l'}}) - \tau_{l'}}$. Since $d(M_-) > 0$, we can be sure that $\tau_{l'} = o(n_{l'})$. Set

$$\chi_{l'} := d(b_{n_{l'}}) + \pi(c_{n_{l'}}) + \tau_{l'} + 1, \qquad I^1_{l'} := [\chi_{l'}, n_{l'}], \qquad I^2_{l'} := I^1_{l'} - \left[\tau_{l'} + d(b_{n_{l'}})\right].$$

Then $|I^1_{l'}| = |I^2_{l'}| = \Theta(n_{l'})$ and, since $\chi_{l'} > b_{n_{l'}}$, we have $I^1_{l'} \subset A \cup C$. Thus, since $d(S) = \delta/2$, we have that $|I^1_{l'} \cap A| \sim \delta |I^1_{l'}|$. But furthermore, since π chooses greedily, it must be the case that for every $x \in I^1_{l'} \cap A$, $x - [\tau_{l'} + d(b_{n_{l'}})] \in I^2_{l'} \cap \pi(A)$. Thus we can finally take $I_{l'} = I^2_{l'}$ in this case, and *Claim 1* is proven.

Claim 2:

$$\pi(a_n) \sim n\left(1 + \frac{\alpha_n}{r}\right), \tag{8}$$

$$\pi(b_n) \sim n\left(1 - \frac{\delta - \alpha_n}{s}\right). \tag{9}$$

We have $\pi(a_n) = a_n + d(a_n)$. We already know that $a_n \sim n$. But $\mathcal{U}1$ and the assumption that $d(M_+) = r/2$ imply that $d(a_n) \sim \frac{\alpha_n n}{r}$. This proves (8). The proof of (9) is similar.

By $\mathcal{U}2$ we know that

$$\pi(a_n) - \alpha_n n = \#\{x \in B \cup C : \pi(x) \le \pi(a_n)\}.$$

From Claim 2 we know that

$$\pi(a_n) - \alpha_n n \sim n\left(1 + \frac{\alpha_n}{r} - \alpha_n\right). \tag{10}$$

Set

$$y = y(n) := \max\{x \in B \cup C : \pi(x) \le \pi(a_n)\}.$$

Now $y = \pi(y) + |d(y)|$. Clearly, $\pi(y) \sim \pi(a_n)$. From *Claim 1* and $\mathcal{U}1$ we also see easily that

$$|d(y)| \sim \frac{1}{s}[\pi(a_n) - \alpha_n n - (1-\delta)y],$$

and hence, by (8) and (10), that

$$\frac{y(n)}{n} \sim \frac{\left(1+\frac{1}{s}\right)\left(1+\frac{\alpha_n}{r}\right)-\frac{\alpha_n}{s}}{1+\frac{1-\delta}{s}} = 1+\Theta(1). \tag{11}$$

Relations (10) and (11) imply that

$$\alpha_{y(n)} \sim 1 - \frac{\left(1+\frac{\alpha_n}{r}-\alpha_n\right)\left(1+\frac{1-\delta}{s}\right)}{\left(1+\frac{1}{s}\right)\left(1+\frac{\alpha_n}{r}\right)-\frac{\alpha_n}{s}}. \tag{12}$$

Now let N be some very large fixed positive integer. We define a sequence $(x_{k,N})_{k=1}^{\infty}$ of rational numbers in $(0,1)$ and a sequence $(z_{k,N})_{k=1}^{\infty}$ of natural numbers tending to infinity by

$$\begin{aligned} x_{1,N} &:= \alpha_N, \quad z_{1,N} := y(N), \\ x_{k+1,N} &:= \alpha_{z_{k,N}}, \quad z_{k+1,N} := y(z_{k,N}) \ \ \forall\, k \geq 1. \end{aligned} \tag{13}$$

Lemma 3.2 implies that $x_{k,N} \to \alpha$ as $k \to \infty$. By (11), this in turn implies that $\frac{z_{k+1,N}}{z_{k,N}} \to c$ for some $c > 1$, independent of N. From the proof of Lemma 3.2 we see that the rate of convergence in both cases is determined by the multisets M_+, M_- and S, and the choice of starting point N only. ¿From this it is easy to show that $\alpha_n \sim \alpha$: for sufficiently large n one can compare α_n and $\alpha_{z_{k,N}}$ for some N such that $n/N \approx c^k$ and both k and N are also sufficiently large. We omit any further details.

From the knowledge that α_n converges to α, the whole of Theorem 3.3 follows easily. Indeed, (8) implies (6) and (9) implies (7), so the proof is complete.

Remark Given M and S satisfying the hypotheses of Theorem 3.3, a permutation τ of $\mathbf{N}$ for which $D_{\tau,S} \leq M$ and L, l as in (6) and (7), one may show that

$$\limsup_{n\in\mathbf{N}} \frac{\tau(n)}{n} \geq L, \qquad \liminf_{n\in\mathbf{N}} \frac{\tau(n)}{n} \leq l.$$

This is perhaps not surprising, and since the argument we have in mind to prove it is quite technical, while not adding much to the ideas already introduced in this section, we choose not to include it.

For the remainder of this section, and in Sections 4 and 5 to follow, we assume that $S = \mathbf{N}$. In particular, $\delta = 1$ in Theorem 3.3.

In the special case that $r = s = 2d$, say, then (6) and (7) imply that

$$L - l = \frac{1}{d}. \tag{14}$$

One may check that the fixpoint α is given by

$$\alpha = \frac{1}{2}\left[(1-2d)+\sqrt{1+4d^2}\right] \tag{15}$$

and hence that

$$L = \frac{1}{l}. \tag{16}$$

In particular, these relations hold if M is symmetric with asymptotic density d, in which case (16) also follows directly from the fact (Proposition 3.1(iv)) that π is an involution. In fact, in the symmetric case, we have a converse to Theorem 3.3. We omit the proof of the following proposition, which is similar to, though considerably simpler than, that of the theorem.

Proposition 3.4 Suppose M is finitary and symmetric. Suppose $L := \lim_{n \in A} \frac{\pi(n)}{n}$ exists and that $L > 1$. Then M has asymptotic density $d := (L - \frac{1}{L})^{-1}$.

4. Interspersion arrays

Let $M = (\zeta_n)_{-\infty}^{\infty}$ be a symmetric, greedy multiset with $\zeta_0 < \infty$. We shall describe below two simple, and very similar, algorithms for constructing an interspersion array from $\pi_g^M = \pi_g^{M,\mathbf{N}}$. In the case when $M = \mathbf{Z}$ these will be shown to coincide with the Wythoff array and its dual (and more generally for the corresponding arrays implicit in Fraenkel's paper [2] when $M = m\mathbf{Z}$, for any $m > 0$). When M is finitary, each array will contain infinitely many rows, whereas if $\zeta_k = \infty$, then each array will contain exactly k rows.

Using a suggestive notation and terminology, we shall denote the two arrays by $W = (w_{i,j})$ and $W^* = (w^*_{i,j})$, and refer to them as the *general-difference Wythoff array* and *general-difference dual Wythoff array* respectively[1]. We denote by $\mathcal{A}$ (resp. $\mathcal{A}^*$) the algorithms for producing W (resp. W^*). We shall now proceed with a formal description of $\mathcal{A}$, including proofs that it produces an array with the desired properties. We then give a short description of $\mathcal{A}^*$ and, since it is very similar, we omit details of the equally similar proofs, merely stating the corresponding results.

To describe $\mathcal{A}$, we begin by removing any zeroes from the multiset M. That is, we take $M' = (\zeta'_n)_{-\infty}^{\infty}$ to be the multiset given by $\zeta'_n := \zeta_n$ if $n \neq 0$, and $\zeta'_0 := 0$. Observe that there is a simple relation between π_g^M and $\pi_g^{M'}$, namely

$$\pi_g^M(i) = i \ \text{ for } 1 \le i \le \zeta_0, \quad \pi_g^M(i) = \pi_g^{M'}(i - \zeta_0) + \zeta_0 \ \text{ for } i > \zeta_0. \tag{17}$$

Now set $\pi := \pi_g^{M'}$, $A := A_{M'}$, $B := B_{M'}$. Let $1 = u_1 < u_2 < u_3 < \cdots$ be the elements of A arranged in increasing order. Since M' is symmetric, we have $B = \pi(A)$ and $\mathcal{U}2$ implies that $i < j \Leftrightarrow \pi(u_i) < \pi(u_j)$. The algorithm $\mathcal{A}$ is a recursive procedure for inserting the pairs $(u_i, \pi(u_i))$ one-by-one into the array W. At the n:th step it inserts the pair $(u_n, \pi(u_n))$

[1] The reason why we do not simply call the arrays 'generalised (dual) Wythoff', which seems natural, is that that terminology has already been used by, for example, Fraenkel and Kimberling [3], in a rather different context.

either immediately to the right of an earlier pair, or at the beginning of a new row. We now give the formal rules:

STEP 1: Set $w_{1,1} := u_1$, $w_{1,2} := \pi(u_1)$.

n^{TH} STEP FOR EACH $n > 1$: Each of the pairs $(u_i, \pi(u_i))$, for $1 \leq i < n$, has already been inserted into the array. Denote by W_n the finite array formed by these, and let r_n be the number of its' rows. We must now explain where to insert the pair $(u_n, \pi(u_n))$. Define $\gamma = \gamma(n)$ to be the smallest amongst the numbers appearing at the right-hand edge of each row of W_n (so $\gamma(n) = \pi(u_i)$ for some $n - r_n \leq i < n$). Let $\xi = \xi(n)$ be defined by $u_{\xi(n)} < \gamma(n) < u_{\xi(n)+1}$. Let

$$\theta = \theta_n := \gamma(n) + \left[\pi(u_{\xi(n)}) - u_{\xi(n)}\right].$$

We claim that $\theta_n = u_m$ for some $m = m(n) \geq n$. For the moment, let us assume this. Then the algorithm $\mathcal{A}$ does the following:

(i) If $m > n$ then it assigns $w_{r_n+1,1} := u_n$, $w_{r_n+1,2} := \pi(u_n)$.
(ii) If $m = n$, then suppose $\gamma(n)$ appears in the t:th row, say $\gamma(n) = w_{t,2j}$. Then we assign $w_{t,2j+1} := u_n$, $w_{t,2j+2} := \pi(u_n)$.

To verify that the algorithm is well-defined, it remains to prove the claim above. First we show that $\theta \notin B$. For suppose $\theta = \pi(u_j)$. Since $u_\xi < \gamma$ we have $\theta > \pi(u_\xi)$ and hence $j > \xi$. By definition of ξ, this implies that $u_j > \gamma$. But then $\pi(u_j) - \gamma = \pi(u_\xi) - u_\xi > \pi(u_j) - u_j$, which contradicts property $\mathcal{U}1$.

So now we know that $\theta_n = u_{m(n)}$ for some $m(n)$. It remains to show that $m(n) \geq n$. This, and the accompanying fact that $\mathcal{A}$ is well-defined, are easily achieved by induction on n. Clearly, the result holds for $n = 2$, so suppose $n > 2$ and that $\mathcal{A}$ is well-defined at all previous steps. By definition of $\mathcal{A}$, either $m(n-1) = n-1$, in which case $\gamma(n) > \gamma(n-1)$ and hence $\theta_n > \theta_{n-1}$ and $m(n) > m(n-1)$ as required, or $m(n-1) \geq n$, in which case γ, η and θ are all unchanged at the n:th step and $m(n) = m(n-1) \geq n$, as required.

We now turn to proving the various properties of the array W. The main property of interest is

Theorem 4.1 (i) W is an interspersion array.
(ii) If M is finitary, then W will contain infinitely many non-empty rows. Otherwise, if $\zeta_k = \infty$ then W will contain exactly k non-empty rows.

Proof: Part (ii) follows easily from part (i): see the remarks at the top of page 317 of [5]. We thus concentrate on proving part (i).

Of the four properties of an interspersion array listed in Section 1, the first is obvious, the second follows from the fact that $\theta_n > \gamma(n)$ for any n, and the third is also a simple consequence of the rules followed by $\mathcal{A}$. So it remains to verify the interspersion property. So let $i, j, p, q \in \mathbf{N}$ with $i < j$, and suppose that $w_{i,p} < w_{j,q} < w_{i,p+1}$. We must show that

$w_{i,p+1} < w_{j,q+1} < w_{i,p+2}$. The proof can be divided into four cases, depending on whether each of p and q is odd or even. We present the details in only one case as all the others are similar.

CASE I: p, q both odd. Then $w_{i,p} = u_x$ and $w_{j,q} = u_y$ for some $x \neq y$. The assumption is that

$$u_x < u_y < \pi(u_x), \tag{18}$$

and from this we want to deduce that

$$\pi(u_x) < \pi(u_y) < u_z, \tag{19}$$

where

$$u_z = \pi(u_x) + \pi(u_\xi) - u_\xi \quad \text{and} \quad u_\xi < \pi(u_x) < u_{\xi+1}. \tag{20}$$

The left-hand inequality in (19) follows immediately from the left-hand inequality in (18). For the other side, we observe that the right-hand inequality of (18) implies that $y \leq \xi$ and hence, by $\mathcal{U}1$, that $\pi(u_y) - u_y \leq \pi(u_\xi) - u_\xi$. But then, by (20), we have that $\pi(u_y) - u_y \leq u_z - \pi(u_x)$, which suffices to give the right side of (19).

This completes the proof of Theorem 4.1.

We now briefly describe the construction of the dual array W^*. The algorithm $\mathcal{A}^*$ first constructs an array $\Omega = (\omega_{i,j})$ which will need to be modified very slightly to produce W^* if $\zeta_0 > 0$. Namely, $\mathcal{A}^*$ begins by setting $\omega_{1,2j-1} = \omega_{1,2j} = j$ for $1 \leq j \leq \zeta_0$. This time we let $u_1 < u_2 < u_3 < \cdots$ denote, in increasing order, the sequence of elements of $A_M \backslash \{1, ..., \zeta_0\}$. $\mathcal{A}^*$ now proceeds to insert the pairs $(u_i, \pi(u_i))$ into the array Ω according to exactly the same rules as $\mathcal{A}$, with the only difference being that, this time, the function $\xi(n)$ is defined by

$$u_{\xi(n)-1} < \gamma(n) < u_{\xi(n)}.$$

The array W^* may now only differ from Ω in the first row. Namely, we take

$$w^*_{i,j} := \begin{cases} \omega_{i,j}, & \text{if } i > 1, \\ j, & \text{if } i = 1,\ 1 \leq j \leq \zeta_0, \\ \omega_{1,j+\zeta_0}, & \text{if } i = 1,\ j > \zeta_0. \end{cases}$$

We omit the proof of the following result:

Proposition 4.2 W^* is an interspersion array. It has infinitely many rows if M is finitary and exactly k rows if $\zeta_k = \infty$.

Remark There is in fact a whole family of interspersion arrays which can be constructed from a given symmetric M, of which W and W^* are the two 'extremes', in the following sense. Let the notation be as in the definition of the algorithm $\mathcal{A}$. Fix n and a choice of an integer $\Delta_n \in [\delta_{\xi(n)}, \delta_{\xi(n)+1}]$. If we take $\theta_n := \gamma(n) + \Delta_n$ then the same argument as before gives that $\theta_n = u_{m(n)}$ for some $m(n) \geq n$. Hence, provided we don't vary our choice of Δ_n as long as $m(n) > n$, one can insert the pairs $(u_i, \pi(u_i))$ in an array according to the same rules as for $\mathcal{A}$. The proof of Theorem 4.1 can be run through to show that this will be always be an interspersion array (as long as we make the appropriate adjustments regarding ζ_0). We omit further details. Clearly, W and W^* correspond respectively to the choices $\Delta_n = \delta_{\xi(n)}$ (resp. $\Delta_n = \delta_{\xi(n)+1}$) for all n.

We close this section by proving:

Proposition 4.3 If $M = \mathbf{Z}$ then W is the Wythoff/Zeckendorff array and W^* is its dual.

Proof: We give the proof for W only; the proof for W^* is similar.

Let $\pi := \pi_g^{M'}$, $A := A_{M'}$. From (2) and (17) it easily follows that

$$\pi(u) = \lceil \phi u \rceil \quad \text{for every } u \in A. \tag{21}$$

By [8], Theorems 1 and 4, in order to show that W is the Wythoff array, it thus suffices to prove the following two facts:

(i) for each $i > 1$, $w_{i,1}$ is the smallest natural number not appearing in the previous rows,

(ii) for every $i \geq 1$ and $j \geq 3$, $w_{i,j} = w_{i,j-1} + w_{i,j-2}$.

Now (i) is a trivial consequence of the rules for the algorithm $\mathcal{A}$, so we concentrate on (ii). We consider two cases, depending on whether j is odd or even.

Case I: j odd. Then there exist $u_1 \leq u_2 \leq u_3 \in A$ such that $w_{i,j-2} = u_1$, $w_{i,j-1} = \pi(u_1)$ and $w_{i,j} = u_3 = \pi(u_1) + [\pi(u_2) - u_2]$, where $u_2 = \max\{A \cap [1, \pi(u_1))\}$. Since $M = \mathbf{Z}$, it is clear that $u_2 = \pi(u_1) - 1$ (i.e.: no two consecutive integers can lie in $B = \pi(A)$). Thus $u_3 = \pi[\pi(u_1) - 1] + 1$ and we need to show that

$$\pi[\pi(u_1) - 1] + 1 = u_1 + \pi(u_1).$$

But this follows from (21) and [8], Lemma 1.3.

Case II: j even. The proof is similar, just a bit more technical, and makes use of [8], Lemma 1.4. We omit further details.

This completes the proof of Proposition 4.3.

Remark One may equally well show that for any $m \geq 1$, if $M = m\mathbf{Z}$, then $W = W_m$ coincides with the generalisation of the Wythoff/Zeckendorff array implicit in Fraenkel's paper [2]. The verification of the recurrence $w_{i,j} = m w_{i,j-1} + w_{i,j-2}$ for $i \geq 1$ and $j \geq 3$, for which one uses (2) and (4), seems rather messy however, so we do not include it.

5. The multisets $\mathcal{M}_{m,p}$

Let $m, p \geq 1$ be any fixed positive integers. We now seek further results for the multiset $\mathcal{M}_{m,p} = (\zeta_n^{m,p})$ where $\zeta_n^{m,p} := p$ if $m|n$ and $\zeta_n^{m,p} = 0$ otherwise. We denote $\pi_{m,p} := \pi_g^{\mathcal{M}_{m,p},\mathbf{N}}$.

$\mathcal{M}_{m,p}$ has density p/m and is finitary and symmetric. Hence, by Proposition 3.1, $\pi_{m,p}$ is an involution, and by Theorem 3.3 the limits L and l exist and are given by

$$L^2 - \frac{m}{p}L - 1 = 0 \;\Rightarrow\; L = \frac{m + \sqrt{m^2 + 4p^2}}{2p}, \tag{22}$$

$$l = \frac{1}{L} = L - \frac{m}{p}. \tag{23}$$

p-Blocking m-Wythoff Nim For want of something better, this is the name we have chosen for a generalisation of the m-Wythoff game of Section 1 for which the P-positions are precisely the pairs $(n-1, \pi_{m,p}(n)-1)$ for $n \geq 1$. The rules of the game are just as in the m-Wythoff game, with one exception. Before each move is made, the previous player is allowed to 'block' some of the possible moves of Type III. More precisely, if the current configuration is (k, l), then before the next move is made, the previous player is allowed to choose up to $p-1$ distinct, positive integers $c_1, ..., c_{p-1} \leq \min\{k, l\}$ and declare that the next player may not move to any configuration $(k - c_i, l - c_i)$.

For $m = 1$ and any p, it is not hard to see that, by property $\mathcal{U}1$, the P-positions of the game are precisely the configurations $(n-1, \pi_{1,p}(n)-1)$. Combining with the methods of [2], one obtains the same result for all m and p. We omit further details. The interest of the game lies in it being a Muller twist, in the sense of [10], of m-Wythoff Nim.

Beatty sequences There is a simple reason why, for any $p > 1$, it won't be possible to express the pairs $(n, \pi_{m,p}(n))$ as $(\lceil nr \rceil, \lceil ns \rceil)$ for any real r and s satisfying (1), and depending only on m and p. Let us say that an ordered pair (x, y) of real numbers is *in standard form* if $x \leq y$. Two ordered pairs $(n_1, \pi_{m,p}(n_1))$ and $(n_2, \pi_{m,p}(n_2))$, in standard form, are said to be *consecutive* if $n_1 < n_2$ and there is no pair $(n_3, \pi_{m,p}(n_3))$ in standard form such that $n_1 < n_3 < n_2$.

Now the point is that, for any $p > 1$, there may exist consecutive pairs $(n_1, \pi_{m,p}(n_1))$ and $(n_2, \pi_{m,p}(n_2))$ for which $n_2 - n_1$ is any integer in $\{1, ..., p+1\}$. On the other hand, for any real α and integer n, the difference $\lceil (n+1)\alpha \rceil - \lceil n\alpha \rceil$ can attain one of only two possible values.

Nevertheless, there does appear to be a close relationship between all the permutations $\pi_{m,p}$ and Beatty sequences. Here we content ourselves with conjecturing a weak form of this relationship:

Conjecture 5.1 Fix $m, p \geq 1$ and let L and l be given by (22) and (23). Then there exists an integer $c = c_{m,p} > 0$, depending only on m and p, such that for each $n \geq 1$, $\pi_{m,p}(n)$ differs from one of the numbers $\lfloor nL \rfloor$ and $\lfloor nl \rfloor$ by at most $c_{m,p}$.

One may check that (2) and (4) imply that $c_{m,1} = m-1$ for all m (more precisely, $\pi(n) = \lfloor nl \rfloor$ or $\lfloor nL \rfloor - j$ for some $0 \le j < m$ in this case). The conjecture is supported by numerical evidence, which even suggests perhaps that the constant $c_{m,p}$ can be made independent of p. For example, for $m = 1$ and $p \le 5$, we have checked that, for all $n \le 10,000$, $\pi_{1,p}(n)$ is one of the four numbers $\lfloor nL \rfloor$, $\lceil nL \rceil$, $\lfloor nl \rfloor$, $\lceil nl \rceil$.

A thorough analysis of the connection between the permutations $\pi_{m,p}$ and Beatty sequences is left for future work.

6. The case $S = k\mathbf{N}$

We now briefly return to the setting of more general subsets S of $\mathbf{N}$. Whenever we can compute the asymptotics of π_g, i.e.: the limits L and l, it makes sense to ask if there is a closer relationship between the sequences $(\pi_g(n))_{n \in A}$ and $(\pi_g(n))_{n \in B}$, and the sequences $\lfloor nL \rfloor$ and $\lfloor nl \rfloor$ respectively (which are Beatty sequences unless L and/or l are rational). For the example introduced earlier ($M = \mathbf{Z}, S = 2\mathbf{N}$), we shall show below (Theorem 6.1) that this is indeed the case, and state a more general conjecture (Conjecture 6.4) which extends Conjecture 5.1. However, as our method of proof for Theorem 6.1 will be seen to already be very technical, we are unable to shed much light here on the more general hypothesis.

Before stating the theorem, we need some further notation. For any positive integer n we denote

$$\epsilon_n := \sqrt{3}n - \lfloor \sqrt{3}n \rfloor.$$

Set

$$\eta := 2 - \sqrt{3},$$

and observe that, for all n,

$$\epsilon_n - \epsilon_{n+1} \equiv \eta \pmod 1. \tag{24}$$

Let

$$0 = n_0 < n_1 < n_2 < \cdots$$

denote the sequence of non-negative integers for which $\epsilon_{n_i} < \eta$. The interval $[2n_{i-1}, 2n_i)$ will be called the i:th *period*.

Theorem 6.1 Let $M = \mathbf{Z}$, $S = 2\mathbf{N}$, $\pi := \pi_g^{M,S}$. Define a function $f = f_2 : \mathbf{N} \to \mathbf{N}$ as follows:

(I) for any $n \ge 1$, $f(2n-1) := \min\{t : t \ne f(i) \text{ for any } i \le 2n-2\}$.

(II) for any $n \geq 1$,

$$f(2n) := n + \lfloor \sqrt{3}n \rfloor, \quad \text{if } \epsilon_n > \eta \text{ and } \epsilon_{n-1} > \eta,$$

$$f(2n) := n + \lfloor \sqrt{3}n \rfloor, \quad \text{if } \epsilon_n < \eta \text{ and } \lfloor \sqrt{3}n \rfloor \in \{f(i) : i < 2n\},$$

$$f(2n) := \lfloor \sqrt{3}n \rfloor, \quad \text{if } \epsilon_n < \eta \text{ and } \lfloor \sqrt{3}n \rfloor \notin \{f(i) : i < 2n\},$$

$$f(2n) := n + \lfloor \sqrt{3}n \rfloor, \quad \text{if } \epsilon_{n-1} < \eta \text{ and } f(2n-2) = \lfloor \sqrt{3}(n-1) \rfloor,$$

$$f(2n) := \lfloor \sqrt{3}n \rfloor + 2, \quad \text{otherwise, i.e.: iff } \epsilon_{n-1} < \eta \text{ and } \lfloor \sqrt{3}(n-1) \rfloor \in \{f(i) : i < 2n-2\}.$$

Then $f \equiv \pi$.

Remark: It is clear that the function f is a well-defined permutation of $\mathbf{N}$. Since, for this pair M, S, we have $r = s = 1$ and $\delta = \frac{1}{2}$, Theorem 3.3 says that $L = \frac{1+\sqrt{3}}{2}$, $l = L - \frac{1}{2} = \frac{\sqrt{3}}{2}$. Thus Theorem 6.1 asserts that, for all $n \in A$, $\pi(n) = \lfloor nL \rfloor$, and for all $n \in B$, $\pi(n) = \lfloor nl \rfloor$ or $\lfloor nl \rfloor + 2$. The behaviour of $\pi(n)$ for $n \in C$ seems to be a bit more erratic, though from $\mathcal{U}2$ we can deduce, for example, that $|\pi(n) - \lfloor nl \rfloor| \leq 2$ for all $n \in C$.

In the proof to follow, the sets A, B and C will refer to π and have their usual meaning. The corresponding sets for f will be denoted A_f, B_f and C_f. We begin with a lemma which follows immediately from the definition of f:

Lemma 6.2 Let $2m_1 < 2m_2$ be two consecutive numbers in A_f. Then either (i) or (ii) holds, where
(i)

$$m_2 = m_1 + 1, \quad \epsilon_{m_1} > \eta, \quad \epsilon_{m_2} > \eta \text{ and } f(2m_2) = f(2m_1) + 3. \tag{25}$$

(ii)

$$m_2 = m_1 + 2, \quad \epsilon_{m_2} > \eta, \quad \epsilon_{m_1} < \eta \text{ or } \epsilon_{m_1+1} < \delta, \text{ and } f(2m_2) - f(2m_1) = 5. \tag{26}$$

Our idea is to prove by induction on $k > 0$ that $f(n) = \pi(n)$ for all n in the k-th period. One may verify by hand that the two functions coincide over the first 3 periods say ($n_3 = 11$). Now let $k > 3$ and suppose that $f \equiv \pi$ over the first $k - 1$ periods. Note that, by definition,

$$\text{If } n \text{ is odd, then } f(i) = \pi(i) \; \forall \, i < n \Rightarrow f(n) = \pi(n). \tag{27}$$

The main tool in our proof (which does not depend on the induction hypothesis) is the following:

Lemma 6.3 Suppose $\epsilon_n < \eta$. Then there are precisely $2n - \lfloor \sqrt{3}n \rfloor$ values of $m < n$ such that $f(2m) \geq \lfloor \sqrt{3}n \rfloor$, unless perhaps $f(2m) = \lfloor \sqrt{3}n \rfloor - 1$ for some $m < n$ where $2m \in A_f$.

Proof: Let $1 \le m < n$ be even such that $f(2m) > \lfloor\sqrt{3}n\rfloor$. Then $2m \in A_f$ and $f(2m) = m + \lfloor\sqrt{3}m\rfloor$. Thus

$$f(2m) > \lfloor\sqrt{3}n\rfloor \Leftrightarrow m > \frac{\sqrt{3}n - \epsilon_n}{1+\sqrt{3}}. \tag{28}$$

Set

$$m^0 := \frac{\sqrt{3}n - \epsilon_n}{1+\sqrt{3}}. \tag{29}$$

After a little manipulation, we find that

$$m^0 = \frac{3n - \lfloor\sqrt{3}n\rfloor}{2} - \frac{\sqrt{3}}{2}\epsilon_n.$$

Set $m_0 := \lfloor m^0 \rfloor$. Since $\epsilon_n < \eta$, it is easily checked that $m_0 = m^0 - \epsilon$, where

$$\epsilon = \begin{cases} 1 - \frac{\sqrt{3}}{2}\epsilon_n, & \text{if } 3n - \lfloor\sqrt{3}n\rfloor \in 2\mathbf{Z}, \\ \frac{1}{2} - \frac{\sqrt{3}}{2}\epsilon_n, & \text{if } 3n - \lfloor\sqrt{3}n\rfloor \notin 2\mathbf{Z}. \end{cases} \tag{30}$$

Since $2m \in A_f$, we have to count the number of elements of A_f in the interval $(2m_0, 2n)$. Since $\epsilon_n < \eta$, there are precisely $2n - \lfloor\sqrt{3}n\rfloor$ elements of B_f in the interval $(1, 2n)$, one for each period. Similarly, there are $2m_0 - \lfloor\sqrt{3}m_0\rfloor + \phi$ elements of B_f in the interval $[1, 2m_0]$, where $\phi = 0$ unless $\epsilon_{m_0} < \eta$ and $2m_0 \in B_f$, in which case $\phi = 1$. Hence the total number of elements of A_f in $(2m_0, 2n)$ is

$$\begin{aligned}(n - m_0 - 1) - \Big[(2n - \lfloor\sqrt{3}n\rfloor) - (2m_0 - \lfloor\sqrt{3}m_0\rfloor + \phi)\Big] \\ = \Big(\lfloor\sqrt{3}n\rfloor - \lfloor\sqrt{3}m_0\rfloor\Big) - (n - m_0) - 1 + \phi \\ = (\sqrt{3}-1)(n - m^0) + (\sqrt{3}-1)\epsilon + (\epsilon_{m_0} - \epsilon_n) + (\phi - 1).\end{aligned}$$

Using (29) and the fact that $(1+\sqrt{3})\eta = \sqrt{3} - 1$, this becomes

$$2n - \lfloor\sqrt{3}n\rfloor + \Delta,$$

where

$$\Delta = (\sqrt{3}-1)\epsilon + \epsilon_{m_0} - \sqrt{3}\epsilon_n + \phi - 1. \tag{31}$$

We shall now show that $\Delta = 0$ unless $\epsilon_{m_0} < \eta$ and $f(2m_0) = \lfloor\sqrt{3}n\rfloor - 1$, in which case $\Delta = -1$. This will suffice to prove the lemma. The analysis can be divided into two cases, suggested by (30). We present in detail the case $\epsilon = \frac{1}{2} - \frac{\sqrt{3}}{2}\epsilon_n$, which is the only one in which the possibility that $\Delta = -1$ can arise. The other case is treated similarly but is technically simpler.

The value of ϵ implies that

$$m_0 = \frac{3n - \lfloor\sqrt{3}n\rfloor}{2} - \frac{1}{2}.$$

A little computation shows that

$$\sqrt{3}m_0 = \left(\lfloor\sqrt{3}n\rfloor - m_0\right) + \gamma, \tag{32}$$

where

$$\gamma = \left(\frac{3+\sqrt{3}}{2}\right)\epsilon_n - \frac{\sqrt{3}+1}{2}.$$

Since $\epsilon_n < \eta$, one checks readily that $\gamma \in \left(\frac{-\sqrt{3}-1}{2}, -1+\eta\right) \subset$ $(-2+\eta, -1+\eta)$. Hence there are the following two possibilities: either

$$\epsilon_{m_0} > \eta \quad \text{and} \quad \epsilon_{m_0} = \frac{3-\sqrt{3}}{2} + \left(\frac{3+\sqrt{3}}{2}\right)\epsilon_n, \tag{33}$$

or

$$\epsilon_{m_0} < \eta \quad \text{and} \quad \epsilon_{m_0} = \frac{1-\sqrt{3}}{2} + \left(\frac{3+\sqrt{3}}{2}\right)\epsilon_n. \tag{34}$$

If (33) holds, then $\phi = 0$ also. Substituting everything into (31) in this case, one readily computes that $\Delta = 0$, independent of ϵ_n, as required. If (34) holds, then substituting everything into (31) one finds that $\Delta = -1 + \phi$. If $2m_0 \in B_f$, then $\phi = 1$ and $\Delta = 0$ again, as required. Otherwise, $\Delta = -1$ and $2m_0 \in A_f$. But then, from (32) and (34), we find that

$$f(2m_0) = m_0 + \lfloor\sqrt{3}m_0\rfloor = m_0 + \left(\lfloor\sqrt{3}n\rfloor - m_0 - 1\right) = \lfloor\sqrt{3}n\rfloor - 1,$$

and the lemma is proved.

Now let us perform the induction step. To simplify notation, set $N := n_k$. Note that $\mathcal{U}2$, together with Lemmas 6.2 and 6.3, imply that if $\lfloor\sqrt{3}N\rfloor - 1 \in f(A_f)$, then $f(2N-1) = \lfloor\sqrt{3}N\rfloor$. Let $m_0 = m_0(N) = \lfloor m^0 N\rfloor$ be as in (29) ff.

The k:th period is either $[2N, 2N+5]$ or $[2N, 2N+7]$ according as to whether $\epsilon_{N+3} < \eta$ or not respectively. Clearly,

$$\epsilon_{N+3} < \eta \Leftrightarrow \epsilon_N < 4\eta - 1. \tag{35}$$

It is required to show that $f(2N+i) = \pi(2N+i)$ for $i \in [0,5]$ or $i \in [0,7]$, as appropriate. The first and crucial observation is that Lemma 6.3, together with the induction hypothesis and the definition of f, imply the result for $i = 0$. By (27) it also suffices to treat the case of even i. We now divide the remainder of the proof into two cases:

CASE I: $2N \in A_f$.

Lemma 6.3 and its proof imply that, in CASE I, either

(i) $2m_0 \in A_f$, $\epsilon_{m_0} < \eta$, $f(2m_0) = \lfloor\sqrt{3}N\rfloor - 1$ and $f(2N-1) = \lfloor\sqrt{3}N\rfloor$, or
(ii) $f(2m_1) = \lfloor\sqrt{3}N\rfloor$ for some $2m_1 \in A_f$. In this case, it is clear from (29) that $m_1 = m_0 + 1$ and $\epsilon_{m_1} = \frac{3+\sqrt{3}}{2}\epsilon_N$.

$i = 2$: Since $2N \in A_f$, the definition of f implies that $2(N+1) \in B_f$, and that $f(2N+2) = \lfloor\sqrt{3}N\rfloor + 2$. We have to show that $2(N+1) \in B$. If not, it can only be because the number $\lfloor\sqrt{3}N\rfloor + 2$ was already chosen by π, and hence also by f (because of the induction hypothesis), and hence lies in $f(A_f)$, by Lemma 6.3. But if (i) holds, then this is impossible by (26), and if (ii) holds, it is impossible by (25).

$i = 4$: This time, it is required to show that $2(N+2) \notin B$. If it were, since the numbers $\lfloor\sqrt{3}N\rfloor + j$, $j = 1, 2$, have already been chosen in positions $2N + j$, $j = 1, 2$, the avoidance property of π leaves as the only option that $\pi(2N+4) = \lfloor\sqrt{3}N\rfloor + 3$. But then this number was not already chosen in position $2N+3$, which is only possible if it already appeared in $f(A_f)$, i.e: it cannot but already have appeared somewhere, and hence π will not choose it again.

$i = 6$: Once again, it needs to be shown that $2(N+3) \notin B_f$. The analysis of the $i = 4$ case, together with (27), shows that all numbers up to and including $\lfloor\sqrt{3}N\rfloor + 3$ have already appeared in the first $2N+3$ positions. By a similar analysis, either the number $\lfloor\sqrt{3}N\rfloor + 4$ has already appeared in $f(A_f)$ by then, or it appears in position $2N+5$. That leaves as the only option, if indeed $2(N+3) \in B$, that $\pi(2N+6) = \lfloor\sqrt{3}N\rfloor + 5$. Our analysis shows moreover that this can only happen if the numbers $\lfloor\sqrt{3}N\rfloor + j$, $j = 1, 2, 3, 4$, have appeared in positions $2N + j'$, where $j' = 1, 2, 3, 5$ respectively. In particular, this means that none of the numbers $\lfloor\sqrt{3}N\rfloor + l$, $l = 1, 2, 3, 4, 5$, appears in $f(A_f)$. This contradicts Lemma 6.2.

CASE II: $2N \in B_f$.

Lemma 6.3 and $\mathcal{U}2$ imply that $\lfloor\sqrt{3}N\rfloor - 1$ does not appear in $f(A_f)$. The analysis is very similar to CASE I, but for $i = 6$ becomes considerably more technical. We present just this part of the proof. Note that, by (35), we may henceforth assume that $\epsilon_N > 4\gamma - 1$.

$i = 6$: It is required to show that $2N+6 \in A$. If not, one easily sees by going through the analysis for the values of $i < 6$ that we must, a priori, have $\pi(2N+6) = \lfloor\sqrt{3}N\rfloor + j$, where $j = 4$ or 5. If $j = 4$ then we will derive the contradiction that none of the six consecutive numbers $\lfloor\sqrt{3}N\rfloor + l$, $l = -1, 0, 1, 2, 3, 4$, appears in $f(A_f)$.

Thus we may assume that $j = 5$. Here we can still deduce that exactly one of the seven consecutive numbers $\lfloor\sqrt{3}N\rfloor + l$, $l = -1, 0, 1, 2, 3, 4, 5$, appears in $f(A_f)$. By Lemma 6.2, the correct value of l must be $1, 2$ or 3. Suppose $f(2m) = \lfloor\sqrt{3}N\rfloor + l$. Clearly, $m = m_1$ or $m = m_1 + 1$, where $m_1 = m_0 + 1$, as above. By (29), we have that

$$(1+\sqrt{3})m_1 = \lfloor\sqrt{3}N\rfloor + \epsilon^*,$$

where

$$\epsilon^* = \begin{cases} (\sqrt{3}+1)\frac{\sqrt{3}}{2}\epsilon_N, & \text{if } 3N - \lfloor\sqrt{3}N\rfloor \in 2\mathbf{Z}, \\ (\sqrt{3}+1)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\epsilon_N\right), & \text{if } 3N - \lfloor\sqrt{3}N\rfloor \notin 2\mathbf{Z}. \end{cases}$$

We examine the two possibilities separately:

First suppose

$$\epsilon^* = \left(\sqrt{3}+1\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\epsilon_N\right).$$

Since $4\eta - 1 < \epsilon_N < \eta$, we easily compute that $\epsilon^* \in (1+2\eta, 2)$. Thus $\epsilon_{m_1} > 2\eta$ and $\lfloor(1+\sqrt{3})m_1\rfloor = \lfloor\sqrt{3}N\rfloor + 1$, hence $\lfloor(1+\sqrt{3})(m_1+1)\rfloor = \lfloor\sqrt{3}N\rfloor + 4$. It follows that $2m_1 \in A_f$ and $2(m_1+1) \in B_f$. But this is contradicted by (25), (26) and the fact that $\epsilon_{m_1} > 2\eta \Rightarrow \epsilon_{m_1+1} > \eta$.

Finally, suppose

$$\epsilon^* = \left(\sqrt{3}+1\right)\frac{\sqrt{3}}{2}\epsilon_N.$$

Then $\epsilon^* = \epsilon_{m_1}$ and $\lfloor(1+\sqrt{3})m_1\rfloor = \lfloor\sqrt{3}N\rfloor$. Thus $l = 2$ or 3 in this case. But in either case, we have at least three consecutive numbers to the left of $\lfloor\sqrt{3}N\rfloor + l$, none of which is appears in $f(A_f)$. By (26), this forces either

(i) $l = 3$, $\epsilon_{m_0} < \eta$, or
(ii) $l = 2$, $\epsilon_{m_1} < \eta$.

But (i) is impossible, since one easily checks that $\epsilon_N \in (0,\eta) \Rightarrow \epsilon_{m_1} \in (\eta, 1-\eta) \Rightarrow \epsilon_{m_0} \in (2\eta, 1)$.
And (ii) is impossible since Lemma 6.2 would then imply that $\lfloor\sqrt{3}N\rfloor + 5$ also appeared in $f(A_f)$.

Thus we have completed the proof that $f = \pi$ over the k:th period, and thus the induction step, and hence the proof of Theorem 6.1, is complete.

We finish the paper with a natural extension of Conjecture 5.1:

Conjecture 6.4 Let m, p, k be any three positive integers. Let $M := \mathcal{M}_{m,p}$ and take $S = \mathcal{S}_k := k\mathbf{N}$. Let $\pi = \pi^k_{m,p} := \pi^{M,S}_g$ and let L, l be as in (6), (7). Then there exists a positive integer $c = c_{m,p,k}$, depending only on m, p and k, such that, for all $n \in \mathbf{N}$, $\pi(n)$ differs from one of the numbers $\lfloor nL\rfloor$ and $\lfloor nl\rfloor$ by at most $c_{m,p,k}$.

As already remarked, Theorem 6.1 implies that we can take $c_{1,1,2} = 2$.

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2-PILE NIM WITH A RESTRICTED NUMBER OF MOVE-SIZE IMITATIONS

Urban Larsson
Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden
urban.larsson@chalmers.se

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Abstract

We study a variation of the combinatorial game of 2-pile Nim. Move as in 2-pile Nim but with the following constraint: Suppose the previous player has just removed say $x > 0$ tokens from the shorter pile (either pile in case they have the same height). If the next player now removes x tokens from the larger pile, then he imitates his opponent. For a predetermined natural number p, by the rules of the game, neither player is allowed to imitate his opponent on more than $p - 1$ consecutive moves. We prove that the strategy of this game resembles closely that of a variant of Wythoff Nim—a variant with a blocking manoeuvre on $p - 1$ diagonal positions. In fact, we show a slightly more general result in which we have relaxed the notion of what an imitation is. The paper includes an appendix by Peter Hegarty, Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, hegarty@chalmers.se.

1. Introduction

A finite impartial game is usually a game where

- there are 2 players and a starting position,
- there is a finite set of possible positions of the game,
- there is no hidden information,
- there is no chance-device affecting how the players move,
- the players move alternately and obey the same game rules,

- there is at least one final position, from which a player cannot move, which determines the winner of the game and
- the game ends in a finite number of moves, no matter how it is played.

If the winner of the game is the player who makes the final move, then we play under normal play rules, otherwise we play a misère version of the game.

In this paper *a game*, say G, is always a finite impartial game played under normal rules. The player who made the most recent move will be denoted by *the previous player*. A position from which the previous player will win, given best play, is called a *P-position*, or just P. A position from which the *next player* will win is called an *N-position*, or just N. The set of all P-positions will be denoted by $\mathcal{P} = \mathcal{P}(G)$ and the set of all N-positions by $\mathcal{N} = \mathcal{N}(G)$.

Suppose A and B are the two piles of a 2-pile take-away game, which contain $a \geq 0$ and $b \geq 0$ tokens respectively. Then the *position* is (a, b) and a *move* (or an *option*) is denoted by $(a, b) \to (c, d)$, where $a - c \geq 0$ and $b - d \geq 0$ but not both $a = c$ and $b = d$. All our games are symmetric in the sense that (a, b) is P if and only if (b, a) is P. Hence, to simplify notation, when we say (a, b) is P (N) we also mean (b, a) is P (N). Throughout this paper, we let $\mathbb{N}_0$ denote the non-negative integers and $\mathbb{N}$ the positive integers.

1.1. The game of Nim

The classical game of Nim is played on a finite number of piles, each containing a non-negative finite number of tokens, where the players alternately remove tokens from precisely one of the non-empty piles—that is, at least one token and at most the entire pile—until all piles are gone. The winning strategy of Nim is, whenever possible, to move so that the "Nim-sum" of the pile-heights equals zero, see for example [Bou02] or [SmSt02, page 3]. When played on one single pile there are only next player winning positions except when the pile is empty. When played on two piles, the pile-heights should be equal to ensure victory for the previous player.

1.2. Adjoin the P-positions as moves

A possible extension of a game is ($\star$) *to adjoin the P-positions of the original game as moves in the new game*. Clearly this will alter the P-positions of the original game.

Indeed, if we adjoin the P-positions of 2-pile Nim as moves, then we get another famous game, namely Wythoff Nim (a.k.a. Corner the Queen), see [Wyt07]. The set of moves are: Remove any number of tokens from one of the piles, or remove the same number of tokens from both piles.

The P-positions of this game are well known. Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. Then (x, y) is a P-position if and only if

$$(x, y) \in \left\{ \left(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor \right) \mid n \in \mathbb{N}_0 \right\}.$$

We will, in a generalized form, return to the nice arithmetic properties of this and other sequences in Proposition 2 (see also [HeLa06] for further generalizations).

Other examples of ($\star$) are the Wythoff-extensions of n-pile Nim for $n \geq 3$ discussed in [BlFr98, FrKr04, Sun, SuZe04] as well as some extensions to the game of 2-pile Wythoff Nim in [FrOz98], where the authors adjoin subsets of the Wythoff Nim P-positions as moves in new games.

1.3. Remove a game's winning strategy

There are other ways to construct interesting extensions to Nim on just one or two piles. For example we may introduce a so-called *move-size dynamic* restriction, where the options in some specific way depend on how the previous player moved (for example how many tokens he removed), or "pile-size dynamic"[1] restrictions, where the options depend on the number of tokens in the respective piles.

The game of "Fibonacci Nim" in [BeCoGu82] (page 483) is a beautiful example of a move-size dynamic game on just one pile. This game has been generalized, for example in [HoReRu03]. Treatments of two-pile move-size dynamic games can be found in [Col05], extending the (pile-size dynamic) "Euclid game", and in [HoRe05].

The games studied in this paper are move-size dynamic. In fact, similar to the idea in Section 1.2, there is an obvious way to alter the P-positions of a game, namely ($\star\star$) *from the original game, remove the next-player winning strategy.* For 2-pile Nim this means that we remove the possibility to *imitate* the previous player's move, where imitate has the following interpretation:

Definition 1 Given two piles, A and B, where $\#A \leq \#B$ and the number of tokens in the respective pile is counted before the previous player's move, then, if the previous player removed tokens from pile A, the next player *imitates* the previous player's move if he removes the same number of tokens from pile B as the previous player removed from pile A.

We call this game *Imitation Nim.* The intuition is, given the position (a, b), where $a \leq b$, Alice can prevent Bob from going to (c, d), where $c < a$ and $b - a = d - c$, by moving $(a, b) \to (c, b)$. We illustrate with an example:

[1]We understand that pile-size dynamic games are not 'truly' dynamic since for any given position of a game, one may determine the P-positions without any knowledge of how the game has been played up to that point.

Example 1 Suppose the position is $(1,3)$. If this is an initial position, then there is no 'dynamic' restriction on the next move so that the set $\{(1,2),(1,1),(1,0),(0,3)\}$ of Nim options is identical to the set of Imitation Nim options. But this holds also, if the previous player's move was

$$(1,x) \to (1,3),$$

or

$$(x,3) \to (1,3) \tag{1}$$

where $x \geq 4$. For these cases, the imitation rule does not apply since the previous player removed tokens from the larger of the two piles. If, on the other hand, the previous move was as in (1) with $x \in \{2,3\}$ then, by the imitation rule, the option $(1,3) \to (1,3-x+1)$ is prohibited.

Further, $(3,3) \to (1,3)$ is a losing move, since, as we will see in Proposition 1 (i), $(1,3) \to (1,2)$ is a winning move. But, by the imitation rule, $(2,3) \to (1,3)$ is a winning move, since for this case $(1,3) \to (1,2)$ is forbidden.

This last observation leads us to ask a general question for a move-size dynamic game, roughly: *When does the move-size dynamic rule change the outcome of a game?* To clarify this question, let us introduce some non-standard terminology, valid for any move-size dynamic game.

Definition 2 Let G be a move-size dynamic game. A position $(x,y) \in G$ is

1. *dynamic*: if, in the course of the game, we cannot tell whether it is P or N without knowing the history, at least the most recent move, of the game;
2. *non-dynamic*

   P: if it is P regardless of any previous move(s),

   N: ditto, but N.

Remark 1 Henceforth, if not stated otherwise, we will think of a (move-size dynamic) game as a game where the progress towards the current position is memorized in an appropriate manner. A consequence of this approach is that each (dynamic) position is P or N.

In light of these definitions, we will now characterize the winning positions of a game of Imitation Nim (see also Figure 1). This is a special case of our main theorem in Section 2. Notice, for example, the absence of Wythoff Nim P-positions that are dynamic, considered as positions of Imitation Nim.

Proposition 1 Let $0 \le a \le b$ be integers. Suppose the game is Imitation Nim. Then (a, b) is

(i) non-dynamic P if and only if it is a P-position of Wythoff Nim;

(ii) non-dynamic N if and only if

(a) there are integers $0 \le c \le d < b$ with $b - a = d - c$ such that (c, d) is a P-position of Wythoff Nim, or

(b) there is a $0 \le c < a$ such that (a, c) is a P-position of Wythoff Nim.

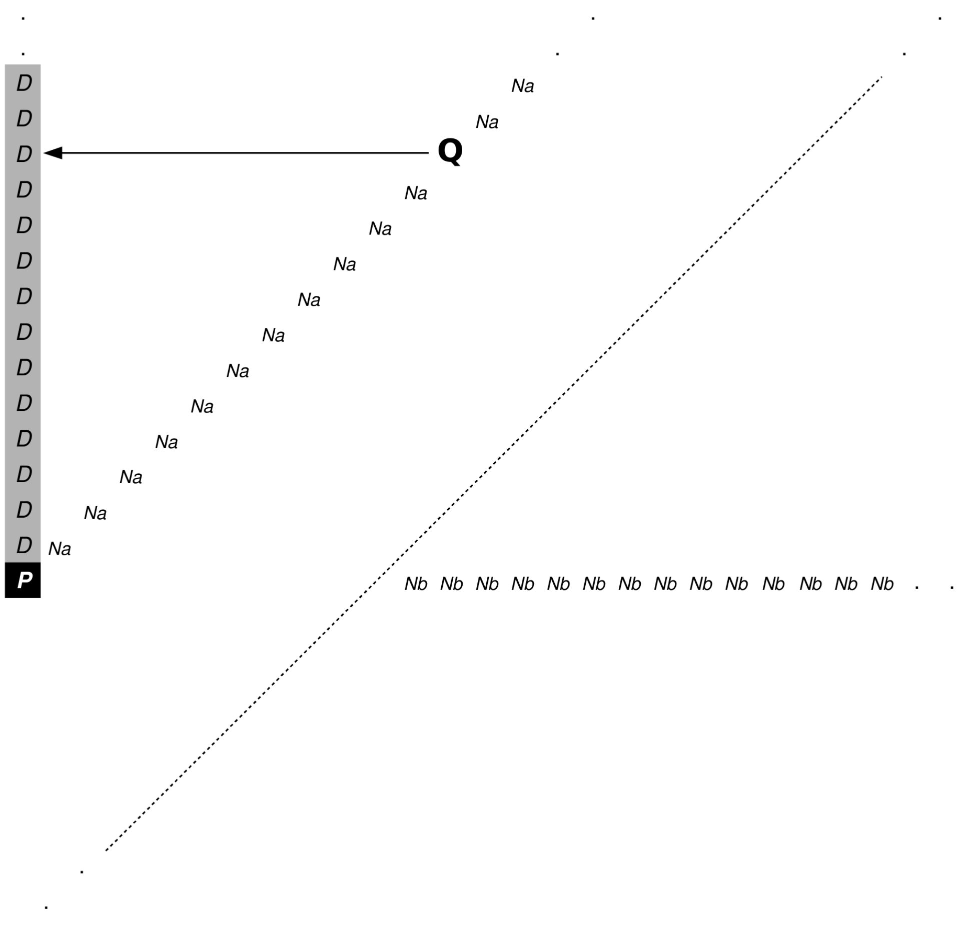

Figure 1: The strategy of Imitation Nim. The P is a (Wythoff Nim) P-position north of the main diagonal. The D's are dynamic positions. The arrow symbolizes a winning move from **Q**. The Na's are the positions of type (iia) in Proposition 1, the Nb's of type (iib).

Remark 2 Given the notation in Proposition 1, it is well-known (see also Figure 1) that if there is an $x < a$ such that (x, b) is a P-position of Wythoff Nim, then this implies the statement in (iia). One may also note that, by symmetry, there is an intersection of type (iia) and (iib) positions, namely whenever $a = d$, that is whenever $c < a < b$ is an arithmetic progression.

By Proposition 1 and Remark 2, (c, b) is a dynamic position of Imitation Nim if and only if there is a P-position of Wythoff Nim, (c, d), with $c \leq d < b$. Further, with notation as in (iia), we get that (c, b) is dynamic P if and only if the previous player moved $(a, b) \to (c, b)$, for some $a > c$.

Recall that the first few P-positions of Wythoff Nim are

$$(0, 0), (1, 2), (3, 5), \ldots.$$

Hence, in Example 1, a (non-dynamic) P-position of Imitation Nim is $(1, 2)$. The position $(1, 3)$ is (by Example 1 or by the comment after Remark 2) dynamic. The positions $(2, 3)$ and $(3, 4)$ are, by Proposition 1 (iia), non-dynamic N. As examples of non-dynamic N-positions of type (iib) we may take $(2, x)$ with $x \geq 3$.

By the comment after Remark 2, we get:

Corollary 1 Treated as initial positions, the P-positions of Imitation Nim are identical to those of Wythoff Nim.

Remark 3 For a given position, the rules of Wythoff Nim allow more options than those of Nim, whereas the rules of Imitation Nim give fewer. Nevertheless, the P-positions are identical if one only considers starting positions. Hence, one might want to view these variants of 2-pile Nim as each other's "duals."

1.4. Two extensions of Imitation Nim and their "duals"

We have given a few references for the subject of move-size dynamic games, of which the first is [BeCoGu82]. But literature on our next topic, *games with memory*, seems to appear only in a somewhat different context[2] from that which we shall develop.

[2]The following discussion on this subject is provided by our anonymous referee:

Kalmár [Kal28] and Smith [Smi66] defined a *strategy in the wide sense* to be a strategy which depends on the present position and on all its antecedents, from the beginning of play. Having defined this notion, both authors concluded that it seems logical that it suffices to consider a *strategy in the narrow sense*, which is a strategy that depends only on the present position (analogous to a *Markov chain*, where only the present position determines the next). They then promptly restricted attention to strategies in the narrow sense.

Let us define a *strategy in the broad sense* to be a strategy that depends on the present position v and on *all* its predecessors $u \in F^{-1}(v)$, whether or not such u is a position in the play of the game. This notion, if anything, seems to be even less needed than a strategy in the wide sense.

Yet, in [FrYe82], a strategy in the broad sense was employed for computing a winning move in polynomial time for annihilation games. It was needed, since the counter function associated with γ (=generalized

1.4.1. A game with memory

A natural extension of Imitation Nim is, given $p \in \mathbb{N}$, to allow $p-1$ consecutive imitations by one and the same player, but to prohibit the p:th imitation. We denote this game by $(1,p)$-*Imitation Nim*.

Remark 4 This rule removes the winning strategy from 2-pile Nim if and only if the number of tokens in each pile is $\geq p$.

Example 2 Suppose the game is $(1,2)$-Imitation Nim, so that no two consecutive imitations by one and the same player are allowed. Suppose the starting position is $(2,2)$ and that Alice moves to $(1,2)$. Then, if Bob moves to $(1,1)$, Alice will move to $(0,1)$, which is P for a game with this particular history. This is because the move $(0,1) \to (0,0)$ would have been a second consecutive imitation for Bob and hence is not permitted. If Bob chooses instead to move to $(0,2)$, then Alice can win in the next move, since $2 > 1$ and hence the imitation rule does not apply.

Indeed, Alice's first move is a winning move, so $(2,2)$ is N (which is non-dynamic) and $(1,2)$ is P. But if $(1,2)$ had been an initial position, then it would have been N, since $(1,2) \to (1,1)$ would have been a winning move. So $(1,2)$ is dynamic. Clearly $(0,0)$ is non-dynamic P. Otherwise the 'least' non-dynamic P-position is $(2,3)$, since $(2,2)$ is N and $(2,1)$ or $(1,3) \to (1,1)$ would be winning moves, as would $(2,0)$ or $(0,3) \to (0,0)$.

1.4.2. The dual of (1,p)-Imitation Nim

In [HeLa06, Lar] we put a *Muller twist* or *blocking manoeuvre* on the game of Wythoff Nim. A nice introduction to games with a Muller twist (Comply/Constrain games) is given in [SmSt02]. Variations on Nim with a Muller twist can also be found, for example, in [GaSt04] (which generalizes a result in [SmSt02]), [HoRe] and [Zho03].

Fix $p \in \mathbb{N}$. The rules of the game which we shall call $(1,p)$-*Wythoff Nim* are as follows. Suppose the current pile position is (a,b). Before the next player removes any tokens, the previous player is allowed to announce $j \in \{1,2,\ldots,p-1\}$ positions, say $(a_1,b_1),\ldots,(a_j,b_j)$ where $b_i - a_i = b - a$, to which the next player may not move. Once the next player has moved, any blocking manoeuvre is forgotten. Otherwise move as in Wythoff Nim.

We will show that as a generalization of Corollary 1, if X is a starting position of $(1,p)$-Imitation Nim then it is P if and only if it is a P-position of $(1,p)$-Wythoff Nim. A general-

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Sprague-Grundy function) was computed only for a small subgraph of size $O(n^4)$ of the game-graph of size $O(2^n)$, in order to preserve polynomiality. This suggests the possibility that a polynomial strategy in the narrow sense may not exist; but this was not proved. It is only reported there that no polynomial time strategy in the narrow sense was found.

ization of Proposition 1 also holds, but let us now move on to our next extension of Imitation Nim.

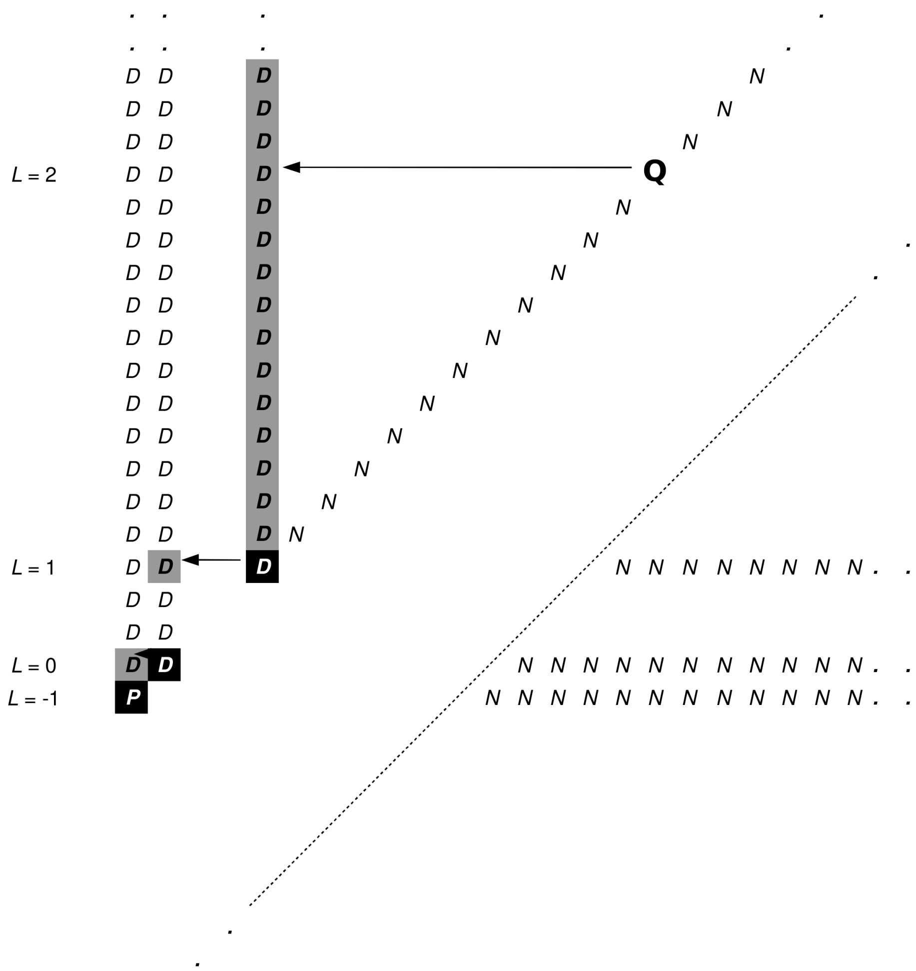

Figure 2: The strategy of $(1,3)$-Imitation Nim. The P is a non-dynamic P-position north of the main diagonal. The black positions are all P-positions of $(1,3)$-Wythoff Nim on one and the same SW-NE diagonal. The D's are dynamic positions. The arrows symbolize three consecutive winning moves from a position **Q**. The N's are non-dynamic N-positions.

1.4.3. A relaxed imitation

Let $m \in \mathbb{N}$. We relax the notion of an imitation to an *m-imitation* (or just imitation) by saying: provided the previous player removed x tokens from pile A, with notation as in Definition 1, then the next player m-imitates the previous player's move if he removes $y \in \{x, x+1, \ldots, x+m-1\}$ tokens from pile B.

Definition 3 Fix $m, p \in \mathbb{N}$. We denote by (m,p)-*Imitation Nim* the game where no p consecutive m-imitations are allowed by one and the same player.

Example 3 Suppose that the game is $(2,1)$-Imitation Nim, so that no 2-imitation is allowed. Then if the starting position is $(1,2)$ and Alice moves to $(0,2)$, Bob cannot move, hence $(1,2)$ is an N-position and it must be non-dynamic since $(1,2) \to (0,2)$ is always an option regardless of whether there was a previous move or not.

1.4.4. The dual of (m,1)-Imitation Nim

Fix a positive integer m. There is a generalization of Wythoff Nim, see [Fra82], here denoted by $(m,1)$-Wythoff Nim, which (as we will show in Section 2) has a natural P-position correspondence with $(m,1)$-Imitation Nim. The rules for this game are: remove any number of tokens from precisely one of the piles, or remove tokens from both piles, say x and y tokens respectively, with the restriction that $|\, x - y\,| < m$.

And indeed, to continue Example 3, $(1,2)$ is certainly an N-position of $(2,1)$-Wythoff Nim, since here $(1,2) \to (0,0)$ is an option. On the other hand $(1,3)$ is P, and non-dynamic P of $(1,2)$-Imitation Nim. For, in the latter game, if Alice moves $(1,3) \to (0,3)$ or $(1,0)$, it does not prevent Bob from winning and $(1,3) \to (1,2)$ or $(1,1)$ are losing moves, since Bob may take advantage of the imitation rule.

In [Fra82], the author shows that the P-positions of $(m,1)$-Wythoff Nim are so-called "Beatty pairs" (view for example the appendix, the original papers in [Ray94, Bea26] or [Fra82], page 355) of the form $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$, where $\beta = \alpha + m$, n is a non-negative integer and

$$\alpha = \frac{2 - m + \sqrt{m^2 + 4}}{2}. \tag{2}$$

1.4.5. The P-positions of (m,p)-Wythoff Nim

In the game of (m,p)-Wythoff Nim, originally defined in [HeLa06] as p-blocking m-Wythoff Nim, a player may move as in $(m,1)$-Wythoff Nim and block positions as in $(1,p)$-Wythoff Nim. From this point onwards, whenever we write Wythoff's game or $W = W_{m,p}$, we are referring to (m,p)-Wythoff Nim.

The P-positions of this game can easily be calculated by a minimal exclusive algorithm (but with exponential complexity in succinct input size) as follows: Let X be a set of non-negative integers. Define $\operatorname{mex}(X)$ as the least non-negative integer not in X, formally $\operatorname{mex}(X) := \min\{x \mid x \in \mathbb{N}_0 \setminus X\}$.

Definition 4 Given positive integers m and p, the integer sequences (a_n) and (b_n) are:

$$\begin{aligned} a_n &= \operatorname{mex}\{a_i, b_i \mid 0 \le i < n\}; \\ b_n &= a_n + \delta(n), \end{aligned}$$

where $\delta(n) = \delta_{m,p}(n) := \left\lfloor \frac{n}{p} \right\rfloor m$.

The next result follows almost immediately from this definition. See also [HeLa06] (Proposition 3.1 and Remark 1) for further extensions.

Proposition 2 [HeLa06] Let $m, p \in \mathbb{N}$.

(a) The P-positions of (m,p)-Wythoff Nim are the pairs (a_i, b_i) and (b_i, a_i), $i \in \mathbb{N}_0$, as in Definition 4;

(b) The sequences $(a_i)_{i\ge 0}^{\infty}$ and $(b_i)_{i\ge p}^{\infty}$ partition $\mathbb{N}_0$ and for $i \in \{0, 1, \ldots, p-1\}$, $a_i = b_i = i$;

(c) Suppose (a, b) and (c, d) are two distinct P-positions of (m,p)-Wythoff Nim with $a \le b$ and $c \le d$. Then $a < c$ implies $b - a \le d - c$ (and $b < d$);

(d) For each $\delta \in \mathbb{N}$, if $m \mid \delta$ then $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = p$, otherwise $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = 0$.

The (m,p)-Wythoff pairs from Proposition 2 may be expressed via Beatty pairs if and only if $p \mid m$. In that case one can prove via an inductive argument that the P-positions of (m,p)-Wythoff Nim are of the form

$$(pa_n,\, pb_n),\ (pa_n + 1,\, pb_n + 1),\ \ldots,\ (pa_n + p - 1,\, pb_n + p - 1),$$

where (a_n, b_n) are the P-positions of the game $(\frac{m}{p}, 1)$-Wythoff Nim (we believe that this fact has not been recognized elsewhere, at least not in [HeLa06] or [Had]).

For any other m and p we did not have a polynomial time algorithm for telling whether a given position is N or P, until recently. While reviewing this article there has been progress on this matter, so there is a polynomial time algorithm, see [Had]. See also a conjecture in [HeLa06], Section 5, saying in a specific sense that the (m,p)-Wythoff pairs are "close to" the Beatty pairs $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$, where $\beta = \alpha + \frac{m}{p}$ and

$$\alpha = \frac{2p - m + \sqrt{m^2 + 4p^2}}{2p},$$

which is settled for the case $m = 1$ in the appendix. In the general case, as is shown in [Had], the explicit bounds for a_n and b_n are

$$(n - p + 1)\alpha \le a_n \le n\alpha$$

and

$$(n-p+1)\beta \leq b_n \leq n\beta.$$

A reader who, at this point, feels ready to plough into the main idea of our result, may move on directly to Section 2. There we state how the winning positions of (m,p)-Imitation Nim coincide with those of (m,p)-Wythoff Nim and give a complete proof for the case $m=1$. In Section 3 we finish off with a couple of suggestions for future work.

1.4.6. Further Examples

In this section we give two examples of games where $p > 1$ and $m > 1$ simultaneously, namely $(2,3)$- and $(3,3)$-Imitation Nim respectively. The style is informal.

In Example 4 the winning strategy (via the imitation rule) is directly analogous to the case $m=1$. In Example 5 we indicate how our relaxation of the imitation rule changes how a player may take advantage of it in a way that is impossible for the $m=1$ case. We illustrate why this does not affect the nice coincidence between the winning positions of Imitation Nim and Wythoff's game. Hence these examples may be profitably studied in connection with (a second reading of) the proof of Theorem 1.

Example 4 The first few P-positions of $(2,3)$-Wythoff Nim are

$$(0,0),(1,1),(2,2),(3,5),(4,6),(7,9),$$

$$(8,12),(10,14),(11,15),(13,19).$$

For the moment assume that the first few non-dynamic P-positions of $(2,3)$-Imitation Nim are $(0,0),(3,5),(8,12),(13,19)$. Clearly, a player should at any point aim at moving to such a position. If this is not possible, one could try and move to a P-position of Wythoff's game. But this is not necessarily a good strategy, in particular if by doing so one imitates the other player's move.

Suppose Alice's first move is $(13,18) \to (11,18)$. Then Bob can move to a P-position of Wythoff's game, namely $(11,18) \to (11,15)$. But this is a 2-imitation. Then Alice may move $(11,15) \to (10,15)$ and once again, provided Bob wants to move to a P-position of Wythoff's game, his only choice is $(10,15) \to (10,14)$, which again is a 2-imitation. At this point he has used up the number of permitted 2-imitations and hence Alice may move $(10,14) \to (8,14)$ and she is assured that Bob will not reach the non-dynamic P-position $(8,12)$.

So, returning to his first move, he investigates the possibility of removing tokens from the pile with 11 tokens. But, however he does this, Alice will be able to reach a P-position of Wythoff's game *without* imitating Bob. Namely, suppose Bob moves to $(x,18)$ with $x \leq 10$.

Then Alice's next move would be to $(x, y) \in \mathcal{P}(W)$, for some $y \leq x + 4$. Clearly this move is not a 2-imitation since $18 - y - (11 - x) = 7 - (y - x) \geq 3$.

By this example we see that the imitation rule is an eminent tool for Alice, whereas Bob is the player who 'suffers its consequences'. In the next example Bob tries to get around his predicament by hoping that Alice would 'rely too strongly' on the imitation rule.

Example 5 The first few P-positions of $(3, 3)$-Wythoff Nim are

$$(0, 0), (1, 1), (2, 2), (3, 6), (4, 7), (5, 8), (9, 15).$$

Suppose, in a game of $(3, 3)$-Imitation Nim, the players have moved

$$\begin{aligned} \text{Alice} &: (6, 9) \to (5, 9) \\ \text{Bob} &: (5, 9) \to (5, 6) \text{ an imitation} \\ \text{Alice} &: (5, 6) \to (4, 6) \\ \text{Bob} &: (4, 6) \to (3, 6) \text{ no imitation.} \end{aligned}$$

Bob will win, in spite of Alice trying to use the imitation rule to her advantage. The mistake is Alice's second move, where she should change her 'original plan' and not continue to rely on the imitation rule.

For the next variation Bob tries to 'confuse' Alice's strategy by 'swapping piles',

$$\begin{aligned} \text{Alice} &: (3, 3) \to (2, 3) \\ \text{Bob} &: (2, 3) \to (2, 1). \end{aligned}$$

Bob has imitated Alice's move once. If Alice continues her previous strategy by removing tokens from the smaller pile, say by moving $(2, 1) \to (2, 0)$, Bob will imitate Alice's move a second time and win. Now Alice's correct strategy is rather to remove token(s) from the larger pile,

$$\begin{aligned} \text{Alice} &: (2, 1) \to (1, 1) \\ \text{Bob} &: (1, 1) \to (0, 1) \\ \text{Alice} &: (0, 1) \to (0, 0). \end{aligned}$$

Here Alice has become the player who imitates, but nevertheless wins.

2. The winning strategy of Imitation Nim

For the statement of our main theorem we use some more terminology.

Definition 5 Suppose the constants m and p are given as in Imitation Nim or in Wythoff's game. Then, if $a, b \in \mathbb{N}_0$,

$$\xi(a,b) = \xi_{m,p}\big((a,b)\big) := \#\big\{\, (i,j) \in \mathcal{P}(W_{m,p}) \mid\; j - i = b - a,\; i < a \,\big\}.$$

Then according to Proposition 2 (d),

$$0 \leq \xi(a,b) \leq p,$$

and indeed, if $(a,b) \in \mathcal{P}(W_{m,p})$ then $\xi(a,b)$ equals the number of P-positions the previous player has to block off (given that we are playing Wythoff's game) in order to win. In particular, $\xi(a,b) < p$ for $(a,b) \in \mathcal{P}(W_{m,p})$.

Definition 6 Let (a,b) be a position of a game of (m,p)-Imitation Nim. Put

$$L(a,b) = L_{m,p}((a,b)) := p - 1$$

if

(A) (a,b) is the starting position, or

(B) $(c,d) \to (a,b)$ was the most recent move and (c,d) was the starting position, or

(C) The previous move was $(e,f) \to (c,d)$ but the move (or option) $(c,d) \to (a,b)$ is not an m-imitation.

Otherwise, with notation as in (C), put

$$L(a,b) = L(e,f) - 1.$$

Notice that by the definition of (m,p)-Imitation Nim,

$$-1 \leq L(a,b) < p.$$

It will be convenient to allow $L(a,b) = -1$, although a player cannot move $(c,d) \to (a,b)$ if it is an imitation and $L(e,f) = 0$. Indeed $L(e,f)$ represents the number of imitations the player moving from (c,d) still has 'in credit'.

Theorem 1 Let $0 \le a \le b$ be integers and suppose the game is (m,p)-Imitation Nim. Then (a,b) is P if and only if

(I) $(a,b) \in \mathcal{P}(W_{m,p})$ and $0 \le \xi(a,b) \le L(a,b)$, or

(II) there is a $a \le c < b$ such that $(a,c) \in \mathcal{P}(W_{m,p})$ but $-1 \le L(a,c) < \xi(a,c) \le p-1$.

Corollary 2 If (a,b) is a starting position of (m,p)-Imitation Nim, then it is P if and only if it is a P-position of (m,p)-Wythoff Nim.

Proof. Put $L(\cdot) = p-1$ in Theorem 1. □

By Theorem 1 (I) and the remark after Definition 6 we get that (a,b) is non-dynamic P if and only if $(a,b) \in \mathcal{P}(W)$ and $\xi(a,b) = 0$. On the other hand, if $(a,b) \in \mathcal{N}(W)$ it is dynamic if and only if there is a $a \le c < b$ such that $(a,c) \in \mathcal{P}(W)$. (See also Figure 2.)

Proof of Theorem 1. We only give the proof for the case $m = 1$. In this way we may put a stronger emphasis on the idea of the game, at the expense of technical details. We will make repeated use of Proposition 2 (a) without any further comment.

Suppose (a,b) is as in (I). Then we need to show that, if (x,y) is an option of (a,b) then (x,y) is neither of form (I) nor (II).

But Proposition 2 (b) gives immediately that $(x,y) \in \mathcal{N}(W)$ so suppose (x,y) is of form (II). Then there is a $x \le c < y$ such that $(x,c) \in \mathcal{P}(W)$ and $L(x,c) < \xi(x,c)$. Since, by (I), $\xi(a,b) \le L(a,b)$ and $L(a,b) - 1 \le L(x,c)$ ($\le L(a,b)$) we get that

$$\xi(a,b) \le L(a,b) \le L(x,c) + 1 \le \xi(x,c),$$

which, in case $c - x = b - a$, is possible if and only if $\xi(a,b) = \xi(x,c)$. But then, since, by our assumptions, $(x,c) \in \mathcal{P}(W)$ and $(a,b) \in \mathcal{P}(W)$, we get $(a,b) = (x,c)$, which is impossible.

So suppose that $c - x \ne b - a$. Then, by Proposition 2 (c), $c - x < b - a$. We have two possibilities:

$y = b$: Then if $(x,b) \to (x,c)$ is an imitation of $(a,b) \to (x,b)$, we get $b - c > a - x = b - c$, a contradiction.

$x = a$: For this case the move $(a,y) \to (a,c)$ cannot be an imitation of $(a,b) \to (a,y)$, since the previous player removed tokens from the larger pile. Then $L(a,c) = p-1 \ge \xi(a,c)$ since, by (II), $(a,c) \in \mathcal{P}(W)$.

Hence we may conclude that if (a,b) is of form (I) then an option of (a,b) is neither of form (I) nor (II).

Suppose now that (a,b) is of form (II). Then $(a,c) \in \mathcal{P}(W)$ is an option of (a,b). But we have $L(a,c) < \xi(a,c)$, so (a,c) is not of form (I). Since $(a,c) \in \mathcal{P}(W)$, by Proposition 2 (b), it cannot be of form (II). But then, since $b > c$, by Proposition 2 (b) and (c), any other option of (a,b), say (x,y), must be an N-position of Wythoff's game. So suppose (x,y) is of form (II). We get two cases:

$y = b$: Then $0 \le x < a$ and there is an option $(x,d) \in \mathcal{P}(W)$ of (x,b) with $x \le d < b$. But by Proposition 2 (b) and (c), we have $d - x \le c - a < b - a$ and hence $(x,b) \to (x,d)$ does not imitate $(a,b) \to (x,b)$. Therefore $L(x,d) = p-1 \ge \xi(x,d)$, which contradicts the assumptions in (II).

$x = a$: Then $0 \le y < b$. If $y > c$, then $(a,c) \in \mathcal{P}(W)$ is an option of (a,y) and two consecutive moves from the larger pile would give $L(a,c) = p-1 \ge \xi(a,c)$. Otherwise, by Proposition 2 (b), there is no option of (a,y) in $\mathcal{P}(W)$. In either case one has a contradiction to the assumptions in (II).

We are done with the first part of the proof.

Therefore, for the remainder of the proof, assume that (α,β), $0 \le \alpha \le \beta$, is neither of form (I) nor (II). Then,

(i) if $(\alpha,\beta) \in \mathcal{P}(W)$, this implies $0 \le L(\alpha,\beta) < \xi(\alpha,\beta) \le p-1$, and

(ii) if there is a $\alpha \le c < \beta$ such that $(\alpha,c) \in \mathcal{P}(W)$, this implies $0 \le \xi(\alpha,c) \le L(\alpha,c) \le p-1$.

We need to find an option of (α,β), say (x,y), of form (I) or (II).

If $(\alpha,\beta) \in \mathcal{P}(W)$, then (ii) is trivially satisfied by Proposition 2 (b). Also, $\xi(\alpha,\beta) > 0$ by (i). Hence, there is a position $(x,z) \in \mathcal{P}(W)$ such that $z - x = \beta - \alpha$ with $x \le z < \beta (= y)$. Then, since $L(\alpha,\beta) < \xi(\alpha,\beta)$, the option (x,β) satisfies (II) (and hence, by the imitation rule, $(\alpha,\beta) \to (x,\beta)$ is the desired winning move).

For the case $(\alpha,\beta) \in \mathcal{N}(W)$ (here (i) is trivially true), suppose $(\alpha,c) \in \mathcal{P}(W)$ with $\alpha \le c < \beta$. Then (ii) gives $L(\alpha,c) \ge \xi(\alpha,c)$, which clearly holds for example if the most recent move wasn't an imitation. In any case this immediately implies (I).

If $c < \alpha$, with $(\alpha,c) \in \mathcal{P}(W)$, then (ii) holds trivially by Proposition 2 (b). Thus (I) holds, because $(\alpha,\beta) \to (\alpha,c)$ isn't an imitation : if it were, then the previous would necessarily have been from the larger pile.

If $c < \alpha$ with $(c,\beta) \in \mathcal{P}(W)$, then the move $(\alpha,\beta) \to (c,\beta)$ isn't an imitation since tokens have been removed from the smaller pile. Hence $p - 1 = L(c,\beta) \ge \xi(c,\beta)$.

By Proposition 2, the only remaining case for (α, β) an N-position of Wythoff's game is whenever there is a position $(x, z) \in \mathcal{P}(W)$ such that $x < \alpha$ and

$$\beta - \alpha = z - x. \tag{3}$$

We may assume there is no $c < \beta$ such that $(\alpha, c) \in \mathcal{P}(W)$, since we are already done with this case. Then (ii) holds trivially and, by Proposition 2 (b), there must be a $c > \beta$ such that $(\alpha, c) \in \mathcal{P}(W)$. But then, by Proposition 2 (c) and (d), we get $\xi(\alpha, \beta) = p > 0$ and so, since for this case we may take (x, z) such that $p - 1 = \xi(x, z)$, we get $L(x, z) \leq p - 2 < \xi(x, z)$. Then, by (3), $(x, \beta) = (x, y)$ is clearly the desired position of form (II). □

3. Final questions

Let us finish off with some questions.

- Consider a slightly different setting of an impartial game, namely where the second player does not have perfect information, but the first player (who has) is not aware of this fact. Similar settings have been discussed, for example, in [BeCoGu82, Owe95]. We may ask, for which games (starting with those we have discussed) is there a simple *second player's strategy* which lets him *learn* the winning strategy of the game *while playing*? By this we mean that if he starts a new 'partie' of the same game at least one move after the first one, he wins.
- Is there a generalization of Wythoff Nim to $n > 2$ piles of tokens (see for example [BlFr98, FrKr04, Sun, SuZe04]), together with a generalization of 2-pile Imitation Nim, such that the P-positions coincide (as starting positions)?
- Are there other impartial (or partizan) games where an imitation rule corresponds in a natural way to a blocking manoeuvre?
- Can one formulate a general rule as to when such correspondences can be found and when not?

Acknowledgements At the Integers 2007 Conference when I introduced Imitation Nim to Aviezri Fraenkel I believe he quickly responded "*Limitation Nim*". As this name emphasises another important aspect of the game, I would like to propose it for the "dual" of Wythoff's original game, that is whenever one wants to emphasise that no imitation is allowed. I would also like to thank A. Fraenkel for contributing with some references and the anonymous referee for the references and the interesting discussion on strategies in a wide and broad sense (footnote 2 on page 6).

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Appendix

Peter Hegarty

The purpose of this appendix is to provide a proof of Conjecture 5.1 of [HeLa06] in the case $m = 1$, which is the most natural case to consider. Notation concerning 'multisets' and 'greedy permutations' is consistent with Section 2 of [HeLa06]. We begin by recalling

Definition Let r, s be positive irrational numbers with $r < s$. Then (r, s) is said to be a *Beatty pair* if

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{4}$$

Theorem Let (r, s) be a Beatty pair. Then the map $\tau : \mathbb{N} \to \mathbb{N}$ given by

$$\tau(\lfloor nr \rfloor) = \lfloor ns \rfloor, \ \forall\, n \in \mathbb{N}, \qquad \tau = \tau^{-1},$$

is a well-defined involution of $\mathbb{N}$. If M is the multiset of differences $\pm\{\lfloor ns \rfloor - \lfloor nr \rfloor : n \in \mathbb{N}\}$, then $\tau = \pi_g^M$. M has asymptotic density equal to $(s - r)^{-1}$.

Proof. That τ is a well-defined permutation of $\mathbb{N}$ is Beatty's theorem. The second and third assertions are then obvious. □

Proposition Let $r < s$ be positive real numbers satisfying (4), and let $d := (s - r)^{-1}$. Then the following are equivalent

(i) r is rational

(ii) s is rational

(iii) d is rational of the form $\frac{mn}{m^2 - n^2}$ for some positive rational m, n with $m > n$.

Proof. Straightforward algebra exercise. □

Notation Let (r, s) be a Beatty pair, $d := (s - r)^{-1}$. We denote by M_d the multisubset of $\mathbb{N}$ consisting of all differences $\lfloor ns \rfloor - \lfloor nr \rfloor$, for $n \in \mathbb{N}$. We denote $\tau_d := \pi_g^{\pm M_d}$.

As usual, for any positive integers m and p, we denote by $\mathcal{M}_{m,p}$ the multisubset of $\mathbb{Z}$ consisting of p copies of each multiple of m and $\pi_{m,p} := \pi_g^{\mathcal{M}_{m,p}}$. We now denote by $M_{m,p}$ the submultiset consisting of all the positive integers in $\mathcal{M}_{m,p}$ and $\overline{\pi}_{m,p} := \pi_g^{\pm M_{m,p}}$. Thus

$$\overline{\pi}_{m,p}(n) + p = \pi_{m,p}(n + p) \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Since $\mathcal{M}_{m,p}$ has density p/m, there is obviously a close relation between $M_{m,p}$ and $M_{p/m}$, and thus between the permutations $\pi_{m,p}$ and $\tau_{p/m}$. The precise nature of this relationship is, however, a lot less obvious on the level of permutations. It is the purpose of the present note to explore this matter.

We henceforth assume that $m = 1$.

To simplify notation we fix a value of p. We set $\pi := \overline{\pi}_{1,p}$. Note that

$$r = r_p = \frac{(2p-1) + \sqrt{4p^2+1}}{2p}, \quad s = s_p = r_p + \frac{1}{p} = \frac{(2p+1) + \sqrt{4p^2+1}}{2p}.$$

Further notation If X is an infinite multisubset of $\mathbb{N}$ we write $X = (x_k)$ to denote the elements of X listed in increasing order, thus strictly increasing order when X is an ordinary subset of $\mathbb{N}$. The following four subsets of $\mathbb{N}$ will be of special interest :

$$\begin{aligned} A_\pi &:= \{n : \pi(n) > n\} := (a_k), \\ B_\pi &:= \mathbb{N}\backslash A_\pi := (b_k), \\ A_\tau &:= \{n : \tau(n) > n\} := (a_k^*), \\ B_\tau &:= \mathbb{N}\backslash A_\tau := (b_k^*). \end{aligned}$$

Note that $b_k = \pi(a_k)$, $b_k^* = \tau(a_k^*)$ for all k. We set

$$\epsilon_k := (b_k - a_k) - (b_k^* - a_k^*) = (b_k - b_k^*) - (a_k - a_k^*).$$

Lemma 1 (i) For every $n > 0$,

$$|M_p \cap [1,n]| = |M_{1,p} \cap [1,n]| + \epsilon,$$

where $\epsilon \in \{0, 1, ..., p-1\}$.
(ii) $\epsilon_k \in \{0, 1\}$ for all k and if $\epsilon_k = 1$ then $k \not\equiv 0 \pmod p$.
(iii) $a_{k+1}^* - a_k^* \in \{1, 2\}$ for all $k > 0$ and cannot equal one for any two consecutive values of k.
(iv) $b_{k+1}^* - b_k^* \in \{2, 3\}$ for all $k > 0$.

Proof. (i) and (ii) are easy consequences of the various definitions. (iii) follows from the fact that $r_p \in (3/2, 2)$ and (iv) from the fact that $s_p \in (2, 3)$. □

Main Theorem For all $k > 0$, $|a_k - a_k^*| \leq p - 1$.

Remark We suspect, but have not yet been able to prove, that $p - 1$ is best-possible in this theorem.

Proof of Theorem. The proof is an induction on k, which is most easily phrased as an argument by contradiction. Note that $a_1 = a_1^* = 1$. Suppose the theorem is false and consider the smallest k for which $|a_k^* - a_k| \geq p$. Thus $k > 1$.

CASE I : $a_k - a_k^* \geq p$.

Let $a_k - a_k^* := p' \geq p$. Let b_l be the largest element of B_π in $[1, a_k)$. Then $b_{l-p'+1}^* > a_k^*$ and Lemma 1(iv) implies that $b_l^* - b_l \geq p'$. But Lemma 1(ii) then implies that also $a_l^* - a_l \geq p' \geq p$. Since obviously $l < k$, this contradicts the minimality of k.

CASE II : $a_k^* - a_k \geq p$.

Let $a_k^* - a_k := p' \geq p$. Let b_l^* be the largest element of B_τ in $[1, a_k^*)$. Then $b_{l-p'+1} > a_k$. Lemma 1(iv) implies that $b_{l-p'+1} - b_{l-p'+1}^* \geq p'$ and then Lemma 1(ii) implies that $a_{l-p'+1} - a_{l-p'+1}^* \geq p' - 1$. The only way we can avoid a contradiction already to the minimality of k is if all of the following hold :

(a) $p' = p$.
(b) $b_i^* - b_{i-1}^* = 2$ for $i = l, l-1, ..., l-p+2$.
(c) $l \not\equiv -1 \pmod{p}$ and $\epsilon_{l-p+1} = 1$.

To simplify notation a little, set $j := l - p + 1$. Now $\epsilon_j = 1$ but parts (i) and (ii) of Lemma 1 imply that we must have $\epsilon_{j+t} = 0$ for some $t \in \{1, ..., p-1\}$. Choose the smallest t for which $\epsilon_{j+t} = 0$. Thus

$$b_j^* - a_j^* = b_{j+1}^* - a_{j+1}^* = \cdots = b_{j+t-1}^* - a_{j+t-1}^* = (b_{j+t}^* - a_{j+t}^*) - 1.$$

From (b) it follows that

$$a_{j+t}^* - a_{j+t-1}^* = 1, \;\; a_{j+\xi}^* - a_{j+\xi-1}^* = 2, \;\; \xi = 1, ..., t-1. \tag{6}$$

Let b_r^* be the largest element of B_τ in $[1, a_j^*)$. Then from (6) it follows that

$$b_{r+t}^* - b_{r+t-1}^* = 3, \;\; b_{r+\xi}^* - b_{r+\xi-1}^* = 2, \;\; \xi = 2, ..., t-1. \tag{7}$$

Together with Lemma 1(iv) this implies that

$$b_{r+p-1}^* - b_{r+1}^* \geq 2p - 3. \tag{8}$$

But since $a_j^* = a_j - (p-1)$ we have that $b_{r+p-1} < a_j$. Together with (8) this enforces $b_{r+p-1}^* - b_{r+p-1} \geq p$, and then by Lemma 1(ii) we also have $a_{r+p-1}^* - a_{r+p-1} \geq p$. Since it is easily checked that $r + p - 1 < k$, we again have a contradiction to the minimality of k, and the proof of the theorem is complete. □

This theorem implies Conjecture 5.1 of [HeLa06]. Recall that the P-positions of $(1, p)$-Wythoff Nim are the pairs $(n-1, \pi_{1,p}(n) - 1)$ for $n \geq 1$.

Corollary With

$$L = L_p = \frac{s_p}{r_p} = \frac{1 + \sqrt{4p^2 + 1}}{2p}, \quad l = l_p = \frac{1}{L_p},$$

we have that, for every $n \geq 1$,

$$\pi_{1,p}(n) \in \{\lfloor nL \rfloor + \epsilon, \lfloor nl \rfloor + \epsilon : \epsilon \in \{-1, 0, 1, 2\}\}. \tag{9}$$

Proof. We have $\pi_{1,p}(n) = n$ for $n = 1, ..., p$, and one checks that (11) thus holds for these n. For $n > p$ we have by (5) that

$$\pi_{1,p}(n) = \pi(n-p) + p, \tag{10}$$

where $\pi = \overline{\pi}_{1,p}$. There are two cases to consider, according as to whether $n - p \in A_\pi$ or B_π. We will show in the former case that $\pi_{1,p}(n) = \lfloor nL \rfloor + \epsilon$ for some $\epsilon \in \{-1, 0, 1, 2\}$. The proof in the latter case is similar and will be omitted.

So suppose $n - p \in A_\pi$, say $n - p = a_k$. Then

$$\pi(a_k) = b_k = a_k + (b_k^* - a_k^*) + \epsilon_k. \tag{11}$$

Moreover $a_k^* = \lfloor kr_p \rfloor$ and $b_k^* = \lfloor ks_p \rfloor$, from which it is easy to check that

$$b_k^* = a_k^* L + \delta, \quad \text{where } \delta \in (-1, 1).$$

Substituting into (11) and rewriting slightly, we find that

$$\pi(a_k) = a_k L + (a_k^* - a_k)(L-1) + \delta + \epsilon_k,$$

and hence by (10) that $\pi_{1,p}(n) = nL + \gamma$ where

$$\gamma = (a_k^* - a_k - p)(L-1) + \delta + \epsilon_k.$$

By Lemma 1, $\epsilon_k \in \{0, 1\}$. By the Main Theorem, $|a_k^* - a_k| \leq p - 1$. It is easy to check that $(2p-1)(L-1) < 1$. Hence $\gamma \in (-2, 2)$, from which it follows immediately that $\pi_{1,p}(n) - \lfloor nL \rfloor \in \{-1, 0, 1, 2\}$. This completes the proof. □

Remark As stated in Section 5 of [HeLa06], computer calculations seem to suggest that, in fact, (9) holds with just $\epsilon \in \{0, 1\}$. So once again, the results presented here may be possible to improve upon.

RESTRICTIONS OF m-WYTHOFF NIM AND p-COMPLEMENTARY BEATTY SEQUENCES

URBAN LARSSON

Abstract. Fix a positive integer m. The game of *m-Wythoff Nim* (A.S. Fraenkel, 1982) is a well-known extension of *Wythoff Nim* (W.A. Wythoff, 1907). The set of P-positions may be represented as a pair of increasing sequences of non-negative integers. It is well-known that these sequences are so-called *complementary Beatty sequences*, that is they satisfy Beatty's theorem. For a positive integer p, we generalize the solution of m-Wythoff Nim to a pair of *p-complementary*—each non-negative integer is represented exactly p times—Beatty sequences $a = (a_n)_{n\in\mathbb{N}_0}$ and $b = (b_n)_{n\in\mathbb{N}_0}$, which, for all n, satisfy $b_n - a_n = mn$. Our main result is that $\{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}$ represents the solution to three new 'p-restrictions' of m-Wythoff Nim—of which one has a certain blocking manoeuvre on the rook-type options. C. Kimberling has shown that the solution of Wythoff Nim satisfies the *complementary equation* $x_{x_n} = y_n - 1$. We generalize this formula to a certain 'p-complementary equation' satisfied by our pair a and b. Further, if $p > 1$, we prove that this pair is unique in the sense that it is the only pair of p-complementary Beatty sequences of which one of the sequences is strictly increasing. We also show that one may obtain our new pair of sequences by three so-called *Minimal EXclusive* algorithms.

1. Introduction and notation

The combinatorial game of *Wythoff Nim* ([Wyt07]) is a so-called (2-player) impartial game played on two piles of tokens. (For an introduction to impartial games see [BeCoGu82, Con76].) As an addition to the rules of the game of Nim ([Bou02]), where the players alternate in removing any finite number of tokens from precisely one of the piles (at most the whole pile), Wythoff Nim also allows removal of the same number of tokens from both piles. The player who removes the last token wins.

This game is more known as 'Corner the Queen', invented by R. P. Isaacs (1960), because the game can be played on a (large) Chess board with one single Queen. Two players move the Queen alternately but with the restriction that, for each move, the (L^1) distance to the lower left corner, position $(0, 0)$, must decrease. (The Queen must at all times remain on the board.) The player who moves to this *final/terminal* position wins.

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In this paper we follow the convention to denote our players with the *next player* (the player who is in turn to move) and the *previous player*. A *P-position* is a position from which the previous player can win (given perfect play). An *N-position* is a position from which the next player can win. Any position is either a P-position or an N-position. We denote *the solution*, the set of all P-positions, of an impartial game G, by $\mathcal{P} = \mathcal{P}(G)$ and the set of all N-positions by $\mathcal{N} = \mathcal{N}(G)$. The positive integers are denoted by $\mathbb{N}$ and the non-negative integers by $\mathbb{N}_0$.

1.1. **Restrictions of m-Wythoff Nim.** Let $m \in \mathbb{N}$. We next turn to a certain m-extension of Wythoff Nim, studied in [Fra82] by A.S. Fraenkel. In the game of *m-Wythoff Nim*, or just mWN (our notation), the Queen's 'bishop-type' options are extended so that $(x, y) \to (x + i, y + j)$ is legal if $|\, i - j \,| < m$. The 'rook-type' options are as in Nim. Hence 1-Wythoff Nim is identical to Wythoff Nim.

In this paper we define three new restrictions of m-Wythoff Nim—here a rough outline:

- The first has a so-called *blocking manoeuvre/Muller Twist* on the rook-type options—before the next player moves, the previous player may announce at most a predetermined number of these options as forbidden (see also [HoRe, SmSt02] and Section 1.2 of this paper);
- The second has a certain congruence restriction on the rook-type options;
- For the third, a rectangle is removed from the lower left corner of the game board (including position $(0, 0)$), so that here we get two terminal positions.

1.2. **A pair of p-complementary Beatty sequences.** A *Beatty sequence* is a sequence of the form $(\lfloor n\alpha + \beta \rfloor)_{n\in\mathbb{N}_0}$, where α is a positive irrational and β is a real number. S. Beatty ([Bea26]) is maybe most known for a (re)[1]discovery of (the statement of) the following theorem: If α and β are positive reals such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ then $(\lfloor n\alpha \rfloor)_{n\in\mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n\in\mathbb{N}}$ split $\mathbb{N}_0$ if and only if they are Beatty sequences. This was proven by [HyOs27] (see also [Fra82]).

A pair of sequences that satisfies Beatty's theorem is *complementary* (see [Fra69, Fra73, Kim07, Kim08]).

In this paper we generalize the notion of complementarity.

Definition 1. Let $p \in \mathbb{N}$. Two sequences (x_i) and (y_i) of non-negative integers are *p-complementary*, if, for each $n \in \mathbb{N}_0$,

$$\#\{i \mid x_i = n\} + \#\{i \mid y_i = n\} = p.$$

As usual, a 1-complementary pair of sequences is denoted *complementary.*

We study the Beatty sequences $a = (a_n)_{n\in\mathbb{N}_0}$ and $b = (b_n)_{n\in\mathbb{N}_0}$, where for all $n \in \mathbb{N}$,

$$a_n = a_n^{m,p} = \left\lfloor \frac{n\phi_{mp}}{p} \right\rfloor \tag{1}$$

[1]This theorem was in fact discovered by J. W. Rayleigh, see [Ray94, Bry03].

and

$$b_n = b_n^{m,p} = \left\lfloor \frac{n(\phi_{mp} + mp)}{p} \right\rfloor, \tag{2}$$

and where

$$\phi_\alpha = \frac{2 - \alpha + \sqrt{\alpha^2 + 4}}{2}. \tag{3}$$

We show that a and b are p-complementary. (Notice also that, for all n, $b_n - a_n = mn$.)

In [Wyt07] W.A. Wythoff proved that the solution of Wythoff Nim is given by $\{\{a_n^{1,1}, b_n^{1,1}\}^2 \mid n \in \mathbb{N}_0\}$, Then in [Fra82] it was shown that the solution of m-Wythoff Nim is

$$\{\{a_n^{m,1}, b_n^{m,1}\} \mid n \in \mathbb{N}_0\}.$$

1.3. **Recurrence.** Let X be a strict subset of the non-negative integers. Then the *Minimal EXclusive* of X is defined as usual (see [Con76]):

$$\operatorname{mex} X := \min(\mathbb{N}_0 \backslash X).$$

For $n \in \mathbb{N}_0$ put

$$x_n = \operatorname{mex}\{x_i, y_i \mid i \in [0, n-1]\} \text{ and } y_n = x_n + mn. \tag{4}$$

With notation as in (4), it was proven in [Fra82] that $(x_n) = (a^{m,1})$ and $(y_n) = (b^{m,1})$. The minimal exclusive algorithm in (4) gives an exponential time solution to mWN whereas the Beatty-pair in (1) and (2) give a polynomial time ditto. (For interesting discussions on complexity issues for combinatorial games, see for example [Fra04, FrPe09].) We show that one may obtain a and b by three minimal exclusive algorithms, which in various ways generalize (4).

It is well-known that the solution of Wythoff Nim satisfies the *complementary equation* (see for example [Kim95, Kim07, Kim08])

$$x_{x_n} = y_n - 1.$$

For arbitrary positive integers m and p, we generalize this formula to a 'p-complementary equation'

$$x_{\varphi_n} = y_n - 1, \tag{5}$$

where $\varphi_n = \frac{x_n + (mp-1)y_n}{m}$, and show that a solution is given by $x = a$ and $y = b$.

1.4. **I.G. Connell's restriction of Wythoff Nim.** In the literature there is another generalization of Wythoff Nim that is of special interest to us. Let $p \in \mathbb{N}$. In [Con59] I.G. Connell studies the restriction of Wythoff Nim, where the the rook-type options are restricted to jumps of precise multiples of p. This game we call Wythoff modulo-p Nim and denote with $\text{WN}^{(p)}$. Hence Wythoff modulo-1 Nim equals Wythoff Nim.

[2] As usual, $\{x, y\}$ denotes unordered pairs (of integers), that is (x, y) and (y, x) are considered the same.

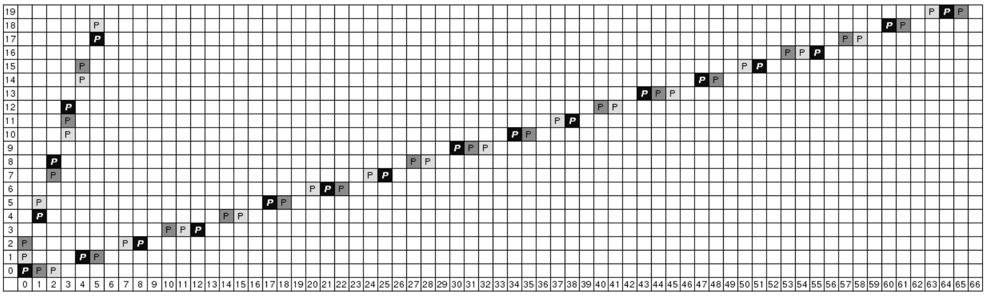

FIGURE 1. The P-positions of Wythoff modulo-3 Nim, $\mathrm{WN}^{(3)}$ are the positions nearest the origin such that there are precisely three positions in each row and column and one position in each NE-SW-diagonal. The black positions represent the (first few) P-positions of 3-Wythoff Nim, namely the positions nearest the origin such that there is precisely one position in each row and one position in every third NE-SW diagonal.

Call the P-positions of $\mathrm{WN}^{(p)}$ $\{\{c_n, d_n\} \mid n \in \mathbb{N}_0\}$, where $c_n = c_n^{(p)}$ and $d_n = d_n^{(p)}$ and let ϕ_α be as in (3). The general solution of $\mathrm{WN}^{(p)}$ is given by

$$c_n = \left\lfloor \frac{n\phi_p}{p} \right\rfloor \text{ and } d_n = c_n + n,$$

a formula which can be derived from [Con59]—from which one may also deduce that (c_i) and $(d_i)_{>0}$ are p-complementary. Notice that, for fixed p and for all n, $a_n^{1,p} = c_n^{(p)}$ and $b_n^{1,p} = d_n^{(p)}$,.

$d_n^{(3)}$	0	1	2	4	5	7	8	10	11	12	14	15	17	18	20	21	22
$c_n^{(3)}$	0	0	0	1	1	2	2	3	3	3	4	4	5	5	6	6	6
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 1. Some values of $c_n^{(3)} = \lfloor \frac{n\phi_3}{3} \rfloor$ and $d_n^{(3)} = c_n^{(3)} + n$.

Remark 1. In Connell's presentation, for the proof of the above formulas, he rather uses p pairs of complementary sequences of integers (in analogy with the discovery of a new formulation of Beatty's theorem in [Sko57]). We have indicated this pattern of P-positions with different shades in Figure 1. In fact, the squares of darkest shade, starting with $(0, 0)$ are P-positions of 3-Wythoff Nim—in general $a_n^{p,1} = c_{pn}^{(p)}$ and $b_n^{p,1} = d_{pn}^{(p)}$— and, as we will see, given a certain game constant, each lighter shade represents the solution of our third variation of this game.

Remark 2. In [BoFr73], Fraenkel and I. Borosh study yet another variation of both m-Wythoff Nim and Wythoff modulo-p Nim which includes a (different from ours) Beatty-type characterization of the P-positions.

1.5. **Exposition.** In Section 2 we define our games, exemplify them and state our main theorem. Roughly: For each of our games, given appropriate game constants, a position is P if and only if it is of the form $\{a_n, b_n\}$, with a and b as in (1) and (2) (so that, in terms of game complexity, the solution of each of our games is polynomial). In Section 3 we generalize Beatty's theorem to p-complementary sequences and prove some arithmetic properties of a and b—most important of which is that (for fixed m and p) a and b are p-complementary. Then, in Section 4, for arbitrary m and $p > 1$, we prove that our new pair of sequences is unique in the sense that it is the only pair of p-complementary Beatty sequences for which one of the sequences is (strictly) increasing. Section 5 is devoted to our p-complementary equation (5) and minimal exclusive algorithms. In Section 6 we prove our game theory results (stated in Section 2) and finally in Section 7 a few questions are posed.

Let us, before we move on to our games, give some more background to the so-called *blocking manoeuvre* in the context of Wythoff Nim.

1.6. **A bishop-type blocking variation of m-Wythoff Nim.** Let $m, p \in \mathbb{N}$. In [HeLa06] we gave an exponential time solution to a variation of m-Wythoff Nim with a 'bishop-type' blocking manoeuvre, denoted by p-Blocking m-Wythoff Nim (and with (m, p)-Wythoff Nim in [Lar09]).

The rules are as in m-Wythoff Nim, except that before the next player moves, the previous player is allowed to block off (at most) $p - 1$ bishop-type—note, not m-bishop-type—options and declare that the next player must refrain from these options. When the next player has moved, any blocked options are forgotten.

The solution of this game is in a certain sense 'very close' to pairs of Beatty sequences (see also the Appendix of [Lar09]) of the form

$$\left(\left\lfloor n\frac{\sqrt{m^2+4p^2}+2p-m}{2p}\right\rfloor\right) \text{ and } \left(\left\lfloor n\frac{\sqrt{m^2+4p^2}+2p+m}{2p}\right\rfloor\right).$$

But we explain why there can be no Beatty-type solution to this game for $p > 1$. However, in [Lar09], for the cases $p \,|\, m$, we give a certain 'Beatty-type' characterisation. For these kind of questions, see also [BoFr84]. However, a recent discovery, in [Had, FrPe09], provides a polynomial time algorithm for the solution of (m, p)-Wythoff Nim (for any combination of m and p).

An interesting connection to 4-Blocking 2-Wythoff Nim is presented in [DuGr08], where the authors give an explicit bijection of solutions to a variation of Wythoff's original game, where a player's bishop-type move is restricted to jumps by multiples of a predetermined positive integer.

For another variation, [Lar09] defines the rules of a so-called move-size dynamic variation of two-pile Nim, (m, p)-Imitation Nim, for which the P-positions, treated as starting positions, are identical to the P-positions of (m, p)-Wythoff Nim.

This discovery of a 'dual' game to (m, p)-Wythoff Nim has in its turn motivated the study of dual constructions of the 'rook-type' blocking manoeuvre in this paper.

2. Three games

This section is devoted to defining and exemplifying our new game rules and to state our main result. We begin by introducing some (non-standard) notation whereby we 'decompose' the Queen's moves into *rook-type* and *bishop-type* ditto.

Definition 2. Fix $m, p \in \mathbb{N}$ and an $l \in \{0, 1, \dots, m\}$.

(i) An (l, p)-*rook* moves as in Nim, but the length of a move must be $ip + j > 0$ positions for some $i \in \mathbb{N}_0$ and $j \in \{0, 1, \dots, l - 1\}$ (we denote a $(0, p)$-rook by a p-rook and a (p, p)-rook simply by a *rook*);

(ii) A m-*bishop* may move $0 \le i < m$ rook-type positions and then any number of, say $j \ge 0$, bishop-type positions (a *bishop* moves as in Chess), all in one and the same move, provided $i + j > 0$ and the L^1-distance to $(0, 0)$ decreases.

2.1. **Game definitions.** As is clear from Definition 2 the rook-type options intersect the m-bishop-type options precisely when $m > 1$. For example, $(2, 3) \to (1, 3)$ is both a 2-bishop-type and a rook-type move. We will make use of this fact when defining the blocking manoeuvre. Therefore, let us introduce some new terminology.

Fix an $m \in \mathbb{N}$. A rook-type option, which is not of the form of the m-bishop as in Definition 2 (ii), is a *roob*(-type)[3] option. Hence, for $m = 2$, $(2, 3) \to (2, 1)$ is a roob option, but $(2, 3) \to (2, 2)$ is not (both are rook options).

Let us define our games.

Definition 3. Fix $m, p \in \mathbb{N}$.

(1) The game of m-*Wythoff* p-*Blocking Nim*, or $m\mathrm{WN}^p$, is a restriction of m-Wythoff Nim with a roob-type blocking manoeuvre.

The Queen moves as in m-Wythoff Nim (that is, as the m-bishop or the rook), but with one exception: Before the next player moves, the previous player may *block off* (at most) $p - 1$ of the next player's roob options. The blocked options are then excluded from the Queen's options. As usual, each blocking manoeuvre is particular to a specific move; that is, when the next player has moved, any blocked options are forgotten and has no further impact on the game. (For $p = 1$ this game equals m-Wythoff Nim.)

(2) Fix an integer $0 \le l < p$. In the game of m-*Wythoff Modulo-*p l-*Nim*, or $m\mathrm{WN}^{(l,p)}$, the Queen moves as the m-bishop or the (l, p)-rook. For $l = 0$ we denote this game by m-*Wythoff Modulo-*p *Nim* or $m\mathrm{WN}^{(p)}$. (In case $m = l = 0$ the game reduces to Wythoff modulo-p Nim, whereas for $l = p$ the game is simply m-Wythoff Nim.)

(3a) Fix an integer $0 \le l < p$. In the game of l-*Shifted* $m{\times}p$-*Wythoff Nim*, or $m{\times}p\mathrm{WN}_l$, the Queen moves as in (mp)-Wythoff Nim (that is, as the (mp)-bishop or the rook), except that, if $l > 0$, it is not allowed to move to a position of the form (i, j), where $0 \le i < ml$ and

[3] Think of 'roob' as 'ROOk minus m-Bishop', or maybe 'ROOk Blocking'

$0 \leq j < m(p-l)$[4]. Hence, for this case, the terminal positions are $(ml, 0)$ and $(0, m(p-l))$. On the other hand $m\times p\mathrm{WN}_0$ is identical to (mp)-Wythoff Nim.

(3b) The game of $m\times p$-Wythoff Nim, $m\times p$WN: Before the first player moves, the second player may decide the parameter l as in (3a). Once the parameter l is fixed, it remains the same until the game has terminated, so that for the remainder of the game, the rules are as in $m\times p\mathrm{WN}_l$.

2.2. **Examples.** Let us illustrate some of our games, where our players are *Alice* and *Bob*—Alice makes the first move (and Bob makes the first blocking manoeuvre in case the game has a Muller twist).

Example 1. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^2$. Then the only bishop-type move is $(0, 2) \to (0, 1)$. There is precisely one roob option, namely $(0, 0)$. Since this is a terminal position Bob will block it off from Alice's options, so that Alice has to move to $(0, 1)$. The move $(0, 1) \to (0, 0)$ cannot be blocked off for the same reason, so Bob wins. If $y \geq 3$ there is always a move $(0, y) \to (0, x)$, where $x = 0$ or 2. This is because the previous player may block off at most one option. Altogether, this gives that $\{0, y\}$ is P if and only if $y = 0$ or 2.

Example 2. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2)}$. Alice can move to $(0, 0)$, since $0 \equiv 2 \pmod 2$, so $(0, 2)$ is N. On the other hand, the position $(0, 3)$ is P since the only options are $(0, 2)$ and $(0, 1)$. (The latter is N since the 2-bishop can move $(0, 1) \to (0, 0)$.)

Example 3. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2,4)}$. Alice cannot move to $(0, 0)$, since $2 - 0 \not\equiv 3, 4 \pmod 4$ and since $(0, 1) \to (0, 0)$ is a 2-bishop-type move $(0, 1)$ is N, so that $\{0, 2\}$ must be P. Then $(0, 3)$ is N and since $(0, y) \to (0, 0)$ is legal if $y = 4$ or 5 we get, by similar reasoning, that $\{0, y\}$ is N for all $y \geq 3$.

Example 4. Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^3$. Then the only bishop-type move is $(0, 4) \to (0, 3)$, so that the roob options are $(0, 0), (0, 1), (0, 2)$. Bob may block off 2 of these positions, say $(0, 0), (0, 2)$. Then if Alice moves to $(0, 1)$ she will loose (since she may not block off (0,0)), so suppose rather that she moves to $(0, 3)$. Than she may not block off $(0, 2)$ so Bob moves $(0, 3) \to (0, 2)$ and blocks off $(0, 0)$. Hence $(0, 4)$ is a P-position.

Example 5. Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^{(3)}$. Alice cannot move to $(0, 0)$ or $(0, 2)$. But $(0, 1) \to (0, 0)$ is a 2-bishop-type option and $(0, 3) \to (0, 0)$ is a 3-rook-type option. This shows that $(0, 4)$ is a P-position.

Notice that, in comparison to Examples 4 and 5, the P-positions in the Examples 1 and 2 are distinct in spite the identical game constants ($m = p = 2$). On the other hand, the P-positions in Examples 1 and 3 coincide.

[4]One may think of the game as if this lower left rectangle is cut out from the game board.

Example 6. If the starting position is $(0,4)$ and the game is $2{\times}3\text{WN}_1$, then Alice cannot move so that Bob wins. If, on the other hand, the game is $2{\times}3\text{WN}_2$, the position $(0,2)$ is terminal and so Alice wins (by moving $(0,4) \to (0,2)$).

Suppose now that the starting position of $2{\times}3\text{WN}_2$ is $(1,8)$. Then, Alice may move to $(0,2)$. But if the starting position of $2{\times}3\text{WN}_0$ is $(1,7)$ Alice may not move to $(0,0)$ and hence Bob wins.

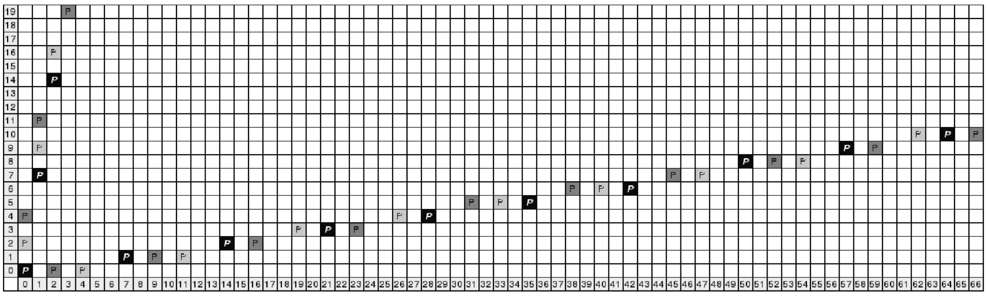

FIGURE 2. P-positions of $2\text{WN}^{(3)}$, 2WN^3, $2\text{WN}^{2,6}$ and $2{\times}3\text{WN}$—the positions nearest the origin such that there are precisely three positions in each row and column and one position in every second NE-SW-diagonal. The palest coloured squares represent P-positions of $2{\times}3\text{WN}_1$. They are of the form (a_{3n+1}, b_{3n+1}) or (b_{3n+2}, a_{3n+2}). The darkest squares, $(\{a^{2,3}_{3i}, b^{2,3}_{3i}\})$, represent the solution of 6WN.

$b^{2,3}_n$	0	2	4	7	9	11	14	16	19	21	23	26	28	31	33	35	38
$a^{2,3}_n$	0	0	0	1	1	1	2	2	3	3	3	4	4	5	5	5	6
$b_n - a_n$	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 2. Some initial values of the Beatty pairs defined in (1) and (2), here $m = 2$ and $p = 3$, together with the differences of their coordinates ($=2n$).

2.3. **Game theory results.** We may now state our main results. We prove them in Section 6, since our proofs depend on some arithmetic results presented in Section 3,4 and 5.

Theorem 2.1. Fix $m, p \in \mathbb{N}$ and let a and b be as in (1) and (2). Then

(i) $\mathcal{P}(m\text{WN}^p) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$;

(ii) (a) $\mathcal{P}(m\text{WN}^{(p)}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$ if and only if $\gcd(m,p) = 1$;

(b) $\mathcal{P}(m\text{WN}^{(m,mp)}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$;

(iii) (a) $\mathcal{P}(m{\times}p\text{WN}_l) = \{(a_{ip+l}, b_{ip+l}) \mid i \in \mathbb{N}_0\} \cup \{(b_{ip-l}, a_{ip-l}) \mid i \in \mathbb{N}\}$

(b) $\mathcal{P}(m{\times}p\text{WN}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$.

3. More on p-complementary Beatty sequences

As we have seen, it is customary to represent the solution of 'a removal game on two heaps' as a sequence of pairs of non-negative integers; or more precisely, as pairs of non-decreasing sequences of non-negative integers. This leads us to a certain extension of Beatty's original theorem, to (a pair of) p-complementary sequences.

In the literature there is a proof of this theorem in [Bry02], where K. O'Bryant uses generating functions (a method adapted from [BoBo93]). Here, we have chosen to include an elementary proof, in analogy to ideas presented in [HyOs27, Fra82].

Theorem 3.1 (O'Bryant)**.** Let $0 < \alpha < \beta$ be real numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Let $p \in \mathbb{N}$. Then we have that $(x_i) = (\lfloor \frac{i\alpha}{p} \rfloor)_{i \in \mathbb{N}_0}$ and $(y_i) = (\lfloor \frac{i\beta}{p} \rfloor)_{i \in \mathbb{N}}$ are p-complementary, that is, for each $n \in \mathbb{N}_0$,

$$p = \#\left\{ i \in \mathbb{N}_0 \mid n = \left\lfloor \frac{i\alpha}{p} \right\rfloor \right\} + \#\left\{ i \in \mathbb{N} \mid n = \left\lfloor \frac{i\beta}{p} \right\rfloor \right\}$$

if and only if α, β are irrational.

Proof. It suffices to establish that exactly p members of the set

$$S = \{0, \alpha, \beta, 2\alpha, 2\beta, \ldots\}$$

is in the interval $[n, n+1)$ for each $n \in \mathbb{N}_0$. But

$$\begin{aligned} \#(S \cap [0, N]) &= \#(\{0, \alpha, 2\alpha, \ldots\} \cap [0, N]) + \#(\{\beta, 2\beta, \ldots\} \cap [1, N]) \\ &= \lfloor pN/\alpha \rfloor + 1 + \lfloor pN/\beta \rfloor, \end{aligned}$$

and since

$$\begin{aligned} pN/\alpha + pN/\beta - 1 &< \lfloor pN/\alpha \rfloor + 1 + \lfloor pN/\beta \rfloor \\ &< pN/\alpha + pN/\beta + 1, \end{aligned}$$

we are done. □

The following result is a special case of the generalization of Beatty's theorem to non-homogeneous sequences in [Sko57, Fra69, Bry03] (so we omit a proof).

Proposition 3.2 (Skolem, Fraenkel)**.** With notation as in Theorem 3.1, for any integer $0 \leq l < p$, the sequences

$$(x_{pi+l}) \text{ and } (y_{pi-l})$$

are complementary. □

The next result is almost immediate by definition of a and b and by Theorem 3.1. It is central to the rest of the paper.

Lemma 3.3. Fix $m, p \in \mathbb{N}$ and let a and b be as in (1) and (2) respectively. Then for each $n \in \mathbb{N}_0$ we have that

(i) a and b are p-complementary;

(ii) $b_n - a_n = mn$;

(iii) if $p = 1$, then

  (a) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = m + 1$, or

  (b) $a_{n+1} - a_n = 2$ and $b_{n+1} - b_n = m + 2$;

(iv) if $p > 1$, then

  (a) $a_{n+1} - a_n = 0$ and $b_{n+1} - b_n = m$, or

  (b) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = m + 1$.

Proof. Since ϕ_x is irrational and $\frac{1}{\phi_x} + \frac{1}{\phi_x + x} = 1$, case (i) is immediate from Theorem 3.1.

For case (ii) put $\nu = \nu_{m,p} = \frac{\phi_{mp}}{p} + \frac{m}{2}$ and observe that

$$b_n - a_n = \left\lfloor n\left(\nu + \frac{m}{2}\right)\right\rfloor - \left\lfloor n(\nu - \frac{m}{2})\right\rfloor.$$

The result follows since

$$\left\lfloor \frac{nm}{2} \right\rfloor - \left\lfloor -\frac{nm}{2} \right\rfloor = \left\lfloor \frac{nm}{2} \right\rfloor + \left\lceil \frac{nm}{2} \right\rceil = mn$$

for all $n \in \mathbb{Z}$.

For case (iii), by [Fra82], we are done. In case $p > 1$, by the triangle inequality, we get

$$\begin{aligned} 0 &< \frac{\phi_{m,p}}{p} \\ &= \frac{1}{p} - \frac{m}{2} + \sqrt{\frac{m}{4} + \frac{1}{p^2}} \\ &< \frac{1}{p} + \frac{1}{p} \\ &\leq 1, \text{ whenever } p > 1, \end{aligned}$$

so that we may estimate

$$a_{n+1} - a_n = \left\lfloor \frac{(n+1)\phi_{mp}}{p} \right\rfloor - \left\lfloor \frac{n\phi_{mp}}{p} \right\rfloor \in \{0, 1\}.$$

Then by (ii) we have

$$\begin{aligned} b_{n+1} - b_n &= a_{n+1} + m(n+1) - a_n - mn \\ &= a_{n+1} - a_n + m, \end{aligned}$$

so that (iv) holds. □

4. A unique pair of p-complementary Beatty sequences

Suppose that, say (y_i), in Theorem 3.1, is strictly increasing. In this case, we may formulate certain 'uniqueness properties' for our pairs of p-complementary Beatty sequences (in case $p = 1$ see also [HeLa06] for extensive generalizations).

Theorem 4.1. Fix an integer $p > 1$. Suppose $x = (x_i) = (x_i)_{i \in \mathbb{N}_0}$ and $y = (y_i) = (y_i)_{i \in \mathbb{N}_0}$ are non-decreasing sequences of non-negative integers such that $x_0 = y_0 = 0$ and, for all n, $x_n \leq y_n$. Then the following items are equivalent:

(i) (x_i) and $(y_i)_{i>0}$ are p-complementary and there is an $m \in \mathbb{N}$ such that, for all n, $y_n - x_n = mn$;
(ii) (x_i) and $(y_i)_{>0}$ are p-complementary Beatty sequences and (y_i) is (strictly) increasing;
(iii) for some fixed $m \in \mathbb{N}$ and for all n, $x_n = a_n^{m,p}$ and $y_n = b_n^{m,p}$;

Proof. By Lemma 3.3 it is clear that (iii) implies (ii) and (i). Hence, it suffices to prove that (i) implies (iii) and (ii) implies (iii).

(i)⇒ (iii): Since x is non-decreasing the condition $y_n - x_n = mn$ clearly implies that y is increasing. Since $p > 1$, by this and by p-complementarity of x and y we get $x_1 - x_0 = 0$ and $y_1 - y_0 = m$. Suppose further that Lemma (3.3) (iv) holds for each of the n first entries of the sequences (x_i) (exchanged for (a_i)) and (y_i) (exchanged for (b_i)) respectively. Then, since these sequences are p-complementary and y is increasing, we get that $x_{n+1} - x_n = 0$ or $x_{n+1} - x_n = 1$ (otherwise the integer $x_n + 1$ would have at most one representation in the sequences x and y, a contradiction). By $y_n - x_n = mn$, we get that Lemma (3.3) (iv) is satisfied for x and y. But, by Lemma (3.3) (i) and (ii) the same inductive argument also holds for the sequences (a_i) and (b_i) (in the sense that $x_{n+1} - x_n = 0$ if and only if $a_{n+1} - a_n = 0$), so we are done.

(ii)⇒ (iii): for each $n \in \mathbb{N}_0$, the 'first difference' of a Beatty sequence $z = (z_i)$ is $z_{n+1} - z_n \in \{\delta(z), \Delta(z)\}$ for some non-negative integers $0 \leq \delta(z) < \Delta(z)$.

By the conditions in (ii) we get that $\delta(x) = 0$. Then if $\Delta(x) > 1$ we must have $\delta(y) = 0$ for otherwise the number of representations of 1 is strictly less then p, which contradicts our assumption, so we must have $\Delta(x) = 1$.

Clearly we may take $\delta(y) = m > 0$ so we must show that $\Delta(y) = m+1$. Suppose that $\Delta(y) > m+1$. Then me may estimate the number of Sturmian words of the successive differences for the sequence x. We already know that (iv a) or (iv b) holds for a Beatty sequence so that $S_x(p(m+1)-1) = p(m+1)$ whenever $\Delta(y) = m+1$, and where S_x is the function that counts the number of words of successive first differences of x of a given length. But exchanging $m+1$ for $m+r$ with $r > 1$ gives all the same words of length $p(m+1)$ and in addition it gives the word $\zeta\zeta\ldots\zeta\eta$ where $\zeta = 00\ldots01$ and $\eta = 00\ldots0$ (where the number of successive ζ:s are m and the number of successive 0:s are $p-1$). Then we get $S_x(p(m+1)-1) = p(m+1)+1$, which contradicts the assumption in (ii) that x is a Beatty sequence.

□

5. Recurrence results

We will next generalize the minimal exclusive algorithm in (4). Since our game rules are three-folded we will study three different recurrences. But first we would like to reveal some more structure of our sequences a and b.

Theorem 5.1. Fix $m, p \in \mathbb{N}$ and let a and b be as in (1) and (2). For each $n \in \mathbb{N}_0$, define

$$\varphi_n = \varphi_n^{m,p} := \frac{a_n + (mp-1)b_n}{m}.$$

Then, for each $n \in \mathbb{N}$, φ_n is the greatest integer such that

$$b_n - 1 = a_{\varphi_n}. \tag{6}$$

Proof. Notice that, for all n,

$$\begin{aligned}\varphi_n &= \frac{a_n + (mp-1)b_n}{m} \\ &= \frac{b_n - mn + (mp-1)b_n}{m} \\ &= \frac{mpb_n - mn}{m} \\ &= pb_n - n, \end{aligned} \tag{7}$$

so that

$$\begin{aligned}\varphi_{n+1} - \varphi_n &= pb_{n+1} - (n+1) - (pb_n - n) \\ &= p(b_{n+1} - b_n) - 1. \end{aligned} \tag{8}$$

For the base case, notice that $b_1 = m$, $a_1 = 0$ and $\varphi_1 = (mp-1)$. Recall that, for each $0 \leq j < m$, there are precisely p representative(s) from a and $b > 0$, (the only representative from b in this interval is $b_0 = 0$ which we by definition do not count). Hence, by $a_0 = 0$, we get that

$$a_{\varphi_1} = a_{mp-1} = m - 1 = b_1 - 1$$

and

$$a_{\varphi_1+1} = a_{mp} = m = b_1.$$

Suppose that (6) holds for all $i \leq n$. Then we need to show that $b_{n+1} - 1 = a_{\varphi_{n+1}}$ and $b_{n+1} = a_{\varphi_{n+1}+1}$.

In case $a_{\varphi_{n+1}} - a_{\varphi_n} = b_{n+1} - b_n$, by $b_n - 1 = a_{\varphi_n}$ and $b_n = a_{\varphi_n+1}$ we get the result, so let us investigate the remaining cases:

(A) $a_{\varphi_{n+1}} - a_{\varphi_n} < b_{n+1} - b_n$;
(B) $a_{\varphi_{n+1}} - a_{\varphi_n} > b_{n+1} - b_n$.

By p-complementarity, the number of representations from a and b in the interval

$$\begin{aligned} I_n &:= (a_{\varphi_n}, a_{\varphi_{n+1}}] \\ &= (a_{\varphi_n}, a_{\varphi_n + p(b_{n+1}-b_n)-1)}] \end{aligned}$$

is $R_n := p(a_{\varphi_{n+1}} - a_{\varphi_n})$, and where the equality is by (8). By assumption, $a_{\varphi_n+1} \in I_n$ so that we have at least $p(b_{n+1} - b_n) - 1$ representations from a in I_n. But also $b_n = a_{\varphi_n} + 1 \in I_n$ so that altogether we have at least $p(b_{n+1} - b_n)$ representations in I_n. Hence

$$\begin{aligned} p(b_{n+1} - b_n) &\leq R_n \\ &= p(a_{\varphi_{n+1}} - a_{\varphi_n}) \end{aligned}$$

which rules out case (A).

Notice that case (B) implies that b_{n+1} lies in I_n so that $a_{\varphi_n+1} = b_n < b_{n+1} \leq a_{\varphi_{n+1}}$. Since both b_n and b_{n+1} lie in I_n we get

$$
\begin{aligned}
2 + \varphi_{n+1} - \varphi_n &= p(b_{n+1} - b_n) + 1 \\
&\leq p(a_{\varphi_{n+1}} - (a_{\varphi_n} + 1)) + 1 \\
&= p(a_{\varphi_{n+1}} - a_{\varphi_n} + 1) - 2p + 1. \qquad (9)
\end{aligned}
$$

By Lemma 3.3 and our assumption it is obvious that $b_{n+2} > a_{\varphi_{n+1}}$. If in addition $a_{\varphi_{n+1}+1} > a_{\varphi_{n+1}}$ we are done, since $p > 0$ together with (9) and p-complementarity give that there is at least one representative to little in I_n.

If on the other hand $a_{\varphi_{n+1}+1} = a_{\varphi_{n+1}}$ this forces $m > 1$ which together with (9) implies that there are two representatives to little, unless also $a_{\varphi_{n+1}+2} = a_{\varphi_{n+1}}$. But this forces $p > 2$ which in its turn implies that there are at least three representatives missing, and so on. $\square$

Remark 3. For arbitrary $m > 0$ and $p = 1$ it is well known that a and b solve $x_{y_n} = x_n + y_n$. This complementary equation is studied in for example [Conn59, FrKi94, Kim07]. However, we have not been able to find any references for the complementary equation $y_n - 1 = x_{y_n - n}$ (by (7), for the cases $p = 1$, a solution is given by $a = x$ and $b = y$).

For the first of our recursive characterizations, we introduce another notation. A *multiset* (or a sequence) X may be represented as (another) sequence of non-negative integers $(\xi^i)_{i \in \mathbb{N}_0}$, where, for each $i \in \mathbb{N}_0$, $\xi^i = \xi^i(X)$ counts the number of occurrences of i in X. For a positive integer p, let $\text{mex}^p(\xi^i)$ denote the least non-negative integer $i \in (\xi^i)$ such that $\xi^i < p$.

Proposition 5.2. Let $m > 0$ and $p \geq 1$ be integers. Then the recursive characterizations (i), (ii) and (iii) are equivalent. In fact, for each $n \in \mathbb{N}_0$, $x_n = a_n^{m,p}$ and $y_n = b_n^{m,p}$ with notation as in (1) and (2).

(i) For $n \geq 0$,

$$x_n = \text{mex}^p(\xi_n^i),$$

where ξ_n is the multiset, where for each $i \in \mathbb{N}_0$,

$$
\begin{aligned}
\xi_n^i &= \#\{j \mid i = x_j \text{ or } i = y_j, 0 \leq j < n\}, \\
y_n &= x_n + mn.
\end{aligned}
$$

(ii) For $n \geq 0$,

$$
\begin{aligned}
& x_n = \text{mex}\{\nu_i^n, \mu_i^n \mid 0 \leq i < n\}, \text{ where} \\
& \nu_i^n = x_i \text{ if } n \equiv i \pmod p, \text{ else } \nu_i^n = \infty, \\
& \mu_i^n = y_i \text{ if } n \equiv -i \pmod p, \text{ else } \mu_i^n = \infty; \\
& y_n = x_n + mn.
\end{aligned}
$$

(iii) For $n \geq 0$ and for each $0 < l < p$,

$$
\begin{aligned}
x_{pn} &= \text{mex}\{x_{pi}, y_{pi} \mid 0 \leq i < n\}, \\
y_{pn} &= x_{pn} + mpn, \\
x_{pn+l} &= \text{mex}\{x_{pi+l}, y_{p(i+1)-l} \mid 0 \leq i < n\}, \\
y_{pn+l} &= x_{pn+l} + m(pn + l).
\end{aligned}
$$

Proof. For $p = 1$ each recurrence is equivalent to (4). Hence let $p > 1$ and, for $x \in \mathbb{Z}$, let $\overline{x}$ denote the congruence class of x modulo p. For each recurrence it is straightforward to check that $(x_i, y_i) = (a_i, b_i) = (0, mi)$ if $0 \le i < p$. Otherwise, by each definition of mex, we must at least have $x_i > 0$.

For case (i), by Theorem 4.1 and by $y_n = x_n + mn$, it suffices to prove that (x_i) is non-decreasing and that (x_i) and (y_i) are p-complementary. But this is immediate by the definition of mex^p.

For case (ii), notice that, for $n \in \mathbb{N}_0$, (see the proof of Theorem 5.1) we have

$$\varphi_n = pb_n - n \equiv -n \pmod{p}. \tag{10}$$

If the assertion does not hold then there is a least $n \ge p$, say n', such that $x_{n'} \ne a_{n'}$. Hence, we have two cases to consider.

(a) $r := x_{n'} < a_{n'}$: By Theorem 3.2 there are two cases to consider.

Case 1: There is an $i \ge 0$ such that $\varphi(i) + p - 1 < n'$ and

$$y_i = x_{\varphi(i)+1} = x_{\varphi(i)+2} = \ldots = x_{\varphi(i)+p-1} = r.$$

But then, by

$$\{\, \overline{-i},\ \overline{-i+1},\ \ldots,\ \overline{-i+p-1}\,\} = \{\, \overline{0},\ \overline{1},\ \ldots,\ \overline{p-1}\,\} \tag{11}$$

and (10), there is a $j \in \{i, \varphi(i)+1, \ldots, \varphi(i)+p-1\}$ such that either $n' \equiv j \pmod{p}$ and $j \in \{\varphi(i)+1, \ldots \varphi(i)+p-1\}$ which implies $\nu_j^{n'} = r$, or $n' \equiv -j \pmod{p}$ and $j = i$ which implies $\mu_j^{n'} = r$. In either case the choice of $x_{n'} = r$ contradicts the definition of mex.

Case 2: There is an $i \ge 0$ such that $i + p - 1 < n'$ and

$$r = x_i = x_{i+1} = x_{i+2} = \ldots = x_{i+p-1}.$$

This case is similar but simpler, since for this case we rather use that

$$\{\, \overline{i},\ \overline{i+1},\ \ldots,\ \overline{i+p-1}\,\} = \{\, \overline{0},\ \overline{1},\ \ldots,\ \overline{p-1}\,\} \tag{12}$$

(b) $r := a_{n'} < x_{n'}$: Then our mex-algorithm has refused r as the choice for $x_{n'}$. But then there must be an indice $0 \le j < n'$ such that either $\nu_j^{n'} = r$ or $\mu_j^{n'} = r$. Hence, we get to consider two cases.

Case 1: $\overline{j} = \overline{n'}$ and $r = x_j$. On the one hand, there is a $p \in \mathbb{N}$ such that $pm + j = n'$ On the other hand, there is a greatest $p' \in \mathbb{N}$ such that $a_{n'-p'} = a_{n'-p'+1} = \ldots = a_{n'}$ and by p-complementarity $0 \le p' < p$. But then, since $n' - p' > n' - pm = j$, we get $a_j < r = x_j$, which contradicts the minimality of n'.

Case 2: $\overline{-j} = \overline{n'}$ and $r = y_j$. Then by Theorem 3.2, $\varphi_j + 1$ is the least indice such that $a_{\varphi_j+1} = a_{n'}$. Then, since (by minimality of n') Theorem 3.2 gives $a_{n'} = b_j$, by p-complementarity we get $n' - (\varphi_j + 1) + 1 \le p - 1$. Then $0 < p' := n' - \varphi_j < p$ and so

$$\overline{-j + p'} = \overline{\varphi(j) + p'} = \overline{n'} = \overline{-j},$$

which is nonsense.

For case (iii), suppose that there is a least indice $n' \geq p$ such that $a_{n'} \neq x_{n'}$. Clearly, there exist unique integers, say t and $0 \leq l < p$, such that $tp + l = n'$.

Suppose that $r := a_{n'} > x_{n'}$. Then, since the mex-algorithm did not choose $x_{n'} = r$, there must be an indice $0 \leq t' < t$ such that either $x_{t'p+l} = r$ or $y_{(t'+1)p-l} = r$. But then, by assumption, either $a_{t'p+l} = x_{t'p+l} = a_{tp+l}$ or $b_{(t'+1)p-l} = y_{(t'+1)p-l} = b_{(t+1)p-l} =$. By Proposition 3.2 both cases are ridiculous so we may assume $a_{n'} \leq x_{n'}$.

If $a_{n'} < x_{n'}$, by Proposition 3.2, there is an indice $0 \leq t' < t$ such that either $a_{t'p+l} = x_{n'}$ or $b_{(t'+1)p-l} = x_{n'}$. But this contradicts the mex-algorithm's choice of $x_{n'} < a_{n'}$. Hence, we get $a_{n'} = x_{n'}$. □

6. Solving our games

Proof of Theorem 2.1. For $p = 1$, the games have identical rules. This case has been established in [Fra82]. The case $m = 1$ has been studied in [Con59] for games of form (ii). (and implicitly for $1{\times}p\mathrm{WN}_l$).

For the rest of the proof assume that $p > 1$. Let us first explain the 'only if' direction of (ii)(a). Denote with $\gamma = \gcd(m, p)$, $p' = p/\gamma$ and $m' = m/\gamma$. Then the positions of the form $(0, mi)$, where $0 \leq i < p'$, are P-positions of $m\mathrm{WN}^{(p)}$. Now, $(0, mp')$ is an N-position because $m'p = mp'$ implies $(0, p'm) \to (0, 0)$. But, by definition, $b_{p'} = mp'$ if $p' < p$ which holds if and only if $\gamma > 1$.

For each game (we need another notation for Case (iii)), we need to prove that, if (x, y)

(A) is of the form $\{a_i, b_i\}$, then none of its options is;
(B) is not of the form $\{a_i, b_i\}$, then there is an option of this form.

By symmetry, we may assume that $0 \leq x \leq y$. Clearly, for our games in (i) and (ii), the final position $(x, y) = (0, 0)$ satisfies (A) but not (B). Hence for these games assume $y > 0$ (and so $i > 0$ for case (A)).

Case (i): Suppose $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{N}_0$. By Lemma 3.3 (i) and (ii), a and b are p-complementary and $b_i - b_j \geq m$ for all $j < i$. Then any roob-type option may be blocked off, unless perhaps $a_j < a_i$ and $b_j = b_i$ for some $j < i$. But this is ridiculous since b is strictly increasing. By Lemma 3.3 (ii) we get that, for $i > j$, $b_i - a_i \pm (b_j - a_j) \geq m$. Then an m-bishop cannot move $(a_i, b_i) \to \{a_j, b_j\}$, This proves (A).

For (B), since $p \geq 2$ and b is strictly increasing, we may assume $x = a_i$ for some i. Then, by Lemma 3.3 (iv): (*) There exists a $j < i$ such that an m-bishop can move $(x, y) \to (a_j, b_j)$ (and this move is not a roob-type move) unless $y - x - (b_j - a_j) \geq m$ for all j such that $a_j \leq x$. But then, since $y \geq x + (m + 1)j > b_j$ for all j such that $a_j = x$, by Lemma 3.3 (i), the previous player cannot block off all p roob-type options of the form $\{a_i, b_i\}$.

Case (iia): For this game, the options of the m-bishop are identical to those in (i). Let us analyze the p-rook.

Hence, suppose $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{N}_0$ and that a p-rook can move to $\{a_j, b_j\}$. Then, since b is strictly increasing, there is a $0 \leq j < i$, such that either $b_i \equiv b_j \pmod p$ and $a_i = a_j$, or $b_i \equiv a_j \pmod p$ and $a_i = b_j$. But then, for the first case, since

$\overline{mj} = \overline{b_j - a_j} = \overline{b_i - a_i} = \overline{mi}$ and $\gcd(m,p) = 1$ we must have $\bar{j} = \bar{i}$. this is ridiculous, since by p-complementarity we have $0 < i - j < p$. For the second case, by Theorem 5.1, we have that

$$\overline{-mj} = \overline{a_j - b_j} = \overline{b_i - a_i} = \overline{mi} = \overline{m(\varphi(j) + t)} = \overline{m(-j + t)},$$

for some $t \in \{1, \ldots, p-1\}$. This implies $\overline{0} = \overline{mt}$ but then again $\gcd(m,p) = 1$ gives a contradiction.

For (B), we follow the ideas in the second part of Case (i) up until (*). Then, for this game, we rather need to show that there is a j such that $y \equiv b_j \pmod p$ and $a_j = x$ or $y \equiv a_j \pmod p$ and $b_j = x$. But this follows directly from the proof of Proposition 5.2 (ii)(a).

Case (iib): Suppose $(x,y) = (a_i, b_i)$ for some $i \in \mathbb{N}_0$ but the (m,mp)-rook can move to some $\{a_j, b_j\}$ (where $j < i$). Then, we have two cases:

Case 1: $b_i \equiv b_j - r \pmod{mp}$ and $a_i = a_j$, for some $r \in \{0, 1, \ldots, m-1\}$. Then $b_i - a_i \equiv b_j - a_j - r \pmod{mp}$ so that $mi \equiv mj - r \pmod{mp}$ and so $m(i-j) \equiv -r \pmod{mp}$. But this forces $r = 0$ and $i - j \equiv 0 \pmod p$ which is impossible since Lemma 3.3 (i) and (iv) implies $i - j \in \{1, 2, \ldots, p-1\}$.

Case 2: $b_i \equiv a_j - r \pmod{mp}$ and $a_i = b_j$, for some $r \in \{0, 1, \ldots, m-1\}$. Then $b_i - a_i \equiv a_j - b_j - r \pmod{mp}$ so that $mi \equiv -mj - r \pmod{mp}$ and so $m(i+j) \equiv -r \pmod{mp}$. By Theorem 5.1 we have that $i = \varphi(j) + s$ for some $s \in \{1, 2, \ldots, p-1\}$. Further, by (10), we have $\varphi(j) \equiv -j \pmod p$, so that $m(\varphi(j) + s + j) = ms \equiv -r \pmod{mp}$. Once again we have reached a contradiction.

For (B), in analogy with (*), it suffices to study the (m,mp)-rook's options where y is such that $y - x - (b_j - a_j) \geq m$ for all j such that $a_j \leq x = a_i$. Hence, we need to show that there are a j and an $r \in \{0, 1, \ldots m-1\}$ such that

$$y \equiv b_j - r \pmod{mp} \quad \text{and} \quad a_j = x,$$

or

$$y \equiv a_j - r \pmod{mp} \quad \text{and} \quad b_j = x.$$

Clearly, we may choose r such that $y - x + r \equiv 0 \pmod m$. Then, for all j, we get $ms := y - x + r \equiv b_j \pm a_j \pmod m$. Hence, it suffices to find a specific j such that

$$j = \frac{b_j - a_j}{m} \equiv s \pmod p \quad \text{and} \quad a_j = x,$$

or

$$-j = \frac{a_j - b_j}{m} \equiv s \pmod p \quad \text{and} \quad b_j = x.$$

But then, by (11) and (12), we are done.

Case (iiia): We may assume that $l > 0$. We have already seen that $(a'_i) := (a_{pi+l})_{i \geq 0}$ and $(b'_i) := (b_{p(i+1)-l})_{i \geq 0}$ are complementary. Our proof will be a straightforward extension of those in [Fra82] (which deals with the case $l = 0$) and [Con59] (which implicitly deals with the case $m = 0$). Observe that $a'_0 = a_l = 0$ and $b'_0 = b_{p-l} = m(p-l)$.

For (A), let $(x,y) = (a_i, b_i)$. In case $i = 0$ (by Definition 3 (3a)), the Queen has no options at all, so assume $i > 0$. Proposition 5.2

(iii) gives that $b'_i - a'_i \pm (b'_j - a'_j) \geq mp$ for all $0 \leq j < i$. Then the mp-bishop cannot move $(x,y) \to (a'_j, b'_j)$ for any $0 \leq j < i$. Since a' and b' are complementary there is no rook-type option $(a'_i, b'_i) \to \{a'_j, b'_j\}$.

For (B), we adjust the statement (*) accordingly: Suppose $x = a'_i$ (and $y \geq b'_0$). By Proposition 5.2 (iii): If the mp-bishop cannot move to (a'_j, b'_j) for any $j < i$ we get that either $i = 0$ or $y - x - (b'_j - a'_j) \geq mp$ for all $j < i$.

But, if $i = 0$ there is a rook-type option to (a'_0, b'_0) (recall here $y > b'_0$), so suppose $i > 0$. But then, since, by Proposition 5.2 (iii), both a' and b' are increasing we get $y \geq b'_j + mp + x - a'_j \geq b'_i + a'_i - a'_j > b'_i$. Hence, for this case, the rook-type move $(x,y) \to (a'_i, b'_i)$ suffices. Suppose on the other hand that $x = b'_i$ with $i \geq 0$. Then, since $y \geq x = b'_i > a'_i$, by complementarity, the Queen may move $(x,y) \to (b'_i, a'_i)$.

Case (iiib): Suppose that the starting position is (a_i, b_i). Then $i = pj + l'$ for some (unique) pair $j \in \mathbb{N}_0$ and $0 \leq l' < p$. The second player should choose $l = l'$. If, on the other hand, the starting position is (b_i, a_i). Then $i = pj - l'$ for some (unique) pair $j \in \mathbb{N}$ and $0 < l' \leq p$. The second player should choose $l = p - l'$. In either case, by Case (iiia), there is no option of the form (a'_i, b'_i).

If the (x,y) is not of this form, again, by Case (iiia), for any (choice of) $0 \leq l < p$, there is a move $(x,y) \to \{a'_i, b'_i\}$ for some $i \geq 0$.

□

7. Questions

Can one find a polynomial time solution of $m\mathrm{WN}^{(l,p)}$ for some integers $l \geq 0$, $m > 0$ and $p > 0$ whenever

- $\gcd(m,p) \neq 1$ and $l = 0$, or
- $0 < l \neq m$ or $m \nmid p$?

If this turns out to be complicated, can one at least say something about its asymptotic behaviour?

Denote the solution of $m\mathrm{WN}^{(l,p)}$ with $\{\{c_i^{(l,m,p)}, d_i^{(l,m,p)}\}\}_{i\in\mathbb{N}_0}$. Let us finish off with two tables of the initial P-positions of such games.

$d_n^{(0,2,2)}$	0	3	6	9	12	15	19	22	25	28	31	34	37	40	43	46	49
$c_n^{(0,2,2)}$	0	0	1	1	2	2	3	4	4	5	5	6	7	7	8	8	9
$d_n - c_n$	0	3	5	8	10	13	16	18	21	23	26	28	30	33	35	38	40
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 3. The first few P-positions of $2\mathrm{WN}^2$ together with the respective differences of their coordinates.

From these tables one may conclude that: The infinite arithmetic progressions of the sequences

$$(b_i^{m,p} - a_i^{m,p})_{i\in\mathbb{N}_0} = (mi)_{i\in\mathbb{N}_0}$$

$d_n^{(1,2,3)}$	0	2	5	7	11	14	16	19	21	26	29	31	36	39	41	44	46
$c_n^{(1,2,3)}$	0	0	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9
$d_n - c_n$	0	2	4	6	9	11	13	15	17	21	23	25	29	31	33	35	37
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 4. The first few P-positions of $2\mathrm{WN}^{(1,3)}$. Notice that (as in Table 3) the successive differences of their coordinates are not in arithmetic progression.

(see also Table 2) are not in general seen among the sequences

$$(d_i^{(l,m,p)} - c_i^{(l,m,p)})_{i\in\mathbb{N}_0}.$$

We believe that the latter sequence is an arithmetic progression if and only if none of the items in our above question is satisfied. We also believe that, for arbitrary constants, $(c_i^{(l,m,p)})$ and $(d_i^{(l,m,p)})_{>0}$ are p-complementary. But the solution of these questions are left for some future work.

Remark 4. We may also define generalizations of $m\mathrm{WN}^p$ and $m \times p\mathrm{WN}_l$:

Fix $l \in \mathbb{N}$. Let $m\mathrm{WN}_l^p$ be as $m\mathrm{WN}^p$ but where the player may only block off *l-roob-type options* (recall, non-l-bishop options). Otherwise, the Queen moves as the m-bishop or the rook. Then obviously $m\mathrm{WN}_m^p = m\mathrm{WN}^p$.

Let $u, v \in \mathbb{N}$ and let $m \times p\mathrm{WN}_{u,v}$ be as $m \times p\mathrm{WN}_l$, but the removed (lower left) rectangle has base u and hight v. Then for this game the final positions are $(u, 0)$ and $(0, v)$. If $l > 0$, $u = ml$ and $v = m(p - l)$ we get $m \times p\mathrm{WN}_{lm,m(p-l)} = m \times p\mathrm{WN}_l$.

We may ask questions in analogy to the above for these variations. For example, we have found a minimal exclusive algorithm satisfying $\mathcal{P}(m\mathrm{WN}_1^p)$ which is related to a polynomial time construction in [Fra98]. Is there an analog polynomial time construction for $\mathcal{P}(m\mathrm{WN}_1^p)$? Another question is if any of these further generalized games conincide via identical set of P-positions? But all this is left for future investigations.

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E-mail address: `urban.larsson@chalmers.se`

Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden